Differential Equations Book Definitions

Explain what a differential equation is using your own words.

A differential equation is an equation relating a function to one or more of its derivatives.

What does it mean to be a solution to a differential equation?

A solution to a differential equation graphs an infinite family of curves called solution curves. A solution is a function $x(t)$ such that $x'(t) = f\,(t, x(t))$

Does a differential equation always have a solution? Explain.

No, some calculations such as the integration for separable equations can create solutions that can't be solved for y and thus don't have an explicit solution. Alternatively, unique solutions are not guaranteed and a differential equation could have multiple solutions making none of those solutions correct.

Explain why solution curves to a differential equation cannot intersect.

Solution curves to a DE follow a unique repeating path, if they were to intersect then the DE's behavior would be changing over time making it unsolvable.

We can use Taylor polynomials to approximate a function $f(x)$ near a point x_0. Explain why this approximation can only be expected to be accurate near x_0.

Since a Taylor series contains an infinite number of powers of x, if the value of x gets large, then those larger powers will grow out of control and overpower the linear part of the Taylor series that we are focused on

What important rule from differential calculus do we use when solving a first-order differential equation?

An **integrating factor** can be multiplied to both sides of a linear differential equation to solve that DE. An integrating factor takes the form $\mu(s) = e^{\int p(x)dx}$

What is a first-order linear differential equation?

A differential equation that only has an order of 1 (y'), and is a linear equation ($y = mx + b$). With those two rules a first-order linear differential equation takes the form $\frac{dy}{dx} + p(t)x = q(t)$ which kinda looks like $y - mx = b$ (linear)

Explain what a bifurcation is in your own words.

A bifurcation gives a representation of the nature of a solution of a Differential Equation. In other words it is a rough draft of what a solution curve might look like that can be found without solving a differential equation

What is the characteristic equation of $ax'' + bx' + cx = 0$?

Replace the x components with lambda: $a\lambda^2 + b\lambda + c = 0$
A characteristic equation relates to a DE and allows you to find the roots of a DE

Suppose that $x_p(t)$ and $x_q(t)$ are two solutions of $ax'' + bx' + cx = g(t)$. How are these two solutions related?

$x_p(t) + x_q(t)$ is a solution of the homogeneous linear differential equation
$ax'' + bx' + cx = 0$
All this is saying is that x_p and x_q are a part of the Homogonous solution of this non-Homogonous equation. Therefore, we can say that combined they are a solution to the Homogonous version of the given DE

Describe the Method of Undetermined Coefficients in your own words.

By creating a logical guess at what the particular solution of a SOLODE looks like, it can then be plugged into the DE to find the actual particular solution by solving for the guesses constant

What does complexification mean?

Complexification is the use of Euler's formula to separate a DE into its real and imaginary parts to make it simpler to find its particular solution

Describe what resonance means in your own words.

When looking at the behavior of an object such as a bridge (can be modeled as $x'' + \omega_0^2 x = A\cos\omega t$) its frequency can be separated into two main sources, the natural frequency (ω_0) which comes from the object itself, and the forcing frequency (ω) which comes from the outside forces acting on the object. If the natural frequency and the forcing frequency are equal then **resonance** occurs and the frequencies reinforce each other.

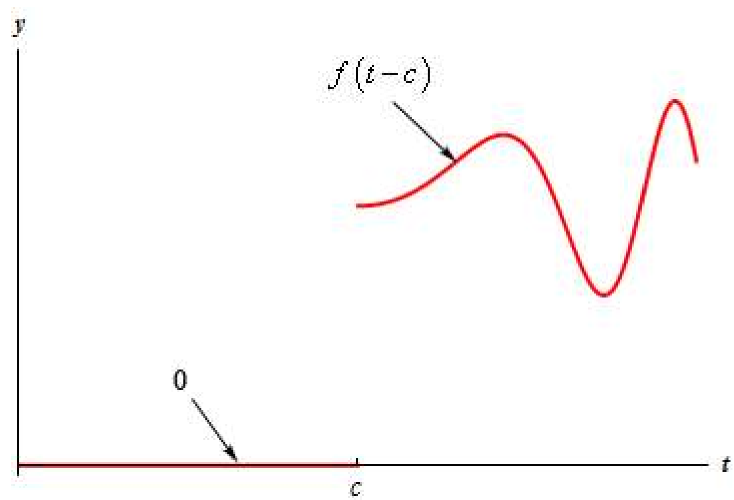
The mid-point of a new bridge moves up and down following the differential equation $x'' + 4x = 4\sin 2t, \quad x(0) = x'(0) = 0$.

Explain the future behavior of the bridge.

Looking at the provided equation we can see that our natural frequency is $\omega_0^2 = 4, \omega_0 = 2$ and our forcing frequency is $\omega = 2$. Therefore since the natural frequency and the forcing frequency are equal the bridge will resonate and continue to oscillate at the same frequency as long as the forcing frequency stays constant.

What is a Heaviside function?

A Heaviside function (or step function) takes a complex DE and simplifies it into a binary function that is easier to manipulate. Meaning, a Heaviside function uses a piecewise to turn a function "on" or equal to the original function for some values of t, and "off" or usually equal to zero for other values of t.



Describe in your own words how the Laplace transform can be used to solve an initial value problem.

Laplace transforms allow a DE to be converted into an *algebraic equation* which can then be solved and then converted back into the desired solution to the DE. This subverts alternative solving methods which don't work for all DE's.

What is an eigenvalue and an eigenvector?

An eigenvalue is a root found from the characteristic polynomial of a system of differential equations. An eigenvector is a vector that scales a particular solution to a system of DE's so that its related eigenvalue matches the original equations

Explain in your own words what it means for two vectors to be linearly independent

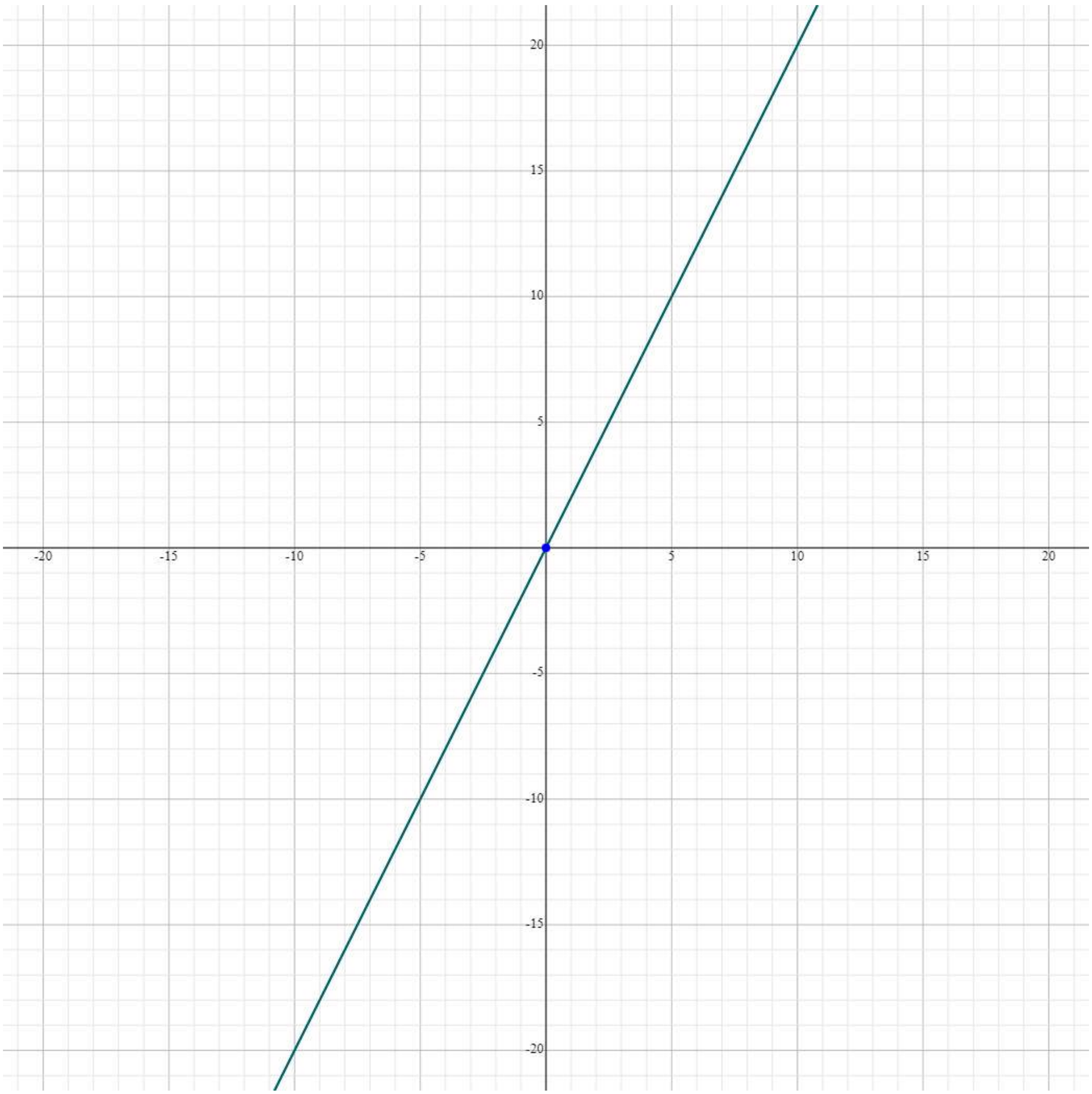
Two vectors are linearly independent if they are not multiples of each other. In other words they do not lie on the same line through the origin. For example, $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ are linearly **dependent** because they are multiples. But $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are linearly **independent** because they are two completely separate vectors.

What is the Principle of Superposition for linear systems of differential equations?

The principal of superposition states that for any System of DE's there are an infinite number of linear combinations of solutions to that DE. In other words, as long as two solutions to a System of DE's share the same eigenvalues and have vectors that are linearly dependent to the other solution, both are viable solutions.
For example, the solutions $y = C_1e^{6t}\begin{bmatrix}1\\2\end{bmatrix} + C_2e^{-4t}\begin{bmatrix}8\\3\end{bmatrix}$ and $y = C_1e^{6t}\begin{bmatrix}2\\4\end{bmatrix} +$
$C_2e^{-4t}\begin{bmatrix}4\\\frac{3}{2}\end{bmatrix}$
could both solutions to the same DE and are both equally correct.

What is a straight-line solution?

A straight line solution is a line that passes through the origin and is scaled by a vector. for example if we have a vector $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ a straight line solution would look like:

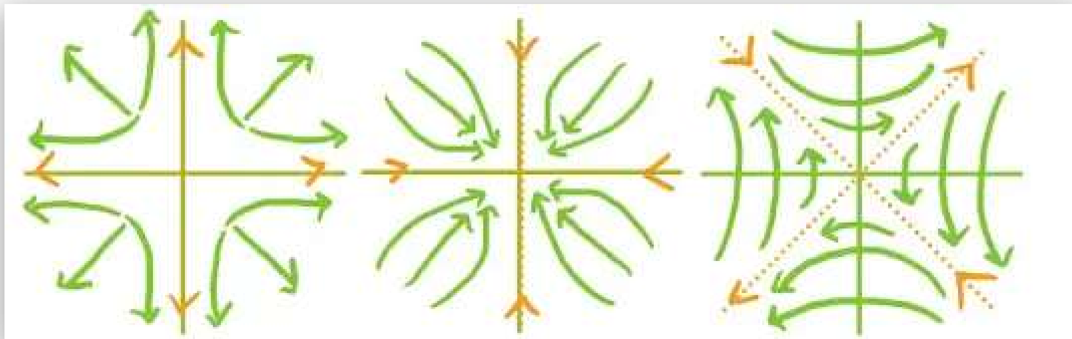


For a 2 × 2 linear system with distinct real eigenvalues, what are the three different possibilities for the phase plane of the system?

The phase plane for real eigenvalues can take the form of a:

1. Sink - both eigenvalues are negative and are directed towards the origin

2. Source - both eigenvalues are positive and are directed away from the origin
3. Saddle - one eigenvalue is positive and the other is negative creating a arcing around the origin



For a 2 × 2 linear system with complex eigenvalues, what are the three different possibilities for the phase plane of the system?

The phase plane for complex eigenvalues can take the form of a:
$(\alpha \pm \beta i)$

1. Spiral Sink - If α is negative the solution spirals inwards towards the origin
2. Spiral Source - If α is positive the solution spirals outward from the origin
3. Node- If α is zero the solution forms circles around the origin

What is Euler's Formula?

$e^{i\beta t} = \cos \beta t + i \sin \beta t$
Euler's formula expands complex numbers into an equation involving sines and cosines

Given a 2 × 2 system with repeated eigenvalues, how many straight-line solutions are there?

Repeated roots only have a single straight line solution because all other solutions depend on the initial eigenvalue

Is it possible for a 3 × 3 linear system of differential equations to have a line of stable solutions and a plane of unstable solutions? Explain your answer.

Yes, a line of stable solutions (solutions that point towards the origin) could exist in the z axis at $(0, 0, z)$ while everywhere else a set of planar solutions at $z = 0$ or $(x, y, 0)$ could be unstable (solutions that point away from the origin). Put more simply, due to the extra degree that a 3×3 matrix provides, different behavior can occur along different axes.

What is the exponential of a matrix A?

$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!}A^k$

That equation may be sufficient as an answer, here is a bit more explanation. The goal of using an exponential of a matrix is to find a more generalized way to solve a system of DE's. The reason an exponential of a matrix can be used is because solutions to systems of DE's take the form $x(t) = e^{tA}x_0$. Therefore, since we know what to do with e^t and x_0 all we need to figure out is what to do with e^A, hence the matrix exponential. Through long proofs and stupid Taylor series the book gets the equation above which can be used in a derived form to solve a system (we don't have to do that). To summarize, the exponential of a matrix A is a calculation using a Taylor series approximation that makes it possible to solve any sized system of DE's in a generalized manner.

What is the linear part of the system of equations below?

$\frac{dx}{dt} = f(x, y) \quad \frac{dy}{dt} = g(x, y)$

When looking at non-linear systems such as the one's given , it is possible to generalize the behavior of the system by only looking at the linear parts of the equations. So for the provided equations you would only look at the terms in each that are of degree 1 (Ex: $2x$, $5y$, NOT $3x^2$) and could then create a linear approximation.

What types of equilibrium solutions are possible in a Hamiltonian system?

The equilibrium points of a Hamiltonian system occur at the critical points of H. Solutions curves of a Hamiltonian system are a level set of H. That is, the solution curves of a Hamiltonian system are equal to the level curves of the Hamiltonian Function.