09 APRIL 2025

QF605 Fixed Income

PROJECT REPORT

Ishani Maitra (01506460)
Muhammad Saqif Bin Juhaimee (01495693)
Lim Fangyi (01340032)
Hong Yang (01502068)
Phoo Pyae Hsu Myat (01520414)
Nandini Agarwal (01518145)
Dragos Horge (01507705)
Nguyen Thanh Binh (01520953)

Part I - Bootstrapping Swap Curve:

TABLE 1: Bootstrapping formulas

Bootstrapping principle		$PV_{fix} = PV_{flt}$
OIS	For 6 months and 1 year	$D_o(0{,}0.5) \times 0.5 \times K = D_o(0{,}0.5)\left[\left(1+\frac{f_0}{360}\right)^{180}-1\right]$ $D_o(0{,}1) \times K = D_o(0{,}1)\left[\left(1+\frac{f_0}{360}\right)^{180}\left(1+\frac{f_1}{360}\right)^{180}-1\right]$ Solve for the forward rates and then the discount factors
	Bootstrapping	$\prod_{i=1}^{N}\left(1 + \Delta_{i-1} f_{0(T_{i-1},T_i)}\right) = 1 + \Delta S_0$
LIBOR	For 6 months and 1 year	$DF(0.5) = \frac{1}{(1 + R_1 * 0.5)}$ $L(6m, 1y) = \frac{\left(R_2 * \left(DF^{OIS(6m)} + DF^{OIS(1y)}\right) - R_1 * DF^{OIS(6m)}\right)}{DF^{OIS(1y)}}$ $DF^{LIBOR(1y)} = \frac{DF^{LIBOR(6m)}}{\left(1 + \frac{1}{2} * L(6m, 1y)\right)}$
	Bootstrapping	$K \sum_{i=1}^{n} \Delta_{i-1} D(0, T_i) = \sum_{i=1}^{n} D(0, T_i) \Delta_{i-1} L(T_{i-1}, T_i)$

The OIS bootstrapping process begins by determining discount factors for initial maturities (e.g., 6 months and 1 year) by finding the respective forward rates. For further steps we iteratively solved for the OIS discount factor that equated the present values of the fixed and floating legs of the swap, ensuring the swap has zero value at inception, which aligns with standard market pricing conventions for OIS swaps. Using the Brentq methoditeratively solve for discount factors that align theoretical swap prices with market rates. Once the curve is built (Fig 1), we apply linear interpolation to estimate intermediate discount factors (e.g., at 0.5 intervals) and compute forward rates, reflecting market expectations of future overnight rates.

Similarly, for LIBOR bootstrapping, we also began by calculating the initial discount factors D(0,6m) and D(0,1y). We follow the same core principle as in OIS: equating the present value of the fixed and floating legs (PV fixed = PV float). The fixed leg consists of periodic collateralized payments at the market IRS rate and discounted using OIS discount factors. The floating leg pays LIBOR, discounted by the LIBOR discount factors. As with OIS, we used the Brentq method to iteratively solve for the discount factors. A continuous LIBOR discount curve was created via interpolation at 0.5-year intervals (Fig. 2). Forward LIBOR rates were calculated based on the derived LIBOR discount factors.

$$S\left(T_i, T_{\{i+m\}}\right) = \frac{\left(0.5 * \sum_{n=i+1}^{i+m} D_{o(0,T_n)} * L\left(T_{\{n-0.5\}}, T_n\right)\right)}{\left(0.5 * \sum_{n=i+1}^{i+m} D_{o(0,T_n)}\right)}$$

We calculated the par swap rates (Table 2). We used the bootstrapped OIS discount factors to discount all cash flows. LIBOR forward rates were used for the floating leg payment calculation. Rates generally rise with longer tenors, reflecting an upward-sloping yield curve and higher interest rate expectations. For example, the 1Y-1Y rate is 3.20%, increasing to 3.84% for 1Y-10Y, and up to 5.35% for 10Y-10Y. This pattern aligns with typical market behaviour, where longer durations carry higher yields due to greater risk and uncertainty.

Table 2: Par Swap Rates

	1Y	2Y	3Y	5Y	10Y
1Y	0.032007	0.033259	0.034011	0.035255	0.038428
5Y	0.039274	0.040075	0.040072	0.041093	0.043634
10Y	0.042189	0.043116	0.044097	0.046249	0.053458

OIS and LIBOR discount factors decline with tenor, reflecting the time value of money. The OIS curve falls more gradually, as it represents near risk-free, collateralized rates. Meanwhile, the steeper LIBOR curve reflects credit and liquidity risk. This divergence grows with maturity, as LIBOR includes risk premiums that compound over time. Since the 2008 crisis, OIS has become the standard for discounting in collateralized markets. At short tenors, differences between the curves are minimal, but at longer tenors, LIBOR discounting significantly lowers present values compared to OIS

OIS DISCOUNT CURVE

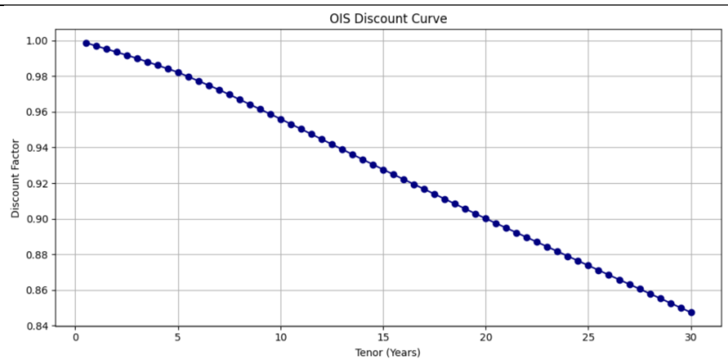

Fig 1: OIS discount curve

LIBOR DISCOUNT CURVE

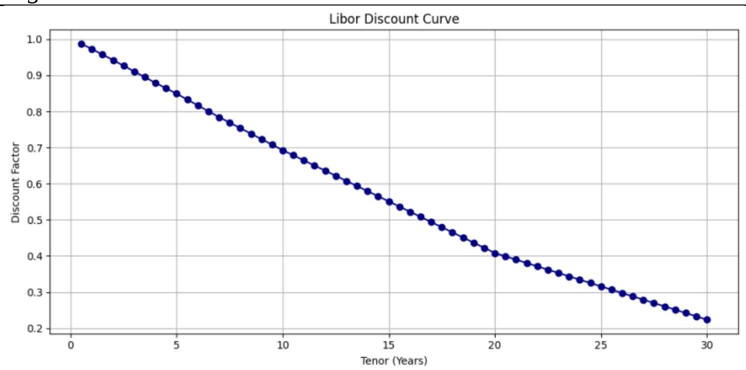

Fig 2: LIBOR discount curve

Part II - Swaption Calibration

Implied volatility is a key parameter for option pricing. The classic Black-Scholes model, assumes a lognormal distribution for the underlying asset price. Thus it prices under the assumption that implied volatility is constant which is inconsistent with the market behaviour. Hence, we require a more robust model that can capture the asymmetry and curvature in a more skewed market. In this analysis, we used displaced diffusion and the SABR model, which were calibrated to the discount factors and forward swap rates generated in part 1.

The displaced diffusion and SABR calibration estimates the model parameters (Table 3) by matching model prices to observed at-the-money implied volatilities for swaptions with the same expiry and tenor but varying strikes. Swaptions are priced using the modified Black76 formula with PVBP as the numeraire, and the corresponding implied volatilities are computed. A least squares approach minimizes the difference between observed and model volatilities for an efficient and robust fit. After calibration, we obtained the implied volatility values for each forward swap to plot the implied volatility smiles (Fig 3&4).

Table 3: Displaced Diffusion (β, σ) and SABR (α , ρ, ν) parameters

		β			
Expiry	1Y	2Y	3Y	5Y	10Y
1Y	1.92E-07	2.60E-07	4.64E-12	0.000004	9.48E-07
5Y	4.96E-12	3.08E-07	1.97E-07	0.000194	5.00E-02
10Y	5.12E-08	4.39E-08	1.23E-07	0.0004	5.23E-04
		σ			
Expiry	1Y	2Y	3Y	5Y	10Y
1Y	0.225	0.2872	0.2978	0.2607	0.2447
5Y	0.2726	0.2983	0.2998	0.266	0.2451
10Y	0.2854	0.2928	0.294	0.2674	0.2437
		α			
Expiry	1Y	2Y	3Y	5Y	10Y
1Y	0.139063	0.184647	0.19685	0.178034	0.171129
5Y	0.166385	0.19833	0.208519	0.189092	0.170995
10Y	0.176344	0.189978	0.197569	0.182759	0.164204
		ρ			
Expiry	1Y	2Y	3Y	5Y	10Y
1Y	-0.63329	-0.52512	-0.48284	-0.41435	-0.26479
5Y	-0.58429	-0.54176	-0.54204	-0.49922	-0.37294
10Y	-0.54135	-0.52757	-0.52421	-0.49427	-0.41753
		ν			
Expiry	1Y	2Y	3Y	5Y	10Y
1Y	2.049661	1.677428	1.438146	1.065046	0.777632
5Y	1.338307	1.058461	0.934483	0.677309	0.531441
10Y	1.002215	0.910132	0.848254	0.705636	0.600422

The β parameter in the Displaced Diffusion model adds flexibility by linking forward rates to volatility non-linearly, enabling it to capture implied volatility smiles or skews—unlike the standard Black model. When β is near 1, the model behaves more like a lognormal model, emphasizing relative price changes and enhancing the curve's shape, especially for deep in- or out-of-the-money swaptions. Shorter-tenor swaptions are particularly sensitive to this effect. β is calibrated to align model-implied volatilities with market data by minimizing errors across strikes. This results in more accurate pricing and risk management, especially in markets with noticeable skew or smile patterns.

In the SABR model for swaptions, the alpha (α) parameter sets the overall level of the implied volatility. A higher α raises implied volatilities across all strikes, though it is the interplay with the other parameters that ultimately shapes the smile. The rho (ρ) parameter reflects the correlation between the underlying rate and its volatility: if $\rho < 0$, there is a steeper left tail (fat left tail) and flatter right tail, making deep in-the-money swaptions more expensive relative to at-the-money; if $\rho > 0$, the right tail is steeper (fat right tail), favoring out-of-the-money swaptions. Finally, the nu (ν) parameter, or volatility of volatility, governs how pronounced the smile's curvature and tails are. As ν increases, implied volatility rises for both in-the-money and out-of-the-money swaptions, resulting in a more pronounced smile and greater sensitivity to market uncertainty.

Using the calibrated parameters for displaced diffusion and SABR model we price 2 swaptions (Table 4) - payer (2year X 8 year) and receiver(8year X 10Y) at the implied volatility strikes of 1%, 2%, 3%, 4%, 5%, 6%, 7% and 8%

Table 4: Swaption prices using Displace Diffusion (DD) and SABR

Option Type	1%	2%	3%	4%	5%	6%	7%	8%
Payer (2y x 10y) DD	0.2881	0.1949	0.1123	0.0513	0.0174	0.0041	0.0006	0.0001
Payer (2y x 10y) SABR	0.2896	0.1983	0.1150	0.0519	0.0215	0.0110	0.0069	0.0049
Receiver (8y x 10y) DD	0.0191	0.0340	0.0567	0.0890	0.1320	0.1861	0.2505	0.3239
Receiver (8y x 10y) SABR	0.0206	0.0393	0.0608	0.0886	0.1286	0.1868	0.2611	0.3445

DISPLACED DIFFUSION IMPLIED VOLATILITY SMILE

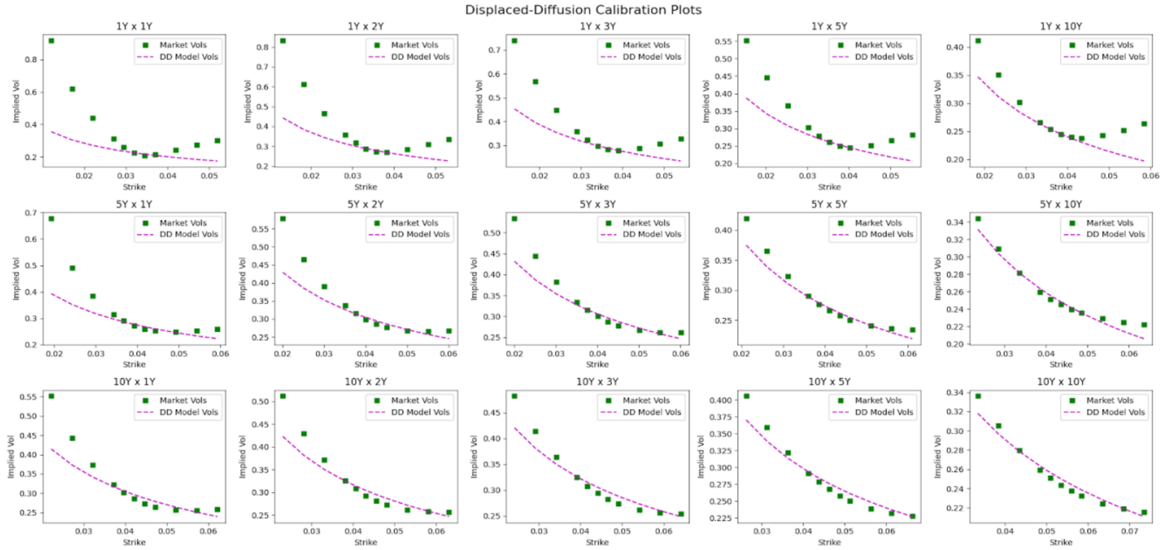

Fig 3: Displaced-Diffusion implied volatility smile

SABR IMPLIED VOLATILITY SMILE

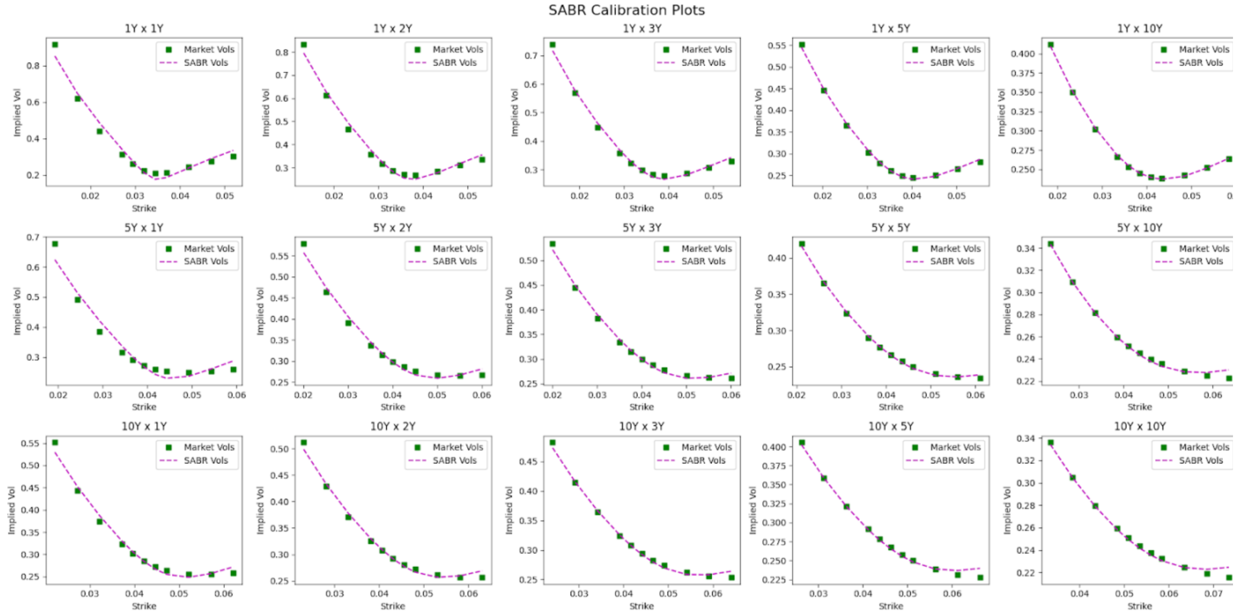

Fig 4: SABR implied volatility smile

Part III - Convexity Correction

Table 5: Formula for CMS

SWAPTION PRICING	$V_{n,N}(0) = D_0(0,T_n) \cdot IRR\big(S_{n,N}(0)\big) \cdot Black76\big(S_{n,N}(0), K, \sigma_{SABR}, T\big)$
STATIC REPLICATION	$\mathrm{E}^T[S_{n,N}(T)] = g(F) + \frac{1}{D_0(0,T)} * \left(\int_0^F h''(K)\ V^{rec}(K)dK + \int_F^\infty h''(K)\ V^{pay}(K)\, dK \right)$
PRESENT VALUE OF CMS LEG	$PV_{CMS10Y} = D(0{,}6m) * 0.5 * CMS_{(6m,10y)}(6m) + D(0{,}1y) * 0.5 * CMS_{(1y,11y)}(1y) + \cdots + D(0{,}5y) * 0.5 * CMS_{(5y,15y)}(5y)$ $PV_{CMS2Y} = D(0{,}3m) * 0.25 * CMS_{(3m,2y3m)}(3m) + D(0{,}6m) * 0.5 * CMS_{(6m,2y6m)}(6m) + \cdots + D(0{,}10y) * 0.25 * CMS_{(10y,12y)}(10y)$

To accurately value CMS instruments, we first interpolated the OIS and LIBOR discount curves to estimate D(0,T)D(0,T)D(0,T) at arbitrary maturities. For SABR model parameters, we applied two-dimensional bivariate spline interpolation, which better captures the joint dependency between expiry and tenor. This approach resolves issues seen in earlier 1D interpolation methods, such as unrealistic curvature and overstated convexity effects, resulting in a smoother and more stable volatility surface. Using these interpolated parameters, we computed SABR-implied volatilities (σSABR(F,K,T)) via the full Hagan formula, with special handling for the at-the-money case (F ≈K) to avoid numerical instability. Swaptions were then priced using the Black76 model under IRR settlement. The convexity-adjusted CMS rates were derived using static replication, and the present value of each CMS leg was calculated by summing these rates over the payment schedule, discounted appropriately using OIS discount factors. We then ran a comparison between CMS rates and forward swap rates (Table 6 and Fig 5-7).

Table 6: Comparison CMS Rates vs Forward Swap Rates

Expiry x Tenor	CMS Rates	Forward Swap Rates
1 x 1	0.032078	0.032007
1 x 2	0.033334	0.033259
1 x 3	0.034080	0.034011
1 x 5	0.035307	0.035255
1 x 10	0.038482	0.038428
5 x 1	0.039693	0.039274
5 x 2	0.040428	0.040075
5 x 3	0.040391	0.040072
5 x 5	0.041350	0.041093
5 x 10	0.043898	0.043634
10 x 1	0.042882	0.042189
10 x 2	0.043721	0.043116
10 x 3	0.044689	0.044097
10 x 5	0.046809	0.046249

Maturity = 1 year	Maturity = 5 year

Fig 5: CMS rate for 1 year maturity

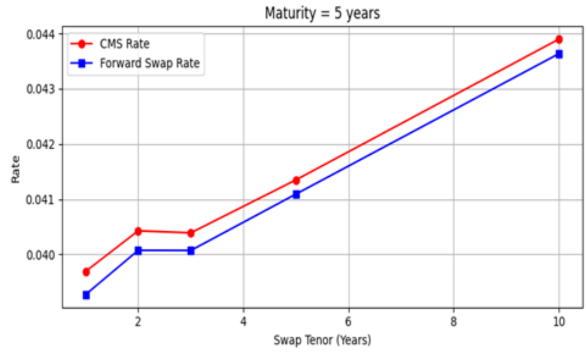

Fig 6: CMS rate for 5 year maturity

Maturity = 1 year	Convexity Premium

Fig 7: CMS rate for 10 year maturity

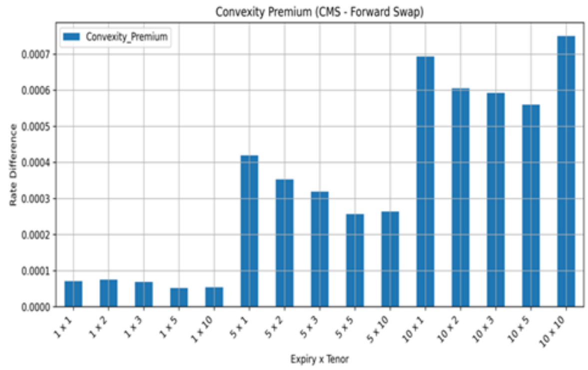

Fig 8: Convexity Premium (CMS rate – Forward Swap)

The effect of maturity and tenor on convexity correction:

- Convexity correction increases with expiry due to rising uncertainty and cumulative volatility.
- It is less sensitive to tenor, since the dominant driver is time to exercise, not swap length.
- In practice, always account for convexity when pricing long-expiry CMS — ignoring it can lead to significant mispricing.

Difference between forward swap rates and CMS rates (convexity correction insights):

Convexity correction is the difference between the CMS rate and the corresponding forward swap rate. It arises because the CMS payoff is nonlinear with respect to the underlying swap rate — and this nonlinearity becomes more pronounced under uncertainty (i.e., volatility). The CMS leg receives a fixed rate based on the future value of a swap rate, which is not a linear function of the underlying rates. Under the risk-neutral measure, we need to take the expectation of a convex function, and due to Jensen's inequality: $E[f(X)] > f(E[X])$ if f is convex. Since the CMS payoff is convex in the swap rate, its expected value is greater than the forward swap rate.

Part IV - Decompounded Options

Table 7: CMS payoff value calculation formulas

g(K) and its derivatives	$g(K) = K^{\frac{1}{4}} - 0.04^{\frac{1}{4}}$ $g'(K) = \frac{1}{4} * K^{-\frac{3}{4}}$ $g''(K) = -\frac{3}{16} * K^{-\frac{7}{4}}$ Note: p= 4, q =2
h(K) and its derivatives	$h(K) = \frac{g(K)}{IRR(K)}$ $h'(K) = \frac{\left(IRR(K) * g'(K) - g(K) * IRR'(K)\right)}{IRR(K)^2}$ $h''(K) = \frac{\left(IRR(K) * g''(K) - IRR''(K) * g(K) - 2 * IRR'(K) * g'(K)\right)}{IRR(K)^2} + \frac{\left(2 * IRR'(K)^2 * g(K)\right)}{IRR(K)^3}$
Caplet Value	$D(0,T)(F - L)^+ + h'(L)V^{pay(L)} + \int_L^{\infty} h''(K)V^{pay(K)dK}$
IRR-settled payer and receiver swaption	$V^{pay}_{(n,N)(0)} = D(0,T_n) * IRR\left(S_{(n,N)(0)}\right) * Black76Call\left(S_{(n,N)(0)}, K, \sigma_{n,N}, T\right)$ $V^{rec}_{(n,N)(0)} = D(0,T_n) * IRR\left(S_{(n,N)(0)}\right) * Black76Put\left(S_{(n,N)(0)}, K, \sigma_{n,N}, T\right)$

Our objective is to compute the present value of these payoffs (Table 8) using static replication with European IRR-settled swaptions.

We use static replication to price the decompounded CMS option, which allows us to express any smooth payoff as a weighted sum of vanilla European swaptions. The core idea is that even exotic payoffs can be replicated by constructing a portfolio of payer and receiver swaptions across different strikes. In this analysis we priced IRR-settled payer and receiver swaptions are priced using the Black-76 model where Sn,N (0) is the 5y × 10y forward swap rate and where σ is the SABR-implied volatility.

To replicate the decompounded CMS payoff, we use the Breeden–Litzenberger result, which relates the second derivative of a payoff function to a portfolio of European options. The second derivative of h(K), i.e., h"(K), is used to assign weights to each strike K. For Case 1: (Non-floored payoff), we replicate both sides of the payoff function using a strip of receiver swaptions for strikes below the forward rate (K < F), and payer swaptions for strikes above the forward rate (K ≥ F):

g(K) captures the direct decompounded value of the forward CMS rate, while the integral terms represent the convexity adjustment through static replication.For Case 2: (Floored payoff), we ignore the negative part of the payoff, which effectively sets a lower bound at the strike L = 0.2^4 = 0.0016. Therefore, the replication uses only payer swaptions starting from this lower strike.

The first term accounts for the discrete value at the boundary (i.e., the lowest strike that contributes to the positive payoff), while the integral accumulates the contributions of all payer swaptions with strikes beyond the floor level. This construction ensures that the replicated portfolio matches the floored payoff shape precisely, while eliminating exposure to negative values.

Table 8: CMS payoff values

Payoff Type	**Payoff**
CMS $10y^{1/p} - 0.04^{1/q}$	0.2322
(CMS $10y^{1/p} - 0.04^{1/q})^{+}$	0.2539

The key reason these two payoffs differ lies in their treatment of negative CMS outcomes. The non-floored payoff is linear in both directions and reflects the full range of potential CMS outcomes — positive and negative — which introduces exposure to receiver swaptions for rates below the strike. This makes the valuation more sensitive to changes across the entire swap rate distribution, especially under high volatility. In contrast, the floored payoff excludes the negative portion, truncating downside risk. As a result, only payer swaptions with strikes above the floor contribute to the value, making the payoff less responsive to low-rate scenarios and inherently lower in value.

Another reason for the difference is that the non-floored payoff benefits more from convexity in a volatile market. Since it captures gains from large deviations on both sides of the forward rate, it effectively embeds optionality in both directions. The floored payoff, by cutting off negative outcomes, gives up this symmetrical exposure and, consequently, yields a lower present value. This is evident in the results: the non-floored CMS leg is valued at 0.2322, while the floored version, despite excluding losses, is higher at 0.2539 due to the floor cutting off potentially large negative contributions. The floored structure also leads to a more skewed contribution from higher strikes, altering the distribution of risk and the hedging strategy needed for replication.