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SUPERVISOR'S USE ONLY

OUTSTANDING SCHOLARSHIP EXEMPLAR

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2019
Calculus

9.30 a.m. Friday 8 November 2019
Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

ASSESSOR'S USE ONLY

QUESTION NUMBER

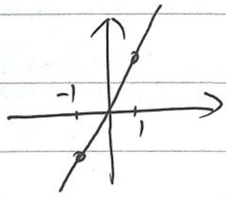
ASSESSOR'S USE ONLY

Q1 a) $x \neq \pm 1$ $\quad f(x) = \frac{(x-1)^2}{(x-1)} + \frac{(x+1)^2}{(x+1)}$

~~$f'(x)$~~

$$\begin{array}{r} x - 1 \\ x-1 \overline{) x^2 - 2x + 1} \\ \underline{x^2 - x} \\ -x + 1 \\ \underline{-x + 1} \\ 0 \end{array} \qquad \begin{array}{r} x + 1 \\ x+1 \overline{) x^2 + 2x + 1} \\ \underline{x^2 + x} \\ x + 1 \end{array}$$

$f(x) = (x-1) + (x+1)$
$= 2x$

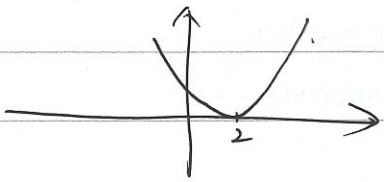


$x < -1 \cap -1 < x < 1 \cap x > 1$

~~b)~~

b) $-(x-2)^2 > 0$ $\quad$ ~~$-(x-2)^2 \neq x$~~

$(x-2)^2 < 0$ $\quad x^2 + 4 - 4x < 0$



$\because x$ is real

$\therefore$ there is no value for $f(x)$ is real.

c) i) A B C

1 2 3

${}^6C_1 \times {}^5C_2 \times {}^3C_3$

$= 60$

QUESTION NUMBER

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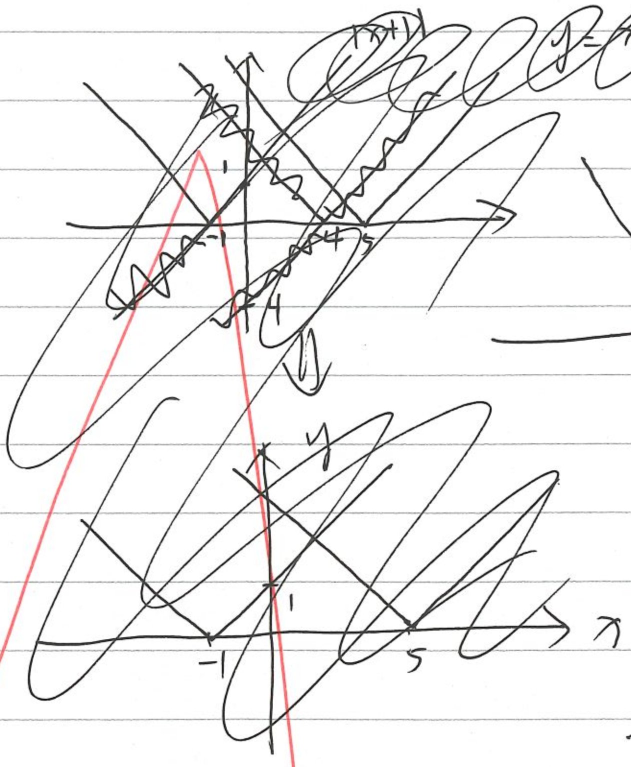
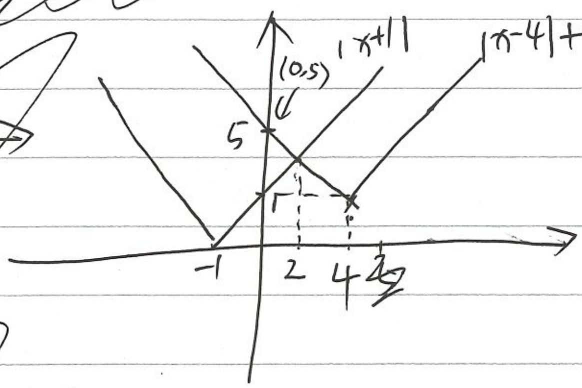
ii) A B C

$${}^6C_2 \times {}^4C_2 \times {}^2C_2 = 90$$

d) $|x+1| - |x-4| \geq 1$

$|x+1| \geq 1 + |x-4|$





$m = \frac{5-1}{0-4} = \frac{4}{-4}$

$= -1$

$\therefore \cancel{y - 0 = -1(x-5)}$

$y - 5 = (-1)x$

$y - 5 = -x$

$y = -x + 5$

$y = -x + 5 = y = x + 1$

$-x + 5 = x + 1$

$4 = 2x$

$x = 2.$

when $x = 2.$ $y = 3$

$\therefore x \geq 2$

QUESTION NUMBER

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e) $\sin^4 A + \cos^4 A = \frac{2}{3}$

$\sin^4 A + (\cos^2 A)^2 = \frac{2}{3}$

$(1-\sin^2 A)^2$

$\sin^4 A + 1 + \sin^4 A - 2\sin^2 A = \frac{2}{3}$

$2\sin^4 A \cancel{\ } - 2\sin^2 A + 1 = \frac{2}{3}$

$2\sin^4 A - 2\sin^2 A + \frac{1}{3} = 0$

$\sin^2 A = x$

$2x^2 - 2x + \frac{1}{3} = 0$

$x = \frac{3+\sqrt{3}}{6}$ ✓ $x = \frac{3-\sqrt{3}}{6}$ ✓

cos

$\because \sin^2 A > 0 \Rightarrow x > 0$

$\cos A < 0$

$90° < A < 180°$ sin $\therefore \sin A > 0$

$\because$ ~~$\sin 2A = 2\sin A\cos A$~~ $\cos^2 A = 1 - \sin^2 A$

~~$= 2\sin A(1$~~ $= \frac{4-\sqrt{3}}{6}$

~~$1 = \cos 2A$~~ or $= \frac{4+\sqrt{3}}{6}$

1° when $\sin^2 A = \frac{3+\sqrt{3}}{6}$ $\cos^2 A = \frac{4-\sqrt{3}}{6}$

$\sin A = \sqrt{\frac{3+\sqrt{3}}{6}}$ $\cos A = -\sqrt{\frac{4-\sqrt{3}}{6}}$

$\therefore \sin 2A = 2\sin A\cos A$

$= 2\sqrt{\frac{3+\sqrt{3}}{6}} \cdot \sqrt{\frac{4-\sqrt{3}}{6}}$

$= -1.09$ (3 s.f.)

Q1

2° when $\sin^2 A = \frac{3-\sqrt{3}}{6}$ $\cos^2 = \frac{4+\sqrt{3}}{6}$

QUESTION NUMBER

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$\sin 2A = 2\sin A\cos A$

$= -2\sqrt{\frac{3-\sqrt{3}}{6}} \times \sqrt{\frac{4+\sqrt{3}}{6}}$ NS

$= -0.899$ (3s.f.)

Q2 $\left(\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}}\right)^2 = \left(\frac{k}{4}\right)^2$

$\frac{x-2}{x} + \frac{x}{x-2} - 2\sqrt{\frac{\cancel{x(x-2)}}{\cancel{x(x-2)}}} = \frac{k^2}{16}$

$\frac{(x-2)^2}{x(x-2)} + \frac{x^2}{x(x-2)} - 2 = \frac{k^2}{16}$

$\frac{x^2+x^2+4-4x}{x(x-2)} - 2 = \frac{k^2}{16}$

$\frac{2x^2-4x+4}{x(x-2)} - \frac{2x(x-2)}{x(x-2)} = \frac{k^2}{16}$

$\frac{\cancel{2x^2} \cancel{-4x} + 4 \cancel{-2x^2} \cancel{+4x}}{x(x-2)} = \frac{k^2}{16}$

$\frac{-4}{x^2-2x} = \frac{k^2}{16}$

$-64 = k^2x^2 - 2k^2x$

$k^2x^2 - 2k^2x + 64 = 0$

$\Delta < 0 \quad b^2 - 4ac < 0$

$4k^4 - (4)(k^2)(+64) < 0$

$4k^4 - 256k^2 < 0 \Rightarrow k^2 < 64$

~~let $k^2 = y$ $4y^2 + 256y < 0$~~ $\because k^2 > 0$

~~$-64 < y < 0$~~

Next page $\Rightarrow$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$\cancel{-64 < k^2 < 0}$

$\because k$ is real $\therefore k^2 > 0$ and $k \neq 0$

$4k^2 + 256 < 0$

$k^2 < -64$

$-64 < k^2 < 0$

$64 > k^2 > 0$

$8 > k > 0$

b) $\log_{10}(x^2+y^2) - \log_{10}13 = 1$

$\log_{10}\left(\frac{x^2+y^2}{13}\right) = \log_{10}10$

$\frac{x^2+y^2}{13} = 10$

$x^2+y^2 = 130$

$x^2+y^2-130 = 0$

$\log_{10}\left(\frac{x+y}{x-y}\right) = \cancel{3}\log_{10}2^3$

$\frac{x+y}{x-y} = 8$

$x+y = 8x-8y$

$0 = 7x-9y$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$\begin{cases} x^2+y^2=130 \\ 7x-9y=0 \end{cases}$$

$7x=9y$

$x=\frac{9}{7}y$

$\frac{81}{49}y^2+y^2=130$

$\frac{\cancel{130}}{49}y^2=\cancel{130}$

$y^2=49 \quad y=\pm 7$

$y=7 \qquad\qquad y=-7$

$x=9 \qquad\qquad x=-9$

impossible

$\because x+y>0 \qquad (x-y)>0$

when ~~$x=7$~~ $y=-7$ $x=-9$

$x+y=-16<0$ (X)

$\therefore x=9 \quad y=7$

QUESTION NUMBER

ASSESSOR USE ONLY

c) $y = r\sin\theta \quad x = r\cos\theta$

$r^2\sin^2\theta = r^2\cos^2\theta - r^4\cos^4\theta$

$r^2\cos^4\theta = \cos^2\theta - \sin^2\theta$

$r^2 = \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^4\theta}$

$= \sec^2\theta - \frac{1-\cos^2\theta}{\cos^4\theta}$

$= \sec^2\theta - \frac{1}{\cos^4\theta} + \frac{1}{\cos^2\theta}$

$= 2\sec^2\theta - \sec^4\theta$

$4 \times \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2 \, d\theta$

$= 2\int_0^{\frac{\pi}{2}} 2\sec^2\theta - \sec^4\theta \, d\theta$

$= 2\left[2\int_0^{\frac{\pi}{2}} \sec^2\theta \, d\theta - \int_0^{\frac{\pi}{2}} \sec^4\theta \, d\theta\right]$

$2\left\{2[\tan\theta]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (1+\tan^2\theta)\, d\tan\theta\right\}$

$[\tan\theta]_0 + \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{2}}$

$= 2\left\{2[\tan\theta]_0^{\frac{\pi}{2}} - [\tan\theta]_0^{\frac{\pi}{2}} - \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{2}}\right\}$

$= 2\left\{[\tan\theta]_0^{\frac{\pi}{2}} - \left[\frac{1}{3}\tan^3\theta\right]_0^{\frac{\pi}{2}}\right\}$

$=$

QUESTION NUMBER

ASSESSOR'S USE ONLY

c) $y^2 = x^2 - x^4$

$y = \sqrt{x^2 - x^4}$

$= x\sqrt{1-x^2}$

$A = 4\int_0^1 x\sqrt{1-x^2}\,dx$

$= 4\times\frac{1}{2}\int_0^1 \sqrt{1-x^2}\,d(x^2)$

$= -2\int_0^1 \sqrt{1-x^2}\,(-x^2+1)$

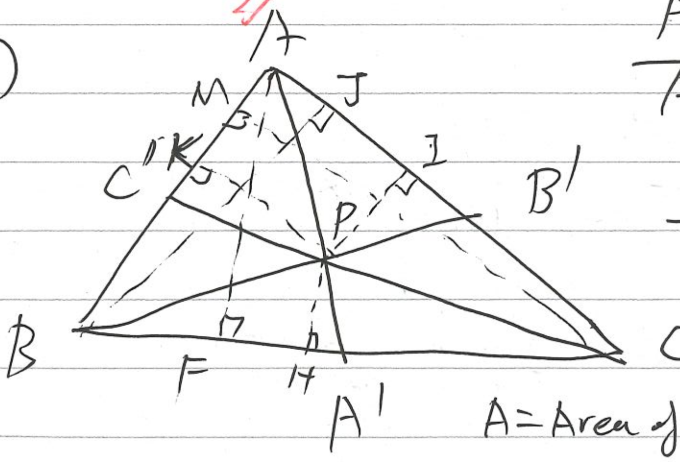
$= -2\left[\frac{2}{3}(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= -\frac{4}{3}\left[(1-x^2)^{\frac{3}{2}}\right]_0^1$

$= -\frac{4}{3}(0 - 1^{\frac{3}{2}})$

$= \frac{4}{3}$

d)



$\frac{PC'}{CC'} = \frac{PK}{CM}$

$\frac{PA'}{AA'} = \frac{PH}{AF}$

$\frac{PB'}{BB'} = \frac{PI}{BJ}$

A = Area of triangle

$CM = \frac{2A}{AB}$ $AF = \frac{2A}{BC}$ $PI = \frac{2A}{AC}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

β, γ

let α ~~become~~ a fraction of Area of triangle

$$PK = \frac{2\alpha A}{AB}$$

same for PH and PJ

define: $\alpha = \dfrac{A_{\Delta APB}}{A_{\Delta ABC}}$

$\beta = \dfrac{A_{BCP}}{A_{\Delta ABC}}$

$\gamma = \dfrac{A_{\Delta ACP}}{A_{\Delta ABC}}$

$$\therefore \frac{PC'}{CC'} = \frac{\frac{\cancel{2}\alpha \cancel{A}}{\cancel{AB}}}{\frac{\cancel{2A}}{\cancel{AB}}} = \alpha.$$

$$PH = \frac{2\beta A}{BC}$$

$$\therefore \frac{PA'}{AA'} = \frac{\frac{\cancel{2}\beta \cancel{A}}{\cancel{BC}}}{\frac{\cancel{2A}}{\cancel{BC}}} = \beta$$

$$\frac{PB'}{BB'} = \frac{\frac{\cancel{2}\gamma \cancel{A}}{\cancel{AC}}}{\frac{\cancel{2A}}{\cancel{AC}}} = \gamma$$

$\therefore$ ~~$\alpha + \beta + \gamma$~~

$$\frac{PC'}{CC'} + \frac{PA'}{AA'} + \frac{PB'}{BB'} = \alpha + \beta + \gamma = 1$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

Q3 a) $f'(4) = \lim_{h\to 0}\left[\frac{f(4+h)-f(4)}{h}\right]$

$f(4) = 9$

$\lim_{h\to 0}\left[\frac{f(4+h)-9}{h}\right]$

$\cancel{= \lim_{h\to 0}\frac{f(4+h)}{h} - \lim_{h\to 0}\frac{9}{h}}$

$f(4+h) = (16+h^2+8h-4(4+h)+3)^2$

$= (\cancel{16}+h^2+8h-\cancel{16}-4h+3)^2$

$(h^2+4h+3)^2$

$\lim_{h\to 0}\left(\frac{(h^2+4h+3)^2-9}{h}\right)$

$\cancel{\frac{h^4(1+\frac{4}{h}+\frac{3}{h^2})^2-\frac{9}{h^2}}{\frac{1}{h}}}$

"$\frac{0}{0}$" type $\Rightarrow \frac{((h^2+4h+3)^2-9)'}{(h)'}$

differentiate upper and low separately

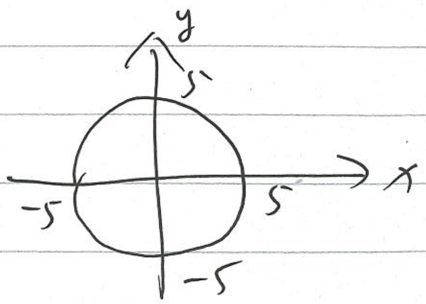
$= \lim_{h\to 0}\frac{2(h+4h+3)(2h+4)}{1}$

$= 2(3)(4) = 24$

QUESTION NUMBER

ASSESSOR'S USE ONLY

b) $x^2+y^2=5^2$



when $P(3,4)$

$$\frac{dy}{dt} = -2.$$

$$\frac{dx}{dt} = ?$$

~~$(x^2+y^2=5^2)'$~~

$$(x^2+y^2)' = (5^2)'$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\cancel{2}x}{\cancel{2}y}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-x}{y}$$

$$= \frac{-2}{\frac{dx}{dt}} = \frac{-x}{y}$$

$$\Rightarrow \frac{-2}{\frac{dx}{dt}} = \frac{-3}{4}$$

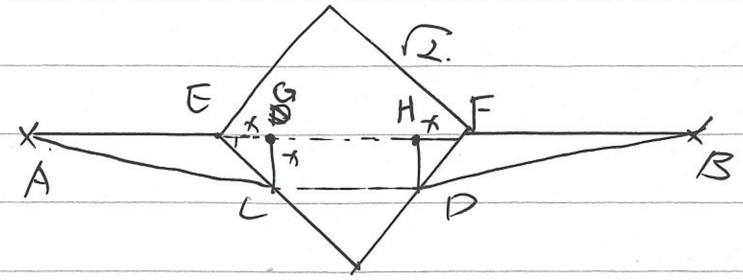
$$\frac{dx}{dt} = \cancel{-}2 \times \cancel{(4)}\ \frac{4}{\cancel{-}3}$$

$$= \cancel{8}\ \frac{8}{3}$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

c)



$EG = x \qquad HF = x$

$EF = \sqrt{2} \cdot \sqrt{2} = 2.$

$\therefore GH = 2 - 2x$

$CD = GH = 2 - 2x$

Time taken for boat: $\dfrac{2-2x}{2.5}$

$\cancel{\therefore AE + FB = 4 - 2 - 2x = 4 - 2 + 2x}$

$\cancel{= 2 + 2x}$

~~Time taken for tramping~~ $\cancel{\dfrac{2+2x}{3}}$

$\cancel{\therefore \text{Time tot} = \dfrac{2+2x}{3} + \dfrac{2-2x}{2.5}}$

$\cancel{\text{Let } y = \dfrac{2+2x}{3} + \dfrac{2-2x}{2.5}}$

$\cancel{y' = \tfrac{1}{3}(2)}$

$\therefore FB = AE = \dfrac{4-2}{2} = 1$

$\therefore AC = \sqrt{(1+x)^2 + x^2} = DB$

$= \sqrt{1 + x^2 + 2x + x^2}$

$= \sqrt{2x^2 + 2x + 1}$

Tot distant travelled by tramping

$= 2\sqrt{2x^2 + 2x + 1}$

QUESTION NUMBER

ASSESSOR USE ONLY

Time taken by tramping

$= \frac{2\sqrt{2x^2+2x+1}}{3}$

$\therefore t_{tot} = \frac{2\sqrt{2x^2+2x+1}}{3} + \frac{2-2x}{2.5}$

$t'_{tot} = \cancel{\frac{2}{3} \times \frac{1}{2} \frac{1}{\sqrt{(2x^2+2x+1)}}}$

$\frac{2}{3} \times \frac{1}{2} \frac{1}{\sqrt{2x^2+2x+1}} (4x+2) \cancel{+ \left(\frac{-2}{2.5}\right)} - \frac{4}{5} = 0$

$\cancel{\frac{2}{6}} \frac{4x+2}{\sqrt{2x^2+2x+1}} = \frac{\cancel{4}\,2}{5} \times 6$

$\frac{4x+2}{\sqrt{2x^2+2x+1}} = \frac{12}{5}$

$(12\sqrt{2x^2+2x+1})^2 = (20x+10)^2$

$144(2x^2+2x+1) = 100 + 400x^2 + 400x$

$288x^2 + 288x + 144 = 100 + 400x^2 + 400x$

$112x^2 + 112x - 44 = 0$

$x = \frac{-7+3\sqrt{14}}{14} \quad x = \frac{-7-3\sqrt{14}}{14}\ (\times)$

$\because x > 0$

boat positioned at $x = \frac{-7+3\sqrt{14}}{14}$ (≈ 0.302)

QUESTION NUMBER		ASSESSOR'S USE ONLY

Q4

$$P(m) = a - \frac{a-b}{t_p} t$$

$$P(f) = 1 - a + \frac{a-b}{t_p} t$$

$$T = \frac{1}{t_p}\left\{\int_0^{t_p} \left(a - \frac{a-b}{t_p} t\right)^2 dt + \int_0^{t_p} \left(1 - a + \frac{a-b}{t_p} t\right) dt\right\}$$

$$-\frac{t_p}{a-b}\int_0^{t_p} \left(a - \frac{a-b}{t_p} t\right)^2 d\left(-\frac{a-b}{t_p} t + a\right)$$

$$-\frac{t_p}{3(a-b)}\left[\left(a - \frac{a-b}{t_p} t\right)^3\right]_0^{t_p}$$

$$= \cancel{\frac{t_p}{3(a-b)}\left(b^3 - a^3\right)} \quad \frac{t_p}{3(a-b)}\left(b^3 - a^3\right)$$

$$= \cancel{\frac{t_p}{3}} \qquad = \frac{t_p}{\cancel{3(a-b)}}\cancel{(b-a)}\left(b^2 + ab + a^2\right)$$

$$= \frac{t_p}{3}\left(b^2 + ab + a^2\right)$$

$$\int_0^{t_p} \left(1 - a + \frac{a-b}{t_p} t\right)^2 dt$$

$$\cancel{= \left[(1-a)t\right]_0^{t_p} + \int_0^{t_p} \frac{a-b}{t_p} t \, dt}$$

$$\cancel{= (1-a)t_p + \frac{(a-b)}{2} t_p \qquad \frac{a-b}{2t_p}\left[t^2\right]_0^{t_p}}$$

$$\cancel{\therefore T = \frac{1}{t_p}\left(\frac{t_p}{3} + (1-a)t_p + \frac{(a-b)t_p}{2}\right)}$$

c. on next page

QUESTION NUMBER

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$$\frac{t_p}{a-b}\int_0^{t_p}\left(1-a+\frac{a-b}{t_p}t\right)^2 d\left(\frac{a-b}{t_p}t+a-1\right)$$

$$=\frac{t_p}{a-b}\,\frac{1}{3}\left[\left(1-a+\frac{a-b}{t_p}t\right)^3\right]_0^{t_p}$$

$$=\frac{t_p}{3(a-b)}\left(\cancel{1-a}\ (1-b)^3-(1-a)^3\right)$$

$$=\frac{t_p}{\cancel{3(a-b)}}\cancel{(1-b-1+a)}\left((1-b)^2+(1-b)(1-a)+(1-a)^2\right)$$

$$1-a-b+ab$$

$$=\frac{t_p}{3}\left(1+b^2-2b+1-a-b+ab+1+a^2-2a\right)$$

$$=\frac{t_p}{3}\left(b^2+a^2-3b+3-3a+ab\right)$$

$$\therefore T=\frac{1}{3}\left(b^2+ab+a^2+b^2+a^2-3b-3a+ab+3\right)$$

$$=\frac{1}{3}\left(2b^2+2a^2+2ab-3a-3b+3\right)$$

$$-6ab+6ab$$

$$=\cancel{\frac{2}{3}(a-b)^2+}\ -4ab$$

$$\frac{1}{3}\left(2b^2+2a^2-4ab+6ab-3a-3b+3\right)$$

$$=\frac{2}{3}(a-b)^2+1+\frac{2ab-b}{}-a.$$

$$=\frac{2}{3}(a-b)^2+1+\frac{b(2a-1)}{\cancel{2b(a-1)}}-a.$$

$$=1-a+b(2a-1)+\frac{2}{3}(a-b)^2$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

b) $\dfrac{dy}{dx} = \dfrac{y^2}{4x^2} - \dfrac{2xy}{4x^2}$.

$\dfrac{dy}{dx} = \dfrac{y^2 - 2xy}{\cancel{4x^2 - 4x^2}\ 4x^2}$

$4x^2 dy = (y^2 - 2xy)dx$

$\cancel{(y^2 - 2xy)dx + (-4x^2)dy = 0}$

~~let y = ux~~

let $y = ux$ $\qquad \dfrac{dy}{dx} = x\dfrac{du}{dx} + u$

$dy = x\,du + u\,dx$

$4x^2(x\,du + u\,dx) = (u^2x^2 - 2x^2u)dx$

$4x^3du + 6x^2u\,dx = u^2x^2dx - \cancel{2x^2u\,dx}$

$4x^3du = -5x^2u^2dx$

$4x\,du = -5u^2dx$

$\dfrac{1}{-5u^2}du = \dfrac{1}{4x}dx$

$-\frac{1}{5}\int u^{-2}du = \frac{1}{4}\int \frac{1}{4x}d(4x)$

$\frac{1}{5}u^{-1} = \frac{1}{4}\ln 4x + C.$

$\dfrac{1}{5u} = \frac{1}{4}\ln 4x + C.$

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$$\frac{1}{5u} = \frac{1}{4}\ln 4x + C$$

$y = ux$

$u = \frac{y}{x}$

$$\frac{1}{5} \cdot \frac{x}{y} = \frac{1}{4}\ln 4x + C \quad \left(\tfrac{1}{4}\ln C\right)$$

$$'' \ \tfrac{1}{4}\ln(4Cx)$$

$$\cancel{\frac{x}{5y}} \quad \frac{5y}{x} = \frac{1}{\frac{1}{4}\ln(4Cx)}$$

$$\cancel{5}y = \frac{4x}{5\cancel{\frac{1}{4}}\ln(4Cx)}$$

$$y = \frac{4x}{5\ln(4Cx)}$$

when $x = 1 \quad y = -6$

$$\frac{-\cancel{6}3}{\cancel{4}2} = \frac{\cancel{4}}{5\ln(4C)}$$

$$-\frac{3}{2} = \frac{1}{5\ln(4C)}$$

$$\cancel{5}\ln(4C) = -\frac{2}{\cancel{3}15}$$

$$4C = e^{-\frac{2}{15}}$$

$$C = \frac{1}{4}e^{-\frac{2}{15}}$$

$$\therefore y = \frac{4x}{5\ln(\cancel{4}\cancel{\tfrac{1}{4}}e^{-\frac{2}{15}}x)}$$

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ASSESSOR'S USE ONLY

$$y = \frac{4x}{5\ln(e^{-\frac{2}{15}}x)}$$

$$= \frac{4x}{5\ln e^{-\frac{2}{15}} + 5\ln x}$$

$$= \frac{4x}{-\frac{2}{3} + 5\ln x}$$

$$= \frac{12x}{-2 + 15\ln x}$$

when $x = 4$.

$$y = \frac{12\times4}{-2+15\ln4} = \frac{12\times4}{-2+30\ln2}$$

$$= 2.55 \text{ (3sf)}$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

Q5 a) $\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$

$=$

~~$w = \cos\theta + i\sin\theta$~~

$\frac{w-1}{w+1} = \frac{(w-1)^2}{w^2-1}$

$\cancel{\cos\theta + i\sin\theta}$

$\frac{(\cos\theta + i\sin\theta - 1)^2}{}$

$\cos^2\theta - \sin^2\theta + 2i\sin\theta\cos\theta - 1$

$\cos^2\theta - \sin^2\theta - \sin^2\theta - \cos^2\theta + \cancel{2i\sin 2\theta}$ $i\sin 2\theta$

$= -2\sin^2\theta + i\cancel{\sin 2\theta}$ $2i\sin\theta\cos\theta$

$\cancel{\cos^2\theta +}$

$\frac{(w-1)^2}{w^2-1} =$

$\frac{w-1}{w+1} = \frac{w+1-2}{w+1} = 1 - \frac{2}{w+1}$

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$$\cancel{1 - \frac{2w-2}{w^2-1}}$$

$$\sin^2 A = \frac{1-\cos 2A}{2}$$

$$\sin^2 \frac{A}{2} = \frac{1-\cos A}{2}$$

$$\cos^2 \frac{A}{2} = \frac{1+\cos A}{2}$$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{1-\cos A}{1+\cos A}}$$

$$\cancel{\frac{w-1}{w+1} = \frac{\cos\theta + i\sin\theta - 1}{\cos\theta + i\sin\theta + 1} \quad \frac{(w-1)^2}{w^2-1}}$$

if $z = \cos\theta + i\sin\theta$

$$2\cos A = z + \frac{1}{z}$$

$$\cos A = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$= \frac{1}{2}\left(\frac{z^2+1}{z}\right)$$

$$\therefore \tan\frac{\theta}{2} = \sqrt{\frac{\frac{2z}{2z} - \frac{z^2+1}{2z}}{\frac{2z}{2z} + \frac{z^2+1}{2z}}}$$

$$= \sqrt{\frac{\frac{2z-z^2-1}{\cancel{2z}}}{\frac{2z+z^2+1}{\cancel{2z}}}}$$

$$= \sqrt{\frac{2z-z^2-1}{2z+z^2+1}} = \sqrt{\frac{-\left(\frac{1-z}{\cancel{z-1}}\right)^2}{(z+1)^2}}$$

$$= i \cdot \frac{1-z}{z+1}$$

$\cancel{z=w}$

$w = z = \cos\theta + i\sin\theta$

$$\therefore i\tan\frac{\theta}{2} = i^2\,\frac{1-w}{w+1} \Rightarrow$$

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$\cancel{\frac{\tan}{i}}$

$\therefore i\tan\frac{\theta}{2} = (-1)\frac{1-w}{w+1}$

$= \frac{w-1}{w+1}$

b) i) $\tan\phi = \frac{b\sin\theta}{a\cos\theta} = \frac{b}{a}\tan\theta$

ii) $\theta - \phi \rightarrow$ greatest

$\tan\phi = \frac{b}{a}\tan\theta \Rightarrow \tan\theta = \frac{a}{b}\tan\phi$

$\tan(\theta-\phi)$ needs to be greatest

$= \cancel{\tan} \frac{\tan\theta - \tan\phi}{1+\tan\theta\tan\phi}$

$\cancel{\tan \frac{\tan\theta - \frac{b}{a}\tan\theta}{1+\frac{b}{a}\tan^2\theta}}$

$\frac{\frac{a}{b}\tan\phi - \tan\phi}{1+\frac{a}{b}\tan^2\phi}$

$\tan(\theta-\phi) = \frac{a\tan\phi - b\tan\phi}{b + a\tan^2\phi} = \frac{\tan\phi(a-b)}{b+a\tan^2\phi}$

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$$\cancel{\tan'(\theta-\phi)}$$

$$\cancel{= \frac{\qquad}{(b+a\tan^2\phi)^2}}$$

$$= \frac{(a-b)}{\frac{b}{\tan\phi} + a\tan\phi}$$

$$\cancel{\phi} \leq \frac{a-b}{2\sqrt{\frac{b}{\cancel{\tan\phi}}\cdot a\cancel{\tan\phi}}}$$

max for $\tan(\theta-\phi)$ $\nearrow$ $\dfrac{a-b}{2\sqrt{ab}}$

when $\dfrac{b}{\tan\phi} = a\tan\phi$

$$b = a\tan^2\phi$$

$$\frac{b}{a} = \tan^2\phi$$

$$\cancel{\tan^2\phi}\ \tan\phi = \sqrt{\frac{b}{a}}$$

$$\phi = \tan^{-1}\left(\sqrt{\frac{b}{a}}\right)$$

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93202A

Annotated Exemplar for 93202 Calculus Outstanding Scholarship

Total Score: 35

Question	Mark	Annotation
1	**8**	The candidate showed competence in recognising the domains of different functions: the rational function in **1a,** the log function in **1b** and the trig function in **1e**. The candidate used 'combination' concept in **1c,** rather than listing all the possibilities like many other candidates did. The candidate showed ability in using trig identity appropriately, although the final answer could have been given in exact value.
2	**8**	The candidate checked the validity of the solutions after solving the logarithm simultaneous equations in **2b.** The candidate successfully recognised the ratio of the height of the 2 triangles with the same base is the same as the ratio of their areas in **2d**, a question attempted by few candidates. The candidate made a sign error in manipulating the algebra expressions in **2a**.
3	**7**	In **3a**, the candidate used L'Hospital's Rule rather than First Principle as required. The candidate successful used related rate of change in **3b**, also acknowledged the 'decreasing rate' as a negative number. In **3c**, the candidate should have checked the final answer is a minimum.
4	**5**	The candidate showed elegance in integrating the complicated polynomial in **4a**, instead of expanding the quadratic expression, then integrate each term as most candidates did, he/she used reverse chain rule. In **5b,** the candidate differentiated product function correctly, but failed to substitute it back to the original equation, and simplified the question as a result.
5	**7**	The candidate showed good knowledge in connecting trig functions with complex numbers in **5a**. However, the last step of working needs more discussion, as the square root of a complex number usually have 2 solutions.