Differential Equations Final Cheat Sheet Chapters 1 + 5

Folodes

$x' + x = t$

2 Methods
- Seperable
- Integrating Factor

Seperable

Eqn in form $x\frac{dy}{dx} = y$

Seperate Variables

Integrate

Solve for constant

Integrating Factor

Eqn in form $Ax' + Bx = t$

$P(x) = e^{\int B\,dt}$

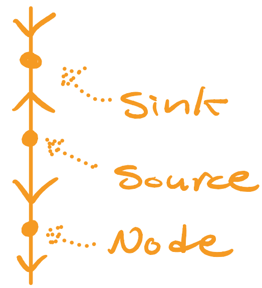
$(x \cdot P(x))' = t \cdot P(x)$

Integrate + Solve

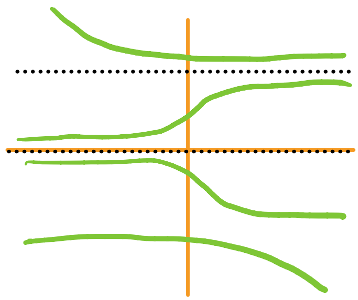
Graphing DE's and Systems of DE's can be graphed

Folodes

Folodes can be represented with a phase line and bifurcation diagram



(Phase Line)



(Bifurcation)

Systems

Systems can be represented with higher order graphs

For complex solns:

center

$0 \pm \beta i$

Spiral Sink

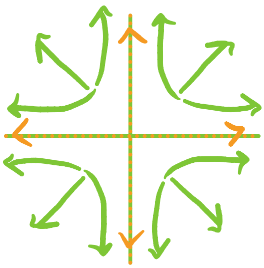
$-\alpha \pm \beta i$

Spiral Source

$+\alpha \pm \beta i$

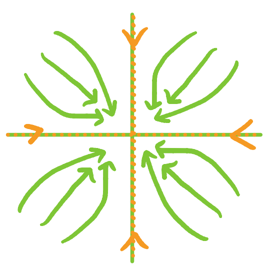
Otherwise:

Eigenvalues and Eigenvectors can be used to graph System Solns



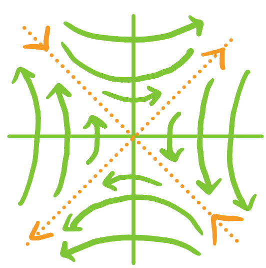
$\lambda_1 = 1 \quad v_1 = [1\ 0]$

$\lambda_2 = 1 \quad v_2 = [0\ 1]$



$\lambda_1 = -1 \quad v_1 = [1\ 0]$

$\lambda_2 = -1 \quad v_1 = [0\ 1]$



$\lambda_1 = 1 \quad v_1 = [1\ 1]$

$\lambda_2 = -1 \quad v_2 = [-1\ -1]$

Differential Equations Final Cheat Sheet — Chapter 2

Solodes

$x'' + x' + x = g(t)$

2 types

- Homogenous
- Non-Homogenous

Homogeneous ($g(t) = 0$)

characteristic eqn:

$x'' + x' + x \Rightarrow \lambda^2 + \lambda + 1$

Solve for λ

3 types of solns:

2 Real Soln:

$x = c_1 e^{\lambda t} + c_2 e^{\lambda t}$

1 Repeated Soln:

$x = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$

2 complex Soln:

$x = e^{\alpha t}(c_1 \cos\beta t + c_2 \sin\beta t)$

Non-Homogeneous

Get x_H

Create guess from $g(t)$

Linear:

$t^3 + t^2 + 1 \Rightarrow Ax^3 + Bx^2 + Cx + D$

Trig:

$\cos(t) \Rightarrow A\cos(t) + B\sin(t)$

Exponential:

$e^{7t} \Rightarrow Ae^{7t}$

Then plug your guess (x) and its derivatives (x', x'') into the equation and Solve for constants

Add Order (t) to guess if needed

Finally Plug the constants into your guess

$x_p = 4x^2 + 3x$

$x = x_H + x_p$

Complexification

For SOLODES w/ $g(t) = \text{Trig}$

Ex: $x'' + x = \sin \alpha t$

First, Real or imaginary?

$g(t) = \cos \Rightarrow$ Real

$g(t) = \sin \Rightarrow$ Imaginary

Next, Make complex guess

$x_c = Ae^{\alpha i t}$

Plug in and solve for constants

$x_c'' + x_c = \sin \alpha t$

Plug constant into guess

$x_c = (a \pm bi) e^{\alpha t}$

Use Eullers theorem

$x_c = (a \pm bi) \cdot (\cos \alpha t + i \sin \alpha t)$

Isolate Real/complex

$x_p = x_{Real} \cos + x_{Real} \sin$

$x_p = x_{imag} \cos + x_{imag} \sin$

Laplace Transforms

Laplace Transforms can be used to solve any DE

Formal Definition

$$\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

Used to find a laplace transform without shortcuts

Table Definitions

Known transforms can be applied from the table by translating variables

IVP's

Solving a DE with laplace requires performing a laplace of the function, reordering it, and then performing an inverse laplace

Given: $y'' + y' + y = t$

$$\mathcal{L}(f(t)) = Ys^2 - y(0)s - y'(0) + Ys - y(0) + Y = \frac{1}{s}$$

Separate Y's

$$Y(s^2+s+1) = \frac{1}{s} - y(0)s - y'(0) - y(0)$$

Solve for Y

$$Y = \frac{1}{s^3+s^2+s} - \frac{y(0)s - y'(0) - y(0)}{s^2+s+1}$$

Inverse laplace for Solution

$$\mathcal{L}^{-1}(Y) = y$$

AAAAAAAAA
AAAAAAAAA
AAAAAAAAA
AAAAAAAAA
AAAAAAAAA
AAAAAAAAA
AAAAAAAAA
AAAAAAAAA
AAAAAAAAH

Partial Fraction Decomposition

When inverse laplacing you may need to use partial fractions

$$\frac{1}{s^2} = \frac{A}{s} + \frac{B}{s^2} \qquad \frac{1}{(s^2+4)(s+1)} \Rightarrow$$

$$\frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} \qquad = \frac{As+B}{s^2+4} + \frac{C}{s+1}$$

Differential Equations Final Cheat Sheet Chapter 5

Systems of DEs Matricies can solve systems of DEs

Eigenvalues

Given:

$$A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$$

Add Lambdas:

$$\begin{bmatrix} a-\lambda & b \\ d & e-\lambda \end{bmatrix}$$

Find Determinant:

$$0 = (a-\lambda)(e-\lambda) - (bd)$$

Solve for λ_1, λ_2

Complex Eigenvalues

Given: $\lambda = \alpha \pm \beta i$ and $A = \begin{bmatrix} a & b \\ d & e \end{bmatrix}$

Find Eigenvector

$$ax + by = (\alpha + \beta i)x$$

Multiply

$$e^{\alpha t}(\cos \beta t + i \sin \beta t)[V]$$

Seperate into Real + Imaginary

$$X(t) = c_1 e^{\alpha t}[R] + c_2 e^{\alpha t}[I]$$

Eigenvectors

Given: λ_1, λ_2

Find Solutions to:

$$ax + by = \lambda_1 x$$

$$ax + by = \lambda_2 x$$

This gives V_1, V_2

Repeated Eigenvectors

Given λ_1 and V_1:

Choose ω (not a multiple of V_1)

Solve for α:

$$(A - \lambda I)\omega = \alpha V_1$$

$$V_2 = \frac{1}{\alpha}\omega$$

Create Solution

$$X = e^{\lambda t}(V_2 + t V_1)$$

Base Solution

Given $\lambda_1, V_1, \lambda_2, V_2$

Follow Pattern:

$$X = c_1 e^{\lambda_1 t}[V_1] + c_2 e^{\lambda_2 t}[V_2]$$

Can also be 2 equations:

$$x = c_1 e^{\lambda_1 t} V_{1x} + c_2 e^{\lambda_2 t} V_{2x}$$

$$y = c_1 e^{\lambda_1 t} V_{1y} + c_2 e^{\lambda_2 t} V_{2y}$$

Exponential can solve higher order DEs

$$e^{tA} = e^{\lambda t}\left(I + t(A-\lambda I) + \frac{t^2}{2!}(A-\lambda I)^2 + \dots + \frac{t^k}{k!}(A-\lambda I)^k\right)$$

where $k = n$ if $A = n \times n$

Solve for $e^{tA} = 3 \times 3$ matrix

n solutions can be made using direction vectors

$$X = c_1 e^{\lambda t}[e^{tA}_x] + c_2 e^{\lambda t}[e^{tA}_y]$$

Laplace Transforms

$f(t) = \mathcal{L}^{-1}(F(s))$	$F(s) = \mathcal{L}(f(t))$
1	$\frac{1}{s} \quad (s > 0)$
e^{at}	$\frac{1}{s-a} \quad (s > a)$
$t^n \quad$ (n is a positive integer)	$\frac{n!}{s^{n+1}} \quad (s > 0)$
$t^a \quad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}} \quad (s > 0)$
$\sin at$	$\frac{a}{(s^2+a^2)} \quad (s > 0)$
$\cos at$	$\frac{s}{(s^2+a^2)} \quad (s > 0)$
$\sinh at$	$\frac{a}{(s^2-a^2)} \quad (s > \|\| a \|\|)$
$\cosh at$	$\frac{s}{(s^2-a^2)} \quad (s > \|\| a \|\|)$
$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2} \quad (s > a)$
$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2} \quad (s > a)$
$t^n e^{at} \quad$ (n is a positive integer)	$\frac{n!}{(s-a)^{n+1}} \quad (s > a)$
$u_c(t)$	$\frac{e^{-cs}}{s} \quad (s > 0)$
$u_c(t) f(t-c)$	$e^{-cs} F(s)$
$e^{ct} f(t)$	$F(s-c)$
$f(ct)$	$(\frac{1}{c}) F(\frac{s}{c}) \quad (c > 0)$
$\delta(t-c) = \delta_c(t)$	e^{-cs}
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$