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SUPERVISOR'S USE ONLY

TOP SCHOLAR

NZQA

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2011
Mathematics with Calculus

9.30 am Saturday 26 November 2011
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–26 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

QUESTION NUMBER

You have three hours to complete this examination.

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1a $\frac{ae^x}{2e^x-1} < 1$.

Consider 2 cases i) when $2e^x-1<0$
ii) when $2e^x-1>0$.

We can dismiss when $2e^x-1=0$ as that is not a solution.
~~Consider~~ as then $\frac{ae^x}{2e^x-1}$ is not defined.

Consider case ii).

$\frac{ae^x}{2e^x-1} < 1$.

$ae^x < 2e^x - 1$.

$1 < (2-a)e^x$

$\ln(1) < \ln(2-a) + \ln e^x$

$2-a$ must be >0 for $\ln(2-a)$ to be defined.
but if $(2-a)e^x > 1$, then $2-a$ is definitely >0

$\therefore -x < \ln(2-a)$.

$x > -\ln(2-a)$, where $(2-a) > 0$ and is a positive constant i.e. $0<a<2$.

Also, $2e^x - 1 > 0$.

$e^x > \frac{1}{2}$.

$x > \ln\left(\frac{1}{2}\right) = -\ln(2)$

but $-\ln(2-a) > -\ln(2)$.

because $2 > 2-a$.

$\therefore x > -\ln(2-a)$, $0<a<2$.

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Consider case i).

Then $2e^x - 1 < 0$

$e^x < \frac{1}{2}$.

$x < -\ln(2)$.

Also, $\frac{ae^x}{2e^x-1} < 1$.

$ae^x > 2e^x - 1$ (because $2e^x - 1 < 0$).

$(a-2)e^x > -1$.

$e^x > \frac{-1}{a-2} = \frac{1}{2-a}$ (If $a-2>0$).

$x > -\ln(2-a)$

$\therefore -\ln(2-a) < x < -\ln(2)$. If $a-2>0$.

However, if $a-2>0$, $2-a<0$ $\therefore \ln(2-a)$ does not exist.

$\therefore (a-2)e^x > -1$. If $a-2<0$.

$e^x < \frac{-1}{a-2} = \frac{1}{2-a}$

$x < -\ln(2-a)$. and $x < -\ln(2)$.

However, if $2e^x - 1 < 0$, $ae^x > 0$, $\therefore \frac{ae^x}{2e^x-1} < 0 < 1$ for all a.

$\therefore x < \ln 2$.

So all real solutions are $x < \ln 2$ and $x > -\ln(2-a)$ when $0 < a < 2$

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b. There are m faces of A_A and n faces of A_B.

$\therefore$ Total surface area $=$ $SA = m\sqrt{2k^2-2k+1} + n\frac{\sqrt{3}}{2}(k^2-k+1)$.

$$\therefore \frac{dSA}{dk} = m\frac{1}{2}(2k^2-2k+1)^{-\frac{1}{2}} \times (4k-2) + n\frac{\sqrt{3}}{2}(2k-1)$$

$$= \cancel{m(4k-2)} \frac{m(2k-1)}{\sqrt{2k^2-2k+1}} + n\frac{\sqrt{3}}{2}(2k-1)$$

$$= (2k-1)\left(\frac{m}{\sqrt{2k^2-2k+1}} + \frac{\sqrt{3}n}{2}\right).$$

$$\therefore 2k-1=0 \quad \text{or} \quad \frac{m}{\sqrt{2k^2-2k+1}} + \frac{\sqrt{3}n}{2} = 0.$$

However, in the second equation, $n>0$ $\therefore \frac{\sqrt{3}n}{2}>0$, and square roots, $\sqrt{2k^2-2k+1}>0$, and $m>0$, $\therefore \frac{m}{\sqrt{2k^2-2k+1}}>0$. $\therefore \frac{m}{\sqrt{2k^2-2k+1}} + \frac{\sqrt{3}n}{2} > 0$ $\therefore$ It does not equal 0.

$$\therefore 2k-1=0$$
$$k=\frac{1}{2}.$$

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To show this is a minimum, take the second derivative.

$$\frac{dSA}{dk} = (2k-1)\left(\frac{m}{\sqrt{2k^2-2k+1}} + \frac{\sqrt{3}n}{2}\right)$$

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$$\frac{d^2SA}{dt^2} = 2\left(\frac{m}{\sqrt{2k^2-2k+1}} + \frac{\sqrt{3}n}{2}\right) + (2k-1)\left(-\frac{1}{2} \times m \times \sqrt{2k^2-2k+1}^{-\frac{3}{2}} \times (4k-2)\right)$$

$$= \frac{2m}{\sqrt{2k^2-2k+1}} + \sqrt{3}n + (2k-1)\left(\frac{-m(2k-1)}{\sqrt{2k^2-2k+1}^{3}}\right)$$

If $k=\frac{1}{2}$, then $\frac{d^2SA}{dt^2} = \frac{2m}{\sqrt{\frac{1}{2}}} + \sqrt{3}n > 0$,

as all ~~the~~ terms are >0.

$\therefore \frac{d^2SA}{dt^2} > 0$ at $k=\frac{1}{2}$ and $\frac{dSA}{dt}=0$ at $k=\frac{1}{2}$.

$\therefore k=\frac{1}{2}$ is a minimum.

c. $\frac{dy}{dx} = y^{m+1}$

$y^{-(m+1)}dy = dx$.

$\therefore \int y^{-(m+1)}dy = \int 1\, dx$

$\frac{y^{-m}}{-m} + c_1 = x + c_2$ (constant).

$\therefore \frac{y^{-m}}{-m} = x + c$ (where $c = c_2 - c_1$).

$y^{-m} = -mx - mc$.

$y^{m} = \frac{-1}{mx+mc}$

$y = \sqrt[m]{\frac{-1}{mx+mc}} = \sqrt[m]{\frac{-1}{mx+k}}$, where $k = mc$.

If m was an even number, then

$y = \pm\sqrt[m]{\frac{-1}{mx+k}}$, as $y^m = \left(\pm\sqrt[m]{\frac{-1}{mx+k}}\right)^m = \frac{-1}{mx+k}$

$\therefore y = \sqrt[m]{\frac{-1}{mx+k}}$ or $\pm\sqrt[m]{\frac{-1}{mx+k}}$ if m is even.

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$\frac{d}{dx}(y^n) = ny^{n-1} \times \frac{dy}{dx}$

$= ny^{n-1} \times y^{m+1}$

$= ny^{n+m}$

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3a. $\cos \frac{7\pi}{12} = \cos\left(\frac{6\pi}{12} + \frac{\pi}{12}\right)$

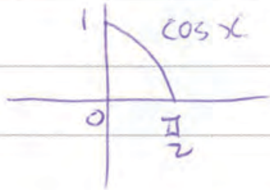
$= \cos\left(\frac{\pi}{2} + \frac{\pi}{12}\right)$

Also, $\cos\left(2 \times \frac{\pi}{12}\right) = 2\cos^2\frac{\pi}{12} - 1 = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}.$

$\therefore 2\cos^2\frac{\pi}{12} = \frac{2+\sqrt{3}}{2}.$

$\cos\frac{\pi}{12} = \pm\frac{\sqrt{2+\sqrt{3}}}{2}$

However



Clearly $\cos\frac{\pi}{12} > 0.$

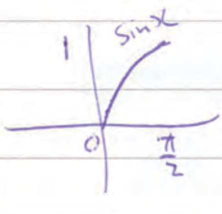
$\therefore \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}.$

~~∴ cos (π/6~~ Also, $\cos\left(2 \times \frac{\pi}{12}\right) = 1 - 2\sin^2\frac{\pi}{12} = \frac{\sqrt{3}}{2}.$

$\frac{2-\sqrt{3}}{2} = 2\sin^2\frac{\pi}{12}.$

$\sin\frac{\pi}{12} = \pm\frac{\sqrt{2-\sqrt{3}}}{2}.$

But



Clearly $\sin\frac{\pi}{12} > 0 \quad \therefore \sin\frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2}.$

$\therefore \cos\left(\frac{\pi}{2} + \frac{\pi}{12}\right) = \cos\frac{\pi}{2}\cos\frac{\pi}{12} - \sin\frac{\pi}{2}\sin\frac{\pi}{12}$

$= 0 - 1 \times \frac{\sqrt{2-\sqrt{3}}}{2}.$

$= -\frac{\sqrt{2-\sqrt{3}}}{2}.$

$= \cos\left(\frac{7\pi}{12}\right).$

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~~b.~~ $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = 20\sqrt{6}$, $\sin\theta = 20\sqrt{6}\cos\theta$

$\cancel{\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}}$

$\tan\frac{\theta}{2} = \dfrac{\pm\sqrt{\frac{1-\cos\theta}{2}}}{\pm\sqrt{\frac{1+\cos\theta}{2}}} = \pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$

$= \pm\sqrt{\dfrac{1-\frac{\sin\theta}{20\sqrt{6}}}{1+\cos\theta}}$

$\sin\frac{\theta}{4} = \pm\sqrt{\dfrac{1-\pm\sqrt{\frac{1-\cos\theta}{2}}}{2}}$

$\cos\frac{\theta}{4} = \pm\sqrt{\dfrac{1+\pm\sqrt{\frac{1+\cos\theta}{2}}}{2}}$

$\therefore$ ~~sin θ/4~~ $\tan\frac{\theta}{4} = \pm\sqrt{\dfrac{1\mp\sqrt{\frac{1-\cos\theta}{2}}}{1\pm\sqrt{\frac{1+\cos\theta}{2}}}}$

If $0<\theta<\frac{\pi}{2}$, ~~all ± are +, except~~
all $\pm$ and $\mp$ are the top sign.

$\therefore \tan\frac{\theta}{4} = \sqrt{\dfrac{1-\sqrt{\frac{1-\cos\theta}{2}}}{1+\sqrt{\frac{1+\cos\theta}{2}}}} = \sqrt{\dfrac{1-\frac{1-\cos\theta}{2}}{1+\sqrt{2(1+\cos\theta)}+\frac{1+\cos\theta}{2}}}$

b. $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = 20\sqrt{6}$. $\therefore \sin\theta = 20\sqrt{6}\cos\theta$.

Since $0<\theta<\frac{\pi}{2}$, all $\pm$ are $+$.

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$\therefore \sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}}$

$\cancel{\therefore \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}}$

$\cancel{\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\sqrt{1-\cos\theta}}$

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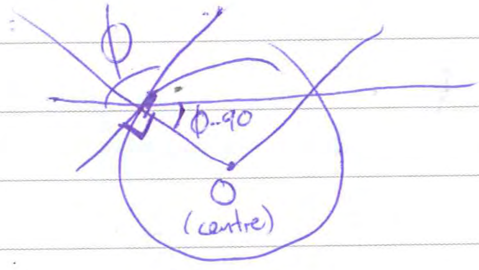
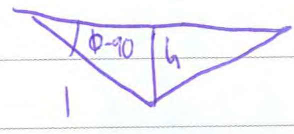
c. Let the radius of the boulder be 1 unit.

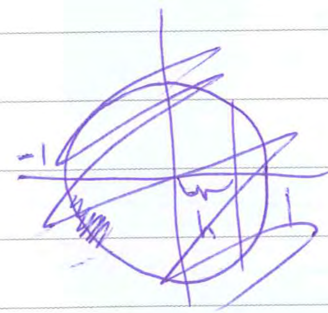


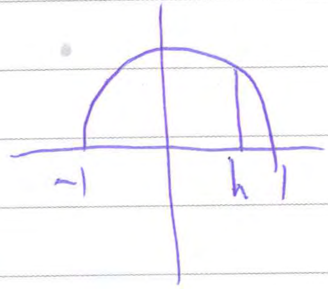
That angle marked $\phi - 90$ is $\phi - 90$ because it is vertically opposite the other one which is $\phi - 90$, since the angle between radius and tangent is $90°$.

$\therefore$



$\therefore \sin(\phi - 90) = h = \cos\phi$.





Here, $y = \sqrt{1-x^2}$.

take the volume of revolution

$$V = \int_{-1}^{h} \pi y^2 \, dx$$

$$= \pi \int_{-1}^{h} (1-x^2)\, dx$$

$$= \pi \left[x - \frac{x^3}{3} \right]_{-1}^{h} = \pi\left(\left(\cos\phi - \frac{\cos^3\phi}{3}\right) - \left(-1 - \frac{-1^3}{3}\right)\right)$$

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$$= \pi\left(\frac{\cos^3\phi}{3} - \cos\phi + \frac{2}{3}\right)$$

Total volume $= \frac{4}{3}\pi \times 1^3 = \frac{4}{3}\pi$.

$\therefore$ Proportion underwater $= \dfrac{\pi\left(\frac{\cos^3\phi}{3} - \cos\phi + \frac{2}{3}\right)}{\frac{4}{3}\pi}$

$$= \frac{\cos^3\phi - 3\cos\phi + 2}{4}.$$

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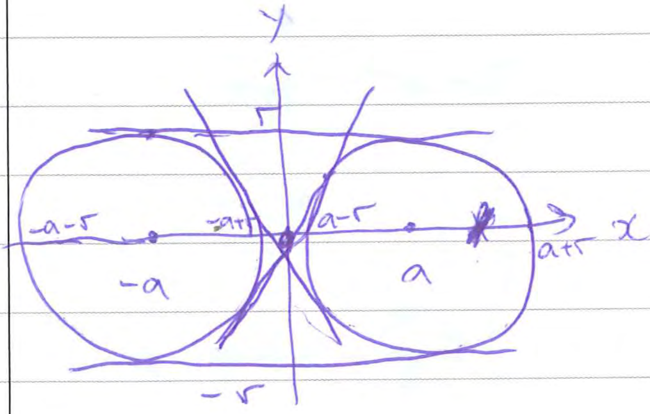
4a.



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It is immediately obvious that 2 of the tangents are $y = \pm r$, and that the other two will be of the form $y = \pm kx$.
Take $y = kx$, where $k > 0$, and consider the $(x-a)^2 + y^2 = r^2$.

~~If $y = kx$, let the point of tangency be (f, kf).~~

$$\therefore (x-a)^2 + k^2x^2 = r^2$$

$$x^2 - 2ax + a^2 + k^2x^2 = r^2$$

$$(k^2+1)x^2 - 2ax + (a^2 - r^2) = 0$$

$$\therefore \Delta = 4a^2 - 4(k^2+1)(a^2-r^2)$$

$$= 4a^2 - 4k^2a^2 + 4k^2r^2 - 4a^2 + 4r^2$$

$$= 4k^2(r^2-a^2) + 4r^2 = 0,$$

because there is only 1 ~~so~~ repeated solution as it is tangent.

$$\therefore k^2(a^2-r^2) = r^2$$

$$k^2 = \frac{r^2}{a^2-r^2}$$

$$k = \frac{\pm r}{\sqrt{a^2-r^2}}$$

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$\therefore$ The 4 tangents are $y = \pm r, \dfrac{\pm r}{\sqrt{a^2 - r^2}} x$

b. $B(x) = I\left(\dfrac{1}{x^2} + \dfrac{1}{(d-x)^2}\right)$

$\therefore B'(x) = I\left(\dfrac{-2}{x^3} + \dfrac{-2}{(d-x)^3} \times -1\right)$

$= 2I\left(\dfrac{1}{(d-x)^3} - \dfrac{1}{x^3}\right)$.

$\therefore 2I\left(\dfrac{1}{(d-x)^3} - \dfrac{1}{x^3}\right) = 0$.

$\dfrac{1}{(d-x)^3} = \dfrac{1}{x^3}$.

$(d-x)^3 = x^3$

$d - x = x$

$x = \dfrac{d}{2}$.

$B''(x) = 2I\left(\dfrac{-3}{(d-x)^4} \times -1 - \dfrac{-3}{x^4}\right)$

$= 2I\left(\dfrac{3}{(d-x)^4} + \dfrac{3}{x^4}\right)$.

If $x = \dfrac{d}{2}$, $B''(x) = 6I\left(\dfrac{1}{\frac{d^4}{16}} + \dfrac{1}{\frac{d^4}{16}}\right) > 0$.

$\therefore$ At $x = \dfrac{d}{2}$, brightness is a minimum.
So it is least brightly at the points equidistant from both bulbs (as $x = d - x$), so that is

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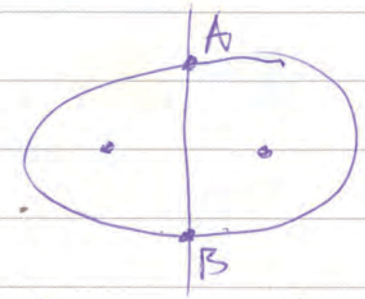

A and B on the diagram.

C. Suppose Observer 1, O_1, ~~hears~~ hears the gunshot t_1 seconds after the gunshot, and O_2 hears it t_2 seconds after.

∴ ~~The~~ Their distances from the gunshot are $340t_1$ and $340t_2$.

~~But t_2 = 340(~~

∴ ~~B~~ The differene between the distances is $340(t_2 - t_1) = 340 \times 1.5 = 510$.

∴ The difference between the two distances is a constant 510m, so this is a hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where ~~the~~ ~~the~~ O_1 and O_2 are at the foci and the gun is on the ~~top~~ hyperbola.

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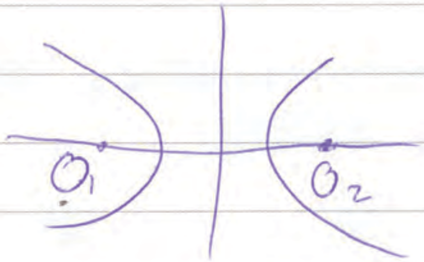
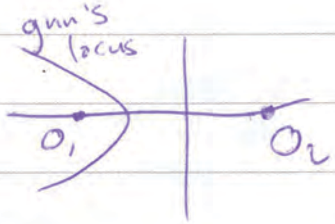


The gun must be on O_1's branch as he hears it first.



$(\pm c, 0)$

~~$(\pm ae, 0)$ is the position of O_1, O_2 and~~
~~$b^2 = a^2(e^2-1) = a^2\left(\frac{c^2}{a^2}-1\right) = c^2 - a^2 = 510^2 - a^2$~~

Let the observers' positions be $(\pm ae, 0)$.

Then $2ae - (ae - a) = ae + a = 510\text{ m}$.

$\therefore\ a(e+1) = 510,\quad a = \frac{510}{e+1}$

$b^2 = a^2(e^2-1) = \cancel{a(e+1)\,a(e-1) = 510\,a(e-1)}$

$= \frac{510^2}{(e+1)^2}(e^2-1)$.

$\therefore$ ~~$\frac{x^2}{a^2} - \frac{y^2}{\ldots}$~~ $\therefore\ \frac{x^2(e+1)^2}{260100} - \frac{y^2(e+1)^2}{260100(e^2-1)} = 1$.

where $e > 1$, $x < 0$.

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5a. $\Gamma\left(\frac{1}{2}\right)\times\Gamma\left(1-\frac{1}{2}\right)=\frac{\pi}{\sin(\frac{1}{2}\pi)}$

$\therefore\ \Gamma\left(\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right)=\pi.$

$\therefore\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$

$\therefore\ \Gamma\left(\frac{1}{2}+1\right)=\left(\frac{1}{2}+1-1\right)\Gamma\frac{1}{2}$

$=\frac{1}{2}\sqrt{\pi}.$

$\therefore\ \Gamma\left(\frac{1}{2}+1+1\right)=\left(\frac{1}{2}+1+1-1\right)\Gamma\left(\frac{1}{2}+1\right)$

$=\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}$

$=\frac{3}{4}\sqrt{\pi}.$

b. $\cancel{\sqrt{2}+\sqrt{3}i=\sqrt{2+3}\ \text{cis}\tan^{-1}\left(\frac{\sqrt{3}}{\sqrt{2}}\right)}$

$\cancel{=\sqrt{5}\ \text{cis}}$

$c^6+(2-k)c^4+(25-2k)c^2-25k=0.$

$(\sqrt{2}+\sqrt{3}i)^6+(2-k)(\sqrt{2}+\sqrt{3}i)^4+(25-2k)(\sqrt{2}+\sqrt{3}i)^2-25k=0.$

$8+6\times4\sqrt{2}\times\sqrt{3}i+15\times4\times-3+20\times2\sqrt{2}\times-3\sqrt{3}i$

$+15\times2\times9+6\times\sqrt{2}\times9\sqrt{3}i+-27=(\sqrt{2}+\sqrt{3}i)^6$

$=(8-180+270-27)+(24\sqrt{6}-120\sqrt{6}+54\sqrt{6})i$

$=71-42\sqrt{6}i$

$(\sqrt{2}+\sqrt{3}i)^4=4+4\times2\sqrt{2}\times\sqrt{3}i+6\times2\times-3+4\times\sqrt{2}\times-3\sqrt{3}i$

$+9$

$=(4+9-36)+(8\sqrt{6}-12\sqrt{6})i$

$=-23-4\sqrt{6}i$

$(\sqrt{2}+\sqrt{3}i)^2=2+2\sqrt{6}i-3=-1+2\sqrt{6}i$

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5a. $\therefore c^6 + (2-k)c^4 + (25-2k)c^2 - 25k$

$= 71 - 42\sqrt{6}i + (2-k)(-23-4\sqrt{6}i) + (25-2k)(-1+2\sqrt{6}i)$

$-25k$

$= 71 - 42\sqrt{6}i - 46 - 8\sqrt{6}i + 23k + 4\sqrt{6}ki - 25 + 50\sqrt{6}i$

$+2k - 4\sqrt{6}ki - 25k$

$= (71-46-25) + (23+2-25)k - \sqrt{6}(42+8-50)i + (4\sqrt{6}-4\sqrt{6})ki$

$= 0$

5b. $x - (\sqrt{2}+\sqrt{3}i) \overline{) x^6 + (2-k)x^4 + (25-2k)x^2 - 25k}$

x^5

$x^6 - (\sqrt{2}-$

If $\sqrt{2}+\sqrt{3}i$ is a root, so must its conjugate be, $\sqrt{2}-\sqrt{3}i$, as all the coefficients are real.

$\therefore (x-(2+\sqrt{3}i))(x-(2-\sqrt{3}i)) = x^2 - 4x + 7$

$x^4 + 4x^3 + (11-k)x^2 + (16-4k)x + (12-25k)$

$\therefore x^2 - 4x + 7 \overline{) x^6 + 0x^5 + (2-k)x^4 + 0x^3 + (25-2k)x^2 + 0x - 25k}$

$x^6 - 4x^5 + 7x^4$

$4x^5 - (k+5)x^4 + 0x^3$

$4x^5 - 16x^4 + 28x^3$

$(11-k)x^4 - 28x^3 + (25-2k)x^2$

$(11-k)x^4 - (44-4k)x^3 + (77-7k)x^2$

$(16-4k)x^3 - (52+5k)x^2 + 0x$

$(16-4k)x^3 - (64-16k)x^2 + (112-28k)x$

$(12-25k)x^2 - (28k-112)x - 25k$

$(12-25k)x^2 - (48-100k)x + 84 - 175k$

$(-72k-64)x + 150k - 84.$

$= 0.$

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$$x^4 + 4x^3 + (11-k)x^2 + (16-4k)x + (21k-116)$$

$$x^2-4x+7 \overline{\big)\, x^6 + 0x^5 + (2-k)x^4 + 0x^3 + (25-2k)x^2 + 0x - 25k}$$

$$x^6 - 4x^5 + 7x^4$$

$$4x^5 - (k+5)x^4 + 0x^3$$

$$4x^5 - 16x^4 + 28x^3$$

$$(11-k)x^4 - 28x^3 + (25-2k)x^2$$

$$(11-k)x^4 - (44-4k)x^3 + (77-7k)x^2$$

$$(16-4k)x^3 + (5k-52)x^2 + 0x$$

$$(16-4k)x^3 + (64-16k)x^2 + (112-28k)x$$

$$(21k-116)x^2 + (28k-112)x - 25k$$

$$(21k-116)x^2 - (84k-464)x + (147k-812)$$

$$(112k-576)x - (172k+812)$$

$$= 0.$$

$\therefore$ ~~$x = \frac{172k+812}{112k-576}$~~

$\therefore (x^2-4x+7)(x^4+ax^3+bx^2+cx+d) = x^6 + (2-k)x^4 + (25-2k)x^2 - 25k$

$\therefore x^6 + ax^5 + bx^4 + cx^3 + dx^2 - 4x^5 - 4ax^4 - 4bx^3 - 4cx^2 - 4dx$
$+ 7x^4 + 7ax^3 + 7bx^2 + 7cx + 7d$

$= x^6 + (a-4)x^5 + (b+7-4a)x^4 + (c+7a-4b)x^3 + (d-4c+7b)x^2$
$+ (7c-4d)x + 7d = x^6 + (2-k)x^4 + (25-2k)x^2 - 25k$

$\therefore a = 4, \quad b-9 = 2-k, \quad 7d = -25k$

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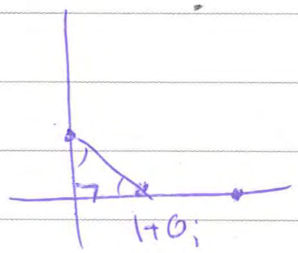
C. ~~$z^8 = 1$~~

As this octagon is regular,

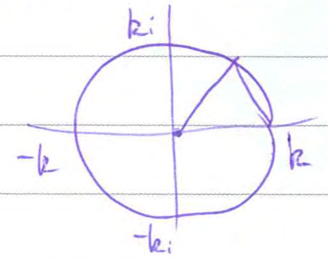


The side lengths are all $\sqrt{1^2+1^2} = \sqrt{2}$.

$\therefore$ Side lengths are all $\sqrt{2}$.

~~$z^8 = 1$ has side lengths all~~

Consider $z^8 = k$.



Then one side length would be

$\sqrt{k^2+k^2-2k^2\cos(45^\circ)} = k\sqrt{2-\sqrt{2}}$

$\therefore$ If $k\sqrt{2-\sqrt{2}} = \sqrt{2}$, then $k = \dfrac{\sqrt{2}}{\sqrt{2-\sqrt{2}}} = \sqrt{\dfrac{2}{2-\sqrt{2}}}$

$= \sqrt{\dfrac{2(2+\sqrt{2})}{2}}$

$= \sqrt{2+\sqrt{2}}$

$\therefore z^8 = \sqrt{2+\sqrt{2}}$, and the solutions are a regular octagon. However, this has been rotated $\frac{45}{2} = 22.5^\circ$ to get the given orientation. $\therefore$ z should be replaced by $\text{cis}\,22.5^\circ z$, ~~so~~ $(\text{cis}\,22.5\,z)^8$

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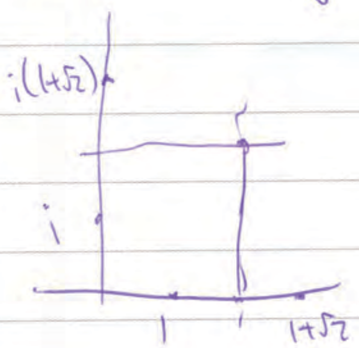
$= \text{cis } 180^\circ z^8$
$= (-1)z^8 = \sqrt{2+\sqrt{2}}$
$\therefore z^8 = -\sqrt{2+\sqrt{2}}$.

the shape has been translated ~~so~~ that
~~so the origin is now~~



Centre is at $x = \frac{1 + 1+\sqrt{2}}{2} = y$

$\therefore$ Centre is $\frac{2+\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i$

$\therefore \left(z - \left(\frac{2+\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i\right)\right)^8 = -\sqrt{2+\sqrt{2}}$
$\therefore p(z) = \left(z - \left(\frac{2+\sqrt{2}}{2} + \frac{2+\sqrt{2}}{2}i\right)\right)^8 + \sqrt{2+\sqrt{2}}$

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2a. We ~~want~~ want $\frac{dx}{dt}$.

$$\frac{dV}{dx} = \frac{dV}{dt} \times \frac{dx}{dt}.$$

$$\frac{dV}{dx} = -25\pi + \pi x^2.$$

A millimetre of rain means 1 mm per square meter. The area of the pond is 25π square meter.

~~∴ There is a~~

$$\therefore \frac{dV}{dt} = \frac{15}{1000} \times 25\pi = 0.375\pi.$$

$$\therefore (-25\pi + \pi \times 3^2) = 0.375\pi \times \frac{dx}{dt}.$$

$$\frac{dx}{dt} = \frac{-25\pi + 9\pi}{0.375\pi} = -42.6̇$$

Therefore it is rising at 42.6̇ metres per hour.

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b i.

$$\frac{d}{dx}\left(A(1-\sqrt{x})^{1.5}(2+\sqrt{3}x)+c\right)$$
$$\cancel{= 1.5A(1-\sqrt{x})^{0.5} \times (2+\sqrt{3}x) + A(1-\sqrt{x})^{1.5} \times \sqrt{3}}$$
$$= 1.5A\sqrt{1-\sqrt{x}} \times (2+\sqrt{3}x) + \sqrt{3}A(1-\sqrt{x})\sqrt{1-\sqrt{x}}$$
$$= 1.5A\,g(x) \times (2+\sqrt{3}x) + \sqrt{3}A(1-\sqrt{x})\,g(x)$$
$$= g(x)\left(3A + 1.5\sqrt{3}Ax + \sqrt{3}A - \sqrt{3}Ax\right)$$
$$= g(x)$$
$$\therefore A(3+0.5\sqrt{3}x+\sqrt{3}) = 1$$

$$\frac{d}{dx}\left(A(1-\sqrt{x})^{1.5}(2+\sqrt{3}x)+c\right)$$
$$= 1.5A(1-\sqrt{x})^{0.5} \times -\frac{1}{2}x^{-0.5} \times (2+\sqrt{3}x) + \sqrt{3}A(1-\sqrt{x})^{1.5}$$
$$= (1.5A \times -\tfrac{1}{2}x^{-\frac{1}{2}})(2+\sqrt{3}x)\sqrt{1-\sqrt{x}} + \sqrt{3}A(1-\sqrt{x})\sqrt{1-\sqrt{x}}$$
$$= g(x)\left(-\tfrac{3}{4}Ax^{-\frac{1}{2}}\right)(2+\sqrt{3}x) + g(x)(\sqrt{3}A - \sqrt{3}A\sqrt{x})$$
$$= g(x)\left(-\tfrac{3}{2}Ax^{-\frac{1}{2}} - \tfrac{3\sqrt{3}}{4}Ax^{\frac{1}{2}} + \sqrt{3}A - \sqrt{3}Ax^{\frac{1}{2}}\right)$$
$$= g(x)A\left(-\frac{3}{2\sqrt{x}} - \frac{3\sqrt{3}}{4}\sqrt{x} + \sqrt{3} - \sqrt{3}\sqrt{x}\right)$$

$$\frac{d}{dx}\,A(1-\sqrt{x})^{1.5}(2+3\sqrt{x})+C$$
$$= 1.5A(1-\sqrt{x})^{0.5} \times -\tfrac{1}{2}x^{-0.5} \times (2+3\sqrt{x}) + \tfrac{3}{2}x^{-\frac{1}{2}}A(1-\sqrt{x})^{1.5}$$
$$= -\tfrac{3}{4}Ax^{-0.5}(2+3\sqrt{x})(\sqrt{1-\sqrt{x}}) + \tfrac{3}{2}x^{-\frac{1}{2}}A(1-\sqrt{x})\sqrt{1-\sqrt{x}}$$
$$= \left(-\tfrac{3}{2}Ax^{-0.5} - \tfrac{9}{4}A\right)g(x) + \left(\tfrac{3}{2}Ax^{-\frac{1}{2}} - \tfrac{3}{2}A\right)g(x)$$
$$= g(x)(-3.75A) = g(x)$$
$$\therefore -3.75A = 1$$
$$A = \frac{-4}{15}$$

ii. Consider the sum of the area bounded by y axis, x axis and g(x), another area bounded by y=1, x=1 and g'(x). These two areas are equal, since g'(x) is only a rotation of g(x), but ~~those~~ their

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sum is $1 + \cancel{A}(A)$, where A is the area we want, since their overlap is A.

$\therefore 2\int_0^1 g(x)\,dx = 1 + F$

$\therefore 2\int_0^1 g(x)\,dx = 2\left[-\frac{4}{15}(1-\sqrt{x})^{1.5}(2+3\sqrt{x})\right]_0^1$

$= 2\left(\frac{-4}{15}(1-\sqrt{1})^{1.5}(2+3\sqrt{1}) - \frac{-4}{15}(1-\sqrt{0})^{1.5}(2+3\sqrt{0})\right)$

$= 2\left(0 + \frac{4}{15}\times 1 \times 2\right)$

$= \frac{16}{15} = 1 + F$

$\therefore F = \frac{1}{15}$. con.