Growth of Functions

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Analogy

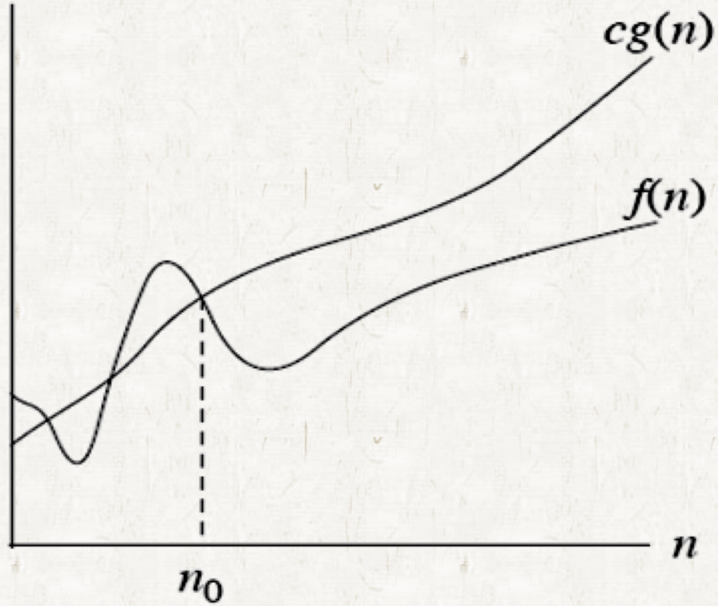
- **Analogy**
  - $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
  - $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
  - $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.

  - Examples

O-notation

- **O-notation**

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.

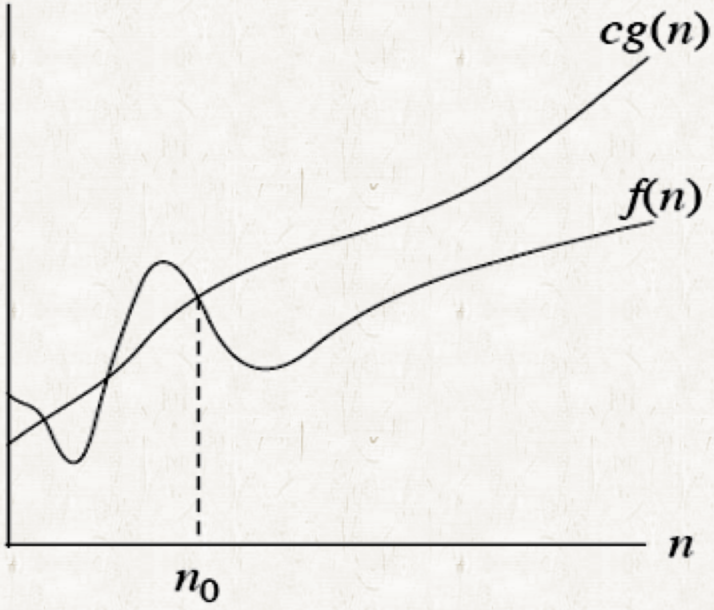


- For all values n to the right of n_0, the value of the function $f(n)$ is on or below $cg(n)$.

O-notation

- **O-notation**

$O(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le cg(n)$ for all $n \ge n_0\}$.



- $g(n)$ is called an ***asymptotic upper bound*** of $f(n)$.
- $f(n) = O(g(n))$ denotes $f(n) \in O(g(n))$.

O-notation

- Example

$$3n + 1 = \mathrm{O}(n^2)$$

- Show there are c and n_o such that $3n + 1 \le cn^2$ for all $n \ \ge \ n_0$.

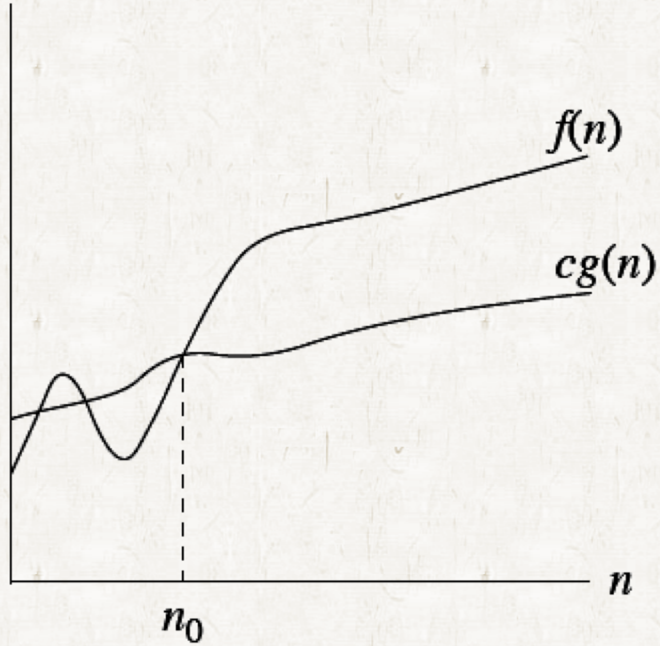
- Dividing by n^2 yields $\dfrac{3}{n} + \dfrac{1}{n^2} \le c.$

- The inequality holds for any $n \ge 1$ and $c \ge 4$.

Ω-notation

- **Ω-notation**

$\Omega(g(n)) = \{f(n)$: there exist positive constants c and n_0 such that $0 \le cg(n) \le f(n)$ for all $n \ge n_0\}$.



- For all values n to the right of n_0, the value of $f(n)$ is on or above $cg(n)$.
- $g(n)$ is called an ***asymptotic lower bound*** of $f(n)$.

Ω-notation

- Example

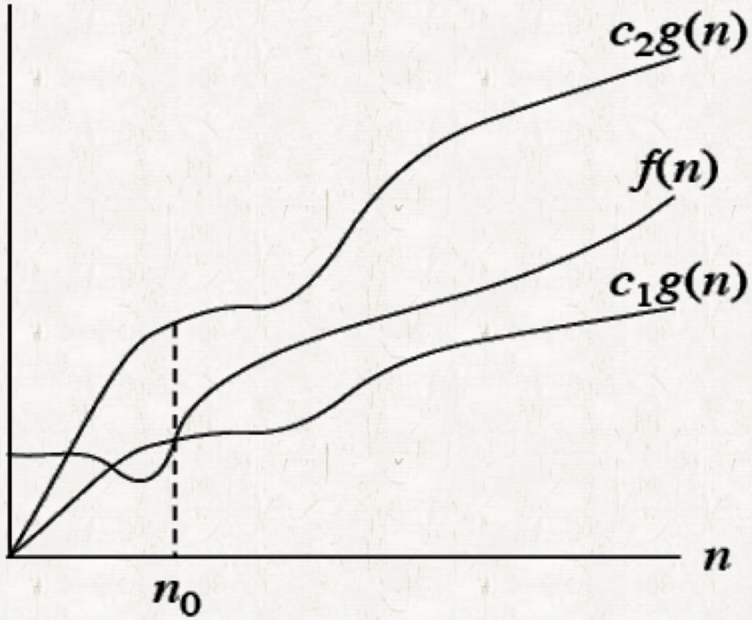
$$3n^2 - 4n + 1 = \Omega(n)$$

- Show there are c and n_o such that $3n^2 - 4n + 1 \geq cn$ for all $n \geq n_0$.
- Dividing by n yields $3n - 4 + \frac{1}{n} \geq c$.
- The inequality holds for any $n \geq 2$ and $c = 2$.

Θ-notation

- **Θ-notation**

$\Theta(g(n)) = \{f(n) :$ there exist positive constants c_1, c_2, and n_0 such that $0 \le c_1 g(n) \le f(n) \le c_2 g(n)$ for all $n \ge n_0\}$.



- For all values of n to the right of n_0, the value of $f(n)$ lies on or above $c_1 g(n)$ and on or below $c_2 g(n)$.
- $g(n)$ is called an a***symptotically tight bound*** for $f(n)$.

Θ-notation

- Example

$$\frac{1}{2}n^2 - 3n = \Theta(n^2)$$

To show there exist positive constants c_1, c_2 and n_o such that

$$c_1 n^2 \le \frac{1}{2}n^2 - 3n \le c_2 n^2 \text{ for all } n \ge n_0.$$

Dividing by n^2 yields $c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$

Θ-notation

- Example

$$c_1 \le \frac{1}{2} - \frac{3}{n} \le c_2.$$

- The right-hand inequality holds for $n \ge 1$ by choosing $c_2 \ge 1/2$.
- The left-hand inequality holds for $n \ge 7$ by choosing $c_1 \le 1/14$.
- Thus, by choosing $c_1 = 1/14$, $c_2 = 1/2$, and $n_0 = 7$,

we can verify that $\frac{1}{2}n^2 - 3n = \Theta(n^2)$

Θ-notation

- Example
  - Consider any quadratic function $f(n) = an^2 + bn + c$, where a, b, and c are constants and $a > 0$.
  - Throwing away the lower-order terms and ignoring the constant yields $f(n) = \Theta(n^2)$.
  - The reader may verify that $0 \le c_1n^2 \le an^2 + bn + c \le c_2n^2$ for all $n \ge n_0$. (Self-study)
  - In general, for any polynomial $p(n) = \sum_{i=0}^{d} a_i n^i$ where the a_i are constants and $a_d > 0$, we have $p(n) = \Theta(n^d)$.

Θ-notation

- **Theorem 3.1**

For any two functions $f(n)$ and $g(n)$, we have $f(n) = \Theta(g(n))$ if and only if $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

Notation

- **Notation in equations and inequalities**
  - $n = O(n^2)$ (set)
  - $T(n) = 2T(n/2) + \Theta(n)$ (element)
  - $2n^2+\Theta(n)=\Theta(n^2)$

Analogy

- **Analogy**
  - $f(n) = \Theta(g(n)) \approx f(n) = g(n)$ in degree.
  - $f(n) = O(g(n)) \approx f(n) \leq g(n)$ in degree.
  - $f(n) = \Omega(g(n)) \approx f(n) \geq g(n)$ in degree.
  - $f(n) = o(g(n)) \approx f(n) < g(n)$ in degree.
  - $f(n) = \omega(g(n)) \approx f(n) > g(n)$ in degree.

o-notation

- The asymptotic upper bound provided by O-notation may or may not be asymptotically tight.

- The bound $2n^2 = O(n^2)$ is asymptotically tight, but the bound $2n = O(n^2)$ is not.

- We use o-notation to denote an upper bound that is not asymptotically tight.

o-notation

- **o-notation**

$o(g(n))= \{f(n)$: for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq f(n) < cg(n)$ for all $n \geq n_0\}$.

- $f(n)$ is called an ***asymptotically smaller*** than $g(n)$.

- For example, $2n = o(n^2)$, but $2n^2 \neq o(n^2)$.

o-notation

- The main difference between O and o.
  - $f(n) = O(g(n))$, the bound $0 \le f(n) \le cg(n)$ holds for *some* constant $c > 0$
  - $f(n) = o(g(n))$, the bound $0 \le f(n) < cg(n)$ holds for *all* constants $c > 0$.

- Intuitively, in the o-notation, the function $f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity; that is,

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0$$

ω-notation

- ω-notation is used to denote a lower bound that is not asymptotically tight.

- **ω-notation**

$\omega(g(n)) = \{f(n)$: for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \leq cg(n) < f(n)$ for all $n \geq n_0\}$

- $f(n)$ is called an ***asymptotically larger*** than $g(n)$.
- So, $f(n) \in \omega(g(n))$ if and only if $g(n) \in o(f(n))$.

ω-notation

- **Example**
  - $n^2/2 = \omega(n)$, but $n^2/2 \neq \omega(n^2)$.

  - The relation $f(n) = \omega(g(n))$ implies that

    $$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

    if the limit exists.

  - That is, $f(n)$ becomes arbitrarily large relative to $g(n)$ as n approaches infinity.

Comparison of functions

- **Comparison of functions**
  - Transitivity
  - Reflexivity
  - Symmetry
  - Transpose symmetry

Comparison of functions

- **Comparison of functions**
  - Transitivity ($=, \leq, \geq, <, >$)
  - Reflexivity ($=, \leq, \geq$)
  - Symmetry ($=$)
  - Transpose symmetry ($\leq \leftrightarrow \geq, < \leftrightarrow >$)

Transitivity

- **Transitivity** ($=, \leq, \geq, <, >$)
  - $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ imply $f(n) = \Theta(h(n))$,
  - $f(n) = O(g(n))$ and $g(n) = O(h(n))$ imply $f(n) = O(h(n))$,
  - $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ imply $f(n) = \Omega(h(n))$,
  - $f(n) = o(g(n))$ and $g(n) = o(h(n))$ imply $f(n) = o(h(n))$,
  - $f(n) = \omega(g(n))$ and $g(n) = \omega(h(n))$ imply $f(n) = \omega(h(n))$.

Reflexivity

- **Reflexivity ($=, \leq, \geq$)**
  - $f(n) = \Theta(f(n))$
  - $f(n) = O(f(n))$
  - $f(n) = \Omega(f(n))$

Symmetry and transpose symmetry

- **Symmetry (=)**
  - $f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$.

- **Transpose symmetry ($\leq \leftrightarrow \geq$, $< \leftrightarrow >$)**
  - $f(n) = O(g(n))$ if and only if $g(n) = \Omega(f(n))$,
  - $f(n) = o(g(n))$ if and only if $g(n) = \omega(f(n))$.

Comparison of functions

- **Trichotomy**
  - For any two real numbers a and b, exactly one of the following must hold: $a < b$, $a = b$, $a > b$.
  - That is, any two numbers are comparable.
  - Are any two functions asymptotically comparable?
    - Is it possible $f(n) \neq O(g(n))$ and $f(n) \neq \Omega(g(n))$?
    - n and $n^{1+\sin n}$

Self-study

- **Exercise 3.1-1**
  - Show $\max(f(n), g(n)) = \Theta(f(n) + g(n))$
- **Exercise 3.1-4**
  - Is $2^{n+1} = O(2^n)$?
  - Is $2^{2n} = O(2^n)$?
- **Problem 3-2 for O, Θ, and Ω.**
  - Use $\lg(n!) = \Theta(n\lg n)$