Tuesday Final Review

Reading Questions   complexification   Repeated Eigens
Bifurcations   Linearity   Existance and Uniqueness
Phase Planes

Repeated Eigenvalues

Consider

$$x' = x + 4y \qquad \begin{bmatrix} 1-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} \qquad (1-\lambda)(-3-\lambda) + 4 = 0$$
$$y' = -x - 3y$$

$$-3 - \lambda + 3\lambda + \lambda^2 + 4 = 0$$
$$\lambda^2 + 2\lambda + 1 = 0$$
$$\lambda = -1, -1$$

For $\lambda = -1$:

$$x + 4y = -1x$$
$$4y = -2x$$
$$v_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

$$x_1 = e^{-t}\begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

Choose $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix} - \begin{bmatrix} 1-(-1) & 4 \\ -1 & -3-(-1) \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} 4 \\ -2 \end{bmatrix} \qquad \alpha = \frac{1}{2}$$

$$\frac{1}{1/2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = v_2$$

$$x_2 = e^{-t}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \end{bmatrix}\right)$$

$$x = c_1 e^{-t}\begin{bmatrix} 4 \\ -2 \end{bmatrix} + c_2 e^{-t}\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -2 \end{bmatrix}\right)$$

Consider

$$x' = 9x + 4y \qquad \begin{bmatrix} 9-\lambda & 4 \\ -9 & -3-\lambda \end{bmatrix} \quad (9-\lambda)(-3-\lambda) + 36 = 0$$
$$y' = -9x - 3y$$

$$-27 - 9\lambda + 3\lambda + \lambda^2 + 36 = 0$$
$$\lambda^2 - 6\lambda + 9 = 0 \qquad \lambda = 3, 3$$

For $\lambda = 3$: $\quad 4y = -6x$

$$9x + 4y = 3x \qquad V_1 = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

Let $\omega = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$X_1 = e^{3t}\begin{bmatrix} 4 \\ -6 \end{bmatrix}$$

$$\begin{bmatrix} 9 & 4 \\ -9 & -3 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -9 \end{bmatrix}$$

$$\begin{bmatrix} 6 \\ -9 \end{bmatrix} = \alpha\begin{bmatrix} 4 \\ -6 \end{bmatrix} \qquad \alpha = \frac{3}{2}$$

$$\frac{2}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 0 \end{bmatrix} = V_2$$

$$X_2 = e^{3t}\left(\begin{bmatrix} 2/3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -6 \end{bmatrix}\right)$$

$$X = e^{3t}\begin{bmatrix} 4 \\ -6 \end{bmatrix} + e^{3t}\left(\begin{bmatrix} 2/3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 4 \\ -6 \end{bmatrix}\right)$$

Consider

$x' = 2x + y \quad x(0) = 0 \qquad \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \quad (2-\lambda)(0-\lambda) + 1 = 0$

$y' = -x \quad y(0) = -5$

$\lambda^2 - 2\lambda + 1 = 0 \quad \lambda = 1, 1 \quad$ For $\lambda = 1$: $2x + y = x \quad y = -x \quad v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

let $w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \alpha = -1$$

$$-1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = v_2$$

$$X = c_1 e^t \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^t \left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

$x = -c_1 e^t - c_2 e^t - c_2 t e^t \qquad 0 = -c_1 - c_2 \qquad c_1 = 0$

$y = c_1 e^t + c_2 t e^t \qquad 0 = c_1 \qquad c_2 = 0$

$X = 0$

Complexification

Consider $y'' + 4y = \cos(3t)$ Find y_p:

$y_c = Ae^{3it}$

$y'_c = 3iAe^{3it}$

$y''_c = -9Ae^{3it}$

$-9Ae^{3it} + 4Ae^{3it} = e^{3it}$

$-5A = 1 \quad A = -\frac{1}{5}$

$y_c = -\frac{1}{5}e^{3it} = -\frac{1}{5}(\cos(3t) + i\sin(3t))$

$$\boxed{y_p = -\frac{1}{5}\cos(3t)}$$

Consider $y'' + 6y' + 7y = 3\sin(2t)$ Find y_p:

$y_c = Ae^{2it}$

$y'_c = 2iAe^{2it}$

$y''_c = -4Ae^{2it}$

$-4Ae^{2it} + 12iAe^{2it} + 7Ae^{2it} = 3e^{2it}$

$-4A + 12iA + 7A = 3$

$A(3+12i) = 3 \quad A = \frac{3}{3+12i} \cdot \frac{3-12i}{3-12i} = \frac{9-36i}{9+144} = \frac{1}{17} - \frac{4}{17}i$

$y_c = \left(\frac{1}{17} - \frac{4}{17}i\right)e^{2it} = \left(\frac{1}{17} - \frac{4}{17}i\right)(\cos(2t) + i\sin(2t))$

$= \frac{1}{17}\cos(2t) + \frac{1}{17}i\sin(2t) - \frac{4}{17}i\cos(2t) + \frac{4}{17}\sin(2t)$

$$\boxed{y_p = \frac{1}{17}\sin(2t) - \frac{4}{17}\cos(2t)}$$

Bifurcations

Consider $y' = c - y^2$

First find Bifurcation Points

$0 = c - y^2 \quad y^2 = c$

If $c > 0$:

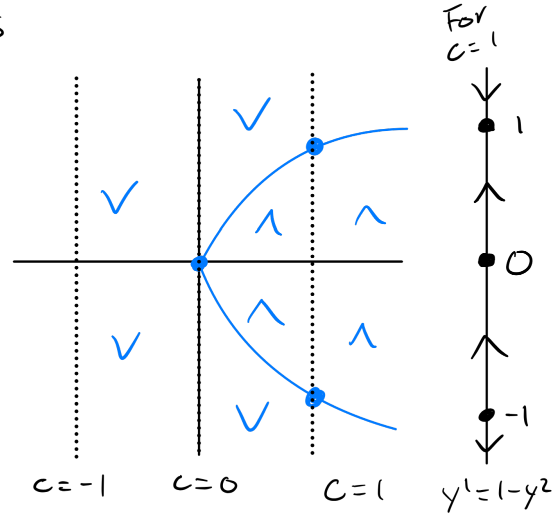
2 soln $\quad y = \pm\sqrt{c}$

If $c = 0$:

1 soln $\quad y = 0$

If $c < 0$:

0 Real soln $\quad y = \sqrt{-c}$



Consider $y' = (c - 4y^2)y$

$0 = (c - 4y^2)y$

Need to find Bifurcation Point

$y = 0, \quad y = \dfrac{0 \pm \sqrt{0 - 4 \cdot 4 \cdot c}}{8}$

If $c > 0$:

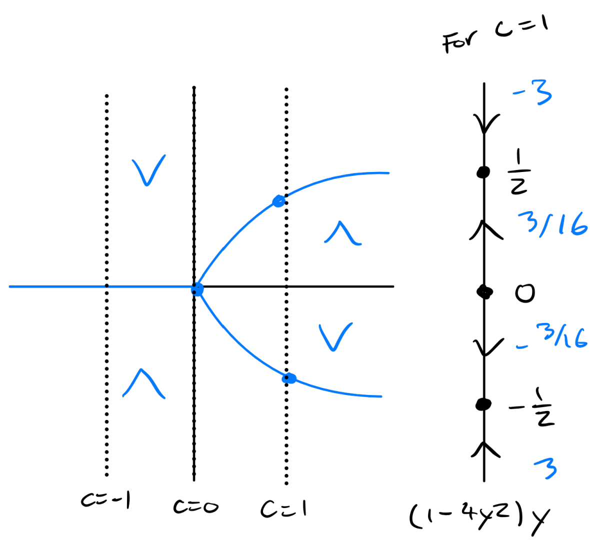
1 soln

If $c < 0$:

3 soln

If $c = 0$:

1 soln



Phase Planes

Make Up Numbers

- 2 Unique Real × 3
- complex × 3
- Repeated × 1

Existence And Uniqueness

— Continuousness

Use the existance and uniqueness theorem to decide if there is a soln

$$\begin{cases} x' = \frac{\sin(tx)}{x^2+t^2} \\ x(0) = 1 \end{cases}$$

continuous ∴ there is a unique soln

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}$$

$f(t,y) = y^{1/3}$ $\quad \frac{\partial y}{\partial f} = \frac{1}{3} y^{-2/3} = \frac{1}{y^{2/3}}$

f is continuous and $\frac{\partial x}{\partial f}$ is continuous away from $y = 0$ so EUT does not work

$$\begin{cases} y' = y^2 \\ y(0) = 1 \end{cases}$$

$f(t, y) = y^2$ $\quad \frac{\partial x}{\partial f} = 2y$

f and $\frac{\partial f}{\partial y}$ are always continuous so unique soln by EUT

$\frac{dy}{dt} = y^2$ $\quad y^{-2} dy = dt$

$-y^{-1} = t + C$ $\quad y = -\frac{1}{t+C}$ $\quad 1 = -\frac{1}{C}$ $\quad C = -1$

$y = -\frac{1}{t-1} = \frac{1}{1-t}$ Soln only defined $(-\infty, 1) \cup (1, \infty)$

Linearity

$$\mathcal{L}\left(t^2 + \cos(t)\right) = \mathcal{L}(t^2) + \mathcal{L}(\cos(t))$$

What else?

Convolution...

FOLODES...

Book Stuff...