



Article

Composition Identities of Chebyshev Polynomials, via 2×2 Matrix Powers

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Abstract: Starting from a representation formula for 2×2 non-singular complex matrices in terms of 2nd kind Chebyshev polynomials, a link is observed between the 1st kind Chebyshev polinomials and traces of matrix powers. Then, the standard composition of matrix powers is used in order to derive composition identities of 2nd and 1st kind Chebyshev polynomials. Before concluding the paper, the possibility to extend this procedure to the multivariate Chebyshev and Lucas polynomials is touched on.

Keywords: Chebyshev polynomials; matrix powers; composition identities

MSC: 33C45; 15A16

1. Introduction

To the authors' knowledge, very few composition identities for Chebyshev polynomials [1,2] are known, apart from the classical one for 1st kind: $T_m[T_n(x)] = T_{mn}(x)$, and others for the 2nd kind, published by Kimberling in [3].

Recently, some convolution equations or sums of finite products of Chebyshev polynomials appeared in literature [4,5], but none of the composition type.

This is the motivation of this article, which aims to fill a gap in the literature on a very well known topic [1,6], which concerns the polynomials of Chebyshev, one of the most used families of orthogonal polynomials.

In what follows, by using a known connection [7] of the 2nd kind Chebyshev polynomials with 2×2 matrix power representation, we derive in a straightforward way composition identities involving 1st and 2nd kind Chebyshev polynomials.

The key result is the useful remark that the trace of matrix powers are simply related to the 1st kind Chebyshev polynomials. Therefore, the standard composition of matrix powers implies the searched composition identities. The considered method could be theoretically used in order to find an infinite number of such compositions, but of course the relative formulas become more and more complicated as the number of nested powers is considered.

The obtained equations could obviously be derived even by using the trigonometric forms of Chebyshev polynomials, but the method proposed in this article seems to be more simple and efficient.

Before concluding the paper, it is mentioned that a similar approach could be used in order to construct composition identities for two variable Chebyshev polynomials and even for the so-called $F_{k,n}$ functions that are solutions of linear recurrence equations, with suitable initial conditions, which include the multivariate case of 2nd kind Lucas polynomials, and are useful in order to represent powers of higher order matrices.

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2. On Powers of a 2×2 Non-Singular Matrix

Let $A = \{a_{h,k}\}_{2\times 2}$ be a non-singular complex matrix, and let

$$u_1 := tr \mathcal{A}, \qquad u_2 = det \mathcal{A} \neq 0,$$
 (1)

respectively the trace and the determinant of A.

The characteristic equation is given by

$$\lambda^2 - u_1 \lambda + u_2 = 0, \tag{2}$$

and denote by λ_1 and λ_2 the eigenvalues of \mathcal{A} .

We start by considering two results connecting matrix powers with Chebyshev polynomials.

The first result is the representation theorem for matrix powers, proved in [7]:

Theorem 1. The integer powers of A, $(n \ge 2)$, are given by:

$$A^{n} = u_{2}^{(n-1)/2} U_{n-1} \left(\frac{u_{1}}{2u_{2}^{1/2}} \right) A - u_{2}^{n/2} U_{n-2} \left(\frac{u_{1}}{2u_{2}^{1/2}} \right) \mathcal{I}, \tag{3}$$

where \mathcal{I} is the 2 × 2 identity matrix, and

$$U_0(x) = 1$$
, $U_1(x) = 2x$, $U_2(x) = 4x^2 - 1$, $U_3(x) = 8x^2 - 4x$,... (4)

are the second kind Chebyshev polynomials.

In particular, we find:

$$A^{2} = u_{2}^{1/2} U_{1} \left(\frac{u_{1}}{2 u_{2}^{1/2}} \right) A - u_{2} U_{0} \left(\frac{u_{1}}{2 u_{2}^{1/2}} \right) \mathcal{I} = u_{1} A - u_{2} \mathcal{I},$$
 (5)

$$A^{3} = u_{2} U_{2} \left(\frac{u_{1}}{2 u_{2}^{1/2}} \right) A - u_{2}^{3/2} U_{1} \left(\frac{u_{1}}{2 u_{2}^{1/2}} \right) \mathcal{I}, \tag{6}$$

$$\mathcal{A}^4 = u_2^{3/2} U_3 \left(\frac{u_1}{2 u_2^{1/2}} \right) \mathcal{A} - u_2^2 U_2 \left(\frac{u_1}{2 u_2^{1/2}} \right) \mathcal{I}, \tag{7}$$

We prove now the second result, which gives the trace of integer matrix powers A^n (n = 0, 1, 2, ...) in terms of first kind Chebyshev polynomials $T_n(x)$:

$$T_0(x) = 1$$
, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^2 - 3x$,... (8)

Theorem 2. For any integer $n \ge 0$, it results:

$$tr \mathcal{A}^n = 2 u_2^{n/2} T_n \left(\frac{u_1}{2 u_2^{1/2}} \right) .$$
 (9)

Proof. Since $tr A^n = \mathbf{i}_1^n + \mathbf{i}_2^n$, we find, in particular:

$$tr A = u_1, \quad tr A^2 = u_1^2 - 2u_2,$$
 (10)

and by the Newton-Girard formulas, we can compute the following ones by the recursion:

$$tr A^n = u_1 (tr A^{n-1}) - u_2 (tr A^{n-2}),$$
 (11)

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which is the same recursion associated with Equation (2).

Putting

$$x = T_1(x) = u_1/2u_2^{1/2}, (12)$$

so that $u_1 = 2u_2^{1/2} x$, and recalling the position (12), we find:

$$tr \mathcal{A} = 2u_2^{1/2} x = 2u_2^{1/2} T_1(x), \quad tr \mathcal{A}^2 = 4x^2 u_2 - 2u_2 = 2u_2 T_2(x).$$
 (13)

Then, Equation (9) is true for n=1 and n=2. Therefore, Equation (9) holds for every $n\geq 0$, since the first and second member in this equation satisfy the same recursion (11) and initial conditions (13), (for n=0 Equation (9) trivially holds). \square

Corollary 1. For any integer $n \ge 0$, assuming position (12), it results in:

$$\frac{\operatorname{tr} A^n}{2 \left(\det A^n \right)^{1/2}} = T_n(x) \,, \tag{14}$$

and therefore the representation formula (3) becomes:

$$A^{n} = u_{2}^{(n-1)/2} U_{n-1}(x) A - u_{2}^{n/2} U_{n-2}(x) \mathcal{I}.$$
(15)

In particular, Equations (6) and (7), by using Equation (12), write:

$$A^{3} = u_{2} U_{2}(x) A - u_{2}^{3/2} U_{1}(x) \mathcal{I},$$
(16)

$$A^{4} = u_{2}^{3/2} U_{3}(x) A - u_{2}^{2} U_{2}(x) \mathcal{I}, \qquad (17)$$

3. Composition of Chebyshev Polynomials

3.1. A Preliminary Example

In this section, we prove some relations involving 1st and 2nd kind Chebyshev polynomials derived by matrix analysis.

Put by definition:

$$\mathcal{B} := \mathcal{A}^2 = u_1 \,\mathcal{A} - u_2 \,\mathcal{I} \,,$$

$$\mathcal{B}^2 = \mathcal{A}^4 = (u_1^2 - 2u_2) \,\mathcal{B} - u_2^2 \,\mathcal{I} \,,$$
(18)

that is:

$$\mathcal{A}^4 = (u_1^3 - 2u_1u_2) \,\mathcal{A} - (u_1^2u_2 - u_2^2) \,\mathcal{I} \,, \tag{19}$$

Note that

$$tr A^2 = 2 u_2 T_2(x), \qquad det A^2 = u_2^2 \neq 0,$$
 (20)

$$tr \mathcal{B}^2 = 2 u_2^2 T_4(x), \qquad det \mathcal{B}^2 = u_2^4 \neq 0.$$
 (21)

Then, we can prove the results:

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Theorem 3. *For any integer n, the following equations hold:*

$$2x U_{n-1}[T_2(x)] = U_{2n-1}(x),$$

$$U_{2n-2}(x) = U_{n-1}[T_2(x)] + U_{n-2}[T_2(x)].$$
(22)

Theorem 4. For any integer n, the following equations hold:

$$U_{4n-1}(x) = U_{n-1}[T_4(x)] U_3(x)$$
,

$$U_{4n-2}(x) = U_{n-1}[T_4(x)] U_2(x) + U_{n-2}[T_4(x)],$$

$$2x U_{2n-1}[T_2(x)] = U_{n-1}[T_4(x)] U_3(x),$$
(23)

$$U_{2n-1}[T_2(x)] + U_{2n-2}[T_2(x)] = U_{n-1}[T_4(x)] U_2(x) + U_{n-2}[T_4(x)],$$

and, as a consequence of Equation (23), we find the identities:

$$U_{4n-1}(x) = 2x U_{2n-1}[T_2(x)],$$

$$U_{4n-2}(x) = U_{2n-1}[T_2(x)] + U_{2n-2}[T_2(x)],$$
(24)

which give back the recurrence relation for $U_n[T_2(x)]$.

Proof of Theorem 3.1. We compute the 2nth powers of \mathcal{A} in two different forms. By using Equations (3) and (18), and recalling the position (12), we find:

$$\mathcal{A}^{2n} = u_2^{(2n-1)/2} U_{2n-1} \left(\frac{u_1}{2u_2^{1/2}} \right) \mathcal{A} - u_2^n U_{2n-2} \left(\frac{u_1}{2u_2^{1/2}} \right) \mathcal{I} =$$

$$= u_2^{(2n-1)/2} U_{2n-1}(x) \mathcal{A} - u_2^n U_{2n-2}(x) \mathcal{I},$$
(25)

$$\mathcal{A}^{2n} = \mathcal{B}^{n} = u_{2}^{n-1} U_{n-1}[T_{2}(x)] \mathcal{B} - u_{2}^{n} U_{n-2}(T_{2}(x)) \mathcal{I} =$$

$$= u_{2}^{n-1} U_{n-1}[T_{2}(x)] [u_{1}\mathcal{A} - u_{2}\mathcal{I}] - u_{2}^{n} U_{n-2}[T_{2}(x)] \mathcal{I} =$$

$$= u_{1}u_{2}^{n-1} U_{n-1}[T_{2}(x)] \mathcal{A} - u_{2}^{n} \{U_{n-1}[T_{2}(x)] + U_{n-2}[T_{2}(x)]\} \mathcal{I}.$$
(26)

Comparing the coefficients of A and I in Equations (23) and (24), we find the two relations:

$$u_2^{(n-1)/2} U_{2n-1}(x) = u_1 u_2^{n-1} U_{n-1}[T_2(x)],$$

$$U_{2n-2}(x) = U_{n-1}[T_2(x)] + U_{n-2}[T_2(x)],$$
(27)

that is, our result, taking into account definition (12). \Box

Proof of Theorem 3.2. Compute the (4n)th power of \mathcal{A} in the following forms:

$$\mathcal{A}^{4n} = \left(\mathcal{A}^4\right)^n = \mathcal{B}^{2n} \,. \tag{28}$$

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We find:

$$\mathcal{A}^{4n} = u_2^{(4n-1)/2} U_{4n-1}(x) \mathcal{A} - u_2^{2n} U_{4n-2}(x) \mathcal{I}, \tag{29}$$

$$\left(\mathcal{A}^4\right)^n = u_2^{2n-2} \, U_{n-1}[T_4(x)] \, \mathcal{A}^4 - u_2^{2n} \, U_{n-2}[T_4(x)] \, \mathcal{I}$$
 ,

and, recalling Equation (7), with the position (12), it results:

$$\mathcal{A}^{4n} = \left(\mathcal{A}^{4}\right)^{n} = u_{2}^{2n-1/2} U_{n-1}[T_{4}(x)] U_{3}(x) \mathcal{A} -$$

$$u_{2}^{2n} \left\{ U_{n-1}[T_{4}(x)] U_{2}(x) + U_{n-2}[T_{4}(x)] \right\} \mathcal{I},$$
(30)

$$\mathcal{A}^{4n} = \mathcal{B}^{2n} = u_2^{2n-1} U_{2n-1}[T_2(x)] \mathcal{B} - u_2^{2n} U_{n-2}[T_2(x)] \mathcal{I}$$
,

and, recalling Equation (5), we find:

$$\mathcal{A}^{4n} = \mathcal{B}^{2n} = u_1 u_2^{2n-1} U_{2n-1}[T_2(x)] \mathcal{A} -$$

$$u_2^{2n} \left\{ U_{2n-1}[T_2(x)] + U_{2n-2}[T_2(x)] \right\} \mathcal{I},$$
(31)

comparing separately the coefficients of \mathcal{A} and \mathcal{I} , in Equations (29)–(30) and (30)–(31), we find Equation (23). \square

3.2. A More General Case

We prove now a more general result.

Theorem 5. For any integers n and m, the following equations hold:

$$U_{mn-1}(x) = U_{n-1}[T_m(x)] U_{m-1}(x),$$

$$U_{mn-2}(x) = U_{n-1}[T_m(x)] U_{m-2}(x) + U_{n-2}[T_m(x)].$$
(32)

Proof of Theorem 3.3. We compute the (mn)th power of \mathcal{A} in different forms. By Equation (15), we have:

$$A^{mn} = u_2^{(mn-1)/2} U_{mn-1}(x) A - u_2^{mn/2} U_{mn-2}(x) \mathcal{I},$$
(33)

but

$$\mathcal{A}^{mn} = (\mathcal{A}^m)^n \ . \tag{34}$$

We find:

$$A^{mn} = (A^m)^n = u_2^{m(n-1)/2} U_{n-1}[T_m(x)] A^m - u_2^{mn/2} U_{n-2}[T_m(x)] \mathcal{I},$$
(35)

where

$$\mathcal{A}^{m} = u_{2}^{(m-1)/2} U_{m-1}(x) \mathcal{A} - u_{2}^{m/2} U_{m-2}(x) \mathcal{I}$$
,

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Then, we find:

$$\mathcal{A}^{mn} = u_2^{(mn-1)/2} U_{n-1}[T_m(x)] U_{m-1}(x) \mathcal{A} -$$

$$u_2^{mn/2} \left\{ U_{n-1}[T_m(x)] U_{m-2}(x) + U_{n-2}[T_m(x)] \right\} \mathcal{I}.$$
(36)

Then, Equation (32) follows by comparing the coefficients of $\mathcal A$ and $\mathcal I$ in Equations (33) and (36). \square

Remark 1. Further extensions of the results of Threorem 3.3 can be obtained by considering the powers of matrix A in the form:

$$\mathcal{A}^{n_1 n_2 n_3} = (\mathcal{A}^{n_1 n_2})^{n_3} = (\mathcal{A}^{n_1})^{n_2 n_3} . \tag{37}$$

An obvious extension is considering a higher number of powers, that is, $n_1n_2 \cdots n_k$. A lot of possible identities can be found using the different form of powers, but the relative equations become more and more complicated as the number k increases.

Remark 2. It is possible to consider an application of the above results to the second kind two-variable Chebyshev polynomials $U_n^{(2)}(u,v)$ [8–10], defined by the recursion:

$$U_0^{(2)}(u,v) = 0, \quad U_1^{(2)}(u,v) = 1, \quad U_2^{(2)}(u,v) = u,$$

$$U_n^{(2)}(u,v) = u U_{n-1}^{(2)}(u,v) - v U_{n-2}^{(2)}(u,v) + U_{n-3}^{(2)}(u,v).$$
(38)

In fact, denoting by $J_1 = tr A$, J_2 , $J_3 = det A$, the invariants of a non-singular 3×3 matrix A, by using the Hamilton–Cayley theorem, in [11], it has been shown that

$$\mathcal{A}^{n} = U_{n-1}^{(2)}(u,v) J_{3}^{\frac{n-2}{3}} \mathcal{A}^{2} + \left[-v U_{n-2}^{(2)}(u,v) + U_{n-3}^{(2)}(u,v) \right] J_{3}^{\frac{n-1}{3}} \mathcal{A} + U_{n-2}^{(2)}(u,v) J_{3}^{\frac{n}{3}} \mathcal{I},$$

$$(39)$$

where $u := J_1 J_3^{-1/3}$, $v := J_2 J_3^{-2/3}$.

Furthermore, in the same article [11], the following expression of $U_n^{(2)}(u,v)$ in terms of the 2nd kind Chebyshev polynomials $U_n(\cdot)$ has been proven:

$$U_n^{(2)}(u,v) = \frac{-(xy)^{(n+2)/2} U_n \left(\frac{x+y}{2\sqrt{xy}}\right) + (xy)^{(n+5)/2} U_{n-1} \left(\frac{x+y}{2\sqrt{xy}}\right) + (xy)^{-n+1}}{1 - xy(x+y) + (xy)^3},$$
(40)

where the variables x, y are related to the u, v by means of the system:

$$\begin{cases} u = x + y + (xy)^{-1} \\ v = xy + (x+y)(xy)^{-1}. \end{cases}$$
 (41)

Then, by using Equation (39), the powers of A can be represented in terms of the second kind ordinary Chebyshev polynomias.

Therefore, the technique used in this article can be exploited in order to derive composition identities for the 2-variable Chebyshev polynomials $U_n(x,y)$.

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Remark 3. Another application could be obtained by recalling the results on powers of $r \times r$ matrices, contained in [12]:

$$\mathcal{A}^{n} = F_{1,n-1}(u_{1}, \dots, u_{r}) \,\mathcal{A}^{r-1} + F_{2,n-1}(u_{1}, \dots, u_{r}) \,\mathcal{A}^{r-2} + \cdots + F_{r,n-1}(u_{1}, \dots, u_{r}) \,\mathcal{I},$$

$$(42)$$

where the $F_{k,n}$ functions of the invariants u_1, \ldots, u_r of the matrix A are defined as the solutions of the linear recursion

$$F_{k,n} = u_1 F_{k,n-1} - u_2 F_{k,n-2} + \dots + (-1)^{r-1} u_r F_{k,n-r},$$
(43)

(k = 1, 2, ..., r), satisfying the initial conditions

$$F_{1,-1} = 0$$
, $F_{1,0} = 0$, ... $F_{1,r-2} = 1$,
 $F_{2,-1} = 0$, $F_{2,0} = 1$, ... $F_{2,r-2} = 0$,
 \vdots \vdots ... \vdots
 $F_{r,-1} = 1$, $F_{r,0} = 0$, ... $F_{r,r-2} = 0$. (44)

The $F_{k,n}$ functions have been used in several frameworks, and in particular in representing matrix exponentials [13,14] and linear dynamical systems [15].

Then, the use of the same technique of this article could be exploited in order to find composition identities for the $F_{k,n}$ functions.

4. Conclusions

It has been shown that the use of a matrix power representation for a 2×2 non-singular matrix can be used in order to derive in a straightforward way composition identities of 2nd and 1st kind Chebyshev polynomials. Further extensions, by using other representation formulas for higher order matrices could be found, but the relative equations involve multivariate Chebyshev polynomials that are not as popular as the classical Chebyshev ones.

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