Exercise 4
Due: 14/01/2021

1. The 4 Bell states are

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$
$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle),$$
$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle),$$
$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

   Find a unitary matrix U that permutes the Bell states as follows: $|\Phi^+\rangle \xrightarrow{U} |\Phi^-\rangle \xrightarrow{U} |\Psi^+\rangle \xrightarrow{U} |\Psi^-\rangle \xrightarrow{U} |\Phi^+\rangle$.

2. Consider the following game. We have three participants: Moe, Larry and Curly, that are sent to different rooms, and each receives a fruit (apple or banana) at random. We are promised that the number of apples is even, (that is, either all got bananas, which happens with probability 1/4, or exactly 1 of them got banana). Their goal is to each output a number in $\{-1, 1\}$, so that if all fruits are bananas the product of their outputs is 1, and otherwise -1.

   Note that the three can devise any strategy they want beforehand, but cannot communicate once they enter their rooms.

   (a) Show that any deterministic strategy cannot succeed with probability 1. (The probability is over the inputs.)
   (b) Find a deterministic strategy that wins with probability 3/4.
   (c) Show that even a randomized strategy, where they can all access the same random string from the rooms, still cannot succeed with probability 1.
   (d) Assume that the three stooges share the following superposition over 3 qubits:

$$\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)\ .$$

   Consider the following protocol: Each stooge applies H on his qubit if he got a banana, and $H' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ i & -1 \end{pmatrix}$ if he got an apple. Then each measures his qubit and chooses the output accordingly. Show that they can win with probability 1.

3. Fix $a \in \{0,1\}^n$, which is unknown to us. Let $f : \{0,1\}^n \to \{0,1\}$ be given by $f(x) = a \cdot x \pmod 2$ where $a \cdot x = \sum_{i=1}^n a_i x_i$. We would like to find a using as few queries to f as possible.

   (a) Prove that a quantum algorithm can reveal a with probability 1 using a *single* query to f. *Hint:* use the Deutsch-Josza algorithm.
   (b) How many queries must a classical protocol perform? Will randomization help?