Noahs amazing guide to 3.2 (step functions)

1) Find the laplace of $f(t) = e^{5t-5} h(t-1)$

Note: $h(t-1) = U_1(t)$

Rewrite $f(t)$: $U_1(t) e^{5(t-1)}$

Chart: $U_c(t) f(t-c) = e^{-cs} F(s)$

Since $U_1(t) = U_c(t)$ our $c = 1$:

$F(s) = e^{-1s} \mathcal{L}(e^{5(t-1)})$

Since $t = t-c$ and $c = 1$ we can replace

$t-1$ with t ☆ Main Point of Step Functions

$F(s) = e^{-1s} \mathcal{L}(e^{5t}) = \boxed{e^{-s}\left(\frac{1}{s-5}\right)}$

2) Find the laplace of $f(t) = 9t \cdot h(t-6)$

$h(t-6) = U_6(t)$

Rewrite: $U_6(t)(9t)$

Chart: $U_c(t) f(t-c) = e^{-cs} F(s)$

$C = 6$

$F(s) = e^{-6s} \mathcal{L}(9t)$

We are looking to replace $t-6$ but only have $9t$. So we must Alter $9t$: $9(t-6) + 54$

This is the same function but now works.

Replace $t-6$ with t

$F(s) = e^{-6s} \mathcal{L}(9t + 54) = \boxed{e^{-6s}\left(\frac{9}{s^2} + \frac{54}{s}\right)}$

③ Find the laplace of $f(t) = \begin{cases} 0 & t < 6 \\ (t-6)^3 & t \geq 6 \end{cases}$

First use the definition of the step function to make an equation

$$f(t) = \begin{cases} \boxed{0} & t < \boxed{6} \\ \boxed{(t-6)^3} & t \geq \boxed{6} \end{cases}$$

$$f(t) = 0 + ((t-6)^3 - 0) U_6(t)$$

Clean up: $f(t) = U_6(t)(t-6)^3$

Now solve as normal

Chart: $U_c(t) f(t-c) = e^{-cs} F(s)$

$c = 6$

$F(s) = e^{-6s} \mathcal{L}((t-6)^3)$

Replace $(t-6)$ with t

$F(s) = e^{-6s} \mathcal{L}(t^3) = \boxed{e^{-6s}\left(\frac{6}{s^4}\right)}$

④ Find the function and laplace of:

$$f(t) = \begin{cases} \boxed{0} & 0 \le t < \boxed{8\pi} \\ \boxed{\sin(t-8\pi)} & t \ge \boxed{8\pi} \end{cases}$$

Use Definition (see #3)

$f(t) = 0 + (\sin(t-8\pi) - 0)U_{8\pi}(t)$

Clean up: $f(t) = U_{8\pi}(t)\sin(t-8\pi)$

Part A

Webwork wants $h(t-8\pi)\sin(t-8\pi)$

Now solve like normal

Chart: $U_c(t)f(t-c) = e^{-cs}F(s)$

$c = 8$

$F(s) = e^{-8s}\mathcal{L}(\sin(t-8\pi))$

Part B

Replace $t-8\pi$ with t

$F(s) = e^{-8s}\mathcal{L}(\sin(t)) = \boxed{e^{-8s}\left(\frac{1}{s^2+1}\right)}$

5 Find the inverse laplace of $F(s) = \frac{e^{-8s}}{s^2+s-6}$

Rewrite the equation so it looks more Familiar

$$F(s) = e^{-8s}\left(\frac{1}{(s-2)(s+3)}\right)$$

Chart: $e^{-cs}F(s) = U_c(t)f(t-c)$

$c = 8$

$$f(t) = U_8(t)\mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+3)}\right)$$

Partial Fraction

$$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$$

$A(s+3) + B(s-2) = 1$

Let $s = -3$: $-5B = 1$

$A = \frac{1}{5} \quad B = -\frac{1}{5}$

Let $s = 2$: $5A = 1$

$$f(t) = U_8(t)\mathcal{L}^{-1}\left(\frac{1}{5}\left(\frac{1}{s-2}\right) - \frac{1}{5}\left(\frac{1}{s+3}\right)\right)$$

$$f(t) = U_8(t)\left(\frac{1}{5}e^{2r} - \frac{1}{5}e^{-3r}\right)$$ NOT t

$r = (t-c) = (t-8)$

$$f(t) = \boxed{U_8(t)\left(\frac{1}{5}e^{2(t-8)} - \frac{1}{5}e^{-3(t-8)}\right)}$$

⑥ $y'' + 25y = g(t)$

$y(0) = 0$

$y'(0) = 0$

$$g(t) = \begin{cases} \boxed{t} & 0 \le t \le \boxed{4} \\ \boxed{0} & \boxed{4} \le t \le \infty \end{cases}$$

Use Definition (see #3)

$f(t) = t + (0 - t)U_4(t)$

Clean up: $f(t) = t - t\,U_4(t)$

Note: there are two laplaces

Chart: $U_c(t)f(t-c) = e^{-cs}F(s)$

$c = 4$

$G(s) = \frac{1}{s^2} - e^{-4s}\mathcal{L}(t)$ Need to Alter

$G(s) = \frac{1}{s^2} - e^{-4s}\mathcal{L}((t-4)+4)$

Replace $t-4$ with t

$G(s) = \frac{1}{s^2} - e^{-4s}\mathcal{L}(t+4) = \frac{1}{s^2} - e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$

Go back to the OG Eq

$y'' + 25y = g(t)$

Plug in $G(t)$

$\mathcal{L}(y'' + 25y) = \frac{1}{s^2} - e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$

$Ys^2 - 0s - 0 + 25Y = \frac{1}{s^2} - e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$

$Y(t)s^2 + 25Y(t) = \frac{1}{s^2} - e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$ PART A

$Y(s^2 + 25) = \frac{1}{s^2} - e^{-4s}\left(\frac{1+4s}{s^2}\right)$ Rewrote for PFD

$Y = \frac{1}{s^2(s^2+25)} - e^{-4s}\left(\frac{1+4s}{s^2(s^2+25)}\right)$ PART B

You now have to do two partial fractions

But they are almost the same (be smart)

$$\frac{1+4s}{s^2(s^2+25)} = \frac{As+B}{s^2+25} + \frac{Cs+D}{s^2}$$

$$(As+B)(s^2)+(Cs+D)(s^2+25) = 1+4s$$

$$As^3+Bs^2+Cs^3+25Cs+Ds^2+25D = 1+4s$$

$A+C=0 \qquad C=\frac{4}{25}$

$B+D=0 \qquad A=-\frac{4}{25}$

$25C=4 \qquad D=\frac{1}{25}$

$25D=1 \qquad B=-\frac{1}{25}$

$$= -\frac{4}{25}\left(\frac{s}{s^2+25}\right)-\frac{1}{25}\left(\frac{1}{s^2+25}\right) + \frac{4}{25}\left(\frac{s}{s^2}\right)+\frac{1}{25}\left(\frac{1}{s^2}\right)$$

Now do the other PFD

$$(As+B)(s^2)+(Cs+D)(s^2+25) = 1$$

$$As^3+Bs^2+Cs^3+25Cs+Ds^2+25D = 1$$

$A+C=0 \qquad C=0$

$B+D=0 \qquad A=0$

$25C=0 \qquad D=\frac{1}{25}$

$25D=1 \qquad B=-\frac{1}{25}$

$$= -\frac{1}{25}\left(\frac{1}{s^2+25}\right)+\frac{1}{25}\left(\frac{1}{s^2}\right)$$

Solving in Parts

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+25)}\right) = -\frac{1}{25(5)}\sin(5r)+\frac{1}{25}r$$

$$\mathcal{L}^{-1}\left(-e^{-4s}\left(\frac{1+4s}{s^2(s^2+25)}\right)\right) = \text{Next page } \ddot{\smile}$$

$$= -U_4(t)\left(-\frac{4}{25}\cos(5r) - \frac{1}{25(5)}\sin(5r) + \frac{4}{25} + \frac{1}{25}r\right)$$

Now, Put them together and let $r = t-4$

$$y = -\frac{1}{125}\sin(5(t-4)) + \frac{1}{25}(t-4) - U_4(t)\left(-\frac{4}{25}\cos(5(t-4))\right.$$
$$\left. - \frac{1}{125}\sin(5(t-4)) + \frac{4}{25} + \frac{1}{25}(t-4)\right)$$

Wow what an easy problem ☺

Webwork wants $U_4(t)$ to be $h(t-4)$

7 and 8

are like 3.1