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SUPERVISOR'S USE ONLY

SCHOLARSHIP EXEMPLAR

NZQA

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2017
Calculus

9.30 a.m. Friday 10 November 2017
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Mark
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

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QUESTION NUMBER

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1 a)

~~$x^4 - y^2 - 71 = 0$~~

Prime number so only factors are 1×71 (as well as -1×-71)

$(x^2 - y)(x^2 + y) = 71$

1. $x^2 - y = 1$ and $x^2 + y = 71$
2. $x^2 - y = 71$ and $x^2 + y = 1$

Solving 1. simultaneously

$-1 + y = -71 + y$

$x^2 = 1 + y = 71 - y \quad 2y = 70 \quad y = 35$

$x = \pm 6$

note as y^2 is part of equation $y = \pm 35$

Solving 2.

$x^2 = 71 + y = 1 - y \quad 2y = 70 \quad y = \pm 35$

Same result

hence only integer solutions are

$x = \pm 6 \qquad y = \pm 35$

QUESTION NUMBER

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1b)

$(p+1)x^2 - (p+1)bx = (p-1)ax + (p-1)c$

$(p+1)x^2 - ((p+1)b + (p-1)a)x - (p-1)c = 0$

$\cancel{x = \dfrac{(p+1)b + (p-1)a \pm \sqrt{((p+1)b + (p-1)a)^2 + 4(p+1)(p-1)c}}{2(p+1)}}$

$\cancel{pb + b + pa - a \pm \sqrt{\ }}$

$(pb + b + pa - a)^2 + 4c(p^2-1) > 0$

N3

$p^2b^2 + 2pb^2 + 2p^2ba + b^2 - 2ab + p^2a^2 - 2pa^2 + a^2 + 4c(p^2-1) > 0$

$b^2(p^2+2p+1) + a^2(p^2-2p+1) + 2ab(p^2-1) + 4c(p^2-1) > 0$

$b^2(p+1)^2 + a^2(p-1)^2 + 2ab(p^2-1) + 4c(p^2-1) > 0$

$p^2(b^2+2ba+a^2) + 2p(b^2-a^2) + (b^2-2ab+a^2) + 4c(p^2-1) > 0$

$p^2(b+a)^2 + 2p(b+a)(b-a) + (b-a)^2 + 4c(p^2-1) > 0$

QUESTION NUMBER

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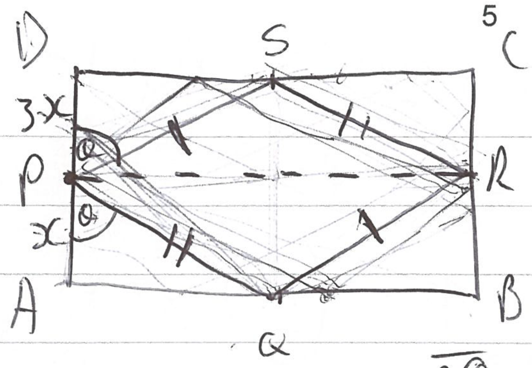
10)

5



Hence PQRS is parallelogram if all angle constraints are true.

QUESTION NUMBER

2a) i.

Perimetre $= 2(\overline{PQ} + \overline{PS})$

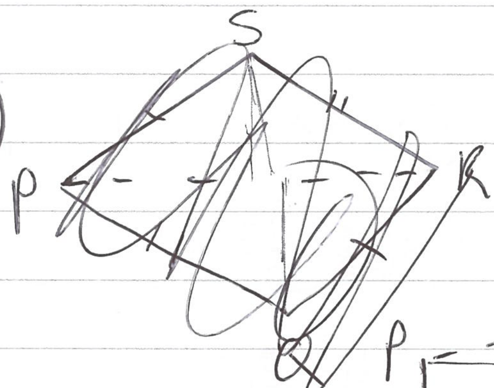
$\overline{PQ} = \frac{x}{\cos\theta}$ $\quad \overline{PS} = \frac{3-x}{\cos\theta}$

$$\text{Perimetre} = 2\left(\frac{x}{\cos\theta} + \frac{3-x}{\cos\theta}\right)$$

$$= 2\left(\frac{3}{\cos\theta}\right)$$

$$= \frac{6}{\cos\theta}$$ hence independant of x.

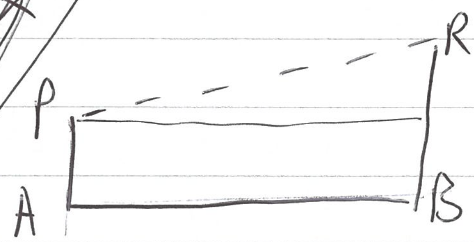
ii)



APRB is a trapezium with lengths $AP = 2$

$RB = 3-2 = 1$

$AB = 3\sqrt{3}$



Hence PR can be found with pythagorism $\triangle$

$$PR^2 = 1^2 + 3\sqrt{3}^2 = 28$$

$$PR = \sqrt{28} = 2\sqrt{7} \simeq 5.3 \text{ units}$$

2c) ① $x+y-z=1$ $\quad z = x+y-1$

~~② $x^2+y^2-z^2+2xy$~~ Subbing z into ② + ③

② $x^2+y^2-(x+y+1)^2 = 5-2xy$

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$$x^2+y^2-x^2-2xy-y^2+2x+2y-1=5-2xy$$

$$x+y=3 \qquad y=3-x$$

So going back to 1

$$3-z=1 \qquad z=2$$

Subbing into 3

$$x^3+y^3-8=43-3xy$$
$$x^3+y^3+3xy=51$$

$$x^3+(3-x)^3+3(3-x)x=51$$

$$x^3+27-18x+3x^2-9x+6x^2-x^3+9x-3x^2=51$$

$$6x^2-18x-24=0$$
$$x=4 \qquad x=-1$$
if $x=4$ $y=-1$ if $x=-1$ $y=4$

Checking against 3 equations both x/y sets of solutions work

Hence solutions to simultaneous are

$$x=4 \quad y=-1 \quad z=2$$

and $x=-1$ $y=4$ $z=2$

here x/y values interchangable due to the nature of equations system.

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3a)

$$\log y = x^x \log x$$

$$\frac{dy}{dx} \log y = x^{x-1} \times \frac{1}{x} = \frac{x^{x-1}}{x} = x^{x-2}$$

$$\frac{dy}{dx} \times \frac{1}{y} = x^{x-2}$$

$$\frac{dy}{dx} = y x^{x-2}$$

$$\frac{dy}{dx} = x^{(x^x)} \cdot x^{x-2}$$

at $x=2$ $\quad \frac{dy}{dx} = 16 \times 2^0 = 16$

$$\frac{dy}{dx}\frac{1}{y} = x^{x-1} \log x + \frac{x^x}{x}$$

$$\frac{dy}{dx} = y\left(x^{x-1} \log x + x^{x-1}\right)$$

$$= x^{(x^x)} \cdot x^{x-1}\left(\log x + 1\right)$$

at $x=2$ $\quad \frac{dy}{dx} = 16 \times 2 \times \left(\log 2 + 1\right)$

$$= 32\left(\log 2 + 1\right) \simeq 41.6$$

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3b)i. $\frac{dy}{dx} = e^x \sin x + \cos x e^x$

$= e^x(\sin x + \cos x)$

$= e^x(\sin x + \sin(x + \frac{\pi}{4}))$ as $\sin\theta = \cos\theta$

$= \cancel{e^x(\sin x + \sin x \frac{\sqrt{2}}{2} + \cos x \cdot \frac{\sqrt{2}}{2}}$

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$\sin(a =$ $\cos(a - \frac{\pi}{}$

3c)

$$\sinh^{-1} x = \frac{1}{\sinh x}$$

$$\frac{d}{dx} \cos x = \sin(a + \frac{\pi}{4} = \frac{2}{e^{x} - e^{-x}}$$

$$\frac{d}{dx} \frac{1}{\sinh x} = \frac{d}{dx} \frac{2}{e^{x} - e^{-x}}$$

RHS →

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4a)

$$\frac{dy}{dx}\ \ln|\cos x| + \frac{1}{2}\ln|\cos 2x| - \frac{1}{3}\ln|\cos 3x| + c$$

$$= \frac{-\sin x}{\cos x} + \frac{-\sin 2x}{\cos 2x} + \frac{\sin 3x}{\cos 3x}$$

$$= -\tan x - \tan 2x + \tan 3x$$

$$= -\tan x - \frac{2\tan x}{1-\tan^2 x} + \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}$$

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4b) $S = \int_{\theta_1}^{\theta_2} \sqrt{a^2(1-\cos\theta)^2 + (a\sin\theta)^2}\ d\theta$

$= \int_{\theta_1}^{\theta_2} \sqrt{a^2 - 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta}\ d\theta$

$= \int_{\theta_1}^{\theta_2} \sqrt{2a^2 - 2a^2\cos\theta}\ d\theta$

$\cancel{= \int_{\theta_1}^{\theta_2} \sqrt{2}a(1-\cos\theta)^{\frac{1}{2}}\ d\theta}$

$S = 2\int_{\pi}^{0} \sqrt{\cancel{2a^2 - 2a^2\cos\theta}}\ \sqrt{2}a(1-\cos\theta)^{\frac{1}{2}}$

$\cancel{= \sqrt{2}\int_{\pi}^{0} 2a^2 - 2a^2\cos\theta\ d\theta}$

$\cancel{= \sqrt{2}\left[2a^2\theta - 2a^2\sin\theta\right]_{\pi}^{0}}$

$\cancel{S = \sqrt{2} \times \left[2a^2\pi\right]}$

$\cancel{= 2\sqrt{2}\pi a^2}$

$\cancel{S = \int_{\pi}^{0} \sqrt{2}a}$ $S = 2\sqrt{2}a\int_{\pi}^{0} (1-\cos\theta)^{\frac{1}{2}}$

$S = 2\sqrt{2}a\int_{\pi}^{0} (2\sin^2\tfrac{1}{2}\theta)^{\frac{1}{2}}$

$S = 4a\int_{\pi}^{0} \sin\tfrac{1}{2}\theta$

$S = 4a \times \left[-2\cos\tfrac{1}{2}\theta\right]_{\pi}^{0}$ MEI

$S = 4a \times (-2 - 0)$

$S = ~~4a~~\ 8a$

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Annotated Exemplar for 93202 Calculus Scholarship		**Total Score**	**22**
Question	**Mark**	**Annotation**	
1	**6**	The candidate recognised the prime factors in **1a** leading to 4 solutions. A start was made on **1b** to form the quadratic but was unable to make sufficient use of sum of roots or discriminant in this case. No attempt at **1c.**	
2	**8**	The candidate has provided evidence in **2a (i)** of competently and succinctly establishing lengths of sides of parallelogram PQRS and hence perimeter. **2a(ii)** used Pythagoras which was not typical of most successful candidates on this question as most used cosine rule for exact answer. Very competent use of algebra in solving simultaneous equations for **2b.**	
3	**0**	The candidate gives a common answer in **3a** – correct first step with logs but then fails to correctly differentiate x^x – a very common error. In **3b(i)** correct differentiation was not sufficient as a substitution using √2 was needed. No attempt at **3b(ii)** and **3b(iii).** Candidate did not take first step to establish derivative in **3c**.	
4	**8**	In **4a** the candidate has differentiated and used correct trig substitution for tan 3x. This was a typical approach.**4b** is well done with only a minor error – good use of trig identity substitution. No attempt at **4c.**	
5	**0**	The candidate has not made any attempt on Question **5**. Possibly ran out of time as they apparently were present for the whole 3 hours.	
		Candidate is typical of those gaining scholarship in that there were good responses to at least 3 questions.	