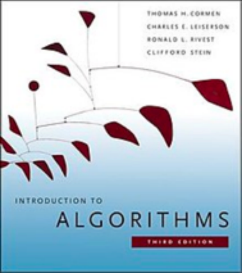
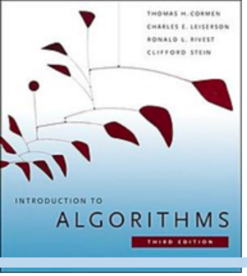


Introduction to Algorithms
14. Competitive Analysis

Hyungsoo Jung

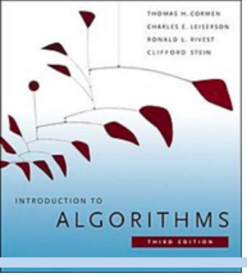
Scalable Computing Systems Laboratory
Hanyang University



Self-organizing lists

List L of n elements

- The operation ACCESS(x) costs $\text{rank}_L(x)$ = distance of x from the head of L.
- L can be reordered by transposing adjacent elements at a cost of 1.



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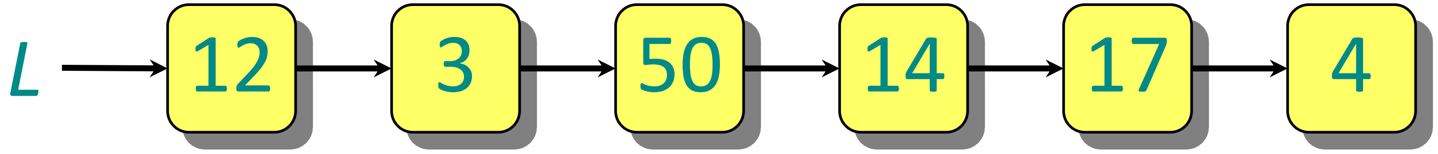

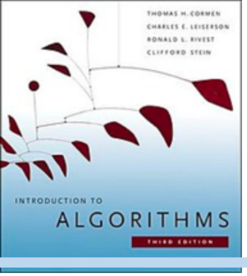

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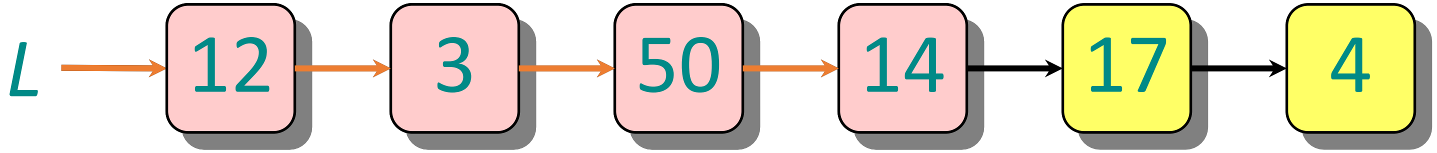
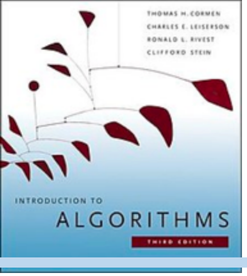


Accessing the element with key 14 costs 4.



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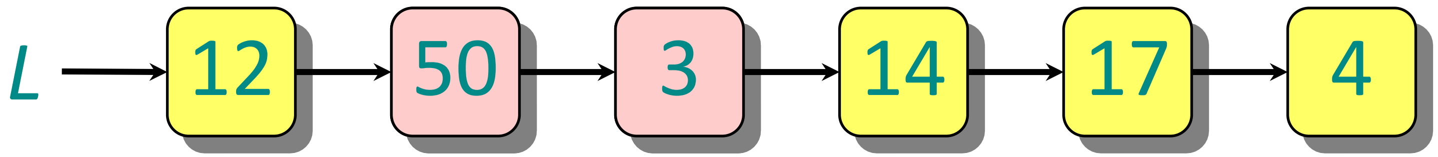


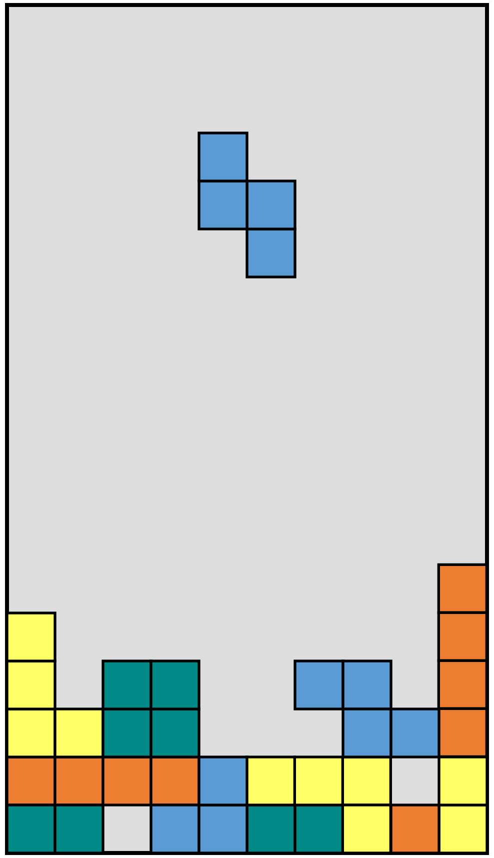
Transposing 3 and 50 costs 1.

On-line and off-line problems

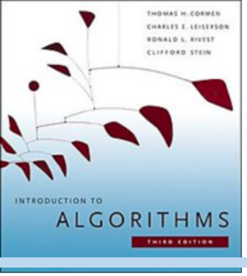
Definition. A sequence S of operations is provided one at a time. For each operation, an ***on-line*** algorithm A must execute the operation immediately without any knowledge of future operations (e.g., *Tetris*).

An ***off-line*** algorithm may see the whole sequence S in advance.

Goal: Minimize the total cost $C_A(S)$.



The game of Tetris

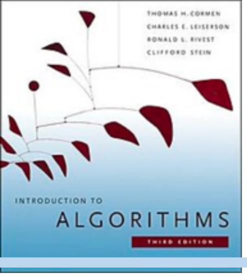


Worst-case analysis of self-organizing lists

An adversary always accesses the tail (nth) element of L. Then, for any on-line algorithm A, we have

$$C_A(S) = \Omega(|S| \times n)$$

in the worst case.



Average-case analysis of self-organizing lists

Suppose that element *x* is accessed with probability *p*(*x*). Then, we have

$$E[C_A(S)] = \sum_{x \in L} p(x) \cdot \mathrm{rank}_L(x),$$

which is minimized when *L* is sorted in decreasing order with respect to *p*.

Heuristic: Keep a count of the number of times each element is accessed, and maintain *L* in order of decreasing count.

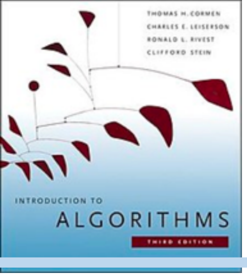
The move-to-front heuristic

Practice: Implementers discovered that the ***move-to-front (MTF)*** heuristic empirically yields good results.

IDEA: After accessing x, move x to the head of L using transposes:

$$\text{cost} = 2 \times \text{rank}_L(x) .$$

The MTF heuristic responds well to locality in the access sequence S.

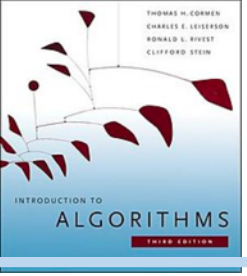


Competitive analysis

Definition. An on-line algorithm A is ***α-competitive*** if there exists a constant k such that for any sequence S of operations,

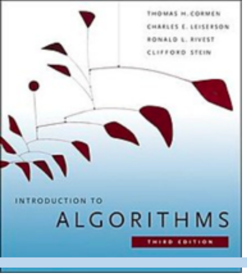
$$C_A(S) \le \alpha \times C_{\mathrm{OPT}}(S) + k\ ,$$

where OPT is the optimal off-line algorithm ("God's algorithm").



MTF is O(1)-competitive

Theorem. MTF is 4-competitive for self-organizing lists.



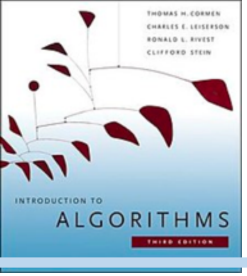
MTF is O(1)-competitive

Theorem. MTF is 4-competitive for self-organizing lists.

Proof. Let L_i be MTF's list after the ith access, and let $L_i{}^*$ be OPT's list after the ith access.

$$
\begin{aligned}
\text{Let}\quad c_i &= \text{MTF's cost for the } i\text{th operation} \\
&= 2 \times \operatorname{rank}_{L_{i-1}}(x) \text{ if it accesses } x; \\
c_i{}^* &= \text{OPT's cost for the } i\text{th operation} \\
&= \operatorname{rank}_{L_{i-1}{}^*}(x) + t_i\,,
\end{aligned}
$$

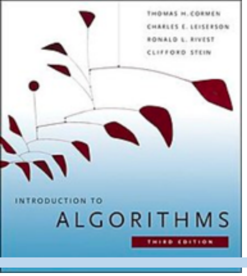
where t_i is the number of transposes that OPT performs.



Potential function

Define the potential function $\Phi : \{L_i\} \to \mathsf{R}$ by

$$
\begin{aligned}
\Phi(L_i) &= 2 \times |\{(x, y) : x \prec_{L_i} y \text{ and } y \prec_{L_i^*} x\}| \\
&= 2 \times \#\,\textbf{\textit{inversions}}\ .
\end{aligned}
$$



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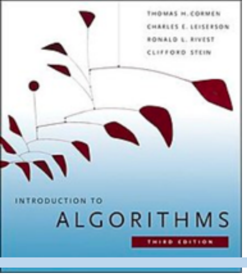
Example.

L_i → E → C → A → D → B

L_i^* → C → A → B → D → E



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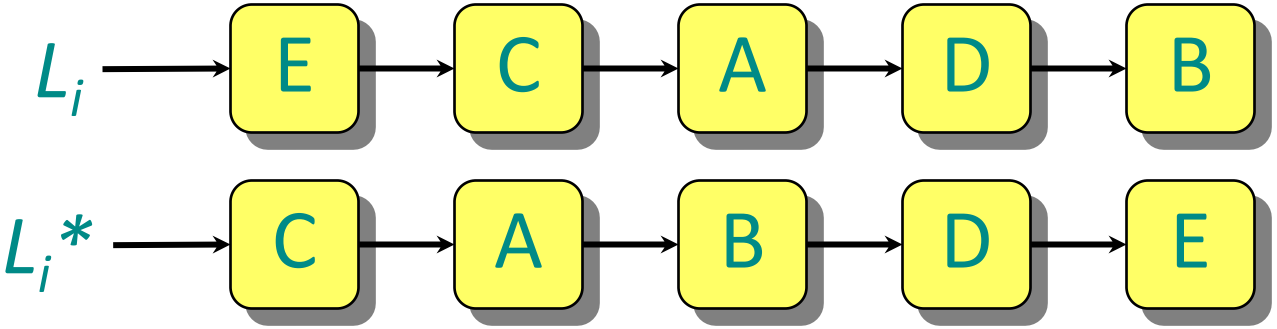
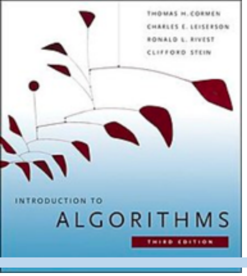


$$\Phi(L_i) = 2 \times |\{\ldots\}|$$



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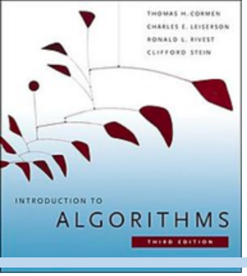
Example.

$L_i \to$ E $\to$ C $\to$ A $\to$ D $\to$ B

$L_i{}^* \to$ C $\to$ A $\to$ B $\to$ D $\to$ E

$$\Phi(L_i) = 2 \times |\{(\mathrm{E},\mathrm{C}), \ldots\}|$$



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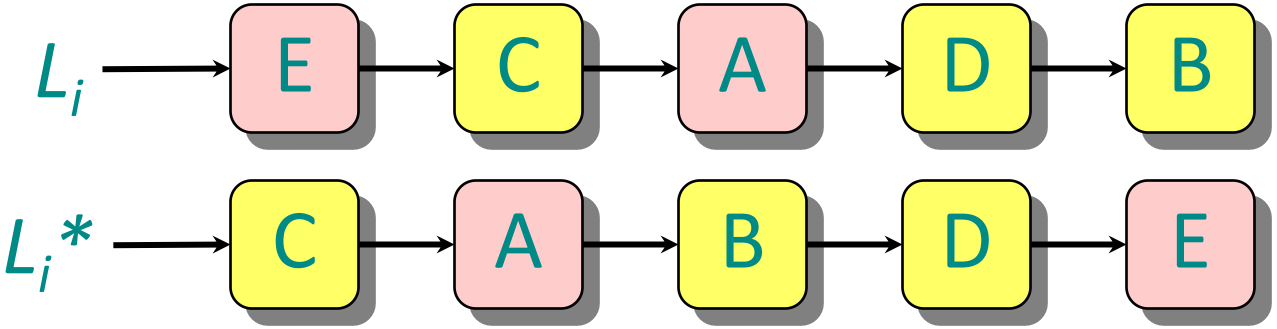
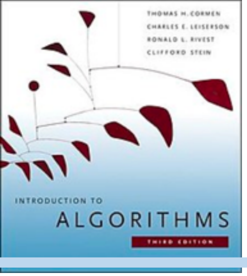


$$\Phi(L_i) = 2 \times |\{(\mathrm{E},\mathrm{C}), (\mathrm{E},\mathrm{A}), \ldots\}|$$



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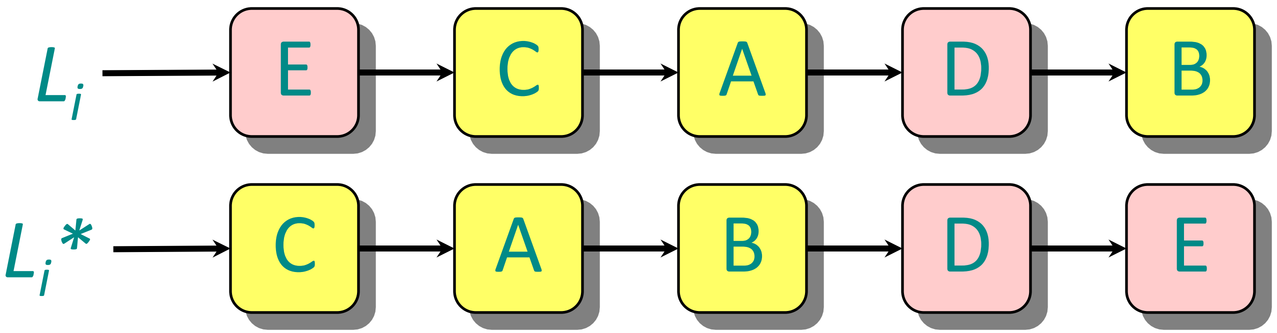
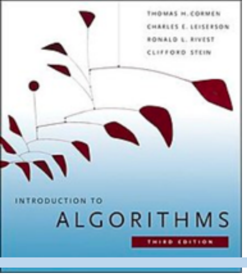


$$\Phi(L_i) = 2 \times |\{(\mathrm{E,C}), (\mathrm{E,A}), (\mathrm{E,D}), \ldots\}|$$



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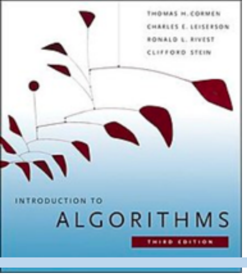

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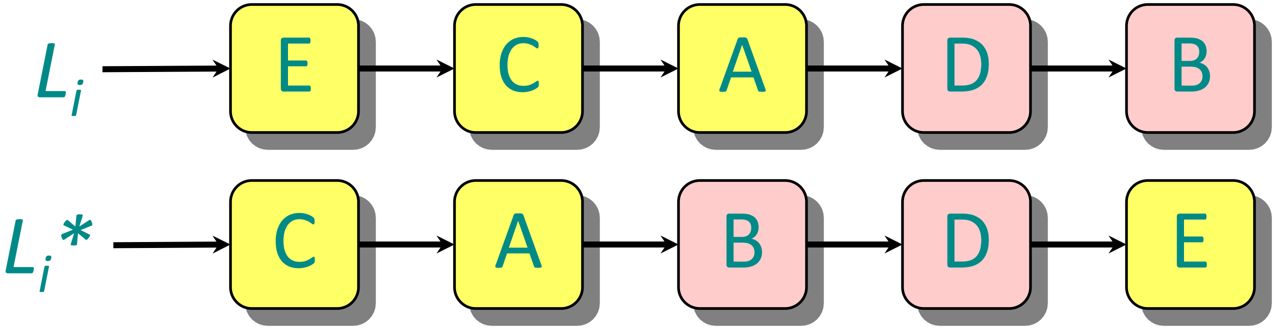

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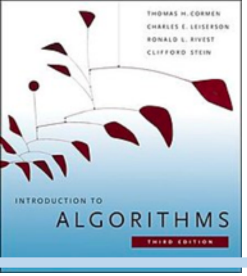

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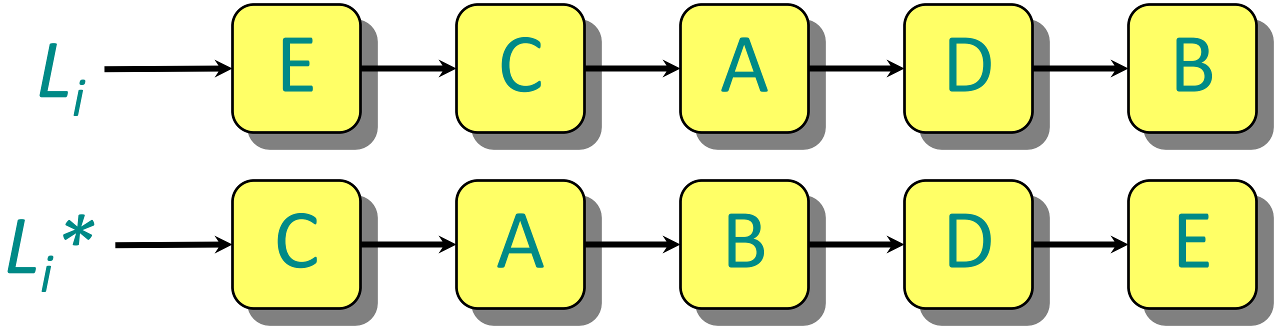
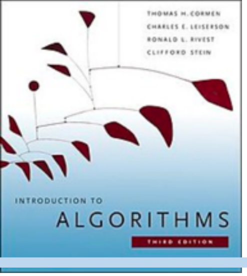


$$\begin{aligned}\Phi(L_i) &= 2 \times |\{(\mathrm{E,C}), (\mathrm{E,A}), (\mathrm{E,D}), (\mathrm{E,B}), (\mathrm{D,B})\}| \\ &= 10\,.\end{aligned}$$



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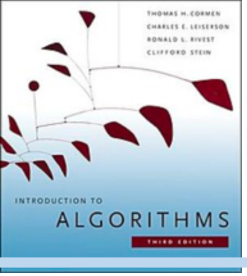

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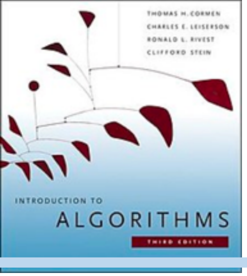
Note that

- $\Phi(L_i) \geq 0$ for $i = 0, 1, \ldots,$
- $\Phi(L_0) = 0$ if MTF and OPT start with the same list.



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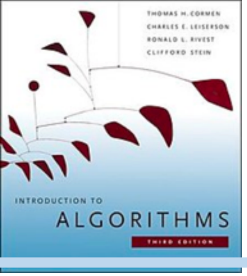
How much does Φ change from 1 transpose?

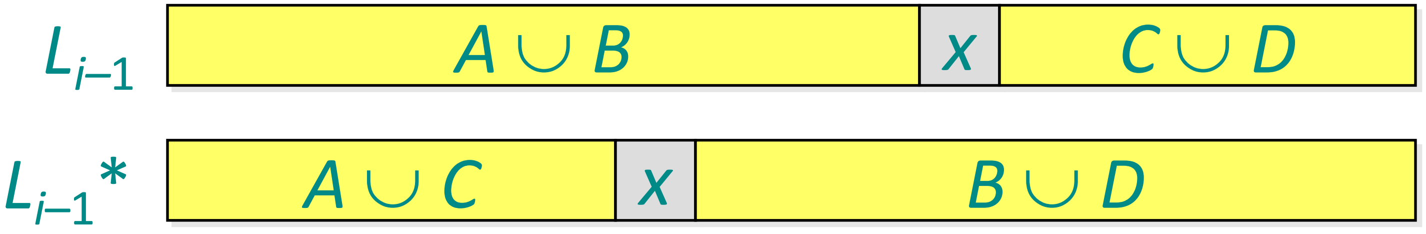
- A transpose creates/destroys 1 inversion.
- $\Delta\Phi = \pm 2$.

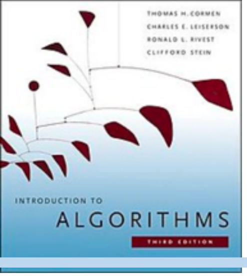


What happens on an access?

Suppose that operation i accesses element x, and define

$$
\begin{aligned}
A &= \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \prec_{L_{i-1}^*} x\}, \\
B &= \{y \in L_{i-1} : y \prec_{L_{i-1}} x \text{ and } y \succ_{L_{i-1}^*} x\}, \\
C &= \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \prec_{L_{i-1}^*} x\}, \\
D &= \{y \in L_{i-1} : y \succ_{L_{i-1}} x \text{ and } y \succ_{L_{i-1}^*} x\}.
\end{aligned}
$$





What happens on an access?

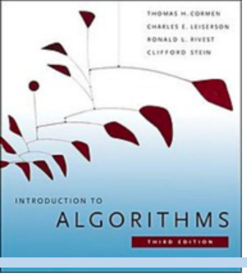
L_{i-1}	$A \cup B$	x	$C \cup D$

$r = \text{rank}_{L_{i-1}}(x)$

$L_{i-1}{}^*$	$A \cup C$	x	$B \cup D$

$r^* = \text{rank}_{L_{i-1}{}^*}(x)$

We have $r = |A| + |B| + 1$ and $r^* = |A| + |C| + 1$.



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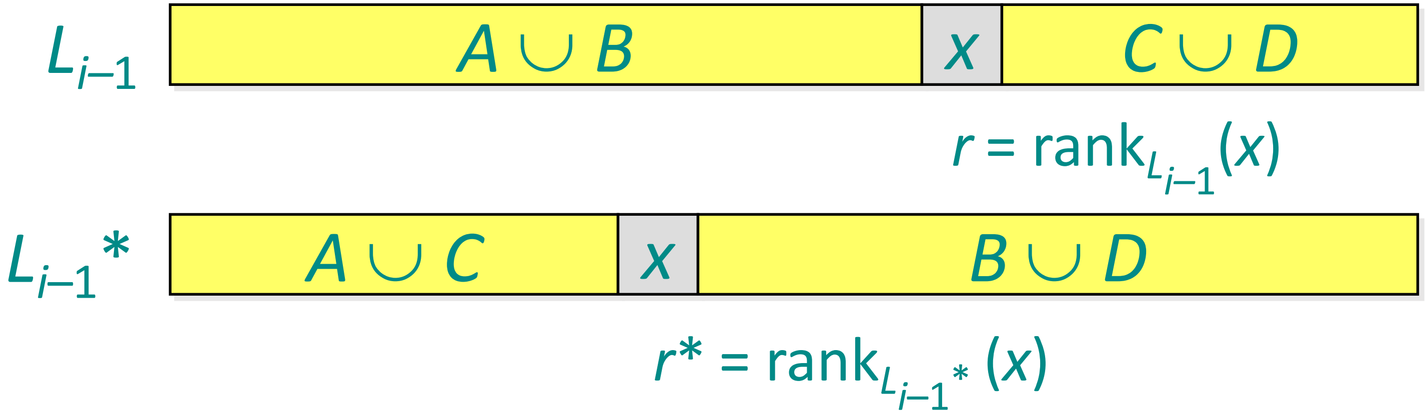

We have $r = |A| + |B| + 1$ and $r^* = |A| + |C| + 1$.

When MTF moves x to the front, it creates $|A|$ inversions and destroys $|B|$ inversions. Each transpose by OPT creates ≤ 1 inversion. Thus, we have

$$\Phi(L_i) - \Phi(L_{i-1}) \leq 2(|A| - |B| + t_i)\ .$$

Amortized cost

The amortized cost for the ith operation of MTF with respect to Φ is

$$\hat{c}_i = c_i + \Phi(L_i) - \Phi(L_{i-1})$$

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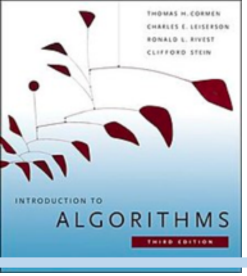
$$\begin{aligned} \hat{c}_i &= c_i + \Phi(L_i) - \Phi(L_{i-1}) \\ &\le 2r + 2(|A| - |B| + t_i) \end{aligned}$$



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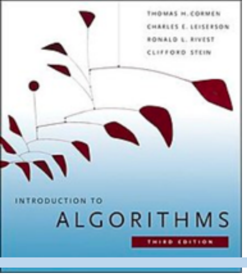
&\le 2r + 2(|A| - |B| + t_i) \\
&= 2r + 2(|A| - (r - 1 - |A|) + t_i)
\end{aligned}$$

(since $r = |A| + |B| + 1$)



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\end{aligned}
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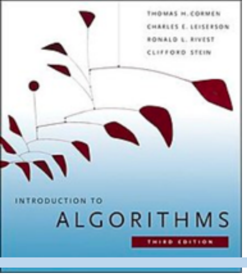

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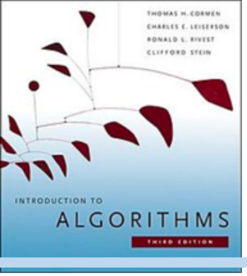
&= 2r + 4|A| - 2r + 2 + 2t_i \\
&= 4|A| + 2 + 2t_i \le 4|A| + 4t_i + 4 - 2 = 4(|A| + t_i + 1) - 2
\end{aligned}
$$

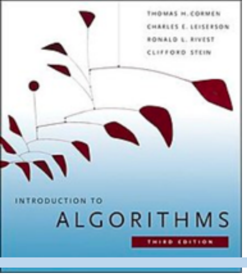


Amortized cost

The amortized cost for the ith operation of MTF with respect to Φ is

$$
\begin{aligned}
\hat{c}_i &= c_i + \Phi(L_i) - \Phi(L_{i-1}) \\
&\le 2r + 2(|A| - |B| + t_i) \\
&= 2r + 2(|A| - (r - 1 - |A|) + t_i) \\
&= 2r + 4|A| - 2r + 2 + 2t_i \\
&= 4|A| + 2 + 2t_i \le 4|A| + 4t_i + 4 - 2 = 4(|A| + t_i + 1)\text{-}2 \\
&\le 4(r^* + t_i)\ \text{-}2
\end{aligned}
$$

(since $r^* = |A| + |C| + 1 \ge |A| + 1$)



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&\le 4(r^* + t_i) - 2 \\
&= 4c_i^* - 2.
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The grand finale

Thus, we have

$$C_{\mathrm{MTF}}(S) = \sum_{i=1}^{|S|} c_i$$

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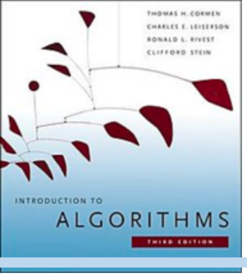
&\le \left(\sum_{i=1}^{|S|} 4c_i{}^* \right) + \Phi(L_0) - \Phi(L_{|S|})
\end{aligned}
$$



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\begin{aligned}
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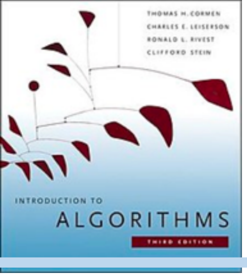
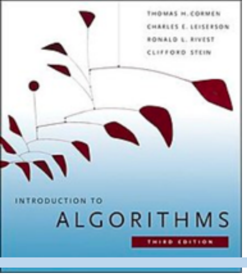
&= \sum_{i=1}^{|S|} \left(\hat{c}_i + \Phi(L_{i-1}) - \Phi(L_i) \right) \\
&\le \left(\sum_{i=1}^{|S|} 4 c_i{}^* \right) + \Phi(L_0) - \Phi(L_{|S|}) \\
&\le 4 \cdot C_{\mathrm{OPT}}(S),
\end{aligned}
$$

since $\Phi(L_0) = 0$ and $\Phi(L_{|S|}) \ge 0$. $\square$



Addendum

If we count transpositions that move x toward the front as “free” (models splicing x in and out of L in constant time), then MTF is 2-competitive.



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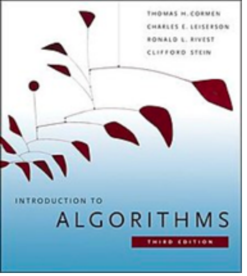

Thank You