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SUPERVISOR'S USE ONLY

TOP SCHOLAR EXEMPLAR

NZQA

NEW ZEALAND QUALIFICATIONS AUTHORITY
MANA TOHU MĀTAURANGA O AOTEAROA

QUALIFY FOR THE FUTURE WORLD
KIA NOHO TAKATŪ KI TŌ ĀMUA AO!

Scholarship 2019
Calculus

9.30 a.m. Friday 8 November 2019
Time allowed: Three hours
Total score: 40

ANSWER BOOKLET

There are five questions in this examination. Answer ALL FIVE questions.

Check that the National Student Number (NSN) on your admission slip is the same as the number at the top of this page.

Write ALL your answers in this booklet.

Make sure that you have Formulae and Tables Booklet S–CALCF.

Show ALL working. Start your answer to each question on a new page. Carefully number each question.

Answers developed using a CAS calculator require **ALL commands to be shown**. Correct answers only will not be sufficient.

Check that this booklet has pages 2–27 in the correct order and that none of these pages is blank.

YOU MUST HAND THIS BOOKLET TO THE SUPERVISOR AT THE END OF THE EXAMINATION.

Question	Score
ONE	
TWO	
THREE	
FOUR	
FIVE	
TOTAL	/40

ASSESSOR'S USE ONLY

QUESTION NUMBER		ASSESSOR'S USE ONLY
1a.	$f(x)$ is not differentiable at its discontinuities, where the denominator is 0 ~~...~~.	
	$x - 1 = 0 \Rightarrow x = 1$	
	$x + 1 = 0 \Rightarrow x = -1.$	
	f is differentiable everywhere else, i.e. $x \in (-\infty, -1) \cup (-1, 1) \cup (1, \infty)$.	
	$f'(x) = \frac{(2x-2)(x-1) - x^2+2x-1}{(x-1)^2} + \frac{(2x+2)(x+1) - x^2-2x-1}{(x+1)^2}.$ by Product Rule.	

QUESTION NUMBER

ASSESSOR'S USE ONLY

1b. ~~f is real whenever~~

$\ln a$ is real ~~iff~~ if and only if $a > 0$.

$\therefore$ for f to be real,

$$(-(x-2)^2) > 0$$

$$(x-2)^2 < 0.$$

But since squares are non-negative,

$$(x-2)^2 \geq 0 \qquad \forall\, x \in \mathbb{R}.$$

So $(x-2)^2 < 0$ is never satisfied, so f is never real.

QUESTION NUMBER		ASSESSOR'S USE ONLY
1ci.	$\binom{6}{1} = 6$ ways to give 1 book to A.	
	5 books remaining so	
	$\binom{5}{2} = 10$ ways to give 2 books to B.	
	C gets the last 3 books ($\binom{3}{3} = 1$ ways).	
	Total number of ways $= 6 \times 10 \times 1 = 60$.	
ii.	$\binom{6}{2} = 15$ ways to give 2 books to A.	
	4 books left, so	
	$\binom{4}{2} = 6$ ways to give 2 books to B.	
	C gets the last 2 books in $\binom{2}{2} = 1$ ways.	
	Total number of ways is ~~6×15~~ $15 \times 6 \times 1 = 90$ ways.	

QUESTION NUMBER | ASSESSOR'S USE ONLY

1d. $|x+1| \geq 1 + |x-4|$

Both sides positive. Squaring does not reverse the inequality.

$$(x+1)^2 \geq (1+|x+4|)^2$$
$$= 1 + (x-4)^2 + 2|x-4|$$
$$x^2+2x+1 \geq 1 + x^2 - 8x + 16 + 2|x-4|$$

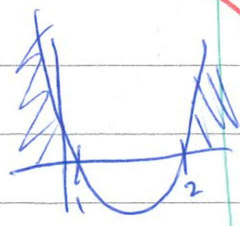
~~$-2|x+4| \geq 6x+16$~~

~~Two cases: $6x+16 \geq 0$ or $6x+16 \leq 0$.~~
~~If $6x+16 \geq 0$,~~
~~$-2|x+4| \geq 6x+16$~~
~~is not possible because LHS~~

$$2|x-4| \leq 10x-16.$$
$$|x-4| \leq 5x-8.$$

Two cases: $5x-8 \geq 0$ and $5x-8 < 0$.
If $5x-8 < 0$, the inequality is not fulfilled because LHS ≥ 0 but RHS < 0, so LHS $>$ RHS.

If $5x-8 \geq 0$, $|x-4| \leq 5x-8$
$$(x-4)^2 \leq (5x-8)^2$$
$$x^2-8x+16 \leq 25x^2-80x+64$$
$$24x^2-72x+48 \geq 0$$
$$x^2-3x+2 \geq 0$$
$$(x-2)(x-1) \geq 0$$



x^2-3x+2 is positive quadratic so $x \geq 2$ or $x \leq 1$.
But $5x-8 \geq 0 \Rightarrow x \geq 8/5$. So $x \geq 2$ is the only solution. So $x \geq 2$.

QUESTION NUMBER

ASSESSOR'S USE ONLY

1e. $\sin^2 A + \cos^2 A = 1$ ... Pythagorean identity.

So since $\sin^4 A + \cos^4 A = 2/3$ ... given

$\sin^4 A + 2\sin^2 A\cos^2 A + \cos^4 A - 2\sin^2 A\cos^2 A = 2/3$

$(\sin^2 A + \cos^2 A)^2 - 2\sin^2 A\cos^2 A = 2/3$

$1^2 - 2\sin^2 A\cos^2 A = 2/3$

$2\sin^2 A\cos^2 A = 1/3$

$\cancel{4}(\sin A\cos A)^2 = 1/6$

$\left(\frac{1}{2}\sin 2A\right)^2 = 1/6.$

$\sin^2 2A = 2/3$

Since $90° < A < 180°$, $180° < 2A < 360°$, so

$\sin 2A < 0.$

$\sin^2 2A = 2/3 \Rightarrow \sin 2A = \pm\sqrt{\frac{2}{3}}$

But $\sin 2A < 0 \Rightarrow \sin 2A = -\sqrt{2/3}$

QUESTION NUMBER

ASSESSOR'S USE ONLY

2a.

$$\sqrt{\frac{x-2}{x}} = \frac{k}{4} + \sqrt{\frac{x}{x-2}}$$

$$\frac{x-2}{x} = \frac{k^2}{16} + \frac{x}{x-2} + \frac{k}{2}\sqrt{\frac{x}{x-2}}$$

$$\frac{k}{2}\sqrt{\frac{x}{x-2}} = \frac{x-2}{x} - \frac{x}{x-2} - \frac{k^2}{16}$$

$$= \frac{(x-2)(x-2) - x^2}{x(x-2)} - \frac{k^2}{16}$$

$$= \frac{-4x+4}{x(x-2)} - \frac{k^2}{16}$$

$$\sqrt{\frac{x}{x-2}} = \cancel{\frac{+8x+8}{2x(x-2)}} - \cancel{\frac{k}{8}} \; \frac{-8x+8}{kx(x-2)} - \frac{k}{8}.$$

$$\frac{x}{x-2} = \left(\frac{-8x+8}{kx(x-2)}\right)^2 + \frac{2x+2}{x(x-2)} + \frac{k^2}{64}$$

$\cancel{kx^2}$ =

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QUESTION NUMBER | ASSESSOR USE ONLY

2b.

$\log_{10}(x^2+y^2) = \log_{10} 10 + \log_{10} 13$

$\cancel{\log}\log_{10}(x^2+y^2) = \log_{10} 130$ $\quad \because \log a + \log b = \log ab$

$x^2+y^2 = 130$

$\log_{10}(x+y) - \log_{10}(x-y) = 3\log_{10} 2$

$\log_{10}\left(\frac{x+y}{x-y}\right) = \log_{10} 8$

$\frac{x+y}{x-y} = 8$

$x+y = 8x - 8y$

$9y = 7x.$

$y = \frac{7x}{9}.$

$x^2 + \left(\frac{7x}{9}\right)^2 = 130$

$x^2 + \frac{49x^2}{81} = 130.$

$\frac{130x^2}{81} = 130$

$x^2 = 81$

$x = \pm 9.$

$y = \frac{7x}{9} = \pm 7.$

For $(x,y) = (9,7)$,

$x^2+y^2 = 130 > 0$

$x+y = 16 > 0$

$x-y = 2 > 0$

So this solution works.

For $(x,y) = (-9,-7)$, $x+y = -16 < 0$. So $\log_{10}(x+y)$ is undefined. Thus $(-9,-7)$ is not a solution.

$\therefore (x,y) = (9,7)$ is the only solution.

QUESTION NUMBER

ASSESSOR'S USE ONLY

2c. ~~By symmetry, this area is 4 times of~~

~~$y = \pm\sqrt{x - x^2}$~~

~~$x = \pm 1$ $y = \pm\sqrt{(\pm 1) - (\pm 1)^2} = 0$~~

At $x = 1$, $y = 1^2 - 1^4 = 0$

At $x = -1$, $y = (-1)^2 - (-1)^4 = 0$.

So these are the bounds of the region.

$y^2 = x^2 - x^4$

$y = \pm\sqrt{x^2 - x^4}$.

By symmetry, then this area is equal to

$= 4\int_0^1 \sqrt{x^2 - x^4}\, dx$

$= 4\int_0^1 x\sqrt{1 - x^2}\, dx$

$= -2\int_0^1 -2x(1 - x^2)^{1/2}\, dx$

$= -2\left[\frac{2}{3}(1 - x^2)^{3/2}\right]_0^1$ ... Reverse Chain Rule

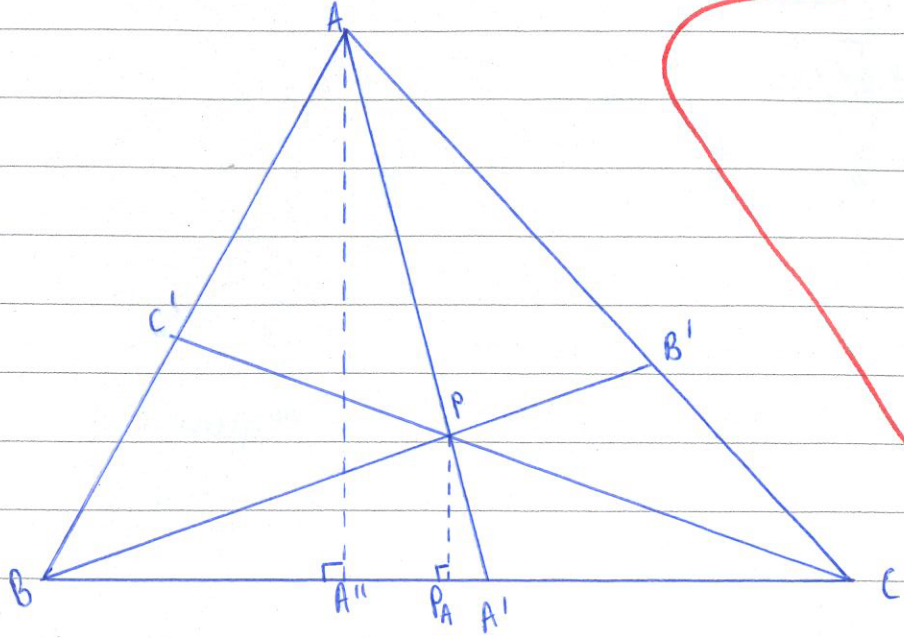
$= 4/3$.

So the area is $4/3$.

QUESTION NUMBER

ASSESSOR USE ONLY

2d. Let $[XYZ]$ be the area of triangle XYZ.



Let the feet of the altitudes from P and A to BC be P_A and A'' respectively.

Then $\angle PA'P_A = \angle AA'A''$ ... same angle

$\angle AA''A' = \angle PP_AA' = 90°$ ... definition

So $\Delta A'PP_A \sim \Delta A'AA''$. ... AA similarity.

$\therefore \dfrac{PA'}{PP_A} = \dfrac{AA'}{AA''}$ ... similarity.

But since PBC and ABC have the same base BC, then $\dfrac{[PBC]}{[ABC]} = \dfrac{PP_A}{AA''} = \dfrac{PA'}{AA'}$.

~~So~~ By a similar argument, $\dfrac{[PAC]}{[BAC]} = \dfrac{PB'}{BB'}$ and $\dfrac{[PAB]}{[CAB]} = \dfrac{PC'}{CC'}$.

So $\dfrac{PA'}{AA'} + \dfrac{PB'}{BB'} + \dfrac{PC'}{CC'} = \dfrac{[PBC]}{[ABC]} + \dfrac{[PAC]}{[BAC]} + \dfrac{[PAB]}{[ABC]}$.

$= \dfrac{[PBC] + [PAC] + [PAB]}{[ABC]}$ $(*)$

But $[PBC] + [PAC] + [PAB] = [ABC]$ because triangles PBC, PAC and PAB make up ABC.

So $(*) = \dfrac{[ABC]}{[ABC]} = 1$ as required.

QUESTION NUMBER

ASSESSOR'S USE ONLY

3a. ~~f'(4)~~

$$f'(4) = \lim_{h\to 0} \frac{((4+h)^2 - 4(4+h)+3)^2 - (4^2-4\times 4+3)^2}{h}$$

$$= \lim_{h\to 0} \frac{(h^2+8h+16-16-4h+3)^2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{(h^2+4h+3)^2 - 9}{h}$$

$$= \lim_{h\to 0} \frac{h^4+4h^3+3h^2+4h^3+16h^2+12h+3h^2+12h+9-9}{h}$$

$$= \lim_{h\to 0} \frac{h^4+8h^3+22h^2+24h+9-9}{h}$$

$$= \lim_{h\to 0} 24 + 22h + 8h^2 + h^3$$

$$= 24.$$

QUESTION NUMBER

ASSESSOR USE ONLY

3b.

$$x^2 + y^2 = 5^2$$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(5^2)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$

Since $\frac{dy}{dt} = 2$ me ... given

$$\frac{dx}{dt} = -\frac{y}{x} \times 2$$

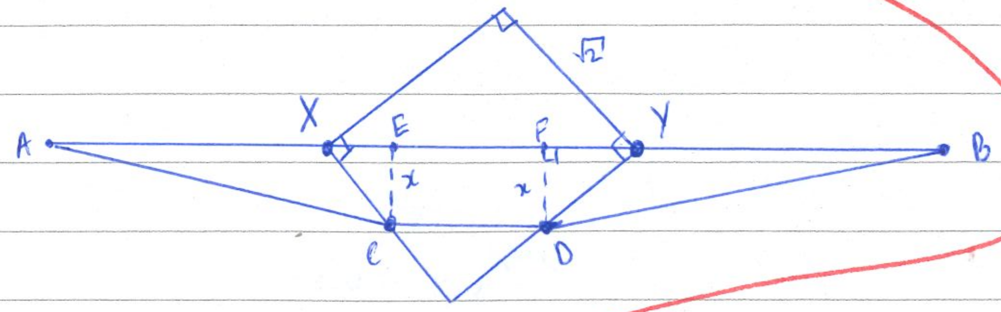
$$= -\frac{4}{3} \times 2$$

$$= -\frac{8}{3}.$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

3c



Label points as shown.

Note $XY^2 = (\sqrt{2})^2 + (\sqrt{2})^2$

$= 4$

$\therefore XY = 2.$

Also $\angle EXC = 45°$ ... diagonal of a square

$\therefore \angle ECX = 180° - 90° - 45° = 45°$

$\therefore \triangle ECX$ isosceles, $EC = EX = x$. ... base angles equal.

By a similar logic/symmetry

$\triangle FDY$ is isosceles, $FD = FY = x$.

Note $AX + YB + XY = 4$

$\Rightarrow AX + YB = 2$

$\Rightarrow 2AX = 2$ ... $AX = YB$, the lake is halfway

$\Rightarrow AX = 1.$

So $AC^2 = AE^2 + CE^2$

$= (1+x)^2 + x^2$

$= 2x^2 + 2x + 1$

By symmetry $BD^2 = 2x^2 + 2x + 1$.

Since CDEF is a rectangle, $CD = EF = XY - EX - FY = 2 - 2x$.

Let the time be t.

[continued next page]

QUESTION NUMBER

ASSESSOR'S USE ONLY

3c. $t = \frac{d}{v}$ for each section so

$t = \frac{AC}{3} + \frac{BD}{3} + \frac{CD}{2.5}$

$= \frac{\sqrt{2x^2+2x+1}}{3} + \frac{\sqrt{2x^2+2x+1}}{3} + \frac{2-2x}{2.5}$

$= \frac{2}{3}\sqrt{2x^2+2x+1} + 0.8 - 0.8x$

$= \frac{2}{3}(2x^2+2x+1)^{1/2} + 0.8 - 0.8x$

$\frac{dt}{dx} = \frac{2}{3}(2x^2+2x+1)^{-1/2}(\frac{1}{2})(4x+2) - 0.8$

$= \frac{4x+2}{3\sqrt{2x^2+2x+1}} - 0.8 = 0$

$\frac{4x+2}{3\sqrt{2x^2+2x+1}} = 0.8$

$4x+2 = 2.4\sqrt{2x^2+2x+1}$

$20x+10 = 12\sqrt{2x^2+2x+1}$

$10x+5 = 6\sqrt{2x^2+2x+1}$

$(10x+5)^2 = 36(2x^2+2x+1)$

$100x^2+100x+25 = 72x^2+72x+36$

$28x^2+28x-11=0$

$x = \frac{-28 \pm \sqrt{28^2-4\times28\times-11}}{2\times28}$

$= \frac{-7 \pm 3\sqrt{14}}{14}$

Since $x>0$, $x = \frac{-7+3\sqrt{14}}{14}$. To show this is a minimum,

$\frac{d^2t}{dx^2} = \frac{d}{dx}\left(\left(\frac{1}{3}\right)\frac{4x+2}{\sqrt{2x^2+2x+1}} - 0.8\right)$

$= \left(\frac{1}{3}\right)\frac{4\sqrt{2x^2+2x+1} - (4x+2)(2x^2+2x+1)^{-1/2}(\frac{1}{2})(4x+2)}{2x^2+2x+1}$

At $x = \frac{-7+3\sqrt{14}}{14}$, $\frac{d^2t}{dx^2} = 0.2794$ (4sf) >0, so it is indeed a minimum. So $x = \frac{-7+3\sqrt{14}}{14}$ or 0.3018 (4sf)

QUESTION NUMBER

ASSESSOR'S USE ONLY

4a. Assuming $p(f) + p(m) = 1$

then $p(f) = 1 - a + \frac{a-b}{t_p} t$.

$$T = \frac{1}{t_p}\int_0^{t_p} \left\{ \left(a - \frac{a-b}{t_p} t\right)^2 + \left(1 - a + \frac{a-b}{t_p} t\right)^2 \right\} dt$$

$$= \frac{1}{t_p}\int_0^{t_p} \Big(a^2 - 2\frac{a^2-ab}{t_p} t + \frac{a^2-2ab+b^2}{t_p^2} t^2$$

$$+ 1 - a + \frac{a-b}{t_p} t - a + a^2 - \frac{a^2-ab}{t_p} t + \frac{a-b}{t_p} t$$

$$- \frac{a^2-ab}{t_p} t + \frac{a^2-2ab+b^2}{t_p} t^2 \Big) dt$$

$$= \frac{1}{t_p}\int_0^{t_p} \Big(1 - 2a + 2a^2 + t\left(\frac{-2a^2+2ab}{t_p} + \frac{a-b}{t_p} - \frac{a^2-ab}{t_p} + \frac{a-b}{t_p} \right.$$

$$\left. - \frac{a^2-ab}{t_p} \right) + t^2 \left(\frac{2a^2-4ab+2b^2}{t_p^2} \right) \Big) dt$$

$$= \frac{1}{t_p}\left[(1-2a+2a^2)t + \frac{t^2}{2}\left(\frac{-4a^2+4ab+2a-2b}{t_p} \right) + \frac{t^3}{3}\left(\frac{2a^2-4ab+2b^2}{t_p^2} \right) \right]_0^{t_p}$$

$$= \frac{1}{t_p}\Big(t_p(1-2a+2a^2) + t_p(-2a^2+2ab+a-b) + \frac{t_p}{3}(2a^2-4ab+2b^2) \Big)$$

$$= 1 - 2a + 2a^2 + 2ab - 2a^2 + a - b + \frac{2}{3}(a-b)^2$$

$$= 1 - a + 2ab - b + \frac{2}{3}(a-b)^2$$

$$= 1 - a + b(2a-1) + \frac{2}{3}(a-b)^2$$ as required.

QUESTION NUMBER | ASSESSOR'S USE ONLY

4b. Let $y = x \cdot u(x)$.

$\frac{dy}{dx} = u(x) + x \frac{du}{dx}$.

$$4x^2\left(u + x\frac{du}{dx}\right) = x^2u^2 - 2x^2u$$
$$4\left(u + x\frac{du}{dx}\right) = u^2 - 2u$$
$$4u + 4x\frac{du}{dx} = u^2 - 2u$$
$$4x\frac{du}{dx} = u^2 - 6u$$
$$\frac{4}{u(u-6)}\,du = \frac{1}{x}\,dx.$$

Let $\frac{4}{u(u-6)} = \frac{A}{u} + \frac{B}{u-6}$ $\forall u$

$4 = A(u-6) + Bu$ $\forall u$.

If $u=6$: $6B = 4$

$B = 2/3$

If $u=0$ $-6A = 4$

$A = -2/3$.

$\therefore$ $\left(-\frac{2}{3u} + \frac{2}{3(u-6)}\right) du = \frac{1}{x}dx$

$$\int\left(-\frac{2}{3u} + \frac{2}{3(u-6)}\right) du = \int \frac{1}{x}dx$$
$$-\frac{2}{3}\ln|u| + \frac{2}{3}(\ln|u-6|) = \ln x + C.$$
$$\frac{2}{3}\ln\left(\frac{u-6}{u}\right) = \ln x + C.$$

Since $u = \frac{f(x)}{x}$, $u(1) = \frac{-6}{1} = -6$.

Sub. $(1,-6)$: $\frac{2}{3}\ln\frac{-6-6}{-6} = \ln 1 + C$

$C = \frac{2}{3}\ln 2$.

QUESTION NUMBER

ASSESSOR'S USE ONLY

4b. Continued.

At $x=4$,

$$\frac{2}{3}\ln\left(\frac{u-6}{u}\right) = \ln 4 + \frac{2}{3}\ln 2$$

$$\ln\left(\frac{u-6}{u}\right) = \frac{3}{2}\ln 4 + \ln 2$$

$$= \ln 8 + \ln 2$$

$$\ln\left(\frac{u-6}{u}\right) = \ln 16$$

$$\frac{u-6}{u} = 16$$

$$u-6 = 16u$$

$$15u = -6$$

$$u = -\frac{2}{5}.$$

Since $f(x) = x \cdot u(x)$, $f(x) = -\frac{2}{5} \times 4$

$$= -\frac{8}{5}.$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

5a.

$$\frac{w-1}{w+1}$$

$$= \frac{\cos\theta - 1 + i\sin\theta}{\cos\theta + 1 + i\sin\theta}$$

$$= \frac{(\cos\theta - 1 + i\sin\theta)(\cos\theta + 1 - i\sin\theta)}{(\cos\theta + 1 + i\sin\theta)(\cos\theta + 1 - i\sin\theta)}$$

$$= \frac{(\cos\theta - (1 - i\sin\theta))(\cos\theta + (1 - i\sin\theta))}{((\cos\theta + 1) + i\sin\theta)((\cos\theta + 1) - i\sin\theta)}$$

$$= \frac{\cos^2\theta - (1 - i\sin\theta)^2}{(\cos\theta + 1) - (i\sin\theta)^2}$$

$$= \frac{\cos^2\theta - (1 - 2i\sin\theta - \sin^2\theta)}{\cos^2\theta + 2\cos\theta + 1 + \sin^2\theta}$$

$$= \frac{\cos^2\theta - 1 + 2i\sin\theta + \sin^2\theta}{2\cos\theta + 2}$$

$$= \frac{1 - 1 + 2i\sin\theta}{2 + 2\cos\theta}$$

$$= i\left(\frac{\sin\theta}{1 + \cos\theta}\right)$$

$$= i\left(\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + 2\cos^2\frac{\theta}{2} - 1}\right)$$

$$= i\left(\frac{\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}\right)$$

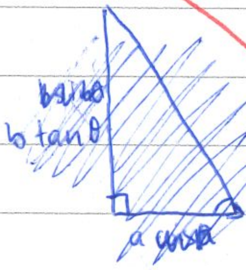
$$= i\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$= i\tan\theta/2$ as required.

QUESTION NUMBER | ASSESSOR'S USE ONLY

5bi.

$$\phi = \tan^{-1}\left(\frac{y}{x}\right)$$
$$= \tan^{-1}\left(\frac{b\sin\theta}{a\cos\theta}\right)$$
$$= \tan^{-1}\left(\frac{b}{a}\tan\theta\right)$$



So $\phi = \tan^{-1}\left(\frac{b}{a}\tan\theta\right)$, or $\tan\phi = \frac{b}{a}\tan\theta$.

ii. Let $y = \phi - \theta$. So $\frac{dy}{d\phi} = 1 - \frac{d\theta}{d\phi}$.

$$\tan\phi = \frac{b}{a}\tan\theta \quad \text{.. from (i)}$$
$$\frac{d}{d\phi}(\tan\phi) = \frac{d}{d\phi}\left(\frac{b}{a}\tan\theta\right) \quad \text{.. implicit differentiation}$$
$$\sec^2\phi = \frac{b}{a}\sec^2\theta \cdot \frac{d\theta}{d\phi}$$
$$\frac{d\theta}{d\phi} = \frac{a\sec^2\phi}{b\sec^2\theta}$$
$$\therefore \frac{dy}{d\phi} = 1 - \frac{a\sec^2\phi}{b\sec^2\theta} = 0$$
$$\frac{a\sec^2\phi}{b\sec^2\theta} = 1$$
$$a\sec^2\phi = b\sec^2\theta$$
$$\sec\phi =$$

next page

QUESTION NUMBER

ASSESSOR'S USE ONLY

5bii cont.

Let $y = \phi - \theta$.

$$\tan\phi = \frac{b}{a}\tan\theta$$

$$a\tan\phi = b\tan\theta.$$

$$\frac{dy}{d\phi} = 1 - \frac{d\theta}{d\phi}.$$

$$\frac{d}{d\phi}(a\tan\phi) = \frac{d}{d\phi}(b\tan\theta)$$

$$a\sec^2\phi = b\sec^2\theta \cdot \frac{d\theta}{d\phi}$$

$$\frac{d\theta}{d\phi} = \frac{a\sec^2\phi}{b\sec^2\theta}$$

So $1 - \frac{a\sec^2\phi}{b\sec^2\theta} = 0$

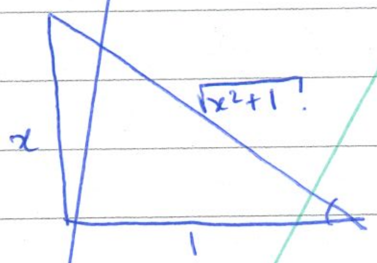
$$\frac{a\sec^2\phi}{b\sec^2\theta} = 1$$

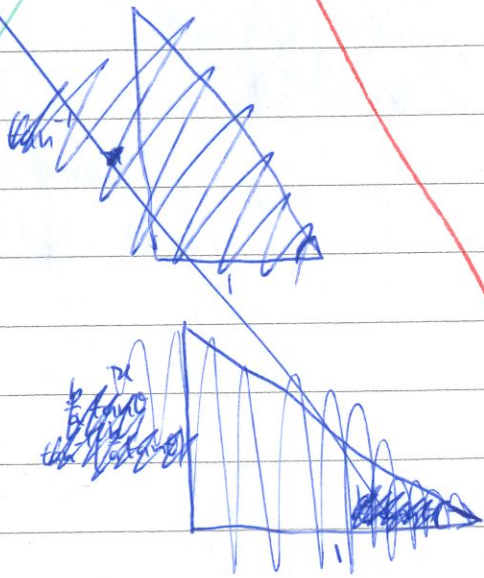
$$a\sec^2\phi = b\sec^2\theta$$

$$\sec\phi = \sqrt{\frac{b}{a}}\sec\theta$$

~~$\sec(\frac{b}{a}\tan\theta)$~~

$$\sec\left(\tan^{-1}\left(\frac{b}{a}\tan\theta\right)\right) = \sqrt{\frac{b}{a}}\sec\theta$$





Note: $\sec(\tan^{-1}(x)) = \left(\frac{1}{\sqrt{x^2+1}}\right)^{-1} = \sqrt{x^2+1}$.

~~So $\sqrt{\left(\frac{b}{a}\tan\theta\right)^2+1} = \sqrt{\frac{b}{a}}\sec\theta$~~

~~$1+\left(\frac{b}{a}\tan\theta\right)^2 = \frac{b}{a}\sec^2\theta$~~

~~$\tan\theta = \sec^2\theta$~~

QUESTION NUMBER

ASSESSOR'S USE ONLY

5bii

$$\cancel{\sqrt{\left(\frac{b}{a}\right)^2 \tan^2\theta + 1} = \sqrt{\frac{b}{a}}\sec\theta}$$

$$\cancel{\frac{b^2}{a^2}\tan^2\theta + 1 = \frac{b}{a}\sec^2\theta}$$

$$\cancel{b^2\tan^2\theta + a^2 = ab\sec^2\theta}$$

$$\phi = \tan^{-1}\left(\frac{b}{a}\tan\theta\right)$$

Let $b = \tan^{-1} a$.

$$\tan b = a$$

$$\frac{d}{da}(\tan b) = \frac{d}{da}(a)$$

$$\sec^2 b \cdot \frac{db}{da} = 1$$

$$\frac{db}{da} = \cos^2 b$$

$$= (\cos \tan^{-1} a)^2$$

$$= \left(\frac{1}{\sqrt{a^2+1}}\right)^2$$

$$\frac{db}{da} = \frac{1}{a^2+1}.$$

$$\frac{d\phi}{d\theta} = \frac{1}{b^2/a^2 \tan^2\theta + 1} \times \frac{b}{a}\sec^2\theta.$$

$$= \frac{b/a \sec^2\theta}{b^2/a^2\tan^2\theta + 1} = \frac{b/a(\tan^2\theta + 1)}{b^2/a^2\tan^2\theta + 1}$$

$$= \frac{ab(\tan^2\theta + 1)}{b^2\tan^2\theta + a^2}$$

Let $y = \phi - \theta$

$$\frac{dy}{d\theta} = \frac{d\phi}{d\theta} - 1 = 0$$

$$\frac{ab(\tan^2\theta + 1)}{b^2\tan^2\theta + 1} = 0 \quad \text{mei}$$

$$ab(\tan^2\theta + 1) = b^2\tan^2\theta + a^2$$

$$ab\tan^2\theta + ab = b^2\tan^2\theta + a^2$$

$$\tan^2\theta(ab - b^2) = 1 - ab$$

$$\tan^2\theta = \frac{a^2 - ab}{ab - b^2} = \frac{a}{b}\left(\frac{a-b}{a-b}\right) = \frac{a}{b}$$

$$\tan\theta = \pm\sqrt{\frac{a}{b}}.$$

QUESTION NUMBER

ASSESSOR'S USE ONLY

5bii. Since ~~θ~~ $\tan\theta = \pm\sqrt{\frac{a}{b}}$,

$$\tan\phi = \frac{b}{a}\tan\theta$$

$$\tan\phi = \frac{b}{a}\left(\pm\sqrt{\frac{a}{b}}\right)$$

$$\tan\phi = \pm\sqrt{\frac{b}{a}}$$

$$\phi = \tan^{-1}\left(\pm\sqrt{\frac{b}{a}}\right) + n\pi \quad \text{where } n \in \mathbb{Z}.$$

QUESTION NUMBER		ASSESSOR'S USE ONLY

2a.

$$\sqrt{\frac{16x-32}{x}} - \sqrt{\frac{16x}{x-2}} = k$$

$$\frac{16x-32}{x} + \frac{16x}{x-2} - 2\sqrt{\left(\frac{16x-32}{x}\right)\left(\frac{16x}{x-2}\right)} = k$$

$$\frac{(16x-32)(x-2)+16x^2}{x(x-2)} - 2\sqrt{\frac{16^2(x-2)(x)}{(x)(x-2)}} = k$$

$$\frac{(16x-32)(x-2)+16x^2}{x(x-2)} - 32 = k$$

[illegible]

$$16(x-2)^2 + 16x^2 - 32x(x-2) = kx(x-2)$$

$$16(x^2-4x+4) + 16x^2 - 32x^2 + 64x = kx^2 - 2kx$$

$$16x^2 - 64x + 64 + 16x^2 - 32x^2 + 64x = kx^2 - 2kx$$

$$kx^2 - 2kx - 64 = 0$$

$$x = \frac{2k \pm \sqrt{4k^2 - (4k)(-64)}}{2k}$$

$$= 1 \pm \sqrt{\frac{4k^2 + 256k}{4k^2}}$$

$$= 1 \pm \sqrt{1 + \frac{64}{k}}.$$

Let $u = k/4$.

$$\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}} = u$$

$$\left(\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}}\right)^2 = u^2$$

$$\frac{x-2}{x} + \frac{x}{x-2} - 2\sqrt{\left(\frac{x-2}{x}\right)\left(\frac{x}{x-2}\right)} = u^2$$

$$\frac{x-2}{x} + \frac{x}{x-2} - 2 = u^2$$

$$\frac{(x-2)^2 + x^2}{x(x-2)} - 2 = u^2$$

$$(x-2)^2 + x^2 - 2x(x-2) = u^2 x(x-2)$$

$$x^2 - 4x + 4 + x^2 - 2x^2 + 4x = u^2x^2 - 2u^2x$$

$$u^2x^2 - 2u^2x - 4 = 0$$

$$x = \frac{2u^2 \pm \sqrt{(2u^2)^2 - 4 \times u^2 \times -4}}{2u^2}$$

$$= 1 \pm \sqrt{1 + \frac{16u^2}{4u^4}}$$

$$= 1 \pm \sqrt{1 + \frac{4}{u^2}}$$

Since $u^2 \geq 0$, then $1 + \frac{4}{u^2} \geq 1$ so roots are never imaginary.

QUESTION NUMBER

ASSESSOR'S USE ONLY

2a.

~~When $x > 2$, $\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}} < 0$, so $u < 0$.~~

~~When $x < 0$, $\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}} > 0$, so $u > 0$.~~

~~$$\text{So } x = \begin{cases} 1 + \sqrt{1+\frac{4}{u^2}} & u < 0 \\ 1 - \sqrt{1+\frac{4}{u^2}} & u > 0. \end{cases}$$~~

When $x > 2$, $\sqrt{\frac{x-2}{x}} < 1$

and $\sqrt{\frac{x}{x-2}} > 1$

so $\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}} < 0$

So $u < 0$

When $x < 0$, $\sqrt{\frac{x-2}{x}} > 1$

and $\sqrt{\frac{x}{x-2}} < 1$

so $\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}} > 0$

So $u > 0$.

~~So $x = \left\{ \sqrt{1+\frac{4}{u^2}} \right.$ $u < 0$~~

$$\text{So } x = \begin{cases} 1 + \sqrt{1+\frac{4}{u^2}} & \ldots\ u < 0 \\ 1 - \sqrt{1+\frac{4}{u^2}} & \ldots\ u > 0. \end{cases}$$

$u \neq 0$ because otherwise

$$\sqrt{\frac{x-2}{x}} - \sqrt{\frac{x}{x-2}} = 0$$

$$\sqrt{\frac{x-2}{x}} = \sqrt{\frac{x}{x-2}}$$

$$(x-2)^2 = x^2$$

$$4x - 4 = 0$$

$$x = 1.$$

But this cannot happen as $x < 0$ or $x > 2$.

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