93202A

932021

S

SUPERVISOR'S USE ONLY

TOP SCHOLAR

NZQA

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

Scholarship 2012
Mathematics with Calculus

9.30 am Saturday 24 November 2012
Time allowed: Three hours
Total marks: 40

ANSWER BOOKLET

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–26 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

QUESTION NUMBER

You have three hours to complete this examination.

ASSESSOR'S USE ONLY

1(a)i) ~~let $\sqrt[3]{a+b}$~~

$$\left(\sqrt[3]{a+b}+\sqrt[3]{a-b}\right)^3$$

$$= a+b+3(a+b)^{\frac{2}{3}}(a-b)^{\frac{1}{3}}+3(a+b)^{\frac{1}{3}}\cdot(a-b)^{\frac{2}{3}}+a-b$$

$$=2a+3(a+b)^{\frac{1}{3}}(a-b)^{\frac{1}{3}}\left((a+b)^{\frac{1}{3}}+(a-b)^{\frac{1}{3}}\right)$$

$$=2a+3\sqrt[3]{a^2-b^2}\left(\sqrt[3]{a-b}+\sqrt[3]{a+b}\right)$$ //.

3

3

3

ii). set $a=\frac{1}{2}$, $b=\frac{1}{6}\sqrt{\frac{23}{3}}$.

Then $\rho^3=2a+3\sqrt[3]{a^2-b^2}\left(\sqrt[3]{a-b}+\sqrt[3]{a+b}\right)$

$$=1+3\sqrt[3]{\frac{1}{4}-\frac{1}{36}\cdot\frac{23}{3}}\cdot\rho$$

$$=1+3\sqrt[3]{\frac{1}{4}-\frac{23}{108}}\ \rho.$$

$$=1+3\sqrt[3]{\frac{1}{27}}\ \rho.$$

$$=1+\rho.$$

Since x^3+x+1 has only one real solution, it is thus ρ. //.

3

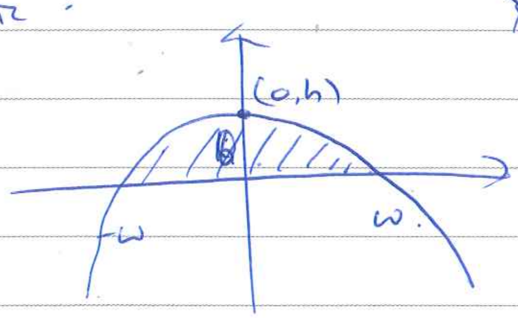
3

3

QUESTION NUMBER

ASSESSOR'S USE ONLY

b) Parboloid : ~~$y = ax^2 + bx + c$~~



~~$y = ax^2 + bx + c$~~

~~$x = 0 \Rightarrow y = h$~~ $\therefore$ $c = h$.

$x = \pm w \Rightarrow y = 0$

$\therefore \quad y = a(x+w)(x-w)$

$x = 0 \Rightarrow y = h = -aw^2$.

$\therefore \quad a = -\dfrac{h}{w^2}$

hence $y = -\dfrac{h}{w^2}(x+w)(x-w)$.

and $V = \displaystyle\int_{-w}^{w} \pi y^2 \, dx$.

$$= 2\pi \int_0^w \frac{h^2}{w^4}(x^2 - w^2)^2 \, dx$$

$$= \frac{2\pi h^2}{w^4} \int_0^w (x^2 - w^2)^2 \, dx$$

$$= \frac{2\pi h^2}{w^4} \cdot \int_0^w (x^4 - 2w^2x^2 + w^4) \, dx$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$= \frac{2\pi h^2}{\omega^4} \left[\frac{x^5}{5} - \frac{2\omega^2 x^3}{3} + \omega^4 x \right]_0^{\omega}$$

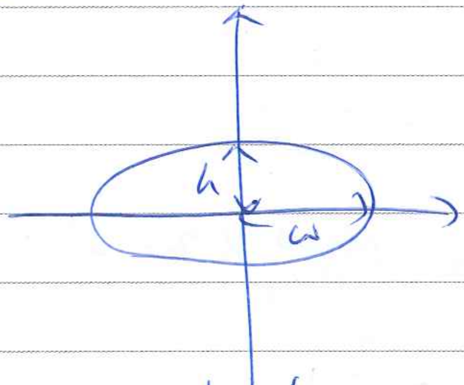
$$= \frac{2\pi h^2}{\omega^4} \left(\frac{\omega^5}{5} - \frac{2\omega^5}{3} + \omega^5 - 0 \right)$$

$$= 2\pi h^2 \omega \cdot \left(\frac{1}{5} - \frac{2}{3} + 1 \right)$$

$$= \boxed{\frac{16\pi h^2 \omega}{15}}$$

a

Elliptic :



$\therefore \; a = \omega, \quad b = h$.

and $\frac{x^2}{\omega^2} + \frac{y^2}{h^2} = 1$.

Thus $y^2 = h^2\left(1 - \frac{x^2}{\omega^2}\right)$.

and $V = 2\pi \int_0^{\omega} y^2 \, dx$.

$$= 2\pi \int_0^{\omega} h^2 \left(1 - \frac{x^2}{\omega^2}\right) dx$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$= \frac{2\pi h^2}{w^2} \int_0^w (w^2 - x^2)\, dx$

$= \frac{2\pi h^2}{w^2} \cdot \left[w^2 x - \frac{x^3}{3} \right]_0^w$

$= \frac{2\pi h^2}{w^2} \cdot \left(w^3 - \frac{w^3}{3} - 0 \right)$

$= 2\pi h^2 \cdot w \cdot \left(1 - \frac{1}{3} \right)$

$= \boxed{\frac{4\pi h^2 w}{3}}$

So the required ratio $= \dfrac{\frac{16\pi h^2 w}{15}}{\frac{4\pi h^2 w}{3}}$

$= \frac{\cancel{3} \times \cancel{16}\ 4}{\cancel{4} \times \cancel{15}\ 5}$

$= \boxed{\frac{4}{5}}$

2

2

2

QUESTION NUMBER | ASSESSOR'S USE ONLY

2a) $x^n = \tan(ny)$.

$\cancel{n x^{n-1}}$

$\Rightarrow \frac{d}{dx}(x^n) = \frac{d}{dx}\left(\tan(ny)\right)$

$\therefore n x^{n-1} = \sec^2(ny) \cdot n \cdot \frac{dy}{dx}$

so $\frac{dy}{dx} = \frac{x^{n-1}}{\sec^2(ny)}$.

But $\cancel{\sec^2\theta + 1}$ $\tan^2\theta + 1 = \sec^2\theta$.

so $\sec^2(ny) = \tan^2(ny) + 1$
$= (x^n)^2 + 1$
$= x^{2n} + 1$.

$\therefore \frac{dy}{dx} = \frac{x^{n-1}}{x^{2n}+1}$ //

3
3
3

b) $2Bx + 2Cy \cdot \frac{dy}{dx} = 0$.

$\Leftrightarrow \frac{dy}{dx} = \frac{-2Bx}{2Cy} = \boxed{\frac{-Bx}{Cy}}$.

To avoid confusion, let $z = f(x)$ be ~~the~~ ~~orthogonal~~ the required orthogonal trajectories //

QUESTION NUMBER | ASSESSOR'S USE ONLY

So $\frac{dz}{dx} \cdot \frac{dy}{dx} = -1$ whenever $y = f(x) = z$. ~~at intersect~~

$\Leftrightarrow \frac{dz}{dx} = \frac{-1}{\frac{dy}{dx}} = \frac{Cy}{Bx} = \frac{Cz}{Bx}$

So $\int \frac{1}{Cz} dz = \int \frac{1}{Bx} dx$

$\Leftrightarrow \frac{1}{C} \ln|z| = \frac{1}{B} \ln|x| + K$ (K is arbitrary)

$\Leftrightarrow \ln|z| = \frac{C}{B} \ln|x| + K$

$\Leftrightarrow |z| = e^{\frac{C}{B}\ln|x| + K}$

$= K \cdot \left(e^{\ln|x|}\right)^{\frac{C}{B}}$

$= K \cdot |x|^{\frac{C}{B}}$

$\Leftrightarrow z = K|x|^{\frac{C}{B}}$.

Thus the required equation is

$y = K|x|^{\frac{C}{B}}$,

where $f(x) = K|x|^{\frac{C}{B}}$

3

3

3

QUESTION NUMBER

ASSESSOR'S USE ONLY

c). In all the cases listed, ~~the change in the~~ ~~the amount~~ ~~amounts~~ the change in a specific variable is directly proportional to itself.
(eg. activity of radioactive substance is directly proportional to the amount left).

So we have $\frac{dy}{dx} \propto y$.

and thus $\frac{dy}{dx} = ky$, $k \in \mathbb{R}$.

$\Leftrightarrow \int \frac{1}{y} dy = \int k\,dx$.

$\Leftrightarrow \ln|y| = kx + c$.

$\Leftrightarrow |y| = e^{kx+c}$.

$\Leftrightarrow |y| = ce^{kx}$.

$\Leftrightarrow y = ce^{kx}$ (c is arbitrary).

which is exactly the same as the equation given: $A(t) = A_0 e^{kt}$.

where the independent variable is time: $x = t$.

2

2

2

QUESTION NUMBER | ASSESSOR'S USE ONLY

3(a) i) $\frac{d}{dx}(x\cos x)$

$= \cos x - x\sin x$

$\therefore x\sin x = \cos x - \frac{d}{dx}(x\cos x)$.

$\Rightarrow \int x\sin x\,dx = \int \cos x\,dx - x\cos x$.

$= \sin x - x\cos x + C$

3 3 3

ii) $\int_0^{n\pi} x\sin x\,dx$

$= \left[\sin x - x\cos x\right]_0^{n\pi}$.

$= \sin n\pi - n\pi\cos n\pi - \sin 0 + 0$.

But $\sin n\pi = 0$; $\cos n\pi = (-1)^n$.

$\therefore \int_0^{n\pi} x\sin x\,dx = -n\pi\cdot(-1)^n$.

$= n\pi(-1)^{n+1}$

3 3 3

QUESTION NUMBER | ASSESSOR'S USE ONLY

b) $\cancel{R = \{\zeta \mid |\zeta|^2 \le 1, \arg\zeta}$

$$R = \{\zeta \mid |\zeta|^2 \le 1,\ \frac{\pi}{12} \le \arg\zeta \le \frac{\pi}{12} + \frac{\pi}{2} = \frac{7\pi}{12};$$
$$\zeta \ne 0.8 \operatorname{cis}\frac{\pi}{6}\}.$$

~~so we have~~

so if we write $\zeta = r\operatorname{cis}\theta$,

$$R = \{\zeta = r\operatorname{cis}\theta \mid r \le 1,\ \frac{\pi}{12} \le \theta \le \frac{7\pi}{12},$$
$$\zeta \ne 0.8\operatorname{cis}\frac{\pi}{6}\}.$$

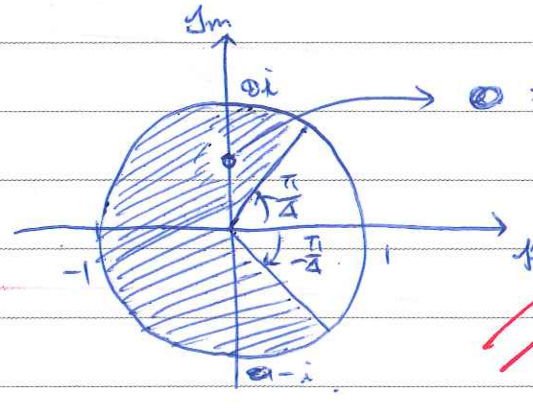
Thus, write $\omega = R\operatorname{cis}\varphi$, noting that $R \le 1$, $\frac{\pi}{12} \le \varphi \le \frac{7\pi}{12}$

$\therefore \quad \omega^3 = R^3 \operatorname{cis}3\varphi.$

and $0 \le R^3 \le 1$, $\frac{\pi}{4} \le 3\varphi \le \frac{7\pi}{4}$

and $\omega \ne 0.8\operatorname{cis}\frac{\pi}{6}$

$$\Rightarrow \omega^3 \ne (0.8)^3 \operatorname{cis}\frac{\pi}{2} = (0.8)^3 i$$
$$= 0.512\, i.$$



2
2
2

QUESTION NUMBER

ASSESSOR'S USE ONLY

Thus the shaded ~~area~~ region is the required region, noting that the point $z = 0.512i$ does not belong to this shaded region. //

QUESTION NUMBER

ASSESSOR'S USE ONLY

4(a). $f(x) = \log_m x + \log_x m.$

$$= \frac{\ln x}{\ln m} + \frac{\ln m}{\ln x}$$

$$= \frac{\ln^2 x + \ln^2 m}{\ln x \ln m}$$

$$= \frac{\ln x + \frac{\ln^2 m}{\ln x}}{\ln m}$$

$\because x, m > 1$

$\therefore \ln x, \ln m > 0.$

Apply AM-GM inequality:

$$f(x) = \frac{\ln x + \frac{\ln^2 m}{\ln x}}{\ln m} \geq \frac{2\sqrt{\ln x \cdot \frac{\ln^2 m}{\ln x}}}{\ln m}$$

$$= \frac{2 \ln m}{\ln m}$$

$$= 2$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

with equality when $\ln x = \dfrac{\ln^2 m}{\ln x}$.

$\Leftrightarrow \ln^2 x = \ln^2 m$.

$\therefore \ln x = \ln m$ (both positive).

$\Leftrightarrow x = m$.

Since equality can be reached, the minimum value of f is thus 2.

3
3
3

b) Substituting in $y^2 = g^2 x^2$ gives:

$$g^4x^4 + (1-g^2)\, g^2x^4 - g^2x^4 + g^2x^2 - g^2x^2 = 0$$

$$\Leftrightarrow g^4x^4 + 0\, g^2x^4 - g^4x^4 - g^2x^4 = 0.$$

which is clearly true.

This implies that $y^2 - g^2x^2$ is a factor of the expression given.

Hence we shall try to factor it out:

$$y^4 + (1-g^2)x^2y^2 - g^2x^4 + g^2x^2 - y^2 = 0$$

$$\Leftrightarrow y^4 + x^2y^2 - g^2x^2y^2 - g^2x^4 - (y^2 - g^2x^2) = 0.$$

$$\Leftrightarrow y^2(y^2 - g^2x^2) + x^2(y^2 - g^2x^2) - (y^2 - g^2x^2) = 0$$

$$\Leftrightarrow (y^2 - g^2x^2)(y^2 + x^2 - 1) = 0.$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$\therefore \quad y^2 - \xi^2 x^2 = 0$

or $\quad y^2 + x^2 - 1 = 0$.

~~And the locus of the given~~

~~Case~~ If $y^2 - \xi^2 x^2 = 0$.

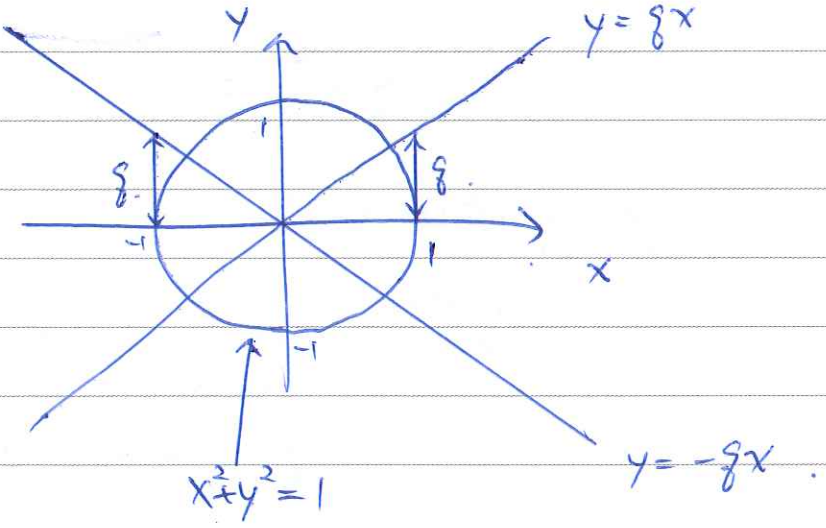
then $\cancel{y = \pm\xi x}$. $y = \xi x$ or $-\xi x$.

If $y^2 + x^2 - 1 = 0$

then $x^2 + y^2 = 1$. ~~⇒ a circle~~ :

⇒ a circle with centre $(0, 0)$

and radius 1.



The required sketch is shown above, noting that the required locus is the union of the two lines $y = \pm\xi x$ and the circle $x^2 + y^2 - 1 = 0$

3

3

3

QUESTION NUMBER

ASSESSOR'S USE ONLY

c)

Note that $\tan 4x = \dfrac{\tan 2x + \tan 2x}{1 - \tan^2 2x}$.

$$= \frac{2\tan 2x}{1 - \tan^2 2x}.$$

and $\tan 2x = \dfrac{\tan x + \tan x}{1 - \tan^2 x}$.

$$= \frac{2\tan x}{1 - \tan^2 x}.$$

So we ~~require~~ have:

$$2\tan(2x)\cdot(\tan^2 x - 1)^2 = \frac{2\tan 2x}{1-\tan^2 2x}\left(\tan^2 x - 2\tan x - 1\right)\left(\tan^2 x + 2\tan x - 1\right)$$

$$\Leftrightarrow (\tan^2 x - 1)^2 = \frac{(\tan^2 x - 2\tan x - 1)(\tan^2 x + 2\tan x - 1)}{1 - \tan^2 2x}.$$

$$\Leftrightarrow (\tan^2 x - 1)^2 = \frac{(\tan^2 x - 2\tan x - 1)(\tan^2 x + 2\tan x - 1)}{1 - \frac{4\tan^2 x}{(1 - \tan^2 x)^2}}$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$$\Leftrightarrow (\tan^2 x - 1)^2 = \frac{(1-\tan^2 x)^2 (\tan^2 x - 2\tan x - 1)(\tan^2 x + 2\tan x - 1)}{(1-\tan^2 x)^2 - 4\tan^2 x}$$

Cancelling out $(1-\tan^2 x)^2$,

we require

$$(1-\tan^2 x)^2 - 4\tan^2 x = (\tan^2 x - 2\tan x - 1)(\tan^2 x + 2\tan x - 1)$$

But setting $a = \tan^2 x - 1$, $b = 2\tan x$,

we can write: $RHS = (a-b)(a+b)$

$$= a^2 - b^2$$

$$= (\tan^2 x - 1)^2 - (2\tan x)^2$$

$$= (1-\tan^2 x)^2 - 4\tan^2 x.$$

$$= LHS.$$

so the above equation is clearly true, thus the given identity is proven and we are done. //

2

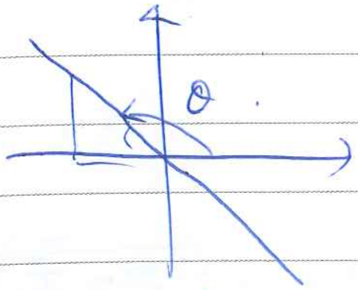
2

2

QUESTION NUMBER

ASSESSOR'S USE ONLY

5a). The angle between two consecutive lines ~~mos~~ must be : ~~$\frac{2\pi}{6}$~~ ~~$\frac{\pi}{3}$~~ $\frac{2\pi}{12} = \frac{\pi}{6}$.



If θ is the angle as shown (ie. angle that rotates the x-axis onto the line).

then the slope $m = \tan\theta$.

~~so~~ ~~we~~ ~~thi~~ so the equations of the lines given is $y = x\tan\theta + c$ ~~(passes through origin)~~

~~(passes through origin so 0 intercept)~~.

$x = 0 \Rightarrow y = 0$. $\therefore c = 0$

and $y = x\tan\theta$.

~~with~~

~~with~~

Since the lines are symmetric in the x-axis, then the line with least positive gradient

has $\theta = \frac{\frac{\pi}{6}}{2} = \frac{\pi}{12}$.

Thus the required values for θ are

$\frac{\pi}{12}$, $\frac{\pi}{12}+\frac{\pi}{6}$, $\frac{\pi}{12}+\frac{\pi}{3}$, $\frac{\pi}{12}+\frac{\pi}{2}$, $\frac{\pi}{12}+\frac{2\pi}{3}$

and $\frac{\pi}{12}+\frac{5\pi}{6}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$\tan\frac{\pi}{12} = 2-\sqrt{3}$.

so $\tan\left(\frac{\pi}{12}+\frac{5\pi}{6}\right) = -\tan\left(\pi - \frac{\pi}{12} - \frac{5\pi}{6}\right) = -\tan\frac{\pi}{12}$

$= -(2-\sqrt{3})$.

$\Rightarrow y = \pm(2-\sqrt{3})x$.

$\cancel{\tan}\ \tan\left(\frac{\pi}{12}+\frac{\pi}{6}\right) = \dfrac{\tan\frac{\pi}{12}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{12}\cdot\tan\frac{\pi}{6}}$

$= \dfrac{2-\sqrt{3}+\sqrt{\frac{1}{3}}}{1-(2-\sqrt{3})\cdot\sqrt{\frac{1}{3}}}$

$= \dfrac{2-\sqrt{3}+\frac{\sqrt{3}}{3}}{1-(2-\sqrt{3})\cdot\frac{\sqrt{3}}{3}}$

$= \dfrac{2-\frac{2\sqrt{3}}{3}}{1-\frac{2\sqrt{3}}{3}+1}$

$= 1$

$\tan\left(\frac{\pi}{12}+\frac{\pi}{6}\right) = \tan\frac{\pi}{4} = 1$.

so $\tan\left(\frac{\pi}{12}+\frac{2\pi}{3}\right) = -\tan\left(\pi - \frac{\pi}{12} - \frac{2\pi}{3}\right) = -\tan\frac{\pi}{4}$

$= -1$.

QUESTION NUMBER

ASSESSOR'S USE ONLY

And the lines for these θ are:

$$y = \pm x.$$

Finally,

$$\tan\left(\frac{\pi}{12}+\frac{\pi}{3}\right) = \tan\left(\frac{5\pi}{12}\right) = 2+\sqrt{3}.$$

So $\tan\left(\frac{\pi}{12}+\frac{\pi}{2}\right) = -\tan\left(\pi-\frac{\pi}{12}-\frac{\pi}{2}\right)$

$$= -\tan\frac{5\pi}{12}$$

$$= -(2+\sqrt{3}).$$

So $y = \pm(2+\sqrt{3})x$.

~~Since the locus~~

For the union, we have:

$$(y+x)(y-x)\left(y+(2-\sqrt{3})x\right)\left(y-(2-\sqrt{3})x\right)\left(y+(2+\sqrt{3})x\right)\left(y-(2+\sqrt{3})x\right)=0$$

$$\cancel{\Leftrightarrow (y^2-x^2)\left(y^2-(4-4\sqrt{3}+3)x^2\right)\left(y^2-(4+4\sqrt{3}+3)x^2\right)=0}$$

$$\cancel{\Leftrightarrow (y^2-x^2)\left(y^2-(7-4\sqrt{3})x^2\right.}$$

$$\cancel{\text{But } \left(y+(2-\sqrt{3})x\right)\left(y-(2+\sqrt{3})x\right)}$$

$$\Leftrightarrow (y^2-x^2)\left(y^2-(7-4\sqrt{3})x^2\right)\left(y^2-(7+4\sqrt{3})x^2\right)=0.$$

$$\Leftrightarrow (x^2-y^2)\left(\frac{y^2}{7-4\sqrt{3}}-x^2\right)\left(\frac{y^2}{7+4\sqrt{3}}-x^2\right)=0$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

$\Leftarrow \cancel{(x^2-y^2)(x^2-}$

But $\dfrac{1}{7-4\sqrt{3}} = \dfrac{7+4\sqrt{3}}{49-16\times 3} = 7+4\sqrt{3}$.

So $\dfrac{1}{7+4\sqrt{3}} = 7-4\sqrt{3}$.

and we have:

$$\left(x^2-y^2\right)\left((7+4\sqrt{3})y^2-x^2\right)\left((7-4\sqrt{3})y^2-x^2\right)=0.$$

$$\Leftrightarrow \left(x^2-y^2\right)\left(x^2-(7+4\sqrt{3})y^2\right)\left(x^2-(7-4\sqrt{3})y^2\right)=0$$

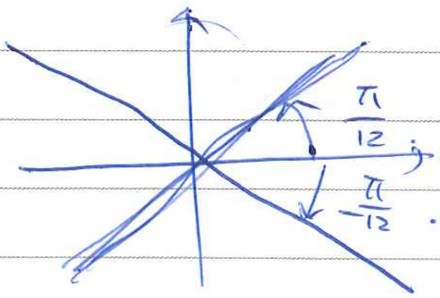
~~as req~~

$$\Leftrightarrow \left(x^2-y^2\right)\left(x^2-(7-4\sqrt{3})y^2\right)\left(x^2-(7+4\sqrt{3})y^2\right)=0$$

as required.

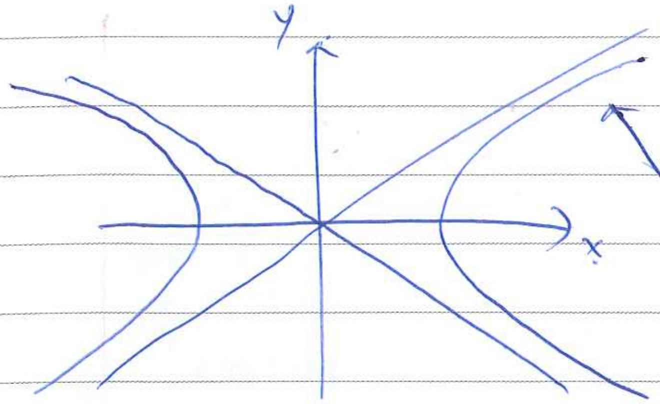
3 3 3

(ii) Consider the two lines:



Suppose we can fit a hyperbola between these two lines, then we are done, ~~because~~ because none of the other lines intersect this region.

QUESTION NUMBER | ASSESSOR'S USE ONLY



required hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

~~Since $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.~~

~~so $\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$.~~

~~$\Leftrightarrow \frac{dy}{dx} = \frac{2x}{a^2} \cdot \frac{b^2}{2y} = \frac{b^2 x}{a^2 y}$.~~

~~We would like $\frac{dy}{dx}$ to tend to the slope of the lines (so that the hyperbola approaches the lines asymptotically but never intersects them).~~

since the equation for the asymptotes of the hyperbola is $y = \pm \frac{b}{a} x$.

we set $\frac{b}{a} = \tan\theta$, $\theta = \frac{\pi}{12}$.

so $\frac{b}{a} = \tan\frac{\pi}{12} = 2 - \sqrt{3}$.

~~Set a to be any positive real value is for simplicity set $a = 1$.~~

and we get $b = (2-\sqrt{3})a$.

QUESTION NUMBER

ASSESSOR'S USE ONLY

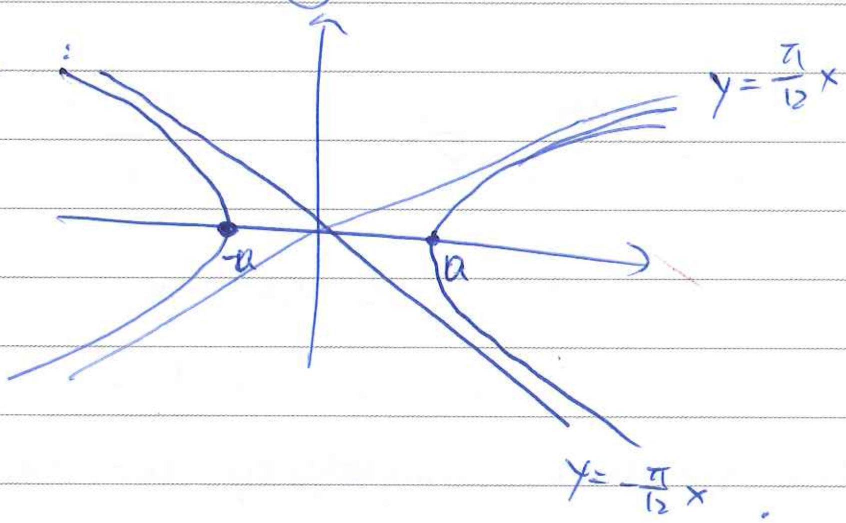
Thus the hyperbola is :

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(2-\sqrt{3})^2} = 1$$

with vertices $(\pm a, 0)$, asymptotes.

$$y = \pm \frac{b}{a} x = \pm (2-\sqrt{3}) x = \pm \tan\frac{\pi}{12} x,$$ which are exactly the two lines drawn.

So :



and thus we are done.

The required equation is ~~$x^2 - \frac{y^2}{(2-\sqrt{3})^2} = 1$~~

(b). $$\frac{x^2}{a^2} - \frac{y^2}{a^2(2-\sqrt{3})^2} = 1$$

A hyperbola with any $a \in \mathbb{R}$ satisfies the conditions.

For example, set $a = 1$,

$$x^2 - \frac{y^2}{(2-\sqrt{3})^2} = 1$$ is one such hyperbola.

3

3

3

Achievement Standard Number 9 3 2 [redacted]

Question Number 5b)   Write your N[redacted]

Supervisor **must** print name & sign here : [redacted]

Let (α, β) and (γ, δ) be any two distinct points on this ellipse.

and let $m = \dfrac{\delta - \beta}{\gamma - \alpha}$.

and any line parallel to the line (α,β), (γ,δ) has equation $y = mx + c$, where c is an appropriate constant.

Now intersect this line with the ellipse.

so $\dfrac{x^2}{a^2} + \dfrac{(mx+c)^2}{b^2} = 1$.

$$\iff x^2\left(\frac{1}{a^2} + \frac{m^2}{b^2}\right) + \frac{2mc}{b^2}x + \left(\frac{c^2}{b^2} - 1\right) = 0$$

Therefore the midpoint of the line segment between the two points of intersection is given by

$$\bar{x} = \frac{x_1 + x_2}{2} = \frac{-\dfrac{\frac{2mc}{b^2}}{\frac{1}{a^2} + \frac{m^2}{b^2}}}{2}$$

$$= -\frac{mc}{\frac{b^2}{a^2} + m^2}$$

$$= -\frac{a^2 cm}{b^2 + a^2 m^2}$$

2

2

This extra answer page MUST be placed inside the plastic bag with your answer booklet and handed in.

Achievement Standard Number 9

Question Number

Write your NSN here

Supervisor **must** print name & sign here :

Similarly, $y = mx + c$

$\Leftrightarrow x = \frac{y-c}{m}$.

so $\frac{\left(\frac{y-c}{m}\right)^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\Leftrightarrow \frac{y^2 - 2cy + c^2}{a^2m^2} + \frac{y^2}{b^2} = 1$.

$\Leftrightarrow y^2\left(\frac{1}{b^2} + \frac{1}{a^2m^2}\right) - \frac{2c}{a^2m^2}y + \left(\frac{c^2}{a^2m^2} - 1\right) = 0$.

so $\bar{y} = \frac{y_1 + y_2}{2} = \frac{1}{2} \cdot \left(-\frac{\frac{-2c}{a^2m^2}}{\frac{1}{b^2} + \frac{1}{a^2m^2}}\right)$

$= \frac{2c}{a^2m^2\left(\frac{1}{b^2} + \frac{1}{a^2m^2}\right)}$

$= \frac{2c}{\frac{a^2m^2}{b^2} + 1}$

$= \frac{2cb^2}{b^2 + a^2m^2}$.

As we vary c, ~~we~~ we vary the points of intersections and ~~to~~ therefore the midpoints.

But $c = \frac{\bar{y}(b^2 + a^2m^2)}{2b^2}$. and $c = \frac{-\bar{x}(b^2 + a^2m^2)}{a^2m}$.

This extra answer page MUST be placed inside the plastic bag with your answer booklet and handed in.

Achievement Standard Number 9 3

Question Number 5b).   Write yo

Supervisor **must** print name & sign her

(Continued).

So $\frac{\bar{y}(b^2+a^2m^2)}{2b^2} = -\frac{\bar{x}(b^2+a^2m^2)}{a^2m}$.

$\because b \neq 0 \Rightarrow b^2+a^2m^2 > 0$.

$\therefore \frac{\bar{y}}{2b^2} = -\frac{\bar{x}}{a^2m}$.

$\Leftrightarrow \bar{y} = -\frac{1}{2a^2b^2m} \cdot \bar{x}$.

This means that ~~no matter which~~ for any fixed m, (ie. the chords are parallel), the midpoints of the chords $(\bar{x}, \bar{y})$ satisfy.

$\bar{y} = -\frac{1}{2a^2b^2m}\bar{x}$, which is just a straight line, and thus the midpoints of such chords must all lie on the same straight line.

This extra answer page MUST be placed inside the plastic bag with your answer booklet and handed in.