Final Exam Review (Worked out Problems)

1a) $y' = ty$, $y(0) = 3$

$\frac{dy}{dt} = ty \quad \frac{1}{y}dy = t\,dt$

$\ln(y) = \frac{1}{2}t^2 + C$

$y = e^{1/2t^2+C} = e^{1/2t^2}C$

$3 = C \quad \boxed{y = 3e^{1/2t^2}}$

1b) $e^{x+y} = y'$, $y(0) = 0$

$e^x e^y = \frac{dy}{dx} \quad e^x dx = e^{-y}dy$

$-e^{-y} = e^x + C \quad e^{-y} = -e^x + C$

$-y = \ln(-e^x + C)$

$y = -\ln(-e^x + C) \quad 0 = -\ln(-1 + C)$

$C = 2 \quad \boxed{y = -\ln(-e^x + 2)}$

1c) $y' = 2 - y$, $y(0) = 3$

$\frac{dy}{dx} = 2 - y \quad \frac{1}{2-y}dy = dx$

$\ln(2-y) = x + C$

$2 - y = e^{x+C} = e^x C$

$y = -e^x C + 2$

$3 = -C + 2 \quad C = -1$

$\boxed{y = e^x + 2}$

1d) $x' = x\tan t + \sin t$, $x(0) = 2$ (Seperable)

$\frac{dx}{dt} = x\tan t + \sin t$ (might not be right)

$dx = (x\tan t + \sin t)dt$

$x = -x\ln(\cos(t)) - \cos t + C$

$x(1 + \ln(\cos(t))) = -\cos t + C$

$x = \frac{-\cos(t) + C}{1 + \ln(\cos(t))} \quad 2 = \frac{-1 + C}{1 + 0}$

$C = 3 \quad \boxed{x = \frac{-\cos(t) + 3}{1 + \ln(\cos t)}}$

1d) $x' = x\tan(t) + \sin(t)$, $x(0) = 2$ (Linear)

$x' - x\tan(t) = \sin(t) \quad \mu = e^{\int -\tan t}$

$\mu = e^{\ln(\cos t)} \quad \mu = \cos t$

$x\cos t = \int \sin(t)\cos(t)\,dt \quad \begin{matrix} u = \sin t \\ du = \cos t \end{matrix}$

$x\cos t = \frac{1}{2}\sin^2(t) + C$

$x = \frac{\sin^2 t}{2\cos t} + \frac{C}{\cos t} \quad 2 = 0 + C \quad C = 2$

$\boxed{x = \frac{\sin^2 t}{2\cos t} + \frac{2}{\cos t}}$

1e) $x' = x\sin(t) + 2te^{-\cos t}$, $x(0) = 1$

$x' - x\sin(t) = 2te^{-\cos t} \quad \mu = e^{\int -\sin(t)\,dt} \quad \mu = e^{\cos t}$

$xe^{\cos t} = \int 2te^{-\cos t}e^{\cos t} = \int 2t \quad xe^{\cos t} = t^2 + C$

$x = \frac{t^2 + C}{e^{\cos t}} \quad 1 = \frac{0 + C}{e^1} \quad C = e^1 \quad \boxed{x = \frac{t^2 + e^1}{e^{\cos t}}}$

1f) $y'' - 3y' + 2y = 0$, $y(0) = 2$, $y'(0) = 1$

$\lambda^2 - 3\lambda + 2 = 0 \quad (\lambda-2)(\lambda-1) \quad \lambda = 2, 1$

$y = c_1 e^{2t} + c_2 e^t \qquad y' = 2c_1 e^{2t} + c_2 e^t$

$2 = c_1 + c_2 \qquad 1 = 2c_1 + c_2 \qquad 1 = 2c_1 + 2 - c_1 \quad -1 = c_1$

$c_2 = 3 \qquad \boxed{y = -1e^{2t} + 3e^t}$

1g) $y'' + 2y' + 2y = 0$, $y(0) = 2$, $y'(0) = 3$

$\lambda^2 + 2\lambda + 2 = 0$

$\lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$

$y = e^{-t}(c_1 \cos t + c_2 \sin t) = c_1 e^{-t}\cos t + c_2 e^{-t} \sin t$

$y' = c_1(-te^{-t}\cos t - e^{-t}\sin t) + c_2(-te^{-t}\sin t + e^{-t}\cos t)$

$2 = c_1 \qquad 3 = c_2$

$\boxed{y = e^{-t}(2\cos t + 3\sin t)}$

1h) $y'' - 2y' + y = 0$, $y(0) = 2$, $y'(0) = -1$

$\lambda^2 - 2\lambda + 1 = 0 \qquad \lambda = 1, 1 \quad$ Repeated

$y = c_1 e^t + c_2 te^t \qquad 2 = c_1$

$y' = c_1 e^t + c_2(e^t + te^t) \quad -1 = c_1 + c_2 \quad c_2 = -3$

$\boxed{y = 2e^t - 3te^t}$

2a) $y'' + 3y' + 2y = 4e^{-3t}$

$y_p = Ae^{-3t} \qquad 9Ae^{-3t} - 9Ae^{-3t} + 2Ae^{-3t} = 4e^{-3t}$

$y'_p = -3Ae^{-3t} \qquad 2A = 4 \quad A = 2$

$y''_p = 9Ae^{-3t} \qquad \boxed{y_p = 2e^{-3t}}$

2b) $y'' + 4y = \cos 3t$ COMPLEXIFICATION

$y_c = Ae^{3it} \qquad -9Ae^{3it} + 4Ae^{3it} = e^{3it} \qquad \boxed{y_p = -\frac{1}{5}\cos 3t}$

$y'_c = 3iAe^{3it} \qquad -5A = 1 \quad A = -\frac{1}{5}$

$y''_c = -9Ae^{3it} \qquad y_c = -\frac{1}{5}(\cos 3t + i\sin 3t)$

2c) $y'' + 3y' + 2y = e^{-t}$

$y_p = Ae^{-t}$ $\qquad Ae^{-t} - 3Ae^{-t} + 2Ae^{-t} = e^{-t}$

$y'_p = -Ae^{-t}$ $\qquad 0 = e^{-t}$

$y''_p = Ae^{-t}$ $\qquad$ Bad Guess

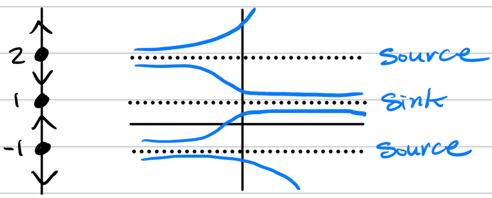
$y_p = Ate^{-t}$ $\qquad A(-2e^{-t} + te^{-t}) + 3A(e^{-t} - te^{-t}) + 2Ate^{-t} = e^{-t}$

$y'_p = A(e^{-t} - te^{-t})$ $\qquad -2Ae^{-t} + Ate^{-t} + 3Ae^{-t} - 3Ate^{-t} + 2Ate^{-t} = e^{-t}$

$y''_p = A(-e^{-t} - (e^{-t} - te^{-t}))$ $\qquad -2A + \cancel{At} + 3A - \cancel{3At} + \cancel{2At} = 1$

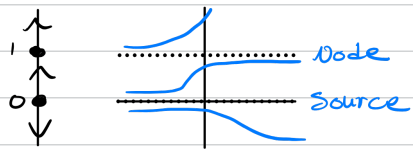
$\boxed{y_p = te^{-t}}$ $\qquad A = 1$

3a) $x' = (x^2 - 1)(x - 2)$ $\qquad x = 2, 1, -1$



3b) $x' = x^3 - 2x^2 + x$ $\qquad x = 0, 1, 1$

$x^2 - 2x + 1 = 0$



4a) $y'' - 2y' - 3y = 0$, $y(0) = 1$, $y'(0) = 0$

$Ys^2 - s - 2Ys + 2 - 3Y = 0$

$Y(s^2 - 2s - 3) = s - 2$ $\qquad Y = \frac{s-2}{(s-3)(s+1)}$ $\qquad \frac{A}{s-3} + \frac{B}{s+1} = \frac{s-2}{(s-3)(s+1)}$

$A(s+1) + B(s-3) = s - 2$

Let $s = -1$: $\qquad$ Let $s = 3$:

$-4B = -3$ $\quad B = \frac{3}{4}$ $\qquad 4A = 1$ $\quad A = \frac{1}{4}$

$\mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{4}\left(\frac{1}{s-3}\right) + \frac{3}{4}\left(\frac{1}{s+1}\right)\right)$

$\boxed{y = \frac{1}{4}e^{3t} + \frac{3}{4}e^{-t}}$

4b) $y'' + 2y = \cos(2t) \quad y(0) = 0 \quad y'(0) = 0$

$Ys^2 + 2Y = \frac{s}{s^2+4}$

$Y = \frac{s}{(s^2+2)(s^2+4)} \qquad (As+B)(s^2+4) + (Cs+D)(s^2+2) = s$

$As^3 + 4As + Bs^2 + 4B + Cs^3 + 2Cs + Ds^2 + 2D = s$

$A + C = 0 \qquad 4B - 2B = 0 \qquad B = 0$

$B + D = 0 \qquad D = 0$

$4A + 2C = 1 \qquad -4C + 2C = 1$

$4B + 2D = 0 \qquad C = -\frac{1}{2} \quad A = \frac{1}{2}$

$Y = \frac{\frac{1}{2}s}{s^2+2} - \frac{\frac{1}{2}s}{s^2+4} \qquad y = \frac{1}{2}\cos(\sqrt{2}t) - \frac{1}{2}\cos(2t)$

4c) $y'' - y = 2t, \quad y(0) = 0 \quad y'(0) = 0$

$Ys^2 - Y = \frac{2}{s} \qquad Y = \frac{2}{s^2(s^2-1)} = \frac{2}{s^2} \cdot \frac{1}{s^2-1}$

$\mathcal{L}^{-1}(F) = 2t$

$\mathcal{L}^{-1}(G) = \frac{1}{(s-1)(s+1)} \quad \frac{A}{s-1} + \frac{B}{s+1} \qquad A(s+1) + B(s-1) = 1 \qquad A = \frac{1}{2} \quad B = -\frac{1}{2}$

$= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$

$y = \mathcal{L}^{-1}(F) * \mathcal{L}^{-1}(G)$

$= 2t * \frac{1}{2}e^{t} + 2t * -\frac{1}{2}e^{-t}$

$= \boxed{\int_0^t \tau e^{t-\tau}\, d\tau - \int_0^t \tau e^{-(t-\tau)}}$

5a) $x' = -4x + 2y$
$y' = -3x + y$

$\begin{bmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{bmatrix} \qquad (-4-\lambda)(1-\lambda) + 6 = 0$

$-4 + 4\lambda - \lambda + \lambda^2 + 6 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda + 2)(\lambda + 1) = 0 \qquad \lambda = -2, -1$

For -2:
$-4x + 2y = -2x \qquad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

For -1:
$-4x + 2y = -x \qquad v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$\boxed{x = c_1 e^{-2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$

5b) $x' = y$

$y' = -2x + 2y$

$$\begin{bmatrix} 0-\lambda & 1 \\ -2 & 2-\lambda \end{bmatrix} \quad (0-\lambda)(2-\lambda)+2=0$$

$-2\lambda + \lambda^2 + 2 = 0$

$\lambda^2 - 2\lambda + 2 = 0$

$$\lambda = \frac{2 \pm \sqrt{4-4(2)}}{2} = 1 \pm i$$

For $\lambda = 1+i$:

$0x + 1y = (1+i)x \quad V_1 = \begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$x = e^t(\cos(t) + i\sin(t))\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$

$$x = \begin{matrix} e^t\cos t + ie^t\sin t \\ e^t\cos t + ie^t\sin t + ie^t\cos t - e^t\sin t \end{matrix}$$

$$\boxed{x = c_1 e^t \begin{bmatrix} \cos t \\ \cos t - \sin t \end{bmatrix} + c_2 e^t \begin{bmatrix} \sin t \\ \sin t + \cos t \end{bmatrix}}$$

5c) $x' = x + 4y$

$y' = -x - 3y$

$$\begin{bmatrix} 1-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} \quad (1-\lambda)(-3-\lambda)+4=0$$

$-3 - \lambda + 3\lambda + \lambda^2 + 4 = 0$

$\lambda^2 + 2\lambda + 1 = 0 \quad \lambda = -1, -1 \quad x + 4y = -x \quad V_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

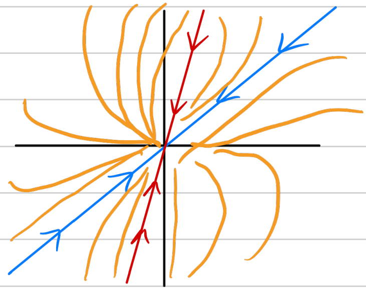
$(A - \lambda I)w = \alpha v$

$$\left(\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 4 \\ -2 \end{bmatrix} \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \alpha\begin{bmatrix} 4 \\ -2 \end{bmatrix} \quad \alpha = 2 \quad V_2 = \frac{1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

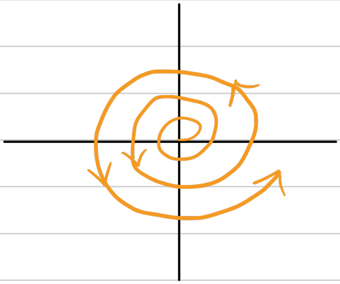
$$\boxed{x = c_1 e^{-t}\begin{bmatrix} 4 \\ -2 \end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \end{bmatrix}\right)}$$

6a) $\boxed{x = c_1 e^{-2t}\begin{bmatrix}1\\1\end{bmatrix} + c_2 e^{-t}\begin{bmatrix}2\\3\end{bmatrix}}$



$-2 > -1$

6b) $\lambda = 1 \pm i$



6c) $\boxed{x = c_1 e^{-t}\begin{bmatrix}4\\-2\end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix}1/2\\0\end{bmatrix} + t\begin{bmatrix}4\\-2\end{bmatrix}\right)}$