No-Homo

$y'' + 16y = 2\sin(2t)$ $\quad y(0) = 1$
$\quad y'(0) = 0$

$\lambda^2 + 16 = 0$

$\lambda = \sqrt{-16} = 0 \pm 4i$

$A \pm Bi = e^{At}(c_1\cos(Bt) + C_2\sin(Bt))$

$y_h = C_1\cos(4t) + C_2\sin(4t)$

Guess: $y_p = A\sin(2t) + B\cos(2t)$
$y'_p = 2A\cos(2t) - 2B\sin(2t)$
$y''_p = -4A\sin(2t) - 4B\cos(2t)$

Plug in: $y'' + 16y = 2\sin(2t)$

$-4A\sin(2t) - 4B\cos(2t) + 16A\sin(2t) + 16B\cos(2t) = 2\sin(2t)$

$12A\sin(2t) + 12B\cos(2t) = 2\sin(2t)$

compare sins and cos

$12A\sin(2t) = 2\sin(2t)$ $\quad 12A = 2 \quad A = \frac{1}{6}$
$12B\cos(2t) = 0 \quad B = 0$

$y_p = \frac{1}{6}\sin(2t)$

$y = C_1\cos(4t) + C_2\sin(4t) + \frac{1}{6}\sin(2t)$
$y' = -4C_1\sin(4t) + 4C_2\cos(4t) + \frac{1}{3}\cos(2t)$

$1 = c_1$

$0 = 4c_2 + \frac{1}{3}$

$c_2 = -\frac{1}{12}$

$$\boxed{y = \cos(4t) - \frac{1}{12}\sin(4t) + \frac{1}{6}\sin(2t)}$$

Homo

$y'' + 6y' + 9y = 0 \quad y(0) = 1 \quad y'(0) = 0$

$r^2 + 6r + 9 = 0$

$(r+3)(r+3) = 0 \quad r = -3, -3$

$y = c_1 e^{-3t} + c_2 t e^{-3t}$

$y' = -3c_1 e^{-3t} + c_2 (t \cdot -3e^{-3t} + 1 \cdot e^{-3t})$

$1 = c_1$

$c_2 = 3$

$0 = -3c_1 + c_2$

$$\boxed{y = e^{-3t} + 3te^{-3t}}$$

Complexification Review

when?

$\rightarrow$ sin or cos right side

how?

$$\rightarrow e^{ait} = \underbrace{\cos(at)}_{\text{Real}} + \underbrace{i\sin(at)}_{\text{Imaginary}}$$

$$x'' + 6x' + 5x = \sin(2t)$$

Looking for x_p

$$\text{guess: } x_c = Ae^{2it}$$

$$x_c' = 2Aie^{2it}$$

$$x_c'' = -4Ae^{2it}$$

$$-4Ae^{2it} + 12Aie^{2it} + 5Ae^{2it} = e^{2it}$$

$$A + 12Ai = 1 \qquad A(1+12i) = 1$$

$$A = \frac{1}{1+12i} \cdot \frac{1-12i}{1-12i} = \frac{1-12i}{145}$$

$$x_c = \left(\frac{1}{145} - \frac{12i}{145}\right)e^{2it}$$

$$x_c = \left(\frac{1}{145} - \frac{12i}{145}\right)\left(\cos(2t) + i\sin(2t)\right)$$

$$x_c = \frac{1}{145}\cos(2t) + \frac{1}{145}i\sin(2t) - \frac{12i}{145}\cos(2t) + \frac{12}{145}\sin(2t)$$

$$\boxed{x_p = \frac{1}{145}\sin(2t) - \frac{12}{145}\cos(2t)}$$

Complexification

$$u'' + 4u' + 20u = -\cos(5t)$$ Find y_p

Looking for Real

$-\cos(5t) - i\sin(5t)$

$-(\cos(5t) + i\sin(5t))$

$-e^{5it}$

Guess: $u_c = Ae^{5it}$

$u_c' = 5iAe^{5it}$

$u_c'' = -25Ae^{5it}$

$$-25Ae^{5it} + 20iAe^{5it} + 20Ae^{5it} = \boxed{-e^{5it}}$$

$$-25A + 20iA + 20A = -1 \qquad A = \frac{-1}{-5+20i} \cdot \frac{-5-20i}{-5-20i}$$

$$A = \frac{5+20i}{425}$$

$$u_c = \left(\frac{5}{425} + \frac{20}{425}i\right)e^{5it}$$

$$u_c = \left(\frac{5}{425} + \frac{20}{425}i\right)(\cos(5t) + i\sin(5t))$$

$$u_c = \underline{\frac{5}{425}\cos(5t)} + \frac{5}{425}i\sin(5t) + \frac{20}{425}i\cos(5t) - \underline{\frac{20}{425}\sin(5t)}$$

$$\boxed{u_p = \frac{1}{85}\cos(5t) - \frac{4}{85}\sin(5t)}$$

No-Homo $\quad y'' - y' - 2y = 4t^2 \quad y(0) = -1 \quad y'(0) = 1$

Homo

$\lambda^2 - \lambda - 2 = 0 \qquad (\lambda-2)(\lambda+1)$

$\lambda = 2, -1$

$y_h = c_1 e^{2t} + c_2 e^{-t}$

Guess: $y_p = At^2 + Bt + C$

$y'p = 2At + B$

$y''p = 2A$

$2A - 2At - B - 2At^2 - 2Bt - 2C = 4t^2 + 0t + 0$

$-2A = 4 \qquad A = -2$

$-2A - 2B = 0 \qquad B = 2$

$2A - 2C - B = 0 \qquad c = -3$

$y_p = -2t^2 + 2t - 3$

$y = c_1 e^{2t} + c_2 e^{-t} - 2t^2 + 2t - 3$

$y' = 2c_1 e^{2t} - c_2 e^{-t} - 4t + 2$

$-1 = c_1 + c_2 - 3 \qquad c_1 = 2 - c_2$

$1 = 2c_1 - c_2 + 2 \qquad 1 = 4 - 2c_2 - c_2 + 2$

$-5 = -3c_2$

$\boxed{y = \frac{1}{3} e^{2t} + \frac{5}{3} e^{-t} - 2t^2 + 2t - 3} \qquad c_2 = \frac{5}{3} \quad c_1 = \frac{1}{3}$

Definition of Laplace

$$f(t) = \sin(t)$$

Reminder: $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

Laplace IVP

$$y'' + 16y = 2\sin(2t) \qquad y(0) = 1,\ y'(0) = 0$$

$$Ys^2 - s - 0 + 16Y = \frac{4}{(s^2+4)}$$

$$Y(s^2+16) = \frac{4}{s^2+4} + s$$

$$Y = \frac{4}{(s^2+16)(s^2+4)} + \frac{s}{(s^2+16)}$$

Save for Later

$$(As+B)(s^2+4) + (Cs+D)(s^2+16) = 4$$

$$As^3 + 4As + Bs^2 + 4B + Cs^3 + 16Cs + Ds^2 + 16D = 4$$

$A + C = 0$ $\quad$ $A = 0$

$B + D = 0$ $\quad$ $B = -\frac{1}{6}$

$4A + 16C = 0$ $\quad$ $C = 0$

$4B + 16D = 4$ $\quad$ $D = \frac{1}{6}$

$$Y = -\frac{1}{6}\left(\frac{1}{s^2+16}\right) + \frac{1}{6}\left(\frac{1}{s^2+4}\right) + \frac{s}{s^2+16}$$

$$\boxed{y = -\frac{1}{12}\sin(4t) + \frac{1}{6}\sin(2t) + \cos(4t)}$$

Convolution Review

Definition: $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$

tau : τ

$f(t) = 4t \rightarrow = 4\tau$

$g(t) = 8t^2 \quad = 8(t-\tau)^2$

Definition #2:

something like: $\frac{s+1}{s(s-2)} = \mathcal{L}^{-1}\left(\frac{s+1}{s}\right) * \mathcal{L}^{-1}\left(\frac{s+1}{s-2}\right)$

Convolution

If $f(t) = t$ and $g(t) = t$ find $f * g$

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First tauify: $f(\tau) = \tau \quad g(t-\tau) = t-\tau$

Make Integral: $\int_0^t \tau(t-\tau)\,d\tau$

$\int_0^t \tau t - \tau^2\,d\tau = \frac{1}{2}\tau^2 t - \frac{1}{3}\tau^3 \Big|_0^t$

$= \frac{1}{2}t^3 - \frac{1}{3}t^3 = \frac{3}{6}t^3 - \frac{2}{6}t^3 = \boxed{\frac{1}{6}t^3}$

Convolution With Laplace

Find the Inverse Laplace of $\frac{1}{(s-1)(s-3)}$

$$= \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) * \mathcal{L}^{-1}\left(\frac{1}{s-3}\right)$$

$$= e^{t} * e^{3t}$$

Tauify:

$$e^{3\tau} * e^{t-\tau}$$

$$\int_0^t e^{3\tau} e^{t-\tau} \, d\tau = \int_0^t e^{t+2\tau} \, d\tau$$

$$\frac{1}{2} e^{t+2\tau} \Big|_0^t \quad \boxed{\frac{1}{2} e^{3t} - \frac{1}{2} e^{t}}$$

Step Function

Find the laplace of $f(t) = u_6(t)9t$

From chart: $U_c(t)f(t-c) = e^{-cs}F(s)$

Our $c=6$ $\qquad 9t \xleftarrow{t-6}$ $\qquad 9(t-6)+54$

$F(s) = e^{-6s}\mathcal{L}(9r+54) \qquad r = t-6$

From chart: $U_c(t)\boxed{f(t-c)} = e^{-cs}\boxed{F(s)}$

$t-c = s$

$t \neq s$

$F(s) = e^{-6s}\left(\frac{9}{s^2} + \frac{54}{s}\right)$

Step from Piecewise

Find laplace of $f(t) = \begin{cases} 0 & t < 6 \\ (t-6)^3 & t \geq 6 \end{cases}$

$\text{Top} + (\text{Bot} - \text{Top})\, u_c(t)$

$0 + ((t-6)^3 - 0)\, U_6(t)$

$f(t) = U_6(t)(t-6)^3$

$c = 6$ $\qquad$ $r = t-6$

$F(s) = e^{-6s}(r)^3$

$F(s) = e^{-6s}\left(\frac{6}{s^4}\right)$