Topological Field Theories and Infinity Categories

MATH8210 Topics in Topology: Final Report

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1 Introduction

Topological field theories (TFT) are used in condensed matter physics to classify phases of matter. We say that a "phase" represents a regime of a system that is invariant under continuous deformation of parameters. Then given the space of all possible systems under certain parameters, such as temperature or pressure, so-called "phase-transitions" denote qualitative, distinct categories of these systems. In particular, we are interested in $\pi_0(X \setminus \Delta)$, where X is our space of possible systems and Δ is the singular locus comprising of phase transitions. If X describes a body of water, our space would have three path components corresponding to the solid, liquid, and gas phases, with certain removed paths demarcating where the material transitions between phases. These transitions occur in many other forms in physics, such as in the quantum hall effect or topological insulators. Particularly in topological domains, TFTs use mathematical results to lend insight in the physical setting.

We want to specify our problem to pertain to quantum mechanical systems. Therefore, we specify that our phases of matter are "gapped", meaning that there is a gap in the spectrum of the Hamiltonian above the ground state energy. In our reference, the adjective "invertible" is also employed, but states that the classification question makes sense without it. We will thus consider it to be outside of the scope of this paper.

The used references consist entirely of the first and fifth chapters of Freed's Lectures on Topology and Field Theory [1]. We will summarize the main 1-category definitions necessary in chapter 1, then motivate ∞-cat tooling. This will lead us to the cobordism hypothesis, which will conclude the paper.

2 The 1-Categorical Framework

We first establish the so-called Atiyah-Segal axiom system for TFTs. The language Freed uses to produce a TFT is via bordism, so the following subsection will briefly introduce the necessary definitions. All manifolds in the following sections are smooth, along with all maps in between manifolds. We'll say that a manifold is *closed* if it is compact without boundary.

2.1 Bordism

Definition 2.1 (Bordism)**.** *Fix $n \in \mathbb{Z}^{\geq 0}$. Let Y_0 and Y_1 be closed $(n-1)$-dimensional manifolds. A* ***bordism*** *from Y_0 to Y_1 is a tuple (X, θ_0, θ_1) consisting of:*

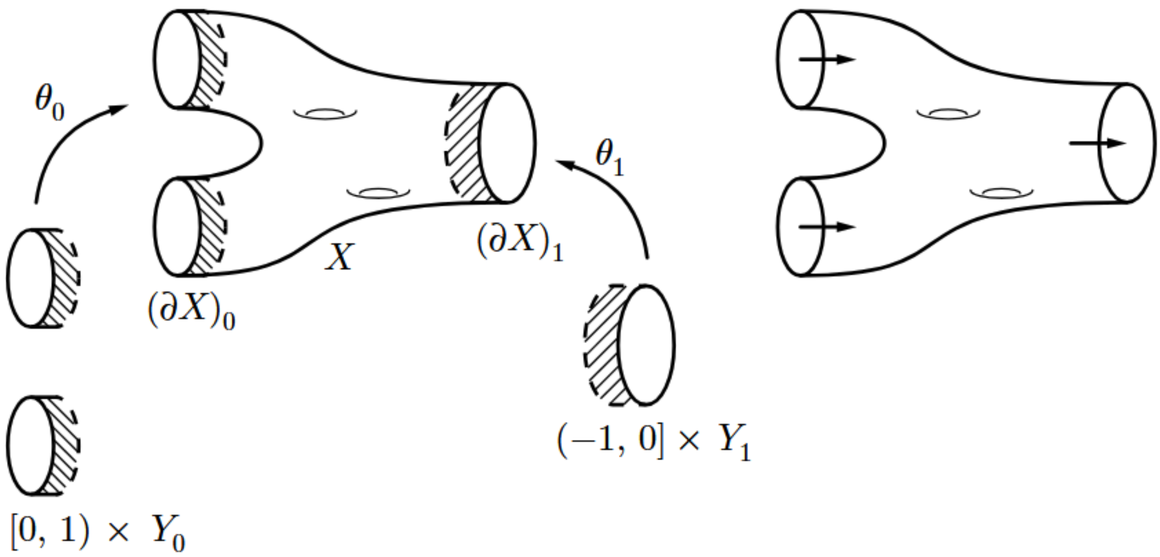

Figure 1: A visual representation of a bordism, taken also from Freed's lectures

1. *A compact n-dimensional manifold X with boundary.*
2. *A partition $p : \partial X \to \{0, 1\}$ of the boundary ∂X.*
3. *Embeddings $\theta_0 : [0, +1) \times Y_0 \to X$ and $\theta_1 : (-1, 0] \times Y_1 \to X$ with disjoint images, so that $\theta_i(0, Y_i) = (\partial X)_i$, $i \in \{0, 1\}$, with $(\partial X)_i = p^{-1}(i)$*

An intuitive picture can be seen in figure 1. With respect to the physics setting, we can think of the n-manifolds as physical configuration of a state, with a bordism encoding a possible time evolution of the state.

We'll say that two closed n-manifolds Y_0, Y_1 are *bordant* if there exists a bordism between them. Furthermore, we'll say that two bordisms are considered equivalent if there exists a diffeomorphism between them preserving the boundary data. Using this equivalence to form classes of manifolds, we find that these classes form a commutative monoid under disjoint union. The next definition upgrades the prior conditions into data.

Definition 2.2. *Let $n \in \mathbb{Z}_{\geq 0}$. The bordism category $Bord_{\langle n-1,n \rangle}$ is the symmetric monoidal category defined by:*

- *Objects are closed $(n-1)$-manifolds Y.*
- *Morphisms are equivalence classes of bordisms $X : Y_0 \to Y_1$.*
- *Composition is given by gluing.*
- *Disjoint union $\amalg$ with unit $\emptyset$, along with identity $[0, 1] \times Y$ for $Y \in Bord_{\langle n-1,n \rangle}$ provide for requisite monoidal structure.*

This definition provides us with all we need to see a rudimentary, 1-category definition of a TFT. Freed calls this an "Axiom System" to disavow any notions one might have about the finality or universality of the definition; ie, one can change various things about the theory in valid ways.

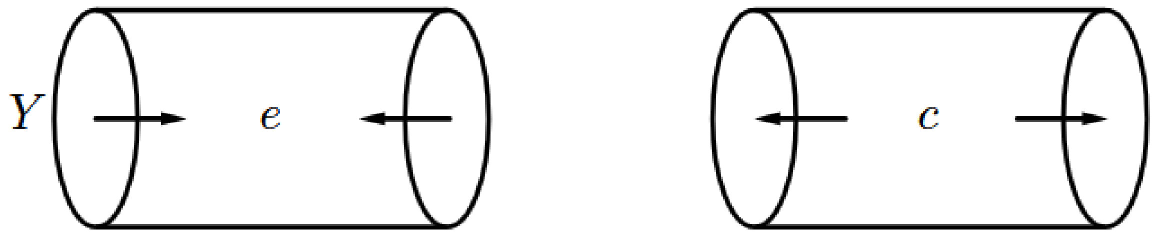

Figure 2: e and c for $Y = S^1$ [1]

Axiom 2.3 (Topological Field Theory)**.** *An n-dimensional TFT is a symmetric monoidal functor:*

$$F : Bord_{\langle n-1,n \rangle} \to Vect_{\mathbb{C}}$$

This is sometimes called the *Atiyah-Segal* axioms.

Some choices are made here that will be (slightly) altered in following sections. For one, the codomain is selected to be the symmetric monoidal category whose objects are complex vector spaces and morphisms linear maps, denoted $\mathrm{Vect}_{\mathbb{C}}$. This encodes two pieces of structure necessary for quantum mechanics: complex linearity. Furthermore, although our bordism category is generally applicable across manifolds, we might find use in restricting the domain to specific types of manifolds. One example might be to oriented manifolds, for instance.

Physical Interpretation:

- $F(Y) := \mathcal{H}$ is the Hilbert Space of global ground states, which we can call the state space.
- As noted before, $X : Y_0 \to Y_1$ is a spacetime trajectory between objects Y_0 and Y_1. $F(X) : F(Y_0) \to F(Y_1)$ is called the partition function.
- The tensor product allows us to combine non-interacting systems side-by-side.

2.2 Dualizability and Finiteness

These axioms encode physical constraints on our state space. We can construct the dual $Y^\vee$ and the following maps, visualized in 2:

- **Co-evaluation:** A bent cylinder $c : \emptyset \to Y \amalg Y^\vee$.
- **Evaluation:** A bent cylinder $e : Y^\vee \amalg Y \to \emptyset$.

This leads us to the following lemma:

Lemma 2.4. *Every object Y in the bordism category is dualizable.*

Since symmetric monoidal functors preserve dualizability, $F(Y)$ must be a dualizable vector space. In $\mathrm{Vect}_{\mathbb{C}}$, an object is dualizable if and only if it is **finite-dimensional**. This results in another specifier for the types of systems we're able to treat with TFTs: for example, a particle-in-a-box resides in an infinite dimensional Hilbert space, so would be disqualified.

3 Extended Locality: Motivating ∞-categories

Physical dynamics should be local. To use this data in our model means to be able to "slice" our manifolds in each dimension into balls, computing on the smaller portions, with the ability to cogently combine information. We therefore require n composition laws for $\mathrm{Bord}_{\langle n-1,n\rangle}$; one for each dimension to cut in. This requirement is called "extended locality".

However, the axiom system already presented fails this criteria. Because $\mathrm{Bord}_{\langle n-1,n\rangle}$ has $n-1$ manifolds as its smallest piece of information, we are unable to "slice" in more than the $n-1$ dimension. However, this is clearly remedied by using (n,n)-categories: our bordism category would then be able to keep track of more granular information and morphisms for gluing, exactly what is necessary.

3.1 Upgrading to (∞, n) categories

Eventually, one can ingrain evaluation of TFTs on smooth families of manifolds into our axiom system. This means we "can and should" promote from n categories to (∞, n) categories, where the higher morphisms in our bordism category are diffeomorphisms, isotopies, isotopies of isotopies, and so on. Freed notes that many interesting examples can be uncovered this way.

4 The Cobordism Hypothesis

The Cobordism Hypothesis describes the complete reconstruction of a TFT from local data.

First, we'll extend our prior Axiom System:

Axiom 4.1. *Choose a symmetric monoidal (∞, n)-category $\mathcal{C}$, An extended n-dimensional TFT on $\mathcal{X}_n$-manifolds with values in $\mathcal{C}$ is a symmetric monoidal functor:*

$$F : Bord_n(\mathcal{X}_n) \to \mathcal{C}$$

where $\mathcal{X}_n$ denotes additional structure put ontop of the manifolds in the bordism category.

We still desire complex linearity in our codomain, which should hold at all levels. Furthermore, the "top two levels" of $\mathcal{C}$ should reduce to $\mathrm{Vect}_{\mathbb{C}}$ or $\mathrm{sVect}_{\mathbb{C}}$.

Definition 4.2.

$$TFT(\mathcal{C})_{\langle n-1,n\rangle} = Hom(Bord_{\langle n-1,n\rangle}, \mathcal{C})$$

Theorem 4.3 (The Cobordism Hypothesis)**.** *Let $\mathcal{C}$ be a symmetric monoidal (∞, n)-category. There is an equivalence of (∞, n)-groupoids:*

$$TFT_n^{fr}(\mathcal{C}) \xrightarrow{\sim} (\mathcal{C}^{fd})^{\sim}$$

, $F \mapsto F(*)$.

TFT_n^{fr} denotes framed topological field theories that do not require rotation invariance. This assumption can be relaxed for a more complex theory; this is a simple case. The codomain $(\mathcal{C}^{fd})^{\sim}$ is the underlying ∞-groupoid of the subcategory of fully dualizable objects.

The condition that $F(\mathrm{pt})$ must be "fully dualizable" is the higher-categorical generalization of the finite-dimensionality condition derived in the Lemma in section two.

Physically, this ensures that local data describes a well-defined, finite system with no singularities, regardless of how we slice the spacetime. If the object were not fully dualizable, constructing the invariant for a closed manifold would result in a divergent trace.

5 Conclusion

The synopsis in this paper outlined the higher-category extension from the Atiyah-Segal axioms to a model that better encodes locality. We concluded with the cobordism hypothesis, a method to classify TFTs by evaluation on a single point, furthermore showing that the resultant data is fully dualizable.

References

[1] D. S. Freed, *Lectures on Field Theory and Topology*, American Mathematical Society, 2019.