Quicksort

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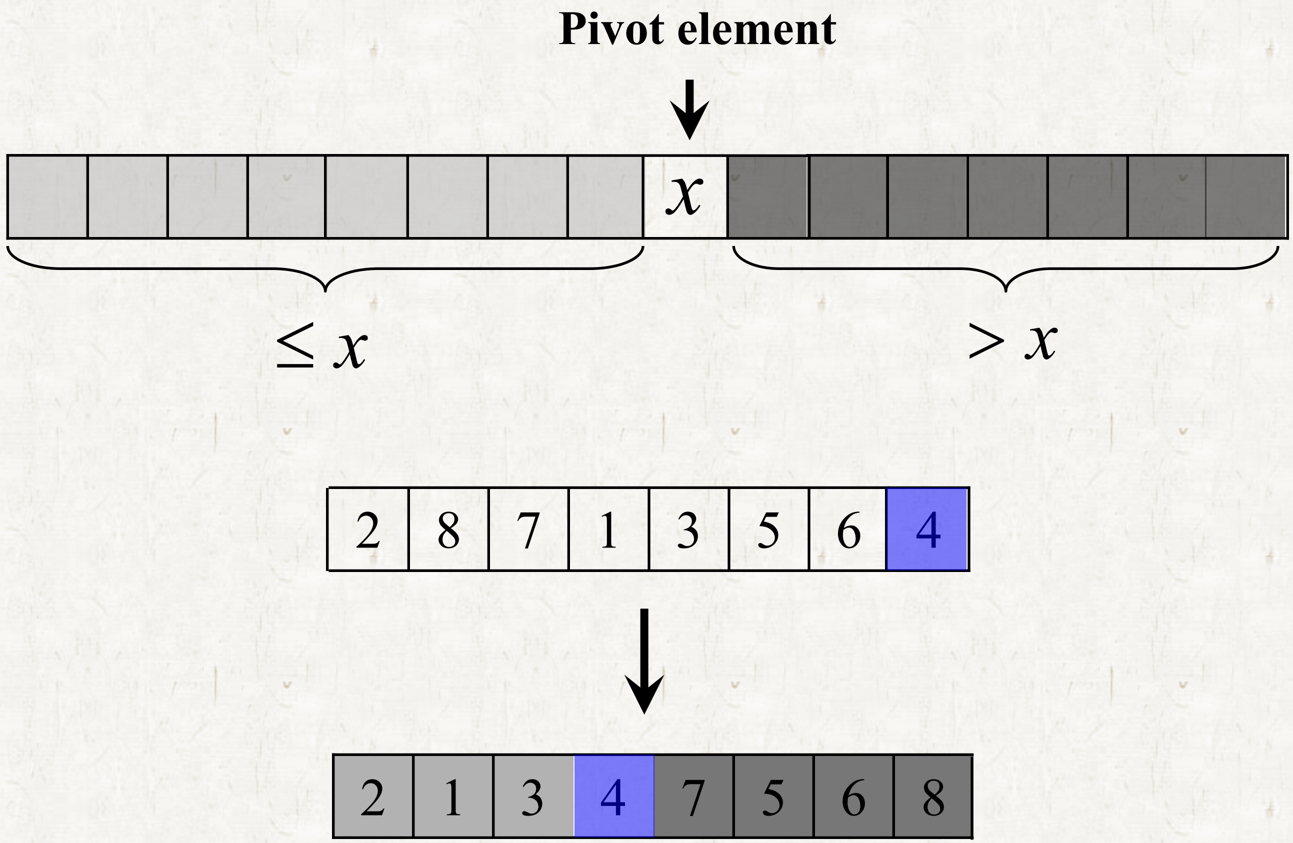
Quicksort

- **Divide-and-Conquer paradigm**

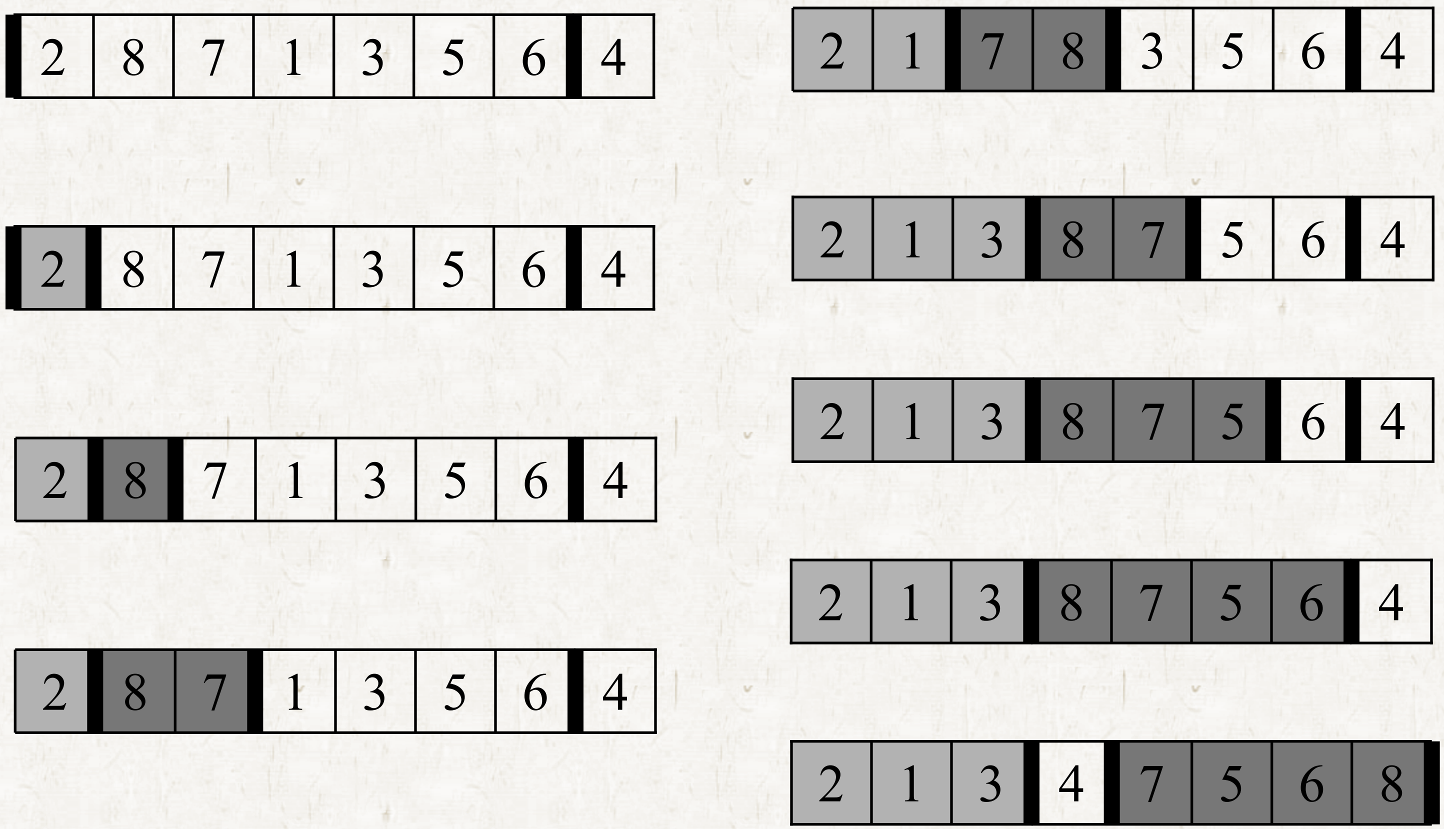
```
QUICKSORT(A, p, r)
 if p < r
      q = PARTITION(A, p, r)
      QUICKSORT(A, p, q - 1)
      QUICKSORT(A, q + 1, r)
```

Quicksort

- **Partition**



Quicksort



2 8 7 1 3 5 6 4
2 8 7 1 3 5 6 4
2 8 7 1 3 5 6 4
2 8 7 1 3 5 6 4
2 1 7 8 3 5 6 4
2 1 3 8 7 5 6 4
2 1 3 8 7 5 6 4
2 1 3 8 7 5 6 4
2 1 3 4 7 5 6 8

Quicksort

- **Partition**
  - $\Theta(n)$ time.

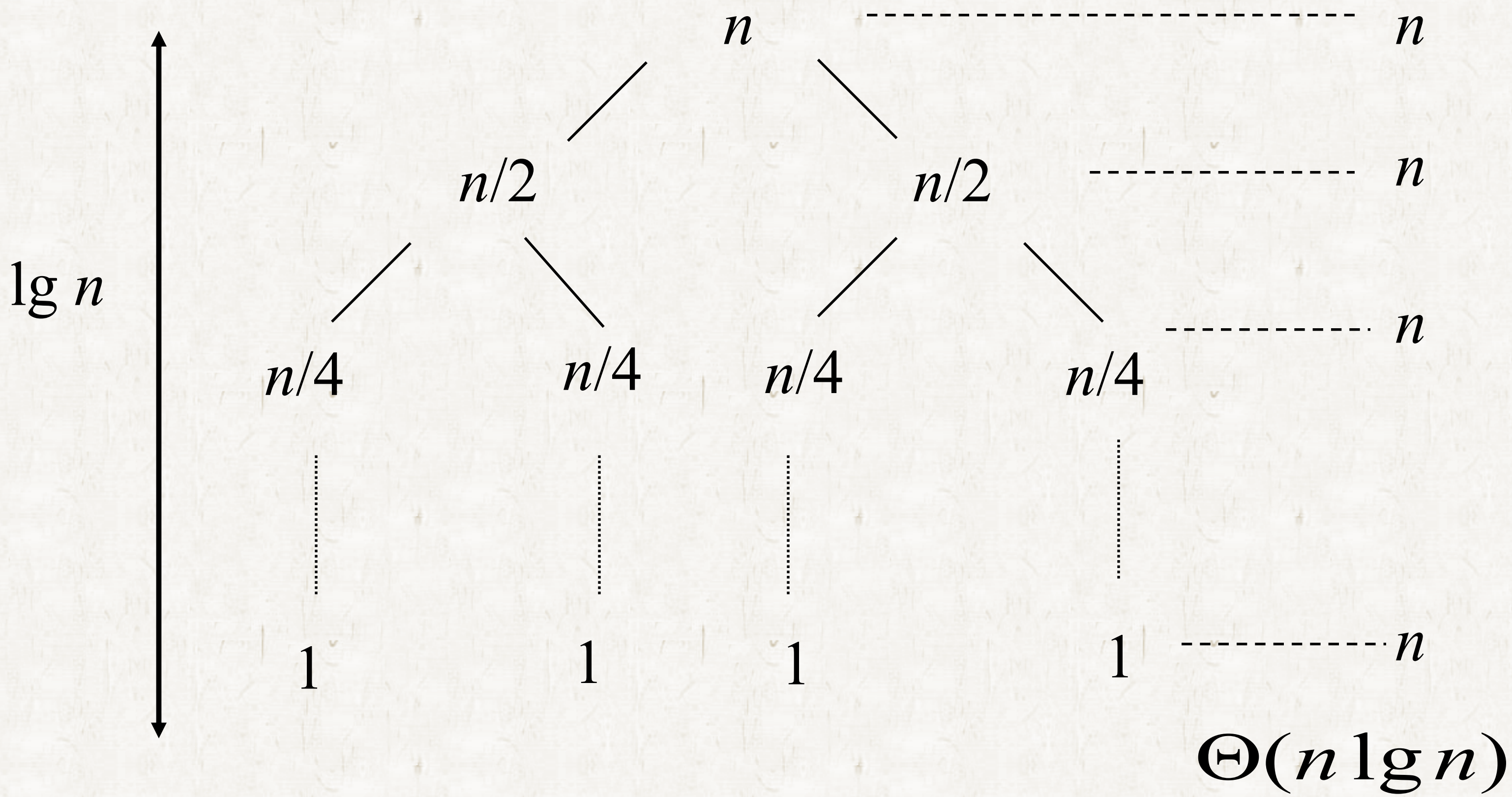
- ***Balanced partitioning*** vs. ***unbalanced partitioning***

Performance of quicksort

- **Balanced partitioning**
  - When PARTITION produces two subproblems of sizes $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil - 1$.
  - $T(n) \leq 2T(n/2) + \Theta(n) = O(n \lg n)$

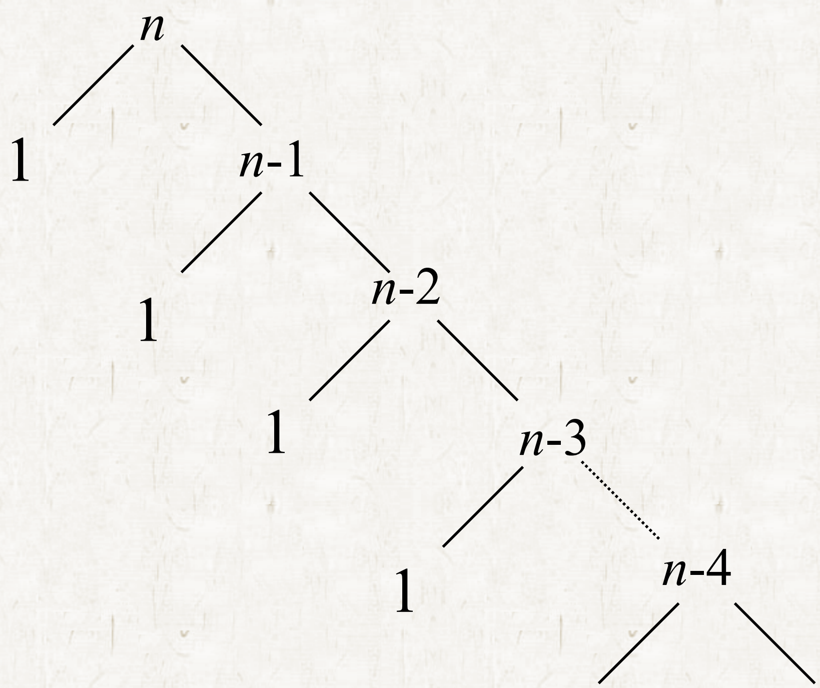
Performance of quicksort

- **Balanced partitioning**



Performance of quicksort

- **Unbalanced partitioning**



Performance of quicksort

- **Unbalanced partitioning**

$$
\begin{aligned}
T(n) &= T(n-1) + \Theta(n) \\
&= \sum_{k=1}^{n} \Theta(k) \\
&= \Theta\left(\sum_{k=1}^{n} k\right) \\
&= \Theta(n^2)
\end{aligned}
$$

Worst-case Analysis

- **Worst-case analysis**
  - Quicksort takes $\Omega(n^2)$ time in worst case.
    - Consider the unbalanced partitioning.
  - Is the unbalanced partitioning the worst case?

Worst-case Analysis

- **Worst-case analysis**
  - Show that the running time of quicksort is $O(n^2)$ by substitution method.

$$T(n) = \max_{0 \le q \le n-1} (T(q) + T(n-q-1)) + \Theta(n)$$

  - Show that $T(n) \le cn^2$ for some constant c.

$$T(n) \le \max_{0 \le q \le n-1} (cq^2 + c(n-q-1)^2) + \Theta(n)$$

$$= c \bullet \max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) + \Theta(n)$$

$$= c \bullet \max_{0 \le q \le n-1} (2q^2 - 2q(n-1) + (n-1)^2) + \Theta(n)$$

$$= c \bullet \max_{0 \le q \le n-1} (2(q - (n-1)/2)^2 + (n-1)^2/2) + \Theta(n)$$

Worst-case Analysis

- **Worst-case analysis**
  - The internal expression is maximized when $q = 0$ or n-1.

$$T(n) \le c \bullet \max_{0 \le q \le n-1} (2(q - (n-1)/2)^2 + (n-1)^2/2) + \Theta(n)$$

$$= c \bullet (n-1)^2 + \Theta(n)$$

$$= cn^2 - c(2n+1) + \Theta(n)$$

$$\le cn^2$$

  - We can pick the constant c large enough so that the $c(2n-1)$ term dominates the $\Theta(n)$ term.
  - Thus, $T(n)=O(n^2)$.

Average-case Analysis

- **Average-case analysis**

$$E[T(n)] = \frac{1}{n}\sum_{q=1}^{n}(E[T(q-1)] + E[T(n-q)]) + \Theta(n)$$
$$= \frac{2}{n}\sum_{q=2}^{n-1} E[T(q)] + \Theta(n)$$

- By substitution method, show $T(n) \le cn\lg n$ for some c.
  - Problem 7-3.

Average-case Analysis II

- **Average Case Analysis II**
  - Let X be the number of comparisons over the entire execution of QUICKSORT on an n-element array.
  - Then the average running time of QUICKSORT is
    - $O(n + \mathrm{E}[X])$.
  - We will not attempt to analyze how many comparisons are made in *each* PARTITION.
  - Rather, we will derive an overall bound on the total number of comparisons.

```
PARTITION(A, p, r)
    x = A[ r ]
    i = p – 1
    for j = p to r – 1
        if A[ j ] ≤ x
            i = i + 1
            exchange A[ i ] with A[ j ]
    exchange A[ i + 1 ] with A[ j ]
    return i + 1
```

Average-case Analysis II

- Let z_i denote the ith smallest element in the sorted array.
- $Z_{ij} = \{z_i, z_{i+1}, \ldots, z_j\}$
- Each pair of elements z_i and z_j is compared at most once.
  - An element is compared only to the pivot element in each PARTITION.
  - The pivot element used in a PARTITION is never again compared to any other elements.

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\}$$

Average-case Analysis II

$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \Pr\{z_i \text{ is compared to } z_j\}$$

- $\Pr\{z_i$ is compared to $z_j\}$
  - $\Pr\{z_i$ or z_j is first pivot chosen from $Z_{ij}\}$

$$= \frac{2}{j-i+1}$$

$$E[x] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

Average-case Analysis II

$k = j - i$, the harmonic series

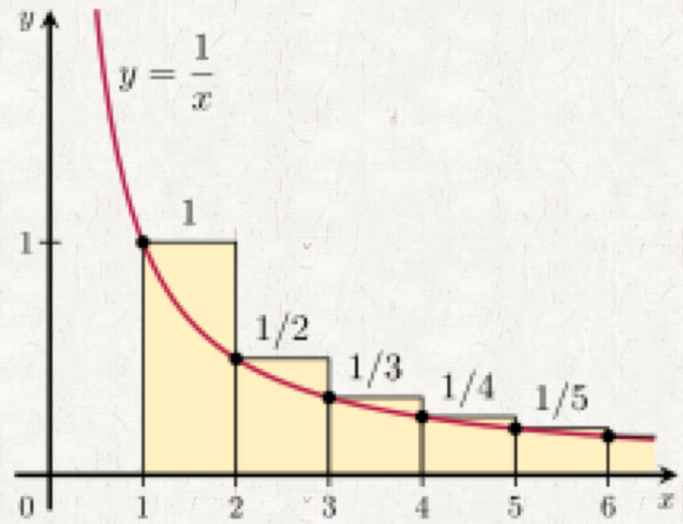
$$E[X] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}$$

$$= \sum_{i=1}^{n\text{-}1} \sum_{k=1}^{n-i} \frac{2}{k+1}$$

$$< \sum_{i=1}^{n\text{-}1} \sum_{k=1}^{n} \frac{2}{k}$$

$$= \sum_{i=1}^{n-1} O(\lg n)$$

$$= O(n \lg n)$$



equation(A.7)

$H_n = \sum_{k=1}^{n} \frac{1}{k} = \lg n + O(1)$

proof: p. 1153

Randomized quicksort

RANDOMIZED-PARTITION(A, p, r)

1. i = RANDOM(p, r)
2. exchange $A[r]$ with $A[i]$
3. **return** PARTITION(A, p, r)

Randomized quicksort

```
RANDOMIZED-QUICKSORT(A, p, r)
if p < r
    q = RANDOMIZED-PARTITION(A, p, r)
    RANDOMIZED-QUICKSORT(A, p, q - 1)
    RANDOMIZED-QUICKSORT(A, q + 1, r)
```

Self-study

- **Exercise 7.1-2**
  - Balanced partition with same elements
- **Exercise 7.2-4**
  - Sorting almost-sorted input

Programming exercise

- **Insertion sort + Quicksort**
  - **Exercise 7.4-5**
- **Improved Quicksort by**
  - **Tail recursion removal and stack depth reduction (Problem 7-4)**
  - **Median of 3 partitioning (Problem 7-5, p162)**