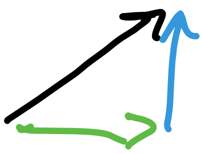
Linear Algebra "Review"

Whats next in Diff Eq?

$$x'' + Ax' + Bx = 0 \Rightarrow \begin{cases} x' = Ax + By \\ y' = Cx + Dy \end{cases}$$

Vectors



$$\vec{v} = \underset{\text{vector}}{\begin{bmatrix} x \\ y \end{bmatrix}} = \begin{bmatrix} x & y \end{bmatrix}$$

$$c \cdot \vec{v} = c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

Vectors and Matricies

$$A = \begin{bmatrix} a & b \\ d & f \end{bmatrix} \quad \vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad A \cdot \vec{v}$$

$$\begin{bmatrix} a & b \\ d & f \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} Ax_1 & By_1 \\ Dx_1 & fy_1 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \quad \text{Find } A \cdot \vec{v}$$

$$= \begin{bmatrix} 2x(t) & 4y(t) \\ 6x(t) & 8y(t) \end{bmatrix}$$

System of DEs to Matrix

$$\begin{cases} x' = \underline{A}\,\underline{x} + \underline{B}\,\underline{y} \\ y' = \underline{D}\,\underline{x} + \underline{F}\,\underline{y} \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} A & B \\ D & F \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{cases} x' = 2x + 4y \\ y' = 6x + 8y \end{cases}$$

$$X = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x' = 3x + 6y$$
$$y' = 9x + 12y$$

Properties of Matricies

Identity Matrix

Scalable 1-0 Matrix with same pattern

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant

cross product

$$\left| \begin{bmatrix} a & b \\ d & f \end{bmatrix} \right| = af - bd$$

$$A = \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix}$$

$$|A| = (8 - 56) = \boxed{-48}$$

Inverse

swap

Negate

$$A = \begin{bmatrix} a & b \\ d & f \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} f & -b \\ -d & a \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix}$$

$$|A| = (8 - 56) = -48$$

$$A^{-1} = \frac{1}{-48}\begin{bmatrix} 4 & -7 \\ -8 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -\frac{4}{48} & \frac{7}{48} \\ \frac{1}{48} & -\frac{2}{48} \end{bmatrix}$$

Linear Independence

Uniqueness for systems

Use determinant to find Linear Independence

Given

$$\vec{x} = [x_1 \; y_1]$$

$$\vec{y} = [x_2 \; y_2]$$

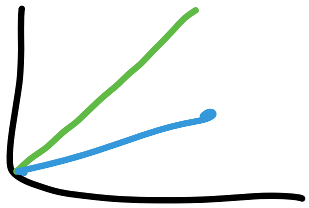
$$\vec{x} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \vec{y} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\vec{x} \cdot \vec{y} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$|\vec{x} \cdot \vec{y}| = (x_1 y_2) - (x_2 y_1)$$

If the determinant $\neq 0$

they are LI



2x2 Matrix Math

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Addition / Subtraction

$$A + B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$A - B = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Multiplication

$$A \times B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

$$A \times B = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

Thats everything we "know"

Eigenvalues

λ = eigenvalues

HSOLODES $\rightarrow \lambda \rightarrow \lambda = 2$
$\lambda = 4$

Finding Eigenvakes

Identity Matrix

$$|A - \lambda I| = 0$$

Matrix eigenvalue determinant

$$\left| \begin{bmatrix} a & b \\ d & f \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} a & b \\ d & f \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left|\begin{bmatrix} a-\lambda & b \\ d & f-\lambda \end{bmatrix}\right| = 0$$

Determinant

$$\boxed{(a-\lambda)(f-\lambda) - bd = 0}$$

→ Always gives two solutions

Try It:

$$\begin{cases} x' = 2x + 3y \\ y' = 4x - y \end{cases}$$

find eigenvalues

Make Matrix:

$$X = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Add eigenvalues:

$$X = \begin{bmatrix} 2-\lambda & 3 \\ 4 & -1-\lambda \end{bmatrix}$$

Find Determinant:

$$\left|\begin{bmatrix} 2-\lambda & 3 \\ 4 & -1-\lambda \end{bmatrix}\right| = (2-\lambda)(-1-\lambda) - 12 = 0$$

Solve:

$$-2 - 2\lambda + \lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - \lambda - 14 = 0$$

$$\lambda = \frac{1 \pm \sqrt{1-4(1)(-14)}}{2}$$

$$\boxed{\lambda = \frac{1}{2} \pm \frac{\sqrt{57}}{2}}$$

Eigenvectors

Given $A = \begin{bmatrix} a & b \\ d & f \end{bmatrix}$ and eigenvalue λ you can form a system of equations

$$ax + by = \lambda x$$

$$dx + fy = \lambda y$$

Try it:

Find the eigenvectors:

$$\begin{cases} x' = 2x - y \\ y' = -x + 2y \end{cases} = X = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left| \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \right| = (2-\lambda)(2-\lambda) - 1 = 0$$

$$4 - 2\lambda - 2\lambda + \lambda^2 - 1 = 0$$

$$\lambda^2 - 4\lambda + 3 = 0$$

$$(\lambda - 3)(\lambda - 1) = 0 \qquad \lambda = 3, 1$$

For $\lambda = 3$:

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$2x - 1y = 3x$$

$$-1x + 2y = 3y$$

Elimination!

$$2x - 1y = 3x$$

$$-2x - 2y = 6y$$

$$-3y = 3x + 6y$$

$3y = -3x$

$y = -x$

$2x + x = 3x \qquad x = 1$

$y = -1$

$$\boxed{V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}} \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

For $\lambda = 1$:

$A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

$2x - y = x$

$-x + 2y = y$

$2x - y = x$

$-2x + 4y = 2y$

$3y = x + 2y$

$x = y$

$x = 1$

$y = 1$

$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$2x - x = x$

$x = x$

Two possible eigenvectors are $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ for $\lambda = 3$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda = 1$

$$\begin{bmatrix} x \\ y \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$$

$$x = c_1 e^{3t} + c_2 e^{t}$$

$$y = -c_1 e^{3t} + c_2 e^{t}$$

Note:

complex can exist