Chapter 5

Multiple Random Variables

Section 5.3: Bivariate Discrete Random Variables

5.1 $$p_{XY}(x, y) = \begin{cases} kxy & x = 1, 2, 3;\ y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

a. To determine the value of k, we have that

$$\sum_x \sum_y p_{XY}(x, y) = 1 = k\sum_{x=1}^{3}\sum_{y=1}^{3} xy = k\sum_{x=1}^{3} x(1+2+3) = 6k\sum_{x=1}^{3} x = 36k \Rightarrow k = \frac{1}{36}$$

b. The marginal PMFs of X and Y are given by

$$p_X(x) = \sum_y p_{XY}(x, y) = \frac{1}{36}\sum_{y=1}^{3} xy = \frac{x}{36}\{1+2+3\} = \frac{6x}{36}$$
$$= \frac{x}{6} \qquad x = 1, 2, 3$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \frac{1}{36}\sum_{x=1}^{3} xy = \frac{y}{36}\{1+2+3\} = \frac{6y}{36}$$
$$= \frac{y}{6} \qquad y = 1, 2, 3$$

c. Observe that $p_{XY}(x, y) = p_X(x)p_Y(y)$, which implies that X and Y are independent random variables. Thus,

$$P[1 \le X \le 2, Y \le 2] = P[1 \le X \le 2]P[Y \le 2] = \{p_X(1) + p_X(2)\} \times \{p_Y(1) + p_Y(2)\}$$
$$= \frac{1}{36}\{1+2\}\{1+2\} = \frac{9}{36} = \frac{1}{4}$$

5.2 The random variable X denotes the number of heads in the first two of three tosses of a fair coin, and the random variable Y denotes the number of heads in the third toss. Let S denote the sample space of the experiment. Then S, X, and Y are given as follows:

S	X	Y
HHH	2	1
HHT	2	0
HTH	1	1
HTT	1	0
THH	1	1
THT	1	0
TTH	0	1
TTT	0	0

Thus, the joint PMF $p_{XY}(x, y)$ of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} \frac{1}{8} & x = 0, y = 0 \\ \frac{1}{8} & x = 0, y = 1 \\ \frac{1}{4} & x = 1, y = 0 \\ \frac{1}{4} & x = 1, y = 1 \\ \frac{1}{8} & x = 2, y = 0 \\ \frac{1}{8} & x = 2, y = 1 \\ 0 & otherwise \end{cases}$$

5.3 The joint PMF of two random variables X and Y is given by

$$p_{XY}(x, y) = \begin{cases} 0.10 & x = 1, y = 1 \\ 0.35 & x = 2, y = 2 \\ 0.05 & x = 3, y = 3 \\ 0.50 & x = 4, y = 4 \\ 0 & otherwise \end{cases}$$

a. The joint CDF $F_{XY}(x, y)$ is obtained as follows:

$$F_{XY}(x, y) = P[X \le x, Y \le y] = \sum_{u \le x} \sum_{v \le y} p_{XY}(u, v)$$

$$F_{XY}(1, 1) = \sum_{u \le 1} \sum_{v \le 1} p_{XY}(u, v) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(1, 2) = \sum_{u \le 1} \sum_{v \le 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(1, 3) = \sum_{u \le 1} \sum_{v \le 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(1, 4) = \sum_{u \le 1} \sum_{v \le 4} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) + p_{XY}(1, 4) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(2, 1) = \sum_{u \le 2} \sum_{v \le 1} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 1) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(2, 2) = \sum_{u \le 2} \sum_{v \le 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(2, 1) + p_{XY}(2, 2) = 0.45$$

$$\begin{aligned} F_{XY}(2, 3) &= \sum_{u \le 2} \sum_{v \le 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) + p_{XY}(2, 1) + p_{XY}(2, 2) + p_{XY}(2, 3) \\ &= p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45 \end{aligned}$$

$$\begin{aligned} F_{XY}(2, 4) &= \sum_{u \le 2} \sum_{v \le 4} p_{XY}(u, v) \\ &= p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) + p_{XY}(1, 4) + p_{XY}(2, 1) + p_{XY}(2, 2) + p_{XY}(2, 3) + p_{XY}(2, 4) \\ &= p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45 \end{aligned}$$

$$F_{XY}(3, 1) = \sum_{u \le 3} \sum_{v \le 1} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(3, 2) = \sum_{u \le 3} \sum_{v \le 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45$$

$$F_{XY}(3, 3) = \sum_{u \le 3} \sum_{v \le 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) + p_{XY}(3, 3) = 0.50$$

$$F_{XY}(3, 4) = \sum_{u \le 3} \sum_{v \le 4} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) + p_{XY}(3, 3) = 0.50$$

$$F_{XY}(4, 1) = \sum_{u \le 4} \sum_{v \le 1} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) + p_{XY}(4, 1) = p_{XY}(1, 1) = 0.10$$

$$F_{XY}(4, 2) = \sum_{u \le 4} \sum_{v \le 2} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) = 0.45$$

$$F_{XY}(4, 3) = \sum_{u \le 4} \sum_{v \le 3} p_{XY}(u, v) = p_{XY}(1, 1) + p_{XY}(2, 2) + p_{XY}(3, 3) = 0.50$$

Thus, the joint CDF of X and Y is given by

$$F_{XY}(x, y) = \begin{cases} 0.00 & x < 1, y < 1 \\ 0.10 & x = 1, y = 1 \\ 0.10 & x = 1, y = 2 \\ 0.10 & x = 1, y = 3 \\ 0.10 & x = 1, y = 4 \\ 0.10 & x = 2, y = 1 \\ 0.45 & x = 2, y = 2 \\ 0.45 & x = 2, y = 3 \\ 0.45 & x = 2, y = 4 \\ 0.10 & x = 3, y = 1 \\ 0.45 & x = 3, y = 2 \\ 0.50 & x = 3, y = 3 \\ 0.50 & x = 3, y = 4 \\ 0.10 & x = 4, y = 1 \\ 0.45 & x = 4, y = 2 \\ 0.50 & x = 4, y = 3 \\ 1.00 & x = 4, y = 4 \end{cases}$$

b.
$$\begin{aligned} P[1 \le X \le 2, Y \le 2] &= p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(2, 1) + p_{XY}(2, 2) = p_{XY}(1, 1) + p_{XY}(2, 2) \\ &= 0.45 \end{aligned}$$

5.4 The joint CDF of X and Y is given by

$$F_{XY}(x, y) = \begin{cases} 1/12 & x = 0, y = 0 \\ 1/3 & x = 0, y = 1 \\ 2/3 & x = 0, y = 2 \\ 1/6 & x = 1, y = 0 \\ 7/12 & x = 1, y = 1 \\ 1 & x = 1, y = 2 \end{cases}$$

We first find the joint PMF of X and Y as follows:

$$F_{XY}(0, 0) = \frac{1}{12} = \sum_{u \le 0}\sum_{v \le 0} p_{XY}(u, v) = p_{XY}(0, 0) \Rightarrow p_{XY}(0, 0) = \frac{1}{12}$$

$$F_{XY}(0, 1) = \frac{1}{3} = \sum_{u \le 0}\sum_{v \le 1} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(0, 1) \Rightarrow p_{XY}(0, 1) = \frac{1}{3} - \frac{1}{12} = \frac{1}{4}$$

$$F_{XY}(0, 2) = \frac{2}{3} = \sum_{u \le 0}\sum_{v \le 2} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(0, 1) + p_{XY}(0, 2) = \frac{1}{3} + p_{XY}(0, 2)$$

$$p_{XY}(0, 2) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

$$F_{XY}(1, 0) = \frac{1}{6} = \sum_{u \le 1}\sum_{v \le 0} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(1, 0) = \frac{1}{12} + p_{XY}(1, 0) \Rightarrow p_{XY}(1, 0) = \frac{1}{12}$$

$$F_{XY}(1, 1) = \frac{7}{12} = \sum_{u \le 1}\sum_{v \le 1} p_{XY}(u, v) = p_{XY}(0, 0) + p_{XY}(0, 1) + p_{XY}(1, 0) + p_{XY}(1, 1) = \frac{5}{12} + p_{XY}(1, 1)$$

$$p_{XY}(1, 1) = \frac{7}{12} - \frac{5}{12} = \frac{1}{6}$$

$$F_{XY}(1, 2) = 1 = \sum_{u \le 1}\sum_{v \le 2} p_{XY}(u, v) = F_{XY}(1, 1) + p_{XY}(0, 2) + p_{XY}(1, 2) = \frac{11}{12} + p_{XY}(1, 2)$$

$$p_{XY}(1, 2) = \frac{1}{12}$$

a. $P[0 < X < 2,\ 0 < Y < 2] = p_{XY}(1, 1) = 1/6$

b. To obtain the marginal CDFs of X and Y, we first obtain their marginal PMFs:

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} p_{XY}(0,0) + p_{XY}(0,1) + p_{XY}(0,2) & x = 0 \\ p_{XY}(1,0) + p_{XY}(1,1) + p_{XY}(1,2) & x = 1 \end{cases}$$

$$= \begin{cases} \frac{2}{3} & x = 0 \\ \frac{1}{3} & x = 1 \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} p_{XY}(0,0) + p_{XY}(1,0) & y = 0 \\ p_{XY}(0,1) + p_{XY}(1,1) & y = 1 \\ p_{XY}(0,2) + p_{XY}(1,2) & y = 2 \end{cases}$$

$$= \begin{cases} \frac{1}{6} & y = 0 \\ \frac{5}{12} & y = 1 \\ \frac{5}{12} & y = 2 \end{cases}$$

Thus, the marginal CDFs are given by

$$F_X(x) = \sum_{u \le x} p_X(u) = \begin{cases} 0 & x < 0 \\ \frac{2}{3} & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

$$F_Y(y) = \sum_{v \le y} p_Y(v) = \begin{cases} 0 & y < 0 \\ \frac{1}{6} & 0 \le y < 1 \\ \frac{7}{12} & 1 \le y < 2 \\ 1 & y \ge 2 \end{cases}$$

c. $P[X = 1, Y = 1] = p_{XY}(1, 1) = 1/6$

5.5 The joint PMF of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} 1/12 & x = 1, y = 1 \\ 1/6 & x = 1, y = 2 \\ 1/12 & x = 1, y = 3 \\ 1/6 & x = 2, y = 1 \\ 1/4 & x = 2, y = 2 \\ 1/12 & x = 2, y = 3 \\ 1/12 & x = 3, y = 1 \\ 1/12 & x = 3, y = 2 \\ 0 & otherwise \end{cases}$$

a. The marginal PMFs of X and Y are given by

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(1, 2) + p_{XY}(1, 3) & x = 1 \\ p_{XY}(2, 1) + p_{XY}(2, 2) + p_{XY}(2, 3) & x = 2 \\ p_{XY}(3, 1) + p_{XY}(3, 2) + p_{XY}(3, 3) & x = 3 \end{cases}$$

$$= \begin{cases} 1/3 & x = 1 \\ 1/2 & x = 2 \\ 1/6 & x = 3 \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) & y = 1 \\ p_{XY}(1, 2) + p_{XY}(2, 2) + p_{XY}(3, 2) & y = 2 \\ p_{XY}(1, 3) + p_{XY}(2, 3) + p_{XY}(3, 3) & y = 3 \end{cases}$$

$$= \begin{cases} 1/3 & y = 1 \\ 1/2 & y = 2 \\ 1/6 & y = 3 \end{cases}$$

b. $P[X < 2.5] = p_X(1) + p_X(2) = 5/6$

c. The probability that Y is odd is $p_Y(1) + p_Y(3) = 0.5$.

5.6 The joint PMF of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} 0.2 & x = 1, y = 1 \\ 0.1 & x = 1, y = 2 \\ 0.1 & x = 2, y = 1 \\ 0.2 & x = 2, y = 2 \\ 0.1 & x = 3, y = 1 \\ 0.3 & x = 3, y = 2 \end{cases}$$

a. The marginal PMFs of X and Y are given by

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(1, 2) & x = 1 \\ p_{XY}(2, 1) + p_{XY}(2, 2) & x = 2 \\ p_{XY}(3, 1) + p_{XY}(3, 2) & x = 3 \end{cases}$$

$$= \begin{cases} 0.3 & x = 1 \\ 0.3 & x = 2 \\ 0.4 & x = 3 \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} p_{XY}(1, 1) + p_{XY}(2, 1) + p_{XY}(3, 1) & y = 1 \\ p_{XY}(1, 2) + p_{XY}(2, 2) + p_{XY}(3, 2) & y = 2 \end{cases}$$

$$= \begin{cases} 0.4 & y = 1 \\ 0.6 & y = 2 \end{cases}$$

b. The conditional PMF of X given Y is given by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

c. To test whether X and Y are independent, we proceed as follows:

$$P[X|Y=1] = p_{X|Y}(x|1) = \frac{p_{XY}(x,1)}{p_Y(1)} = \frac{p_{XY}(x,1)}{0.4} = \begin{cases} \dfrac{0.2}{0.4} & x = 1 \\ \dfrac{0.1}{0.4} & x = 2 \\ \dfrac{0.1}{0.4} & x = 3 \end{cases}$$

$$= \begin{cases} 0.50 & x = 1 \\ 0.25 & x = 2 \\ 0.25 & x = 3 \end{cases}$$

$$P[X|Y=2] = p_{X|Y}(x|2) = \frac{p_{XY}(x,2)}{p_Y(2)} = \frac{p_{XY}(x,2)}{0.6} = \begin{cases} \dfrac{0.1}{0.6} & x = 1 \\ \dfrac{0.2}{0.6} & x = 2 \\ \dfrac{0.3}{0.6} & x = 3 \end{cases}$$

$$= \begin{cases} 1/6 & x = 1 \\ 1/3 & x = 2 \\ 1/2 & x = 3 \end{cases}$$

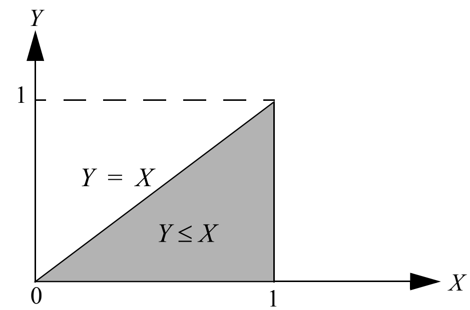
Since $p_{X|Y}(x|1) \neq p_{X|Y}(x|2)$, we conclude that X and Y are not independent. Note also that we could have tested for independence by seeing that $p_X(x)p_Y(y) \neq p_{XY}(x,y)$. Thus, either way we have shown that X and Y are not independent.

Section 5.4: Bivariate Continuous Random Variables

5.7 The joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} kx & 0 < y \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

a. To determine the value of the constant k, we must carefully define the ranges of the values of X and Y as follows:



$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy = \int_{x=0}^{1}\int_{y=0}^{x} kxdydx = \int_{x=0}^{1} kx^2dx = k\left[\frac{x^3}{3}\right]_0^1 = \frac{k}{3}$$

which implies that $k = 3$. Note that we can also obtain k by reversing the order of integration as follows:

$$1 = \int_{y=0}^{1}\int_{x=y}^{1} kxdydx = k\int_{y=0}^{1}\left[\frac{x^2}{2}\right]_y^1 dy = \frac{k}{2}\int_{y=0}^{1}[1-y^2]dy = \frac{k}{2}\left[y-\frac{y^3}{3}\right]_0^1 = \frac{k}{2}\left[1-\frac{1}{3}\right] = \frac{k}{3}$$

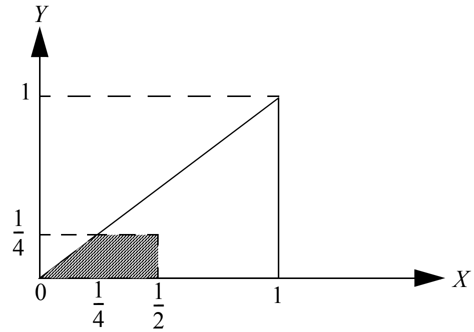
which gives the same result $k = 3$.

b. The marginal PDFs of X and Y are given as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_{y=0}^{x} kxdy = kx^2 = 3x^2 \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_{x=y}^{1} kxdx = k\left[\frac{x^2}{2}\right]_y^1 = \frac{3}{2}[1-y^2] \qquad 0 < y \le 1$$

c. To evaluate $P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right]$, we need to find the region of integration as follows:



Thus, we have that

$$P\left[0 < X < \frac{1}{2}, 0 < Y < \frac{1}{4}\right] = \int_{x=0}^{\frac{1}{4}}\int_{y=0}^{x} kxdydx + \int_{x=\frac{1}{4}}^{\frac{1}{2}}\int_{y=0}^{\frac{1}{4}} kxdydx = \int_{x=0}^{\frac{1}{4}} kx^2dx + \int_{x=\frac{1}{4}}^{\frac{1}{2}} \frac{kx}{4}dx$$

$$= k\left\{\left[\frac{x^3}{3}\right]_0^{1/4} + \left[\frac{x^2}{8}\right]_{1/4}^{1/2}\right\} = k\left\{\frac{1}{3}\left(\frac{1}{64}\right) + \frac{1}{8}\left(\frac{1}{4} - \frac{1}{16}\right)\right\} = \frac{11}{128}$$

5.8 The joint CDF of X and Y is given by

$$F_{XY}(x, y) = \begin{cases} 1 - e^{-ax} - e^{-by} + e^{-(ax+by)} & x \geq 0;\ y \geq 0 \\ 0 & otherwise \end{cases}$$

a. To find the marginal PDFs of X and Y, we first obtain the joint PDF and then obtain the marginal PDFs as follows:

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x \partial y}F_{XY}(x, y) = abe^{-(ax+by)} \qquad x \geq 0;\ y \geq 0$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^{\infty} abe^{-(ax+by)}dy = ae^{-ax}[-e^{-by}]_0^{\infty} = ae^{-ax} \qquad x \geq 0$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_0^{\infty} abe^{-(ax+by)}dx = be^{-by}[-e^{-ax}]_0^{\infty} = be^{-by} \qquad y \geq 0$$

b. Observe that $f_X(x)f_Y(y) = ae^{-ax}be^{-by} = abe^{-(ax+by)} = f_{XY}(x,y)$. That is, the product of the marginal PDFs is equal to the joint PDF. Therefore, we conclude that X and Y are indepencent random variables.

5.9 The joint PDFof X and Y is given by

$$f_{XY}(x,y) = \begin{cases} ke^{-(2x+3y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

a. For $f_{XY}(x,y)$ to be a true joint PDF, we must have that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1 = k\int_0^{\infty} e^{-2x}dx\int_0^{\infty} e^{-3y}dy = k\left(\frac{1}{2}\right)\left(\frac{1}{3}\right) = \frac{k}{6}$$

Thus, $k = 6$.

b. The marginal PDFs of X and Y are given by

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^{\infty} 6e^{-(2x+3y)}dy = 6e^{-2x}\int_0^{\infty} e^{-3y}dy \\ &= 2e^{-2x} \qquad x \geq 0 \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{\infty} 6e^{-(2x+3y)}dx = 6e^{-3y}\int_0^{\infty} e^{-2x}dx \\ &= 3e^{-3y} \qquad y \geq 0 \end{aligned}$$

Another way to obtain the marginal PDFs is by noting that the joint PDF is of the form $f_{XY}(x,y) = k \times a(x) \times b(y)$ in the rectangular region $0 \leq x \leq \infty, 0 \leq y \leq \infty$, where k is a constant, $a(x)$ is the x-factor and $b(y)$ is the y-factor. Therefore, we must have that

$$\begin{aligned} f_X(x) &= Ae^{-2x} \qquad x \geq 0 \\ f_Y(y) &= Be^{-3y} \qquad y \geq 0 \\ 6 &= AB \end{aligned}$$

To find the values of A and B we note that

$$\int_0^{\infty} Ae^{-2x}dx = 1 = A\left[\frac{-e^{-2x}}{2}\right]_0^{\infty} = \frac{A}{2} \Rightarrow A = 2$$

Thus, $B = 6/2 = 3$. From these we obtain $f_X(x) = 2e^{-2x}, x \geq 0$, and $f_Y(y) = 3e^{-3y}, y \geq 0$.

c.

$$P[X < 1, Y < 0.5] = \int_{x=0}^{1}\int_{y=0}^{0.5} f_{XY}(x,y)dydx = \int_{x=0}^{1} 2e^{-2x}dx\int_{y=0}^{0.5} 3e^{-3y}dy = [-e^{-2x}]_0^1[-e^{-3y}]_0^{0.5}$$
$$= (1-e^{-2})(1-e^{-1.5}) = 0.6717$$

5.10 The joint PDF of X and Y is given by

$$f_{XY}(x,y) = \begin{cases} k(1-x^2y) & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

a. To determine the value of the constant k, we know that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1 = k\int_{y=0}^{1}\int_{x=0}^{1}(1-x^2y)dxdy = k\int_{y=0}^{1}\left[x - \frac{x^3y}{3}\right]_0^1 dy = k\int_{y=0}^{1}\left[1-\frac{y}{3}\right]dy$$
$$= k\left[y - \frac{y^2}{6}\right]_0^1 = k\left[1-\frac{1}{6}\right] = \frac{5k}{6}$$
$$k = 6/5 = 1.2$$

b. To find the conditional PDFs, we must first find the marginal PDFs, which are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = k\int_0^1(1-x^2y)dy = k\left[y - \frac{x^2y^2}{2}\right]_{y=0}^1 = 1.2\left(1-\frac{x^2}{2}\right) \qquad 0 \leq x \leq 1$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx = k\int_0^1(1-x^2y)dx = k\left[x - \frac{x^3y}{3}\right]_{x=0}^1 = 1.2\left(1-\frac{y}{3}\right) \qquad 0 \leq y \leq 1$$

Thus, the conditional PDFs of X given Y, $f_{X|Y}(x|y)$, and Y given X, $f_{Y|X}(y|x)$, are given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{1.2(1-x^2y)}{1.2\left(1-\frac{y}{3}\right)} = \frac{3(1-x^2y)}{3-y} \qquad 0 \le x \le 1$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{1.2(1-x^2y)}{1.2\left(1-\frac{x^2}{2}\right)} = \frac{2(1-x^2y)}{2-x^2} \qquad 0 \le y \le 1$$

5.11 The joint PDF of X and Y is given by

$$f_{XY}(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2$$

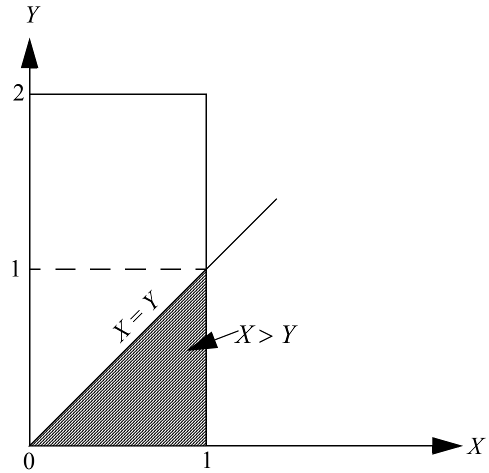
a. To find the CDF $F_X(x)$ of X, we first obtain its PDF as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \frac{6}{7}\int_0^2\left(x^2 + \frac{xy}{2}\right)dy = \frac{6}{7}\left[x^2y + \frac{xy^2}{4}\right]_0^2 = \frac{6}{7}(2x^2 + x) \qquad 0 < x < 1$$

Thus, the CDF of X is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du = \frac{6}{7}\int_0^x (2u^2 + u)du = \frac{6}{7}\left[\frac{2u^3}{3} + \frac{u^2}{2}\right]_0^x$$
$$= \begin{cases} 0 & x < 0 \\ \frac{6}{7}\left(\frac{2x^3}{3} + \frac{x^2}{2}\right) & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

b. To find $P[X > Y]$, we proceed as follows. First, we need to define the region of integration through the following figure.



Thus,

$$\begin{aligned}P[X>Y] &= \int_{x=0}^{1}\int_{y=0}^{x} f_{XY}(x,y)\,dy\,dx = \frac{6}{7}\int_{x=0}^{1}\int_{y=0}^{x}\left(x^2+\frac{xy}{2}\right)dy\,dx = \frac{6}{7}\int_{x=0}^{1}\left[x^2y+\frac{xy^2}{4}\right]_{y=0}^{x}dx \\ &= \frac{6}{7}\int_{x=0}^{1}\left[x^3+\frac{x^3}{4}\right]dx = \left(\frac{6}{7}\times\frac{5}{4}\right)\int_{x=0}^{1}x^3\,dx = \left(\frac{6}{7}\times\frac{5}{4}\right)\left[\frac{x^4}{4}\right]_0^1 \\ &= \frac{15}{56}\end{aligned}$$

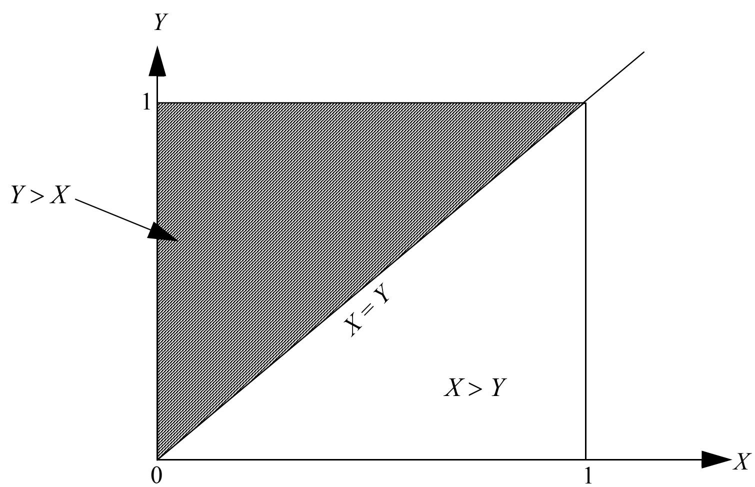
c.

$$\begin{aligned}P\left[Y>\frac{1}{2}\middle|X<\frac{1}{2}\right] &= \frac{P\left[Y>\frac{1}{2}, X<\frac{1}{2}\right]}{P\left[X<\frac{1}{2}\right]} = \frac{P\left[Y>\frac{1}{2}, X<\frac{1}{2}\right]}{F_X\left(\frac{1}{2}\right)} = \frac{\frac{6}{7}\int_{x=0}^{1/2}\int_{y=1/2}^{2}\left(x^2+\frac{xy}{2}\right)dy\,dx}{\left(\frac{5}{28}\right)} = \frac{(138/896)}{(5/28)} \\ &= 0.8625\end{aligned}$$

5.12 The joint PDF of X and Y is given by

$$f_{XY}(x, y) = \begin{cases} ke^{-(x+y)} & x \geq 0, y \geq x \\ 0 & \text{otherwise} \end{cases}$$

a. The value of the constant k can be obtained as follows. First, we determine the region of integration via the following figure:



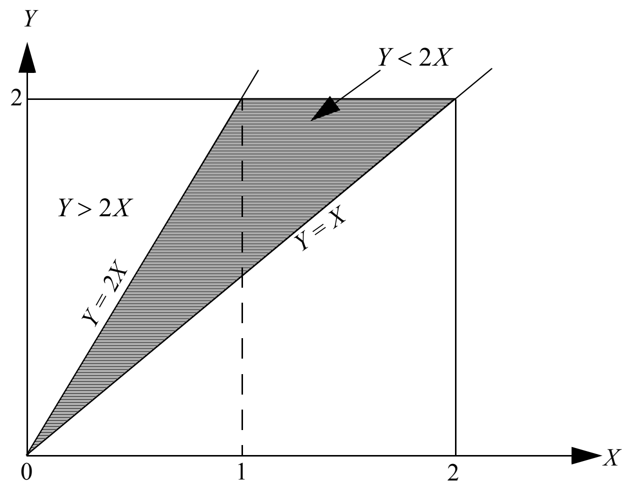
Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)dxdy = 1 = k\int_{y=0}^{\infty}\int_{x=0}^{y} e^{-(x+y)}dxdy = k\int_{y=0}^{\infty} e^{-y}[-e^{-x}]_0^y dy = k\int_{y=0}^{\infty} e^{-y}(1-e^{-y})dy$$

$$= k\left[-e^{-y} + \frac{e^{-2y}}{2}\right]_0^{\infty} = k\left(1 - \frac{1}{2}\right) = \frac{k}{2}$$

$$k = 2$$

b. To find $P[Y < 2X]$, we graph the region of integration as shown in the figure below:



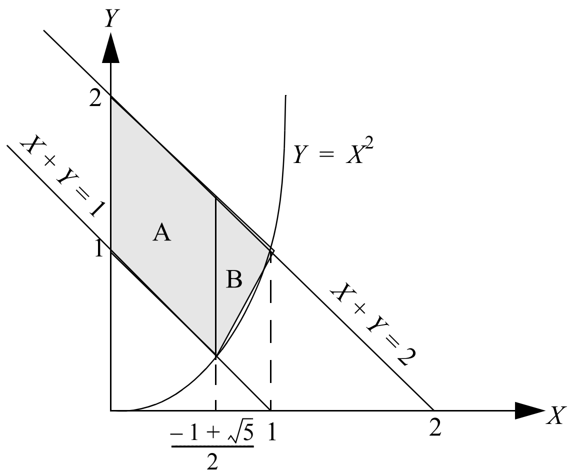
Thus,

$$P[Y<2X] = k\int_{x=0}^{\infty}\int_{y=x}^{2x} e^{-(x+y)}dydx = k\int_{x=0}^{\infty} e^{-x}[-e^{-y}]_{x}^{2x}dx = k\int_{x=0}^{\infty} e^{-x}[e^{-x}-e^{-2x}]dx$$
$$= k\left[-\frac{e^{-2x}}{2}+\frac{e^{-3x}}{3}\right]_0^{\infty} = k\left(\frac{1}{2}-\frac{1}{3}\right) = 2\left(\frac{1}{6}\right) = \frac{1}{3}$$

5.13 The joint PDF of X and Y is given by

$$f_{XY}(x,y) = \frac{6x}{7} \qquad 1 \le x+y \le 2, x \ge 0, y \ge 0$$

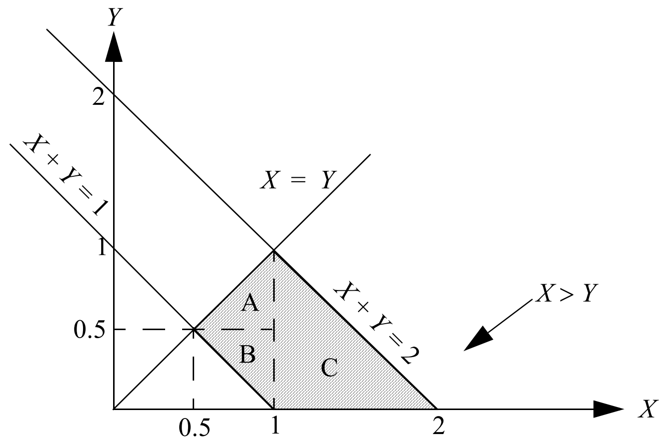
a. To obtain the integral that expresses the $P[Y>X^2]$, we must show the region of integration, as illustrated in the figure below.



From the figure we see that the region of integration is the shaded region, which has been partitioned into two areas labeled A and B. Area A is bounded by the line $x = 0$, which is the y-axis; the line $x = (-1+\sqrt{5})/2$, which is the feasible solution to the simultaneous equations $x+y = 1$ and $y = x^2$; the line $x+y = 1$; and the line $x+y = 2$. Similarly, area B is bounded by the curve $y = x^2$, the line $x+y = 2$, and the line $x = (-1+\sqrt{5})/2$. Thus, the desired result is given by

$$\begin{aligned} P[Y > X^2] &= \int_A f_{XY}(x, y)dxdy + \int_B f_{XY}(x, y)dxdy \\ &= \int_{x=0}^{(-1+\sqrt{5})/2} \int_{y=1-x}^{2-x} \frac{6x}{7}dydx + \int_{x=(-1+\sqrt{5})/2}^{1} \int_{y=x^2}^{2-x} \frac{6x}{7}dydx \end{aligned}$$

b. To obtain the exact value of $P[X > Y]$, we note that the new region of integration is as shown below.



Thus, we obtain three areas labeled A, B, and C; and the desired result is as follows:

$$
\begin{aligned}
P[X > Y] &= \int_A f_{XY}(x, y)dxdy + \int_B f_{XY}(x, y)dxdy + \int_C f_{XY}(x, y)dxdy \\
&= \int_{x=0.5}^{1}\int_{y=0.5}^{x}\frac{6x}{7}dydx + \int_{x=0.5}^{1}\int_{y=1-x}^{0.5}\frac{6x}{7}dydx + \int_{x=1}^{2}\int_{y=0}^{2-x}\frac{6x}{7}dydx \\
&= \frac{6}{7}\int_{x=0.5}^{1} x[x-0.5]dx + \frac{6}{7}\int_{x=0.5}^{1} x[x-0.5]dx + \frac{6}{7}\int_{x=1}^{2} x[2-x]dx \\
&= \frac{6}{7}\left\{\left[\frac{x^3}{3} - \frac{0.5x^2}{2}\right]_{0.5}^{1} + \left[\frac{x^3}{3} - \frac{0.5x^2}{2}\right]_{0.5}^{1} + \left[x^2 - \frac{x^3}{3}\right]_{1}^{2}\right\} = \frac{6}{7}\left\{\frac{5}{48} + \frac{5}{48} + \frac{2}{3}\right\} = \frac{6}{7}\left\{\frac{7}{8}\right\} = \frac{3}{4}
\end{aligned}
$$

5.14 The joint PDF of X and Y is given by

$$
f_{XY}(x, y) = \begin{cases} \frac{1}{2}e^{-2y} & 0 \le x \le 4, y \ge 0 \\ 0 & otherwise \end{cases}
$$

The marginal PDFs of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^{\infty} \frac{1}{2}e^{-2y}dy = \frac{1}{2}\left[-\frac{e^{-2y}}{2}\right]_0^{\infty}$$

$$= \frac{1}{4} \qquad 0 \le x \le 4$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \frac{1}{2}e^{-2y}\int_0^4 dx = 2e^{-2y} \qquad y \ge 0$$

Note that the joint PDF is completely separable in the form $f_{XY}(x, y) = k \times a(x) \times b(y)$ in the rectangular region $0 \le x \le 4, 0 \le y \le \infty$, where k is a constant, $a(x)$ is the x-factor and $b(y)$ is the y-factor. Therefore, we must have that

$$f_X(x) = A \qquad 0 \le x \le 4$$

$$f_Y(y) = Be^{-2y} \qquad y \ge 0$$

$$\frac{1}{2} = AB$$

To find the values of A and B we note that

$$\int_0^{\infty} Be^{-2y}dy = 1 = B\left[\frac{-e^{-2y}}{2}\right]_0^{\infty} = \frac{B}{2} \Rightarrow B = 2$$

Thus, $A = 0.5/2 = 1/4$ as before.

Section 5.6: Conditional Distributions

5.15 Let S denote the sample space of the experiment, R the event that a red ball is drawn, and G the event that a green ball is drawn. Since the experiment is performed with replacement, the probability of a sample point in S is product of the probabilities of those events. More importantly, since $P[R] = 3/5 = 0.6$ and $P[G] = 2/5 = 0.4$, we obtain the following values for the probabilities of the sample points in S together with their corresponding values of X and Y:

S	P[S]	X	Y
RR	0.36	1	1
RG	0.24	1	0
GR	0.24	0	1
GG	0.16	0	0

a. The joint PMF $p_{XY}(x, y)$ of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} 0.16 & x = 0, y = 0 \\ 0.24 & x = 0, y = 1 \\ 0.24 & x = 1, y = 0 \\ 0.36 & x = 1, y = 1 \\ 0 & otherwise \end{cases}$$

b. The conditional PMF of X given Y is given by

$$p_{X|Y}(x|y) = \frac{p_{XY}(x, y)}{p_Y(y)}$$

But the marginal PMF of Y is given by

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} 0.16 + 0.24 & y = 0 \\ 0.24 + 0.36 & y = 1 \end{cases}$$

$$= \begin{cases} 0.4 & y = 0 \\ 0.6 & y = 1 \end{cases}$$

Thus, we obtain the following result:

$$p_{X|Y}(x|0) = \begin{cases} 0.4 & x = 0 \\ 0.6 & x = 1 \end{cases}$$

$$p_{X|Y}(x|1) = \begin{cases} 0.4 & x = 0 \\ 0.6 & x = 1 \end{cases}$$

5.16 The joint PDF of X and Y is $f_{XY}(x, y) = 2e^{-(x+2y)}$, $x \geq 0, y \geq 0$. To find the conditional expectation of X given Y and Y given X, we proceed as follows:

$$E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$$

$$E[Y|X] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy$$

Now,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = \int_0^{\infty} 2e^{-(x+2y)}dy = e^{-x}[-e^{-2y}]_0^{\infty} = e^{-x}$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \int_0^{\infty} 2e^{-(x+2y)}dx = 2e^{-2y}[-e^{-x}]_0^{\infty} = 2e^{-2y}$$

Since $f_X(x)f_y(y) = (e^{-x})(2e^{-2y}) = 2e^{-(x+2y)} = f_{XY}(x, y)$, we conclude that X and Y are independent random variables. Therefore,

$$E[X|Y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \int_{0}^{\infty} x f_X(x)dx = E[X] = 1$$

$$E[Y|X] = \int_{-\infty}^{\infty} y f_{Y|X}(y|x)dy = \int_{0}^{\infty} y f_Y(y)dy = E[Y] = \frac{1}{2}$$

5.17 Let S denote the sample space of the experiment, H the event that a toss came up heads, and T the event that a toss came up tails. Since the experiment is performed with replacement, the probability of a sample point in S is product of the probabilities of those events. Thus S, X, and Y are given as follows:

S	P[S]	X	Y
HHHH	1/16	2	2
HHHT	1/16	2	1
HHTH	1/16	2	1
HHTT	1/16	2	0
HTHH	1/16	1	2
HTHT	1/16	1	1
HTTH	1/16	1	1
HTTT	1/16	0	0
THHH	1/16	1	2
THHT	1/16	1	1
THTH	1/16	1	1
TTHH	1/16	0	2
THTT	1/16	1	0
TTHT	1/16	0	1
TTTH	1/16	0	1
TTTT	1/16	0	0

a. The joint PMF $p_{XY}(x, y)$ of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} \frac{1}{16} & x = 0, y = 0 \\ \frac{1}{8} & x = 0, y = 1 \\ \frac{1}{16} & x = 0, y = 2 \\ \frac{1}{8} & x = 1, y = 0 \\ \frac{1}{4} & x = 1, y = 1 \\ \frac{1}{8} & x = 1, y = 2 \\ \frac{1}{16} & x = 2, y = 0 \\ \frac{1}{8} & x = 2, y = 1 \\ \frac{1}{16} & x = 2, y = 2 \\ 0 & otherwise \end{cases}$$

b. To show that X and Y are independent random variables, we proceed as follows:

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & x = 0 \\ \frac{1}{8} + \frac{1}{4} + \frac{1}{8} & x = 1 \\ \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & x = 2 \end{cases}$$

$$= \begin{cases} \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = 1 \\ \frac{1}{4} & x = 2 \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & y = 0 \\ \frac{1}{8} + \frac{1}{4} + \frac{1}{8} & y = 1 \\ \frac{1}{16} + \frac{1}{8} + \frac{1}{16} & y = 2 \end{cases}$$

$$= \begin{cases} \frac{1}{4} & y = 0 \\ \frac{1}{2} & y = 1 \\ \frac{1}{4} & y = 2 \end{cases}$$

Now,

$$p_X(x)p_Y(y) = \begin{cases} \frac{1}{16} & x = 0, y = 0 \\ \frac{1}{8} & x = 0, y = 1 \\ \frac{1}{16} & x = 0, y = 2 \\ \frac{1}{8} & x = 1, y = 0 \\ \frac{1}{4} & x = 1, y = 1 \\ \frac{1}{8} & x = 1, y = 2 \\ \frac{1}{16} & x = 2, y = 0 \\ \frac{1}{8} & x = 2, y = 1 \\ \frac{1}{16} & x = 2, y = 2 \\ 0 & otherwise \end{cases}$$

Since $p_X(x)p_Y(y) = p_{XY}(x, y)$, we conclude that X and Y are independent.

5.18 The joint PDF of X and Y is given by

$$f_{XY}(x, y) = xye^{-y^2/4} \qquad 0 \le x \le 1, y \ge 0$$

a. The marginal PDFs of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy = x\int_0^{\infty} ye^{-y^2/4}dy$$

Let $z = y^2/4 \Rightarrow dz = ydy/2 \Rightarrow ydy = 2dz$. Thus,

$$f_X(x) = x\int_0^{\infty} ye^{-y^2/4}dy = x\int_0^{\infty} 2e^{-z}dz = 2x[-e^{-z}]_0^{\infty} = 2x \qquad 0 \le x \le 1$$

Similarly, we obtain

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = ye^{-y^2/4}\int_0^1 xdx = ye^{-y^2/4}\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}ye^{-y^2/4} \qquad y \ge 0$$

b. To determine if X and Y are independent we observe that

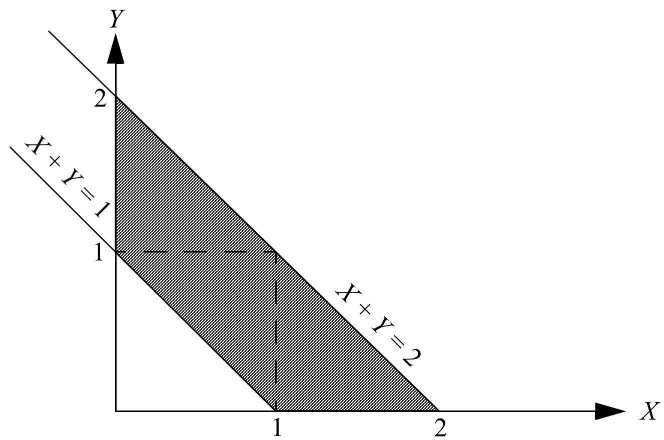
$$f_X(x)f_Y(y) = \{2x\}\left\{\frac{1}{2}ye^{-y^2/4}\right\} = xye^{-y^2/4} = f_{XY}(x, y)$$

Thus, we conclude that X and Y are independent. Note that this can also be observed from the fact that $f_{XY}(x, y)$ can be separated into an x-function and a y-function and the region of interest is rectangular.

5.19 The joint PDF of random variables X and Y is given by

$$f_{XY}(x, y) = \frac{6}{7}x \qquad 1 \le x + y \le 2, x \ge 0, y \ge 0$$

The region of interest is as shown in the following figure:



a. The marginal PDFs of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \begin{cases} \int_{y=1-x}^{2-x} \frac{6}{7}x\,dy & 0 \le x < 1 \\ \int_{y=0}^{2-x} \frac{6}{7}x\,dy & 1 \le x < 2 \end{cases}$$

$$= \begin{cases} \frac{6}{7}x & 0 \le x < 1 \\ \frac{6}{7}x(2-x) & 1 \le x < 2 \end{cases}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)dx = \begin{cases} \int_{x=1-y}^{2-y} \frac{6}{7}x dx & 0 \le y < 1 \\ \int_{x=0}^{2-y} \frac{6}{7}x dx & 1 \le y < 2 \end{cases}$$

$$= \begin{cases} \frac{3}{7}(3 - 2y) & 0 \le y < 1 \\ \frac{3}{7}(2 - y)^2 & 1 \le y < 2 \end{cases}$$

b. From the above results we see that $f_X(x)f_Y(y) \ne f_{XY}(x, y)$. Therefore, we conclude that X and Y are not independent.

Section 5.7: Covariance and Correlation Coefficient

5.20 The joint PMF of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} 0 & x = -1, y = 0 \\ \frac{1}{3} & x = -1, y = 1 \\ \frac{1}{3} & x = 0, y = 0 \\ 0 & x = 0, y = 1 \\ 0 & x = 1, y = 0 \\ \frac{1}{3} & x = 1, y = 1 \end{cases}$$

a. To determine if X and Y are independent we first find their marginal PMFs as follows:

$$p_X(x) = \sum_y p_{XY}(x, y) = \begin{cases} \sum_y p_{XY}(-1, y) & x = -1 \\ \sum_y p_{XY}(0, y) & x = 0 \\ \sum_y p_{XY}(1, y) & x = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{3} & x = -1 \\ \frac{1}{3} & x = 0 \\ \frac{1}{3} & x = 1 \\ 0 & otherwise \end{cases}$$

$$p_Y(y) = \sum_x p_{XY}(x, y) = \begin{cases} \sum_x p_{XY}(x, 0) & y = 0 \\ \sum_x p_{XY}(x, 1) & y = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{3} & y = 0 \\ \frac{2}{3} & y = 1 \\ 0 & otherwise \end{cases}$$

b. From the results we observe that $p_X(x)p_Y(y) \neq p_{XY}(x, y)$. Therefore, we conclude that X and Y are not independent.

c. The covariance of X and Y can be obtained as follows:

$$E[X] = \frac{1}{3}\{-1+0+1\} = 0$$

$$E[Y] = \frac{1}{3}(0) + \frac{2}{3}(1) = \frac{2}{3}$$

$$E[XY] = \sum_x \sum_y xy p_{XY}(x, y) = \frac{1}{3}\{(-1)(1)+(0)(0)+(1)(1)\} = 0$$

$$Cov(X, Y) = \sigma_{XY} = E[XY] - E[X]E[Y] = 0$$

5.21 Two events A and B are such that $P[A] = 1/4$, $P[B|A] = 1/2$, and $P[A|B] = 1/4$. The random variable X has value $X = 1$ if event A occurs and $X = 0$ if event A does not occur. Similarly, the random variable Y has value $Y = 1$ if event B occurs and $Y = 0$ if event B does not occur.

First, we find $P[B]$ and the PMFs of X and Y.

$$P[B|A] = \frac{P[AB]}{P[A]} \Rightarrow P[AB] = P[B|A]P[A] = \frac{1}{8}$$

$$P[A|B] = \frac{P[AB]}{P[B]} \Rightarrow P[B] = \frac{P[AB]}{P[A|B]} = \frac{1}{2}$$

Note that because $P[A|B] = P[A]$ and $P[B|A] = P[B]$ events A and B are independent. Thus, the random variables X and Y are independent. The PMFs of X and Y are given by

$$p_X(x) = \begin{cases} 1 - P[A] \\ P[A] \end{cases} = \begin{cases} \frac{1}{4} & x = 0 \\ \frac{3}{4} & x = 1 \end{cases}$$

$$p_Y(x) = \begin{cases} 1 - P[B] \\ P[B] \end{cases} = \begin{cases} \frac{1}{2} & y = 0 \\ \frac{1}{2} & y = 1 \end{cases}$$

a. The mean and variance of X are given by

$$E[X] = \frac{3}{4}(0) + \frac{1}{4}(1) = \frac{1}{4}$$

$$E[X^2] = \frac{3}{4}(0^2) + \frac{1}{4}(1^1) = \frac{1}{4}$$

$$\sigma_X^2 = E[X^2] - (E[X])^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

b. The mean and variance of Y are given by

$$E[Y] = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$$

$$E[Y^2] = \frac{1}{2}(0^2) + \frac{1}{2}(1^1) = \frac{1}{2}$$

$$\sigma_Y^2 = E[Y^2] - (E[Y])^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

c. As stated earlier, X and Y are independent because the events A and B are independent. Thus, $\sigma_{XY} = 0$ and $\rho_{XY} = 0$, which means that X and Y are uncorrelated.

5.22 The random variable X denotes the number of 1's that appear in three tosses of a fair die, and Y denotes the number of 3's. Let A denote the event that an outcome of the toss is neither 1 nor 3. Then the sample space of the experiment and the values of X and Y are shown in the following table.

S	P[S]	X	Y
111	$(1/6)^3$	3	0
113	$(1/6)^3$	2	1
11A	$(1/6)^2(2/3)$	2	0
1A1	$(1/6)^2(2/3)$	2	0
131	$(1/6)^3$	2	1
1AA	$(1/6)(2/3)^2$	1	0

S	P[S]	X	Y
1A3	$(1/6)^2(2/3)$	1	1
133	$(1/6)^3$	1	2
13A	$(1/6)^2(2/3)$	1	1
333	$(1/6)^3$	0	3
33A	$(1/6)^2(2/3)$	0	2
331	$(1/6)^3$	1	2
3A3	$(1/6)^2(2/3)$	0	2
313	$(1/6)^3$	1	2
3AA	$(1/6)(2/3)^2$	0	1
3A1	$(1/6)^2(2/3)$	1	1
311	$(1/6)^3$	2	1
31A	$(1/6)^2(2/3)$	1	1
AAA	$(2/3)^3$	0	0
AA1	$(2/3)^2(1/6)$	1	0
AA3	$(2/3)^2(1/6)$	0	1
A1A	$(2/3)^2(1/6)$	1	0
A3A	$(2/3)^2(1/6)$	0	1
A11	$(1/6)^2(2/3)$	2	0
A13	$(1/6)^2(2/3)$	1	1
A33	$(1/6)^2(2/3)$	0	2
A31	$(1/6)^2(2/3)$	1	1

The PMFs of X and Y are given by

$$p_X(x) = \begin{cases} (1/6)^3 + 3(1/6)^2(2/3) + 3(1/6)(2/3)^2 + (2/3)^3 & x = 0 \\ 3(1/6)^3 + 4(1/6)^2(2/3) + 3(1/6)(2/3)^2 & x = 1 \\ 3(1/6)^3 + 3(1/6)^2(2/3) & x = 2 \\ (1/6)^3 & x = 3 \end{cases}$$

$$= \begin{cases} \frac{125}{216} & x = 0 \\ \frac{75}{216} & x = 1 \\ \frac{15}{216} & x = 2 \\ \frac{1}{216} & x = 3 \end{cases}$$

$$p_Y(y) = \begin{cases} (1/6)^3 + 3(1/6)^2(2/3) + 3(1/6)(2/3)^2 + (2/3)^3 & y = 0 \\ 3(1/6)^3 + 4(1/6)^2(2/3) + 3(1/6)(2/3)^2 & y = 1 \\ 3(1/6)^3 + 3(1/6)^2(2/3) & y = 2 \\ (1/6)^3 & y = 3 \end{cases}$$

$$= \begin{cases} \frac{125}{216} & y = 0 \\ \frac{75}{216} & y = 1 \\ \frac{15}{216} & y = 2 \\ \frac{1}{216} & y = 3 \end{cases}$$

Finally, the joint PMF of X and Y is given by

$$p_{XY}(x, y) = \begin{cases} \frac{64}{216} & x = 0, y = 0 \\ \frac{48}{216} & x = 0, y = 1 \\ \frac{12}{216} & x = 0, y = 2 \\ \frac{1}{216} & x = 0, y = 3 \\ \frac{48}{216} & x = 1, y = 0 \\ \frac{24}{216} & x = 1, y = 1 \\ \frac{3}{216} & x = 1, y = 2 \\ \frac{12}{216} & x = 2, y = 0 \\ \frac{3}{216} & x = 2, y = 1 \\ \frac{1}{216} & x = 3, y = 0 \\ 0 & otherwise \end{cases}$$

Thus, the correlation coefficient of X and Y, ρ_{XY}, can be obtained as follows:

$$E[X] = E[Y] = \frac{0(125)+1(75)+2(15)+3(1)}{216} = \frac{1}{2}$$

$$E[X^2] = E[Y^2] = \frac{0^2(125)+1^2(75)+2^2(15)+3^2(1)}{216} = \frac{2}{3}$$

$$\sigma_X^2 = \sigma_Y^2 = E[X^2]-(E[X])^2 = \frac{2}{3}-\frac{1}{4} = \frac{5}{12}$$

$$E[XY] = \frac{(1)(1)(24)+(1)(2)(3)}{216} = \frac{1}{6}$$

$$\sigma_{XY} = E[XY]-E[X]E[Y] = \frac{1}{6}-\frac{1}{4} = -\frac{1}{12}$$

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X\sigma_Y} = \frac{-(1/12)}{(5/12)} = -0.2$$

Section 5.9: Multinomial Distributions

5.23 Let p_A denote the probability that a chip is from supplier A, p_B the probability that it is from supplier B and p_C the probability that it is from supplier C. Now,

$$p_A = \frac{10}{40} = 0.25$$

$$p_B = \frac{16}{40} = 0.40$$

$$p_C = \frac{14}{40} = 0.35$$

Let K be a random variable that denotes the number of times that a chip from supplier B is drawn in 20 trials. Then K is a binomially distributed random variable with the PMF

$$p_K(k) = \binom{20}{k}p_B^k(1-p_B)^{20-k} \qquad k = 0, 1, \ldots, 20$$

Thus, the probability that a chip from vendor B is drawn 9 times in 20 trials is given by

$$p_K(9) = \binom{20}{9}(0.4)^9(0.6)^{11} = 0.1597$$

5.24 With reference to the previous problem, the probability p that a chip from vendor A is drawn 5 times and a chip from vendor C is drawn 6 times in the 20 trials is given by

$$p = \binom{20}{5\ 9\ 6}(0.25)^5(0.4)^9(0.35)^6 = \frac{20!}{5!9!6!}(0.25)^5(0.4)^9(0.35)^6 = 0.0365$$

5.25 Let p_E denote the probability that a professor is rated excellent, p_G the probability that a professor is rated good, p_F the probability that a professor is rated fair, and p_B the probability that a professor is rated bad. Then we are given that

$$\begin{aligned} p_E &= 0.2 \\ p_G &= 0.5 \\ p_F &= 0.2 \\ p_B &= 0.1 \end{aligned}$$

Given that 12 professors are randomly selected from the college,

a. the probability P_1 that 6 are excellent, 4 are good, 1 is fair, and 1 is bad is given by

$$P_1 = \binom{12}{6\ 4\ 1\ 1}(0.2)^6(0.5)^4(0.2)^1(0.1)^1 = \frac{12!}{6!4!1!1!}(0.2)^6(0.5)^4(0.2)^1(0.1)^1 = 0.0022$$

b. the probability P_2 that 6 are excellent, 4 are good, and 2 are fair is given by

$$P_2 = \binom{12}{6\ 4\ 2\ 0}(0.2)^6(0.5)^4(0.2)^2(0.1)^0 = \frac{12!}{6!4!2!0!}(0.2)^6(0.5)^4(0.2)^2(0.1)^0 = 0.0022$$

c. the probability P_3 that 6 are excellent and 6 are good is given by

$$P_3 = \binom{12}{6\ 6\ 0\ 0}(0.2)^6(0.5)^6(0.2)^0(0.1)^0 = \frac{12!}{6!6!}(0.2)^6(0.5)^6 = \frac{12!}{6!6!}(0.1)^6 = 0.000924$$

d. the probability P_4 that 4 are excellent and 3 are good is the probability that 4 are excellent, 3 are good, and 5 are either bad or fair with a probability of 0.3, and this is given by

$$P_4 = \binom{12}{4\ 3\ 5}(0.2)^4(0.5)^3(0.3)^5 = \frac{12!}{4!3!5!}(0.2)^4(0.5)^3(0.3)^5 = 0.0135$$

e. the probability P_5 that 4 are bad is the probability that 4 are bad and 8 are not bad, which is given by the following binomial distribution:

$$P_5 = \binom{12}{4}(0.1)^4(0.9)^8 = \frac{12!}{4!8!}(0.1)^4(0.9)^8 = 0.0213$$

f. the probability P_6 that none is bad is the probability that all 12 are not bad with probability 0.9, which is given by the binomial distribution

$$P_6 = \binom{12}{0}(0.1)^0(0.9)^{12} = (0.9)^{12} = 0.2824$$

5.26 Let p_G denote the probability that a toaster is good, p_F the probability that it is fair, p_B the probability that it burns the toast, and p_C the probability that it can catch fire. We are given that

$$\begin{aligned} p_G &= 0.50 \\ p_F &= 0.35 \\ p_B &= 0.10 \\ p_C &= 0.05 \end{aligned}$$

Given that a store has 40 of these toasters in stock, then

a. the probability P_1 that 30 are good, 5 are fair, 3 burn the toast, and 2 catch fire is given by

$$P_1 = \binom{40}{30\ 5\ 3\ 2}(0.50)^{30}(0.35)^5(0.10)^3(0.05)^2 = \frac{40!}{30!5!3!2!}(0.50)^{30}(0.35)^5(0.10)^3(0.05)^2 = 0.000026$$

b. the probability P_2 that 30 are good and 4 are fair is the probability that 30 are good, 4 are fair, and 6 are either bad or can catch fire, which is given by

$$P_2 = \binom{40}{30\ 4\ 6}(0.50)^{30}(0.35)^4(0.15)^6 = \frac{40!}{30!4!6!}(0.50)^{30}(0.35)^4(0.15)^6 = 0.000028$$

c. the probability P_3 that none catches fire is given by the binomial distribution

$$P_3 = \binom{40}{0}(0.05)^0(0.95)^{40} = (0.95)^{40} = 0.1285$$

d. the probability P_4 that none burns the toast and none catches fire is given by

$$P_4 = \binom{40}{0}(0.15)^0(0.85)^{40} = (0.85)^{40} = 0.0015$$

5.27 Let p_B denote the probability that a candy goes to a boy, p_G the probability that a candy goes to a girl, and p_A the probability that it goes to an adult. Then we know that

$$p_B = \frac{8}{20} = 0.40$$

$$p_G = \frac{7}{20} = 0.35$$

$$p_A = \frac{5}{20} = 0.25$$

Given that 10 pieces of candy are given out at random to the group, we have that

a. the probability P_1 that 4 pieces go to the girls and 2 go to the adults is the probability that 4 pieces go to the boys, 4 go to the girls, and 2 go to the adults, which is given by

$$P_1 = \binom{10}{4\ 4\ 2}(0.40)^4(0.35)^4(0.25)^2 = \frac{10!}{4!4!2!}(0.40)^4(0.35)^4(0.25)^2 = 0.0756$$

b. The probability P_2 that 5 pieces go to the boys is the probability that 5 pieces go to the boys and the other 5 pieces go to either the girls or the adults, which is given by

$$P_2 = \binom{10}{5}(0.40)^5(0.60)^5 = \frac{10!}{5!5!}(0.40)^5(0.60)^5 = 0.2006$$