Single-Source Shortest Paths

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Definition

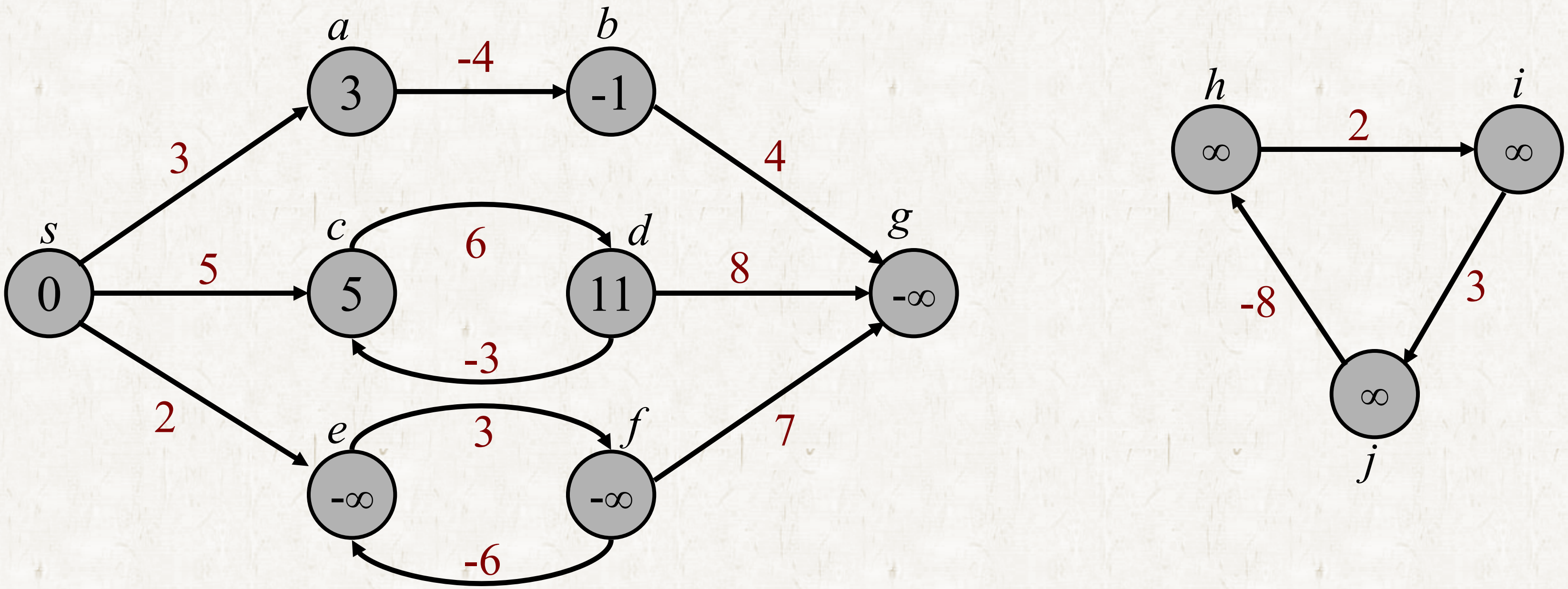
- ***Edge weight***
- ***Path weight***
  - The sum of all edge weights in the path.
- ***A Shortest path*** from u to v.
  - A path from u to v whose weight is the smallest.
  - Vertex u is the ***source*** and v is the ***destination***.
- ***The Shortest-path weight*** from u to v.
  - The weight of a shortest-path from u to v
  - $\delta(u,v)$

Definition

- **Shortest-path problems**
  - Single-source & single-destination
  - Single-source
  - Single-destination
  - All pairs

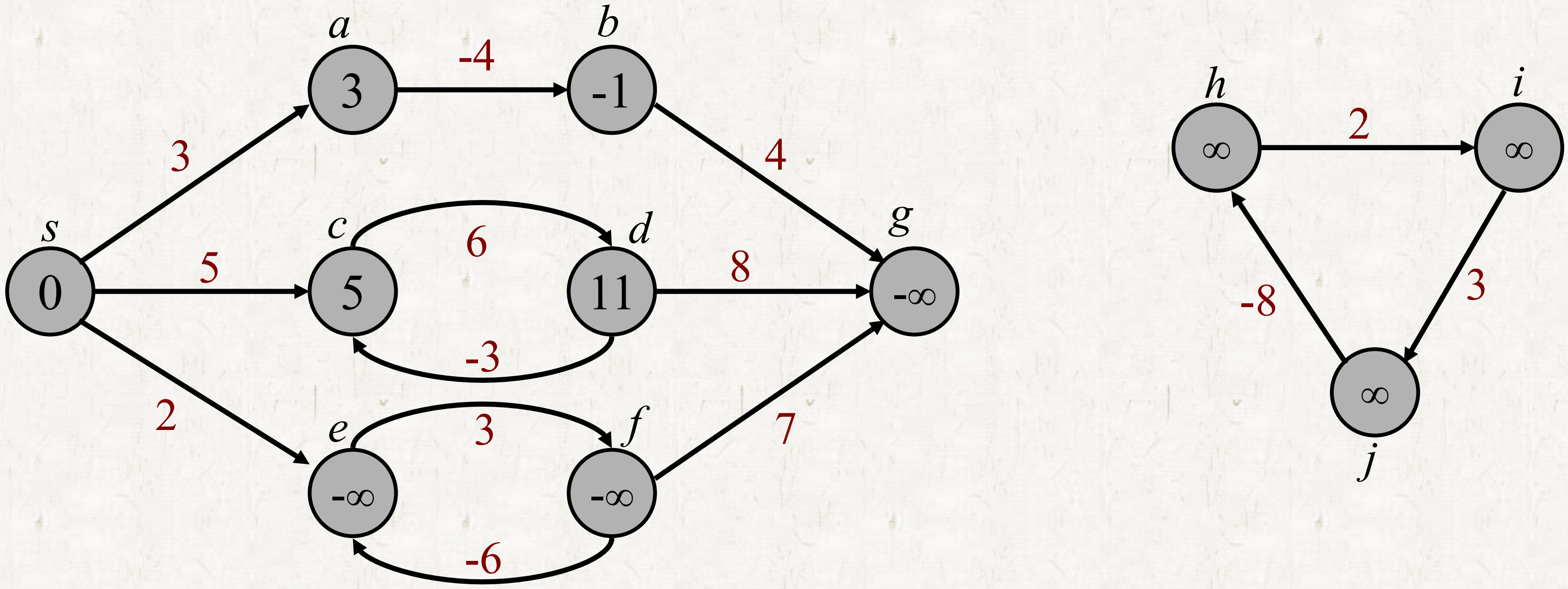
Negative-weight edges

- What is a shortest path from *s* to *g*?



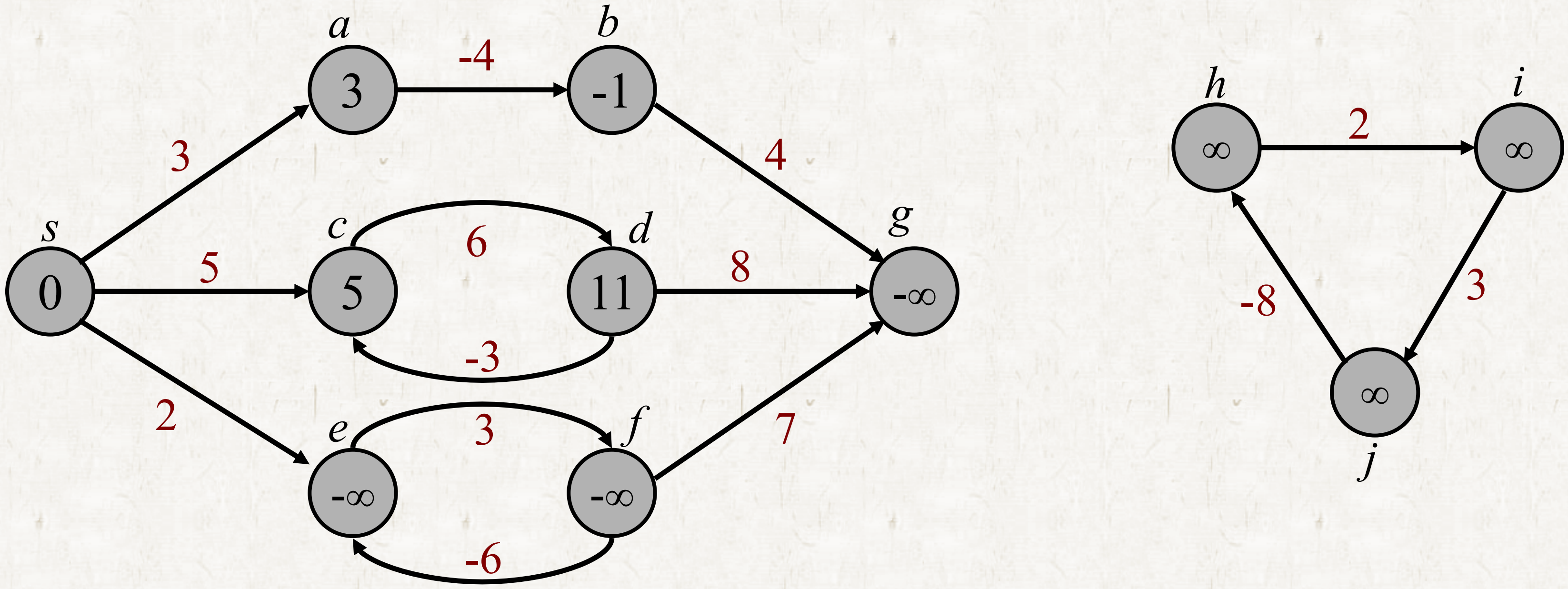
Negative-weight edges

- Do all negative-weight edges make problems?
- Do all negative-weight cycles make problems?



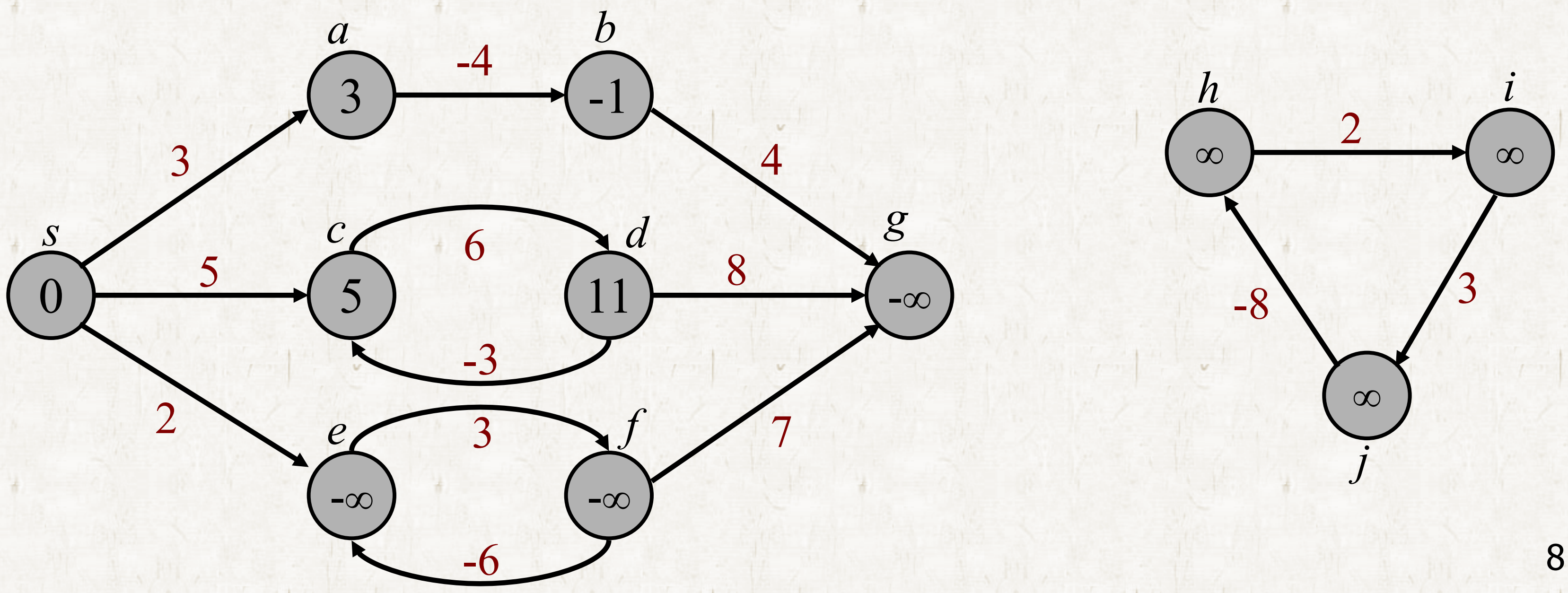
Negative-weight edges

- Do all negative-weight cycles reachable from the source make problems?



Negative-weight edges

- Single-source shortest paths can be defined if there are not any ***negative-weight cycles reachable from the source.***

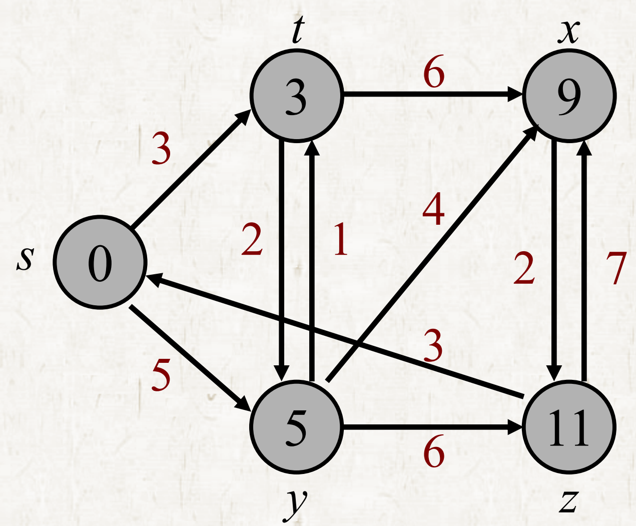


Cycles

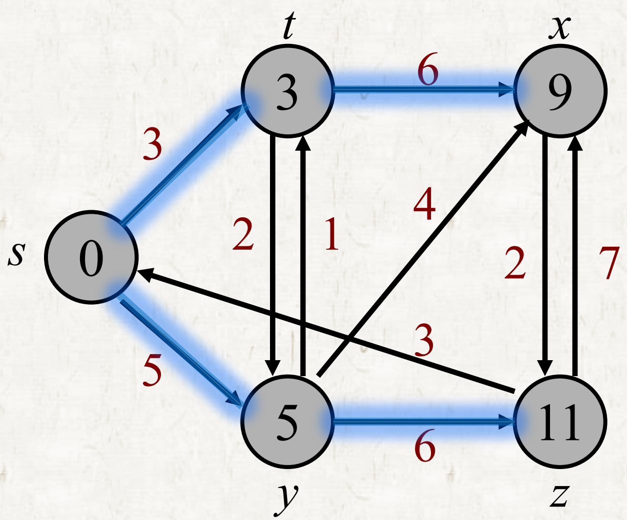
- **Cycles**
  - A shortest path does not include cycles.
  - A shortest-path length is at most $|V|$-1.

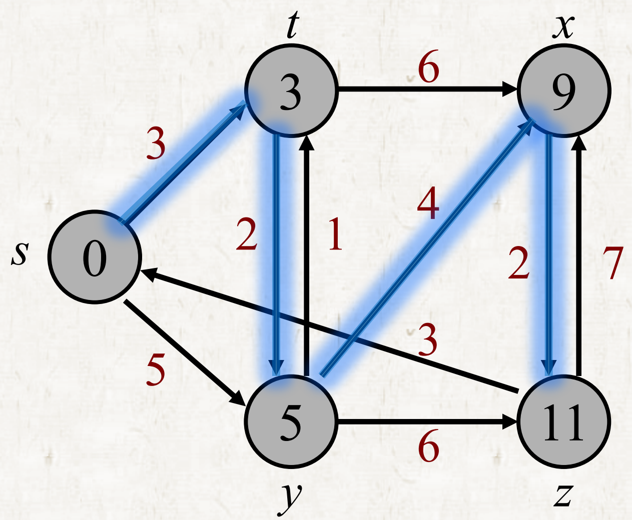
Predecessor subgraph

- **Predecessor subgraph**
  - Shortest-path tree
  - Optimal substructure



Predecessor subgraph

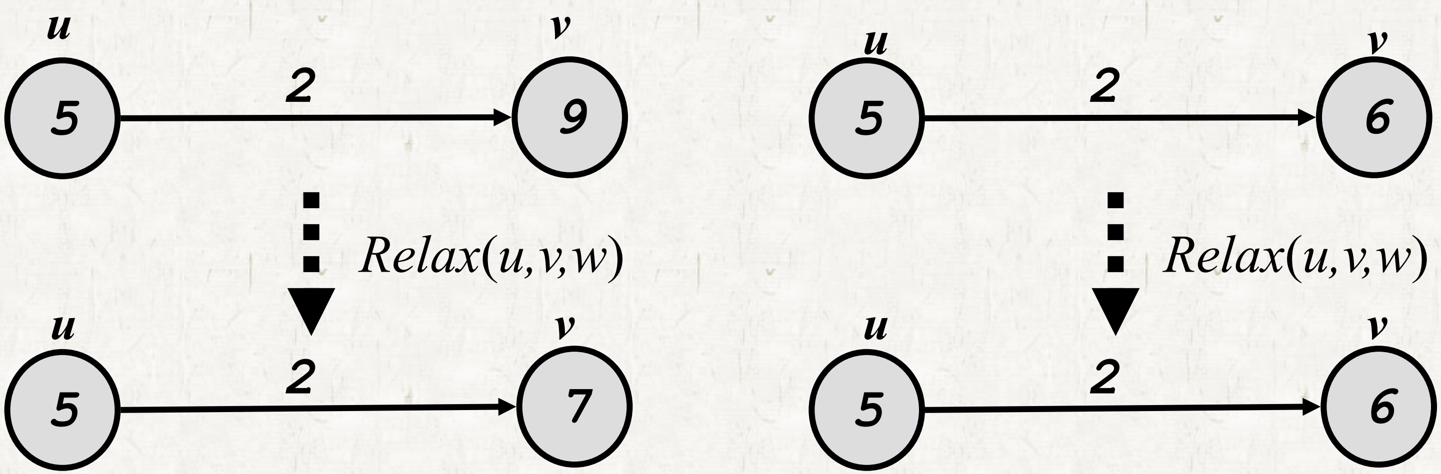




Relaxation

- **Relaxation on (*u*, *v*)**

  if (d[u] + w[u, v] < d[v])
      d[v] = d[u] + w[u, v];
      π[v] = u;



Properties of shortest paths and relaxation

Triangle inequality (Lemma 24.10)
For any edge (u,v) in E, we have δ(s,v) <= δ(s,u) + w(u,v).

Upper-bound property (Lemma 24.11)
We always have v.d >= δ(s,v) for all vertices v in V, and once v.d achieves the value δ(s,d), it never changes.

No-path property (Corollary 24.12)
If there is no path from s to v, then we always have v.d = δ(s,d) = ∞.

Convergence property (Lemma 24.14)
If s -..-> u -> v is a shortest path in G for some u, v in V, and if u.d = δ(s,u) at any time prior to relaxing edge (u,v), then v.d= δ(s,v) at all times afterward.

Properties of shortest paths and relaxation

Path-relaxation property (Lemma 24.15)
If $p = \langle v_0, v_1, \ldots, v_k \rangle$ is a shortest path from $s=v_0$ to v_k, and we relax the edges of p in the order (v_0,v_1), (v_1,v_2), …, (v_{k-1},v_k), then $v_k.d = \delta(s, v_k)$. This property holds regardless of any other relaxation steps that occur, even if they are intermixed with relaxations of the edges of p.

Predecessor-subgraph property (Lemma 24.17)
Once $v.d = \delta(s,v)$ for all v in V, the predecessor subgraph is a shortest-paths tree rooted at s.

Dijkstra's algorithm

- **Dijkstra's algorithm**
  - It works properly when all edge weights are ***nonnegative***.

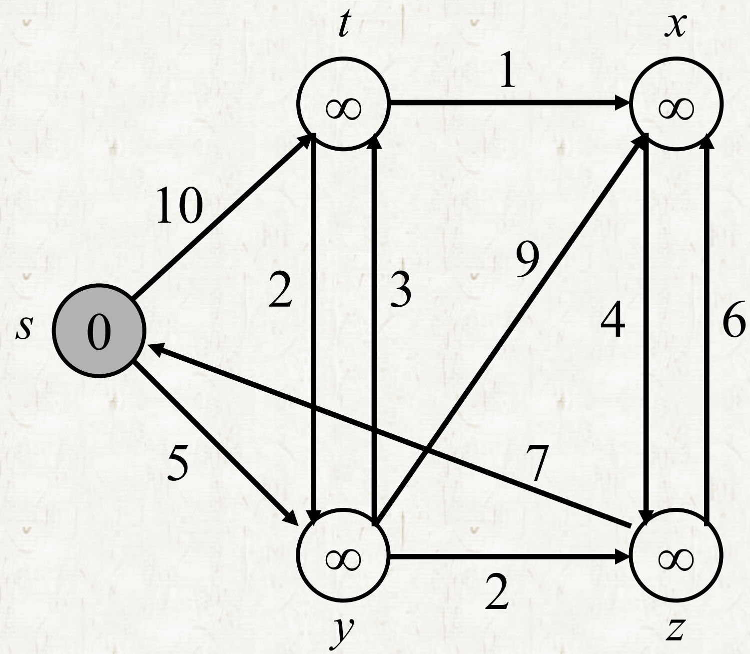
Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S ← Ø
Q ← V[G]
while Q ≠ Ø
   do u ← EXTRACT-MIN(Q)
         S ← S ∪ {u}
         for each vertex v ∈ Adj[u]
               do RELAX(u, v, w)
```

Dijkstra's Algorithm

Q

s	*t*	*y*	*x*	*z*
0	∞	∞	∞	∞

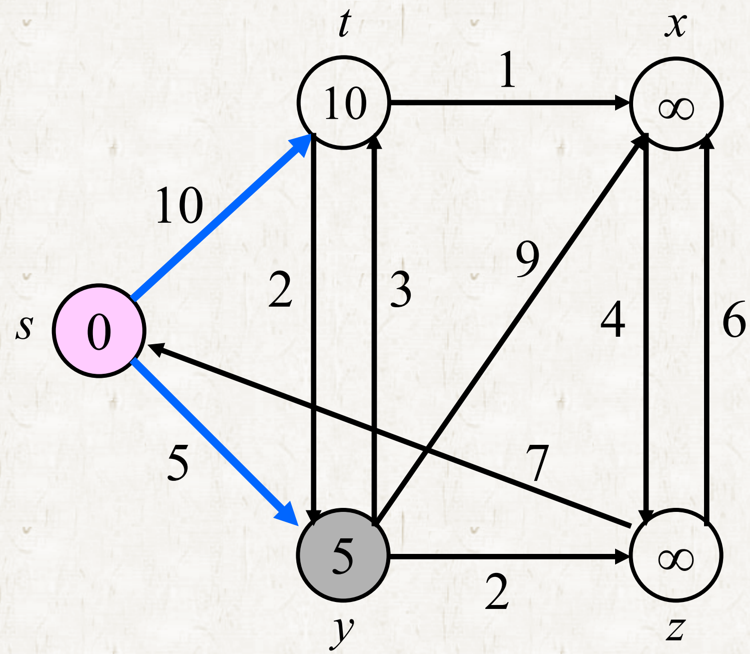


S

Dijkstra's Algorithm

Q

s	t	y	x	z
0	∞	∞	∞	∞
	10	5	-	-

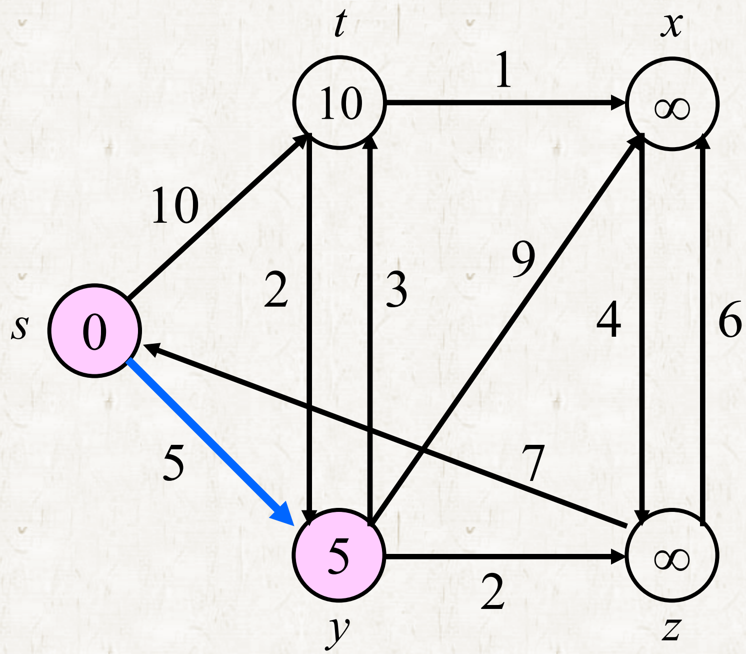


$\boldsymbol{S}=\{s\}$

Dijkstra's Algorithm

Q

s	t	y	x	z
0	∞	∞	∞	∞
	10	5	-	-

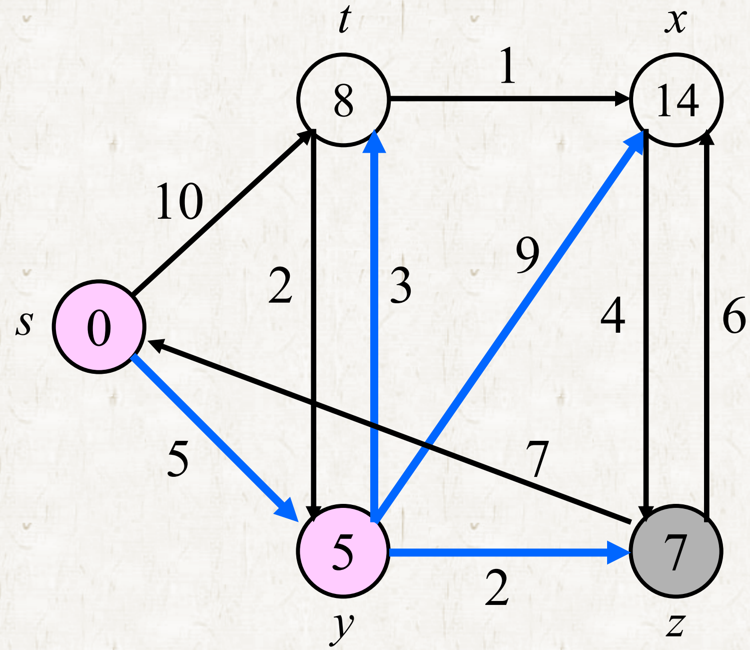


$\boldsymbol{S}=\{s, y\}$

Dijkstra's Algorithm

Q

s	*t*	*y*	*x*	*z*
0	∞	∞	∞	∞
	10	5	-	-
	8		14	7

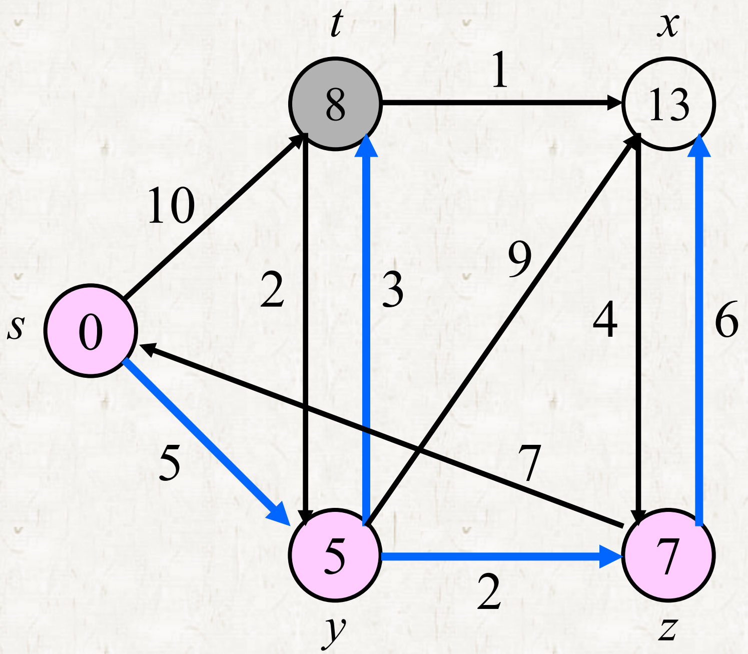


$\boldsymbol{S} = \{s, y\}$

Dijkstra's Algorithm

Q

s	*t*	*y*	*x*	*z*
0	∞	∞	∞	∞
	10	5	-	-
	8		14	7
	8		13	

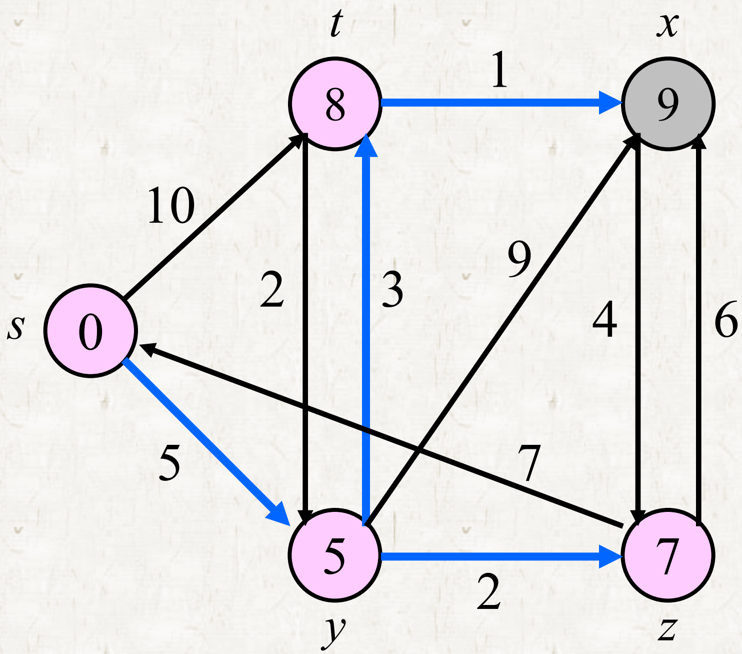


S$=\{s, y, z, t\}$

Dijkstra's Algorithm

Q

s	t	y	x	z
0	∞	∞	∞	∞
	10	5	-	-
	8		14	7
	8		13	
			9	

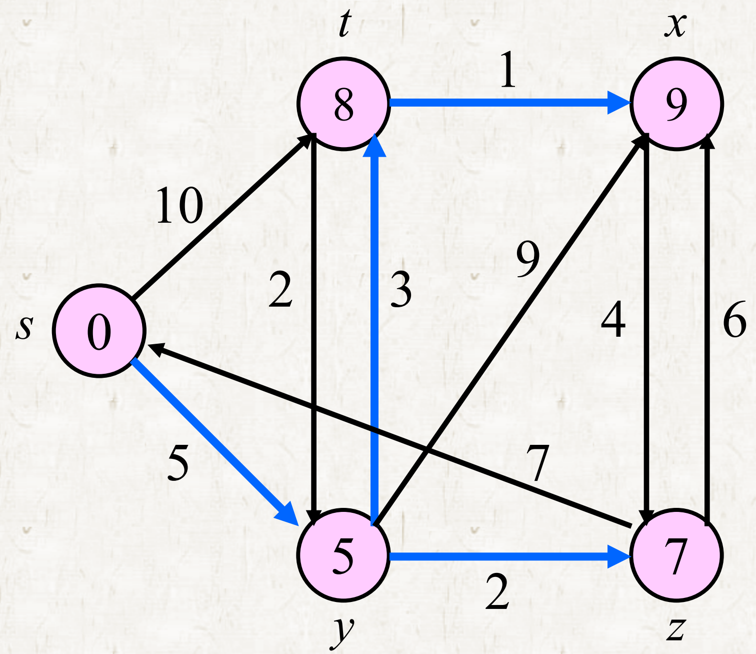


S$=\{s, y, z, t\}$

Dijkstra's Algorithm

Q

s	t	y	x	z
0	∞	∞	∞	∞
	10	5	-	-
	8		14	7
	8		13	
			9	



S $=\{s, y, z, t, x\}$

Dijkstra's algorithm

```
DIJKSTRA(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
S ← Ø
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while Q ≠ Ø
   do u ← EXTRACT-MIN(Q)
          S ← S ∪ {u}
          for each vertex v ∈ Adj[u]
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```

Dijkstra's algorithm

- **Running time**
  - $O(V^2)$ if we use an array
  - $O(V \lg V + E \lg V)$ if we use a heap
  - $O(V \lg V + E)$ if we use a Fibonacci heap.

Dijkstra's algorithm

Definitions

- length of a path: sum of edge weights along the path
- distance from u to v, $\delta(u, v)$: minimum length

Problem: Given a directed graph with NONNegative edge weights $G = (V, E)$, and a special source vertex $s \in V$, determine the distance from the source vertex to every vertex in G.

- $d[v]$: estimate the shortest path
- $\pi[v]$: predecessor pointer of the path

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Dijkstra's algorithm

Principle Observation

• Any subpath of a shortest path must also be a shortest path. Maintain an Estimate of shortest path for each vertex $d[v]$

• Initially, $d[s] = 0$ and $d[v] = \infty$

• $d[v] \geq \delta(s, v)$: As the algorithm goes on, it updates $d[v]$ until all $d[v]$ converge to $\delta(s, v)$ (This update process is called relaxation.)

```
if (d[u] + w[u, v] < d[v])
    d[v] = d[u] + w[u, v];
    π[v] = u;
```


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Dijkstra's algorithm

- Maintain a subset of vertices $S \subseteq V$, for which we claim we "know" the shortest distance, $d[u] = \delta(s, u)$.

- Initially, $S = \{\}$ and one by one we selected vertices from $V - S$ to add S at each stage.

- We select the vertex whose $d[u]$ is minimum. We implement this on a *priority* queue where every operation (Insert, Delete_min, Decrease_key) can be done in $O(\log n)$ time.
- At each stage
  - select a vertex u, which has the smallest d_u among all the unknown vertices.
  - declare that the shortest path from s to u is known
  - update d_v: $d_v = d_u + w_{u,v}$ if this value for d_v is an improvement. (decide if it is a good idea to use u on the path to v.)

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Correctness

Lemma When a vertex u is added to S, $d[u] = \delta(s, u)$.

Proof: We assume all edge weights are STRICTLY positive. Suppose the algorithm FIRST attempts to add a vertex u to S for which $d[u] \neq \delta(s, u)$, so $d[u] > \delta(s, u)$. Consider the situation JUST PRIOR to the insertion of u. Consider the true shortest path from s to u. Since $s \in S$ and $u \in V - S$, at some point this path takes a jump out of S. Let (x, y) be the edge taken by the path where $x \in S$ and $y \in V - S$. We argue $y \neq u$. (Why? Since $d[x] = \delta(s, x)$ and we applied relaxation when we add x, we would have set $d[u] = d[x] + w(x, u) = \delta(s, u)$, but we assumed this is not the case.) Now since y appears midway on the path from s to u and all subsequent edges are positive, we have $\delta(s, y) < \delta(s, u)$, and thus, $d[y] = \delta(s, y) < \delta(s, u) < d[u]$. Thus, y would have been added BEFORE u, in contradiction to our assumption.

The Bellman-Ford algorithm

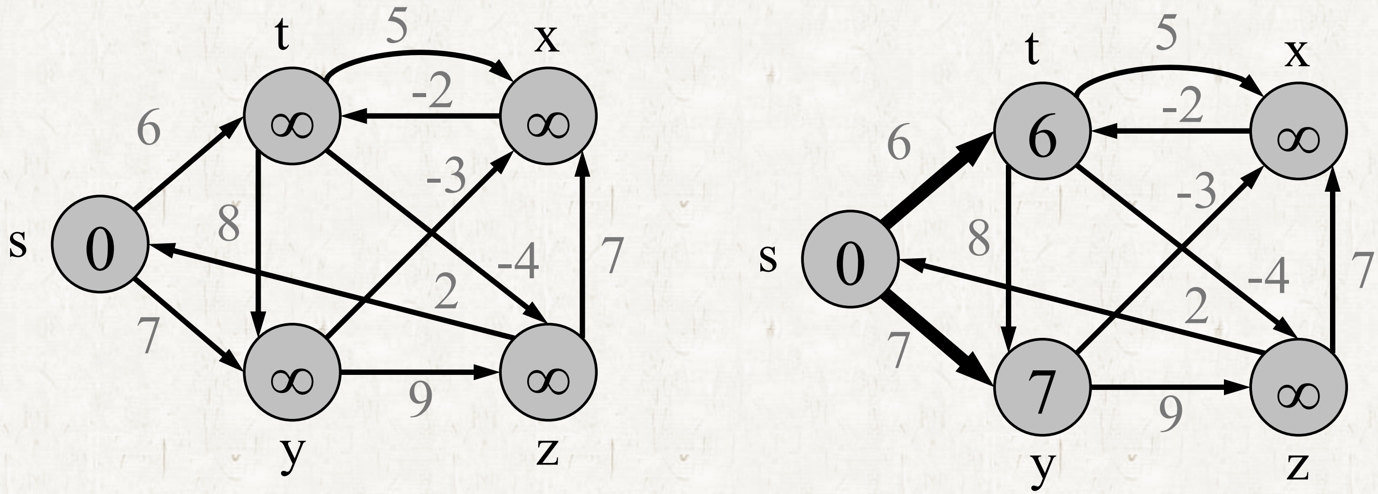
- **The Bellman-Ford algorithm**
  - it solves the single source shortest-paths problem in the general case in which edge weights may be negative.

The Bellman-Ford algorithm

```
BELLMAN-FORD(G, w, s)
INITIALIZE-SINGLE-SOURCE(G, s)
For i ←1 to |V[G]|-1
    do for each edge(u, v) ∈ E[G]
            do RELAX(u, v, w)
for each edge(u, v) ∈ E[G]
    do if d[u] + w(u, v) < d[v]
            then return FALSE
return TRUE
```

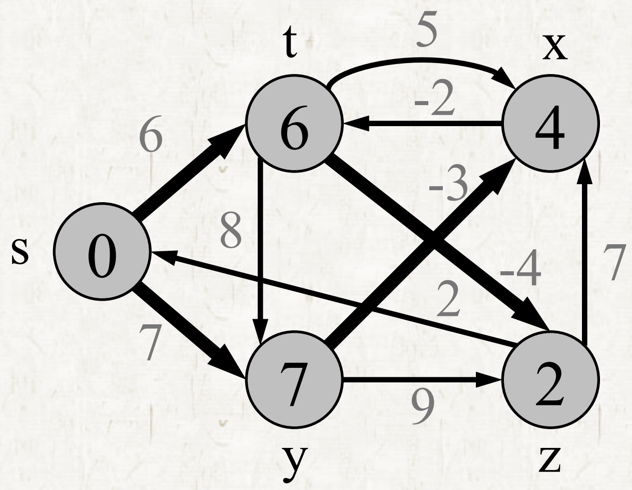
The Bellman-Ford algorithm

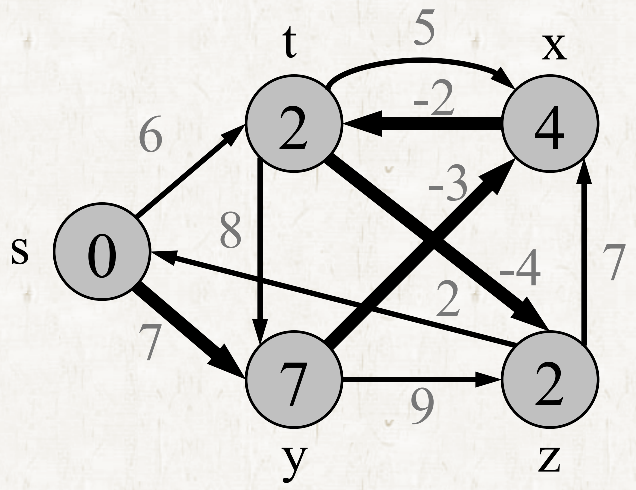
- Relaxation order
  - (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)



The Bellman-Ford algorithm

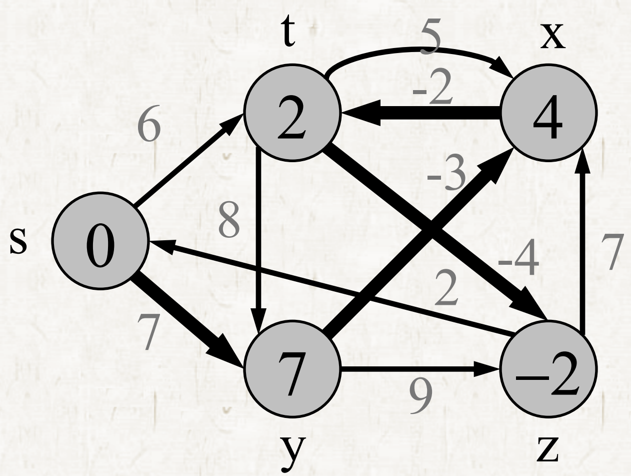
- Relaxation order
  - (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)





The Bellman-Ford algorithm

- Relaxation order
    - (t,x), (t,y), (t,z), (x,t), (y,x), (y,z), (z,x), (z,s), (s,t), (s,y)



The Bellman-Ford algorithm

- **The Bellman-Ford algorithm**
  - Running time : $O(VE)$

The Bellman-Ford algorithm

Assume B-F returns True but there is a negative weight cycle $<v_0, v_1, ..., v_k>$.

$$\sum_{i=1}^{k} d[v_i] \le \sum_{i=1}^{k} \left(d[v_{i-1}] + w(v_{i-1}, v_i) \right)$$

$$= \sum_{i=1}^{k} d[v_{i-1}] + \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

$$\sum_{i=1}^{k} d[v_i] = \sum_{i=1}^{k} d[v_{i-1}]$$

$$0 \le \sum_{i=1}^{k} w(v_{i-1}, v_i)$$

Single-source shortest paths in directed acyclic graphs

DAG-SHORTEST-PATHS(*G*, *w*, *s*)

topologically sort the vertices of *G*

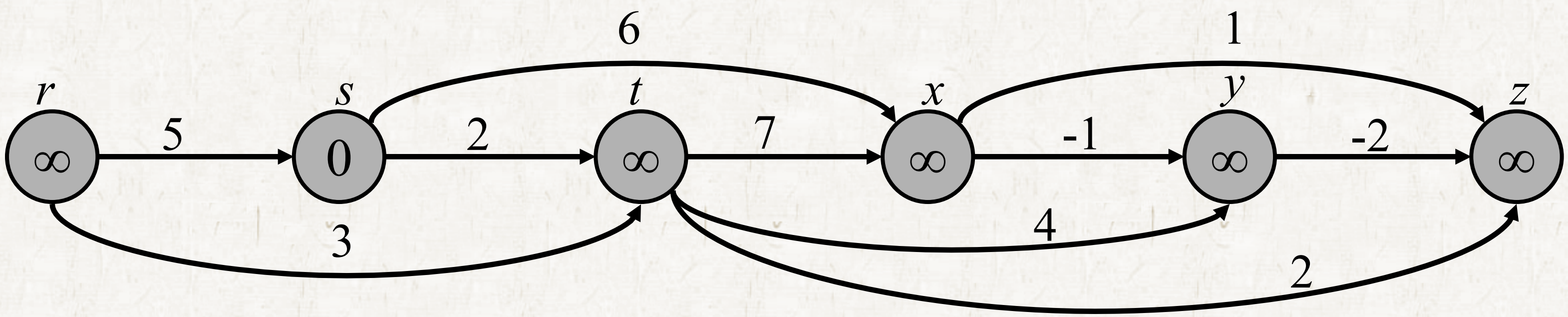
INITIALIZE-SINGLE-SOURCE(*G*, *s*)

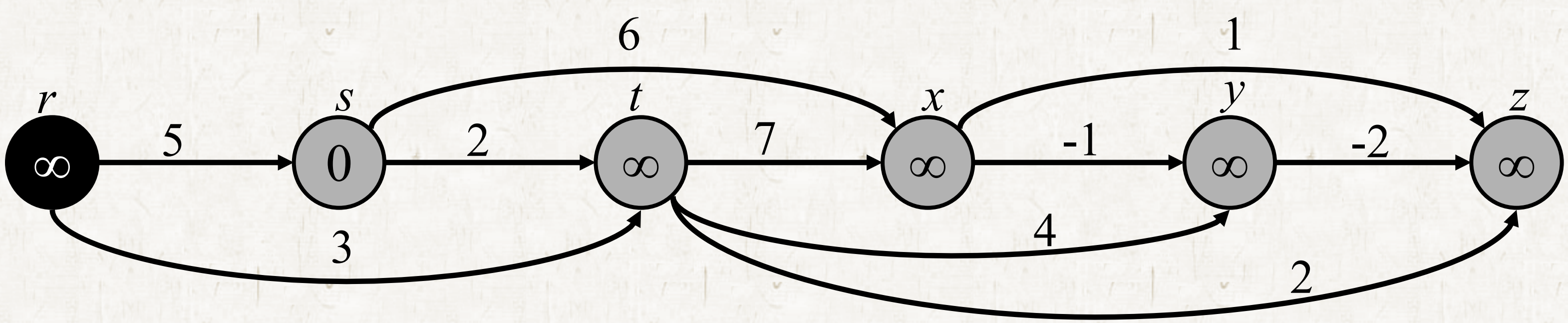
for **each vertex *u*, taken in topologically sorted order**

  do for **each vertex** $v \in Adj[u]$

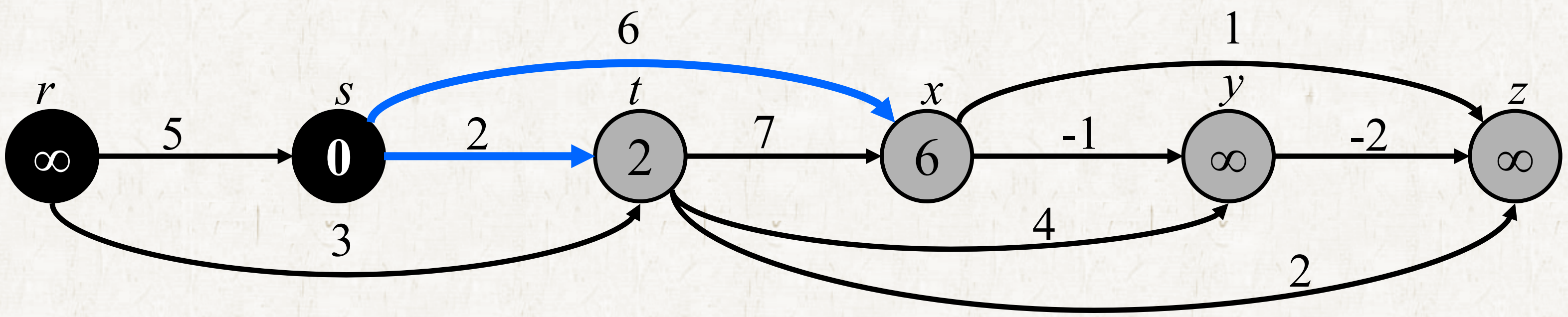
    do **RELAX(*u*, *v*, *w*)**

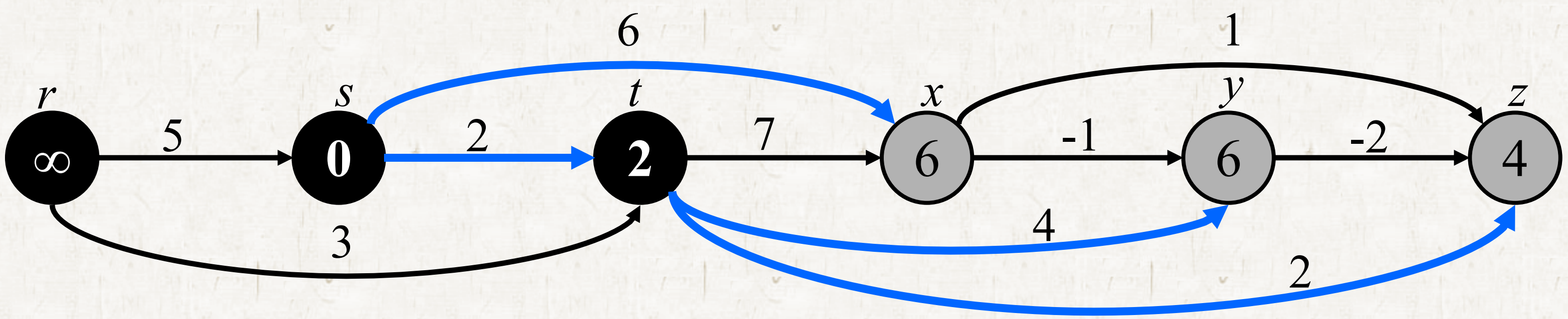
Single-source shortest paths in directed acyclic graphs



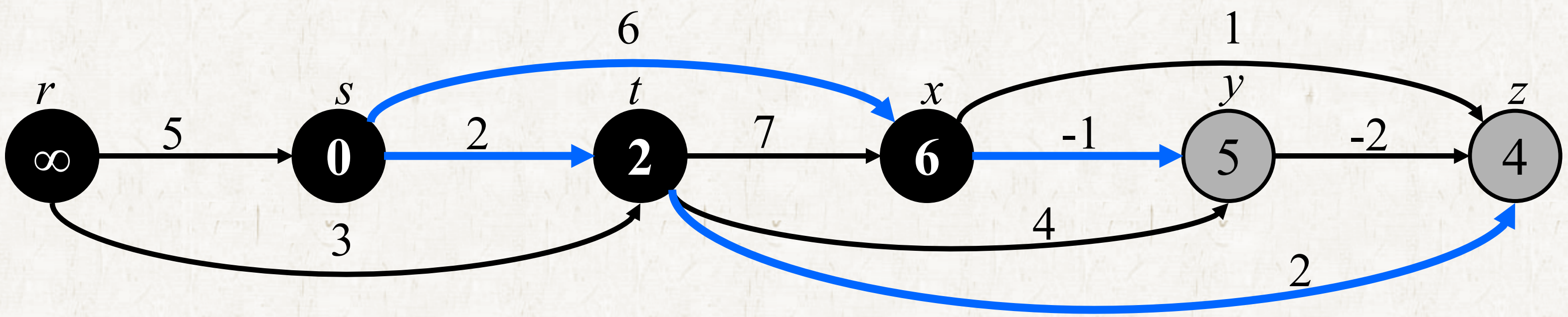


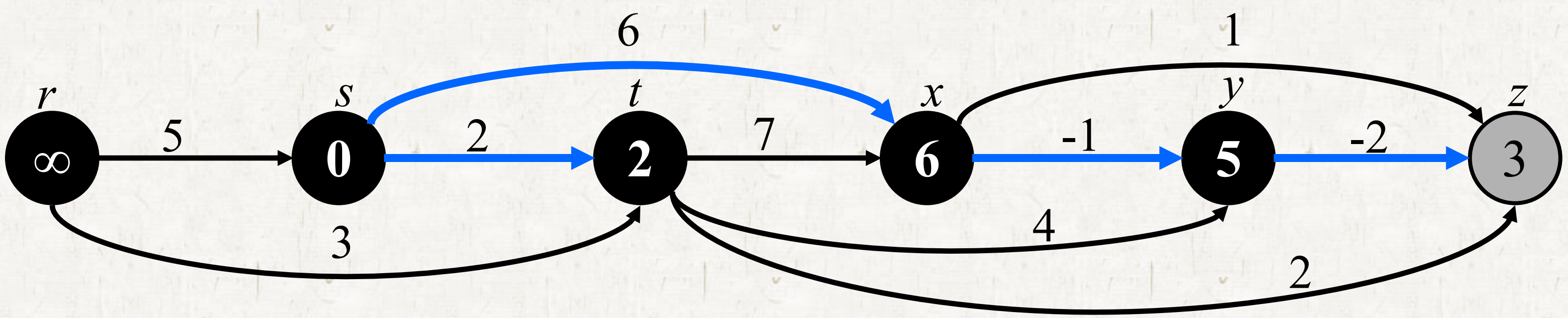
Single-source shortest paths in directed acyclic graphs



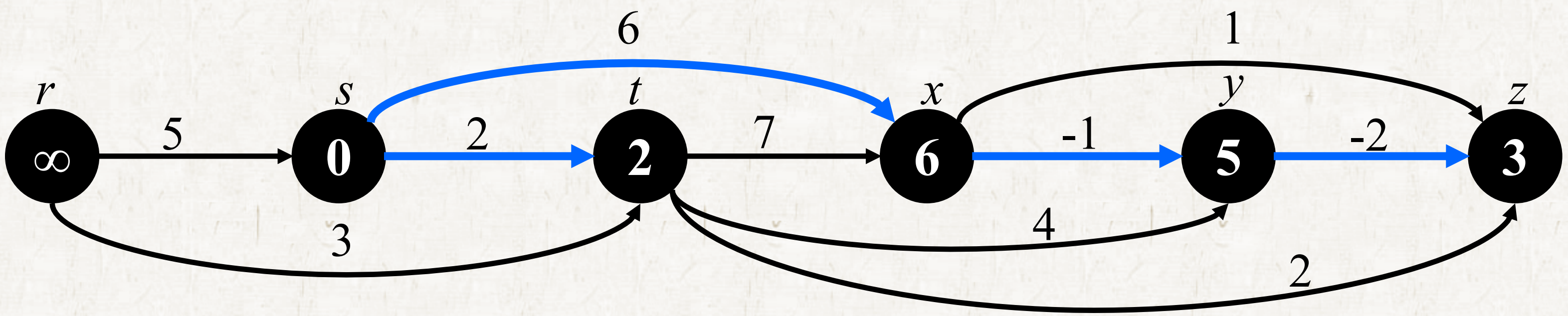


Single-source shortest paths in directed acyclic graphs





Single-source shortest paths in directed acyclic graphs



- Running time: $\Theta(V + E)$ time

PERT chart

- PERT
  - Program evaluation and review technique
  - Edges represent jobs to be performed.
  - Edge weights represent the times required to perform particular jobs.

PERT chart

- PERT
  - If edge (u,v) enters vertex v and edge (v,x) leaves v, then job (u,v) must be performed prior to job (v,x).
  - A path through this dag represents a sequence of jobs that must be performed in a particular order.
  - A ***critical path*** is a longest path through the dag.

PERT chart

- Finding a critical path in a dag

  - Negate the edge weights and run DAG-SHORTEST-PATHS or

  - Run DAG-SHORTEST PATHS, with the modification that we replace "∞" by "-∞" and ">" by "<".