Whats next?

- Systems of DE
- going from $x'' + Ax' + Bx = 0$

  to $\begin{cases} x'' + Ax' + Bx = 0 \\ y'' + Cy' + Dy = 0 \end{cases}$

- Why? Real life is complex and has multiple systems

How to solve systems of DEs?

- Linear Algebra!
- Today is Review not solving

Vectors

- Remember how vectors have components?

  $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ genral $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [x_1, y_1]$

→ Scalar multiplication still works

$$c\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} cx_1 \\ cy_1 \end{bmatrix}$$

→ ∴ A vector is a 2x1 Matrix

Vectors and Matricies

→ Say we have Matrix $A = \begin{bmatrix} A & B \\ D & F \end{bmatrix}$ and vector $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ find $A\vec{v}$

$$\begin{bmatrix} A & B \\ D & F \end{bmatrix}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} Ax_1 & By_1 \\ Dx_1 & Fy_1 \end{bmatrix}$$

→ Try it: $A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$

$$= \begin{bmatrix} 2x(t) & 4y(t) \\ 6x(t) & 8y(t) \end{bmatrix}$$

→ Now lets use our new knowledge on a system of DE's

Consider $\begin{cases} x' = Ax + By \\ y' = Dx + Fy \end{cases}$

Look familiar?

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} A & B \\ D & F \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

→ Try It: $\begin{cases} x' = 2x + 4y \\ y' = 6x + 8y \end{cases}$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = X = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

→ Try it:

$$X = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{cases} x' = 3x + 6y \\ y' = 9x + 12y \end{cases}$$

Properties of Matrices

→ Now that we are more comfortable

→ Identity matrix

→ The identity matrix is a scalable 1-0 matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ldots$$

→ Determinant

→ you've already done this

given $\begin{bmatrix} a & b \\ d & f \end{bmatrix}$ $\det \begin{bmatrix} a & b \\ d & f \end{bmatrix} = af - bd$

→ Its that thing from the cross product!

→ Determinants are exteremely important but super easy

Try it: Given $A = \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix}$

find $|A|$:

$$(8 - 56) = \boxed{-48}$$

→ Inverse

→ If $A = \begin{bmatrix} a & b \\ d & f \end{bmatrix}$

then $A^{-1} = \frac{1}{af - bd} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

→ A matrix and its inverse share the following relationship:

$$A \cdot A^{-1} = I$$ (Identity matrix)

Try it: Given $A = \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix}$

$$\frac{1}{-48} \begin{bmatrix} 2 & 7 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{48} & -\frac{7}{48} \\ -\frac{8}{48} & -\frac{4}{48} \end{bmatrix}$$

→ Linear Independence

→ Defines uniqueness for a system of equations

→ we can use a determinant to check LI

Given

$\vec{x} = [x_1, y_1]$

$\vec{y} = [x_2, y_2]$ determine LI

Make matrix

$$x = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad y = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

$$\det \begin{bmatrix} x_1 & x_2 \\ y_1 & x_2 \end{bmatrix} = x_1 y_2 - y_1 x_2$$

If $x_1 y_2 - y_1 x_2 = 0$

The systems are not LI

If $x_1y_2 - y_1x_2 \neq 0$

The systems are LI

Try it: Given $\vec{x} = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

determine if $\vec{x}$ any $\vec{y}$ are LI

$$\left| \begin{bmatrix} 2 & 4 \\ 6 & 3 \end{bmatrix} \right| = 6 - 24 = \boxed{-18}$$

$\therefore$ LI

2x2 Matrix Math

→ Our basic operations $+, -, \times$ work on 2x2 matrices

Given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

→ Addition / Subtraction

$$A+B = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

$$A-B = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

→ Multiplication

$$A \times B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Think of B as two vectors

$$A \times B = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Add Example if needed

→ Division

→ Not really a thing but you may need to use inverse

So thats everything you were expected to already know...

Eigenvalues

- λ denotes an eigenvalue
- Eigenvalues are a value that help us find a solution to a system of DE's (think of λ from HSOLODES)
- To find eigenvalues from a system of DE's:

determinant

$$|A - \lambda I| = 0$$

Matrix

Eigenvalue

Identity Matrix $\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

Proof - Proof - Proof - Proof

Expand Expression:

$$\left| \begin{bmatrix} a & b \\ d & f \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

$$\left| \begin{bmatrix} a-\lambda & b \\ d & f-\lambda \end{bmatrix} \right| = 0$$

$$\det \begin{bmatrix} a-\lambda & b \\ d & f-\lambda \end{bmatrix} = (a-\lambda)(f-\lambda) - bd$$

$$\boxed{(a-\lambda)(f-\lambda) - bd = 0}$$

$\rightarrow$ Always gives 2 soln

Try it

given $\begin{cases} x' = 2x + 3y \\ y' = 4x - y \end{cases}$

find the eigenvalues

→ Make Matrix:

$$X = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

→ Add eigenvalues:

$$\begin{bmatrix} 2-\lambda & 3 \\ 4 & -1-\lambda \end{bmatrix}$$

→ Find determinant:

$$\left| \begin{bmatrix} 2-\lambda & 3 \\ 4 & -1-\lambda \end{bmatrix} \right| = (2-\lambda)(-1-\lambda) - 12 = 0$$

→ Solve:

$$-2 - 2\lambda + \lambda + \lambda^2 - 12 = 0$$

$$\lambda^2 - \lambda - 14 = 0$$

$$\frac{1 \pm \sqrt{1-4(1)(-14)}}{2}$$

$$\boxed{\lambda = \frac{1}{2} \pm \frac{\sqrt{57}}{2}}$$

Eigenvectors:

→ once you have found an eigenvalue, you can then find its respective eigenvectors

→ Given $A = \begin{bmatrix} a & b \\ d & f \end{bmatrix}$ and eigenvalue λ form a new system of Eqns:

$$\begin{aligned} ax + by &= \lambda x \\ dx + fy &= \lambda y \end{aligned}$$

Then solve for x and y

Try it:

Find Eigenvalues:

$$\begin{cases} x' = 2x - y \\ y' = -x + 2y \end{cases} = X = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\left| \begin{bmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{bmatrix} \right| = (2-\lambda)(2-\lambda) - 1 = 0$$

$4 - 2\lambda - 2\lambda + \lambda^2 - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0 \quad \lambda = 3, 1$

Find Eigenvectors:

For $\lambda = 3$:

$2x - y = 3x$
$-x + 2y = 3y$

Oh my that seems to be Elimination...

$$
\begin{array}{r}
2x - y = 3x \\
-2x + 4y = 6y \\
\hline
3y = 3x + 6y
\end{array}
$$

$V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$3y = -3x$

$y = -x$

$2x + x = 3x$

$x = 1 \quad y = -1$

For $\lambda = 1$:

$2x - y = x$
$-x + 2y = y$

$$\begin{array}{r} 2x - y = x \\ -\ 2x + 4y = 2y \\ \hline 3y = x + 2y \end{array}$$

$y = x$

$2x - x = x$

$x = 1 \quad y = 1$

$$V_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Therefore Two possible eigenvectors are

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for $\lambda = 3$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ for $\lambda = 1$