An Equation for the Moon's Crescent

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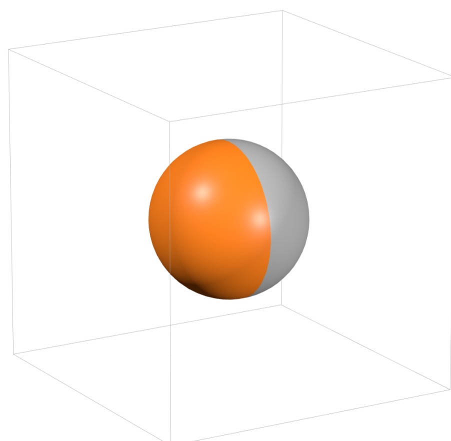
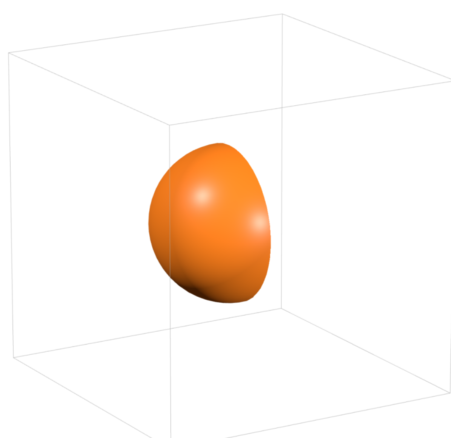
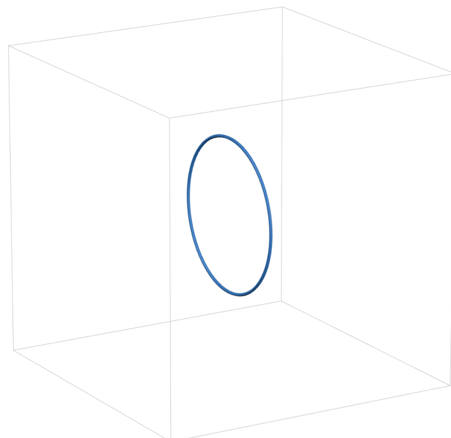
The Moon is evenly lit by the sun.



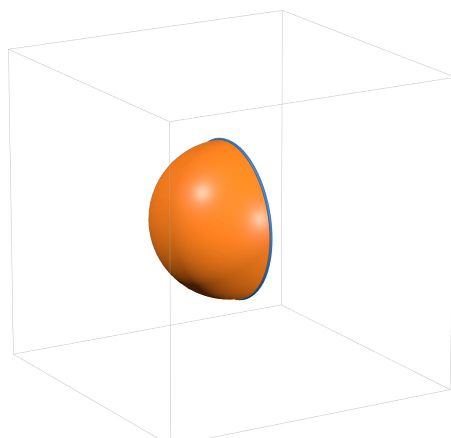
If the Moon is a perfect sphere, then the illuminated portion would be a semi-sphere.



And the boundary between its illuminated and dark portions would be a ring.



The crescent we see is the area between this boundary ring and the edge of the Moon.



The edges of the Moon (assuming it's a perfect sphere) are always a perfect circle, whose equation is well known:

$$f(x) = x^2 + y^2 = r^2$$

The difficult part is finding the equation for the boundary ring as perceived two dimensionally.

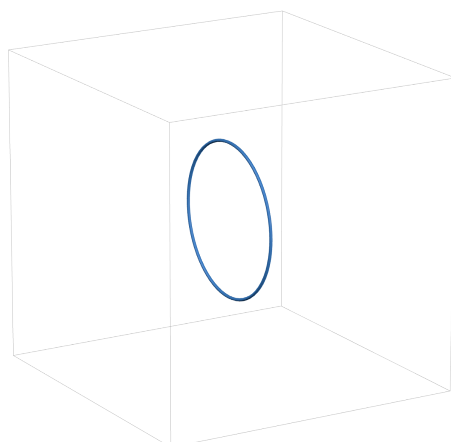
$$f(x) = ?$$

We can start by modeling the ring as a perfect circle in 3D space. The parametric vector expression for this is

$$f(t) = \begin{bmatrix} 0 \\ \cos(t) \\ \sin(t) \end{bmatrix}$$

$$0 \leq t \leq 2\pi$$

You can follow along using desmos' 3D calculator at https://www.desmos.com/3d using formats like $P = (0, \cos(t), \sin(t))$

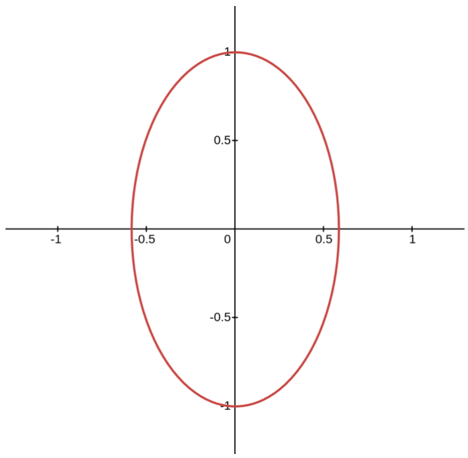


To rotate $f(t)$ we can multiply it by the y-axis rotation matrix $R_y(\theta)$ where θ is the angle we want to rotate it by

$$R_y(\theta)f(t) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ \cos(t) \\ \sin(t) \end{bmatrix} = \begin{bmatrix} \sin(t)\sin(\theta) \\ \cos(t) \\ \sin(t)\cos(\theta) \end{bmatrix}$$

To project this to 2D we can just set the z component to zero and remove it.

$$R_y(\theta)f(t) = \begin{bmatrix} \sin(t)\sin(\theta) \\ \cos(t) \\ \sin(t)\cos(\theta) \end{bmatrix} \to f_{2D}(t) = \begin{bmatrix} \sin(t)\sin(\theta) \\ \cos(t) \end{bmatrix}$$



But we aren't done yet because we want to convert our parametric function $f_{2D}(t)$ to a cartesian function $f(x)$. We can do that with some algebra and trig, but first let's rotate $f_{2D}(t)$ 90 degrees first to avoid absolute value problems later

$$f_{2D}(t) = \begin{bmatrix} \sin(t)\sin(\theta) \\ \cos(t) \end{bmatrix} \to f_R(t) = \begin{bmatrix} \cos(t) \\ \sin(t)\sin(\theta) \end{bmatrix}$$

$$x = \cos(t)$$

$$y = \sin(t)\sin(\theta)$$

$$\frac{y}{\sin(\theta)} = \sin(t)$$

$$x^2 + \frac{y^2}{\sin^2(\theta)} = \cos^2(t) + \sin^2(t)$$

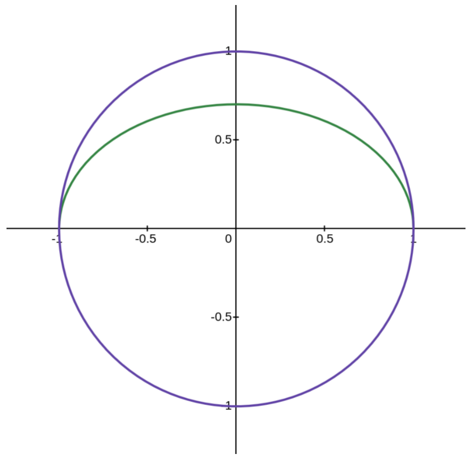
$$x^2 + \frac{y^2}{\sin^2(\theta)} = 1$$

$$x^2\sin^2(\theta) + y^2 = \sin^2(\theta)$$

$$y^2 = \sin^2(\theta) - x^2\sin^2(\theta)$$

$$f(x) = y = \sqrt{\sin^2(\theta) - x^2\sin^2(\theta)}$$

Inscribing this inside the unit circle we get the Moon's perfect crescent!



Latex code for this project can be found at
https://www.overleaf.com/read/ybhwzyqwgfvz#294511.