

# INDIVIDUAL-BASED INTEGRAL PROJECTION MODELS: THE ROLE OF SIZE-STRUCTURE ON EXTINCTION RISK AND ESTABLISHMENT SUCCESS

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ABSTRACT.

- (1) Matrix models or integral projection models (IPMs) are commonly used to study the dynamics of structured populations, where discrete or continuous traits influence survival, growth, or reproduction.
- (2) When a population's size is small, as is often the case for threatened species or potentially invasive species arriving in novel habitats, extinction risk may be substantial due to demographic stochasticity. Branching processes, which are individual-based counterparts to matrix models and IPMs, allow one to quantify these risks of extinction.
- (3) For discretely structured populations, the theory of multi-type branching processes provides analytic methods to compute how extinction risk changes over time and how it depends on the size and composition of the population. Building on prior work on continuous-state branching processes, we extend these analytic methods to individual-based models accounting for any mixture of discrete and continuous population structure.
- (4) The individual-based IPMs are defined by probabilistic update rules at the level of the individual which determine how each individual with a given trait value dies, changes trait values (e.g. grows in size), or produces individuals with the same or other trait values. We show that probabilities of extinction can be analytically determined by probability-generating functionals associated with the

individual-based IPMs. In particular, we present analytical expressions for how extinction probabilities change over time and depend on the initial abundance and trait distribution of the population.

- (5) We illustrate how to numerically implement these methods using data from the short-lived desert shrub species *Cryptantha flava*, and provide a more general discussion of how to implement these methods to other data sets including those involving fluctuating environmental conditions.
- (6) As most IPM studies have the necessary data to parameterize individual-based IPMs, these methods provide a computationally efficient means to explore how continuously structured populations differing in their evolutionary history and environmental context may differ in their vulnerability to extinction or ability to colonize new habitats.

## INTRODUCTION

Computations of extinction probabilities or likelihoods of establishment success lie on opposing sides of a theoretician's coin and have been used to address theoretical and practical issues in conservation biology, restoration ecology, biological invasions and population genetics. Risks of extinction or establishment failure stem from populations consisting of a finite number of individuals, each of which faces a non-zero risk of mortality on any given day. These extinction risks are shaped, in part, by the size and composition of a population whose individuals may differ in age, size, geographical location, or other important characteristics influencing demography. When population structure is finite-dimensional (e.g. a finite number of age classes, stages, geographical locations), multi-type branching processes can model these extinction risks and, thereby, serve as the stochastic, finite-population counterpart of matrix models [Harris, 1963, Athreya and Ney, 2004, Caswell, 2001, Haccou et al., 2005]. These stochastic models have been used successfully to address a

49 diversity of questions concerning fixation probabilities of beneficial alleles [Patwa and Wahl, 2008],  
 50 evolutionary emergence of pathogens [Antia et al., 2003, Park et al., 2013], extinction risk of small  
 51 populations [Boyce, 1992, Gosselin and Lebreton, 2000, Fujiwara and Caswell, 2001, Erickson et al.,  
 52 in press], and establishment success in heterogeneous environments [Haccou and Iwasa, 1996, Haccou  
 53 and Vatunin, 2003, Schreiber and Lloyd-Smith, 2009].

54 To parameterize matrix models or multi-type branching processes, individuals must be discretely  
 55 categorized into a finite number of types. However, when collecting demographic data, researchers  
 56 commonly measure continuous traits (e.g. mass, length, geographical location) about individuals  
 57 and use continuum-based statistics to approximate “fine-grained” discrete-traits. Integral projec-  
 58 tion models (IPMs) allow one to account for this continuous population structure [Easterling et al.,  
 59 2000]. These IPMs can be viewed as infinite-dimensional matrix models which can be numerically  
 60 approximated by finite-dimensional matrix models. Consequently, many of the standard demo-  
 61 graphic concepts and methods for matrix models (e.g. stable state distributions, reproductive  
 62 values, life table response experiments, sensitivity analysis) exist for IPMs [Easterling et al., 2000,  
 63 Ellner and Rees, 2006, 2007, Rees and Ellner, 2009, Coulson, 2012, Ellner and Schreiber, 2012,  
 64 Metcalf et al., 2013, Rees et al., 2014, Merow et al., 2014].

65 Here, we describe individual-based counterparts of IPMs using continuous-state branching pro-  
 66 cesses [Harris, 1963]. For these finite population, stochastic models, we present an analytical method  
 67 for computing extinction probabilities. As these methods are easily implemented numerically, they  
 68 circumvent the need to use individual-based simulations and allow one to efficiently study how ex-  
 69 tinction probabilities or establishment failure depend on continuous as well as discrete population  
 70 structure. We illustrate the application of these methods with an individual-based IPM of the short-  
 71 lived desert shrub *Cryptantha flava* from Utah, USA [Salguero-Gómez et al., 2012, Salguero-Gómez  
 72 et al., 2014].

## THE GENERAL MODELS AND METHODS

**The Individual-Based IPM.** We consider an individual-based model where the set of all possible individual states (e.g. age, size, geographical location, etc.) lies in a compact metric space  $X$ . For a standard size-structured IPM,  $X = [a, b]$  corresponds to the range of sizes measured in the field where  $a$  is the minimal size and  $b$  is the maximal size. For models with a mixture of age and size structure,  $X$  could be given by  $\{1, \dots, T\} \times [a, b]$  where  $T$  corresponds to the maximal age of an individual. For spatially structured populations,  $X = [a, b]$  may correspond to locations along a transect or  $X = [a, b] \times [c, d]$  may correspond to latitude and longitude coordinates of individuals.

Following Harris [1963], we consider finite populations in which the state of the population at any point in time is characterized by the different states  $(x_1, x_2, \dots, x_k)$  of individuals within the population and the number of individuals in each state  $(n_1, n_2, \dots, n_k)$ . Specifically, if there  $n_1$  individuals in state  $x_1$ ,  $n_2$  individuals in state  $x_2$ , ...,  $n_k$  individuals in state  $x_k$ , then the state of the population is given by  $\mathbf{s} = (n_1, n_2, \dots, n_k; x_1, x_2, \dots, x_k)$ . The set of all possible population states is

$$\mathcal{S} = \{(\mathbf{n}; \mathbf{x}) = (n_1, \dots, n_k; x_1, \dots, x_k) : k, n_i \in \mathbb{N}, x_i \in X\} \cup \{0\}$$

where  $\mathbb{N} = \{1, 2, 3, \dots\}$  denotes the natural numbers and 0 is the extinction state corresponding to no individuals in the population.

Let  $\mathbf{s}_t = (\mathbf{n}; \mathbf{x}) \in \mathcal{S}$  be the population state at time  $t$ . The dynamics of  $\mathbf{s}_t$  are determined by a set of probabilistic rules that determine the contribution of each individual in the population to the population in next time step  $t + 1$ . These “contributions” may correspond to an individual surviving and changing state (e.g. growing in size, getting older, dispersing to another geographical location), or an individual having offspring. Consistent with standard branching process theory, each individual updates independently of all other individuals in the population.

The update rule for an individual in state  $x$  is given by a probability measure  $m(x, d\mathbf{s})$  on the state space  $\mathcal{S}$ . Specially, the probability an individual in state  $x$  contributes  $\mathbf{s}$  individuals to the

97 population in the next time step where  $\mathbf{s}$  lies in a subset  $A \subset \mathcal{S}$  equals

$$\mathbb{P}[\mathbf{s}_1 \in A | \mathbf{s}_0 = (1; x)] = \int_A m(x, d\mathbf{s})$$

98 where the left hand side reads “the probability the population state lies in  $A$  at time 1 after initially  
99 having only one individual in state  $x$  at time 0.”

100 If the population state is currently  $\mathbf{s}_t = (n_1, \dots, n_k; x_1, \dots, x_k)$ , then the state  $\mathbf{s}_{t+1}$  is determined  
101 as follows:

- 102 (1) for each of the  $n_1$  individuals in state  $x_1$ , randomly and independently choose the number  
103 of replacement individuals from distribution  $m(x_1, d\mathbf{s})$ ,
- 104 (2) repeat step (1) for the states  $x_2, \dots, x_k$ , and
- 105 (3) determine the new population state  $\mathbf{s}_{t+1}$  by identifying the states of all individuals and  
106 counting the total number of individuals in each of these states.

107 This iterative algorithm can be used to create individual-based simulations of the individual-based  
108 IPM. As with any branching process, stochastic realizations of this process, with probability one,  
109 either go to extinction in finite time, or the population abundance grows without bound. This latter  
110 event is typically interpreted as a population becoming established or persisting.

111 **Probability-generating functionals and extinction probabilities.** We can characterize the  
112 probabilistic state of the system using probability-generating functionals  $\Psi$  (pgfs). Unlike moment-  
113 generating functionals as used by Harris [1963], the pgfs allow us to directly compute how extinction  
114 probabilities change in time as well as compute asymptotic extinction probabilities.

115 To define the pgf  $\Psi$ , we introduce some notation. To motivate this notation, suppose  $h(x)$  equals  
116 the probability that an individual of size  $x$  dies and has no offspring over the next year. If the  
117 current population state is  $\mathbf{s} = (x_1, \dots, x_k; n_1, \dots, n_k)$ , then independence of individuals implies  
118 that  $\prod_{i=1}^k h(x)^{n_i}$  is the probability that population goes extinct by the next time step i.e., all  
119 individuals die and have no offspring. With this motivation in mind, given *any* continuous function

120  $h : X \rightarrow \mathbb{R}$  and population state  $\mathbf{s} \in \mathcal{S}$ , define

$$h^{\mathbf{s}} = \begin{cases} \prod_{i=1}^k h(x_i)^{n_i} & \text{if } \mathbf{s} = (n_1, \dots, n_k; x_1, \dots, x_k) \\ 1 & \text{if } \mathbf{s} = 0. \end{cases}$$

121 The probability-generating functional  $\Psi$  is defined by

$$(1) \quad \Psi(h)(x) = \int h^{\mathbf{s}} m(x, d\mathbf{s})$$

122 and corresponds to the expected value of  $h^{\mathbf{s}_1}$  where  $\mathbf{s}_1$  is the population state at time 1, given the  
 123 population initially consisted of a single individual in state  $x$ . This expectation requires integrating  
 124 over all possible states  $\mathbf{s}$  that the population may take on in the next time step due to the different  
 125 possible contributions of the individual in state  $x$ . Appendix A describes how this definition of  $\Psi$   
 126 simplifies to the pgf for the classical multi-type branching processes.

127 Two facts allow us to use  $\Psi$  to calculate extinction probabilities. First, when  $h_0$  is the zero  
 128 function (i.e.,  $h_0(x) = 0$  for all  $x$ ),  $\Psi(h_0)(x)$  is the probability the population goes extinct in one  
 129 time step, given that initially it consisted of one individual in state  $x$ . That is,

$$(2) \quad \Psi(h_0)(x) = \int_{\{0\}} 1 \times m(x, d\mathbf{s}) + \int_{\mathcal{S} \setminus \{0\}} 0 \times m(x, d\mathbf{s}) = \mathbb{P}[\mathbf{s}_1 = 0 | \mathbf{s}_0 = (1; x)].$$

130 Secondly, we claim that the  $t$ -fold composition of  $\Phi$  applied to  $h_0$  and evaluated at the individual  
 131 state  $x$ ,

$$\Psi^t(h_0)(x) = \underbrace{\Psi(\Psi(\dots \Psi(h_0) \dots))}_{t \text{ times}}(x)$$

132 is the probability of extinction by time  $t$  given that the population initially consisted of one indi-  
 133 vidual in state  $x$ :

$$(3) \quad \Psi^t(h_0)(x) = \mathbb{P}[\mathbf{s}_t = 0 | \mathbf{s}_0 = (1; x)] =: E_t(x).$$

134 To verify this claim, we argue by induction. Equation (2) implies that equation (3) holds for  
 135  $t = 1$ . Now suppose that equation (3) holds at time  $t$ ; we will show it holds at time  $t + 1$ . On the  
 136 event that  $\mathbf{s}_1 = (n_1, \dots, n_k; x_1, \dots, x_k)$  is the population state at time 1, extinction occurs by time  
 137  $t + 1$  only if each of the lineages of the  $n_1 + \dots + n_k$  individuals go extinct in the next  $t$  time steps.  
 138 As the fates of these lineages are independent of one another, it follows that

$$\begin{aligned} \mathbb{P}[s_{t+1} = 0 | \mathbf{s}_0 = (1; x), \mathbf{s}_1 = (n_1, \dots, n_k; x_1, \dots, x_k)] &= \prod_{i=1}^k \mathbb{P}[\mathbf{s}_{t+1} = 0 | \mathbf{s}_1 = (1; x_i)]^{n_i} \\ &= \prod_{i=1}^k (\Psi^t(h_0)(x_i))^{n_i} = \Psi^t(h_0)^{\mathbf{s}_1} \end{aligned}$$

139 where the second equality follows from our inductive hypothesis. By the law of total probability, we  
 140 can integrate the conditional probabilities of extinction over all possible population states at time  
 141  $t = 1$  (i.e.  $\mathbf{s}_1$ ) to get

$$E_{t+1}(x) = \mathbb{P}[s_{t+1} = 0 | \mathbf{s}_0 = (1; x)] = \int [\Psi^t(h_0)]^{\mathbf{s}} m(d\mathbf{s}, x)$$

142 which, by definition, equals  $\Psi^{t+1}(h_0)(x)$  as claimed.

143 Equation (3) can be used to compute extinction probabilities iteratively. Furthermore, as indi-  
 144 viduals update independent of one another, the probability of the population going extinct by time  
 145  $t$  for any initial condition  $\mathbf{s} = (n_1, \dots, n_k; x_1, \dots, x_k)$  equals

$$(4) \quad \mathbb{P}[\mathbf{s}_t = 0 | \mathbf{s}_0 = \mathbf{s}] = E_t^{\mathbf{s}}.$$

146 These analytic expressions allow us to efficiently compute extinction probabilities by constructing  
 147 a numerical approximation of the pgf  $\Psi$  and iterating it with an initial condition of a zero vector  
 148 which corresponds to the numerical approximation of the zero function.

As  $E_0(x) \leq E_1(x) \leq E_2(x) \dots$  for any  $x \in X$  and  $E_t(x) \leq 1$  for all  $t$ , there is a well defined limit corresponding to the probability of eventual extinction:

$$E_\infty(x) := \lim_{t \rightarrow \infty} E_t(x).$$

Using moment-generating functionals with suitable technical hypotheses, Harris [1963] showed that  $E_\infty(x) < 1$  for all  $x$  if the dominant eigenvalue of the mean-field IPM is greater than one, and  $E_\infty(x) = 1$  for all  $x$  otherwise.

#### AN ILLUSTRATION WITH A SHORT-LIVED PERENNIAL

We illustrate these general methods using an individual-based IPM for the yellow-flowered perennial plant, Plateau Yellow Miner's Candle (*Cryptantha flava*), of the borage family (*Boraginaceae*). For populations growing along the Colorado Plateau, USA, Salguero-Gómez et al. [2012] developed an IPM using data collected from 2004 to 2010 [Salguero-Gómez et al., 2014]. Here, we use a subset of this data available in an R package, IPMpack [Metcalf et al., 2014]. All code to for this example is archived at Zenodo [Schreiber and Ross, 2015].

[Figure 1 about here.]

In the model, the state  $x$  of an individual is the size, which equals the total number of vegetative and flowering rosettes. If  $N_t(x)$  denotes the density of individuals of size  $x$  in year  $t$ , then Salguero-Gómez et al. [2012] used an IPM of the form

$$N_{t+1}(x) = \int_a^b [s(y)G(y, x) + e(x)p(y)f(y)N_t(y)] dy$$

where  $s(y)$  is the probability of surviving to the next year for individuals of size  $y$ ,  $G(y, x) dy$  is the infinitesimal probability that a surviving individual of size  $y$  is size  $x$  in the next year,  $p(y)$  is the probability that an individual of size  $y$  flowers,  $f(y)$  is the mean number of offspring produced by



an individual of size  $y$ , and  $e(x)$  is the infinitesimal probability that an offspring is of size  $x$  at the time of the annual census.

Following Salguero-Gómez et al. [2012], we use generalized linear models (GLMs) for most of the functional forms of the demographic kernels. Computations were performed using the base GLM function in R [R Core Team, 2015]. We used logistic regression (i.e. a GLM with the binomial family) for determining  $s(y)$  and  $p(y)$ , and a GLM with the Poisson family for modelling  $f(y)$ . For the growth kernel, linear regression determined the expected size of an individual in the next year and the actual size was assumed to be normally distributed about this mean. The variance of this normal distribution, for simplicity, was assumed to be independent of the current size of an individual. We modeled  $e(y)$  with a gamma distribution fit to the empirical distribution of germinants. Figure 1 shows the data and fits for  $s(y)$ ,  $G(y, x)$ ,  $p(x)$  and  $f(x)$ .

The kernels  $s(x)$ ,  $G(y, x)$ ,  $p(x)$  and  $e(x)$  provide us with all the information that the individual-based IPM requires for probabilistic updating individuals for survival, growth, flowering, reproduction, and size of germinating individuals at first census. The fecundity kernel  $f(y)$ , however, only specifies the mean number of offspring produced, but for an individual-based IPM, we need the distribution of the number of offspring produced by an individual. Fortunately, this information is built into the structure of the GLMs due to the assumptions in our model choice. As the fecundity data was modeled using a Poisson family for the GLM, the mean number  $f(x)$  of offspring also specifies the distribution. More generally, one might use multi-parameter distributions such as a zero-inflated Poisson or a negative binomial, in which case parameters in addition to the mean are needed to specify the distribution of offspring number.

**Deriving the pgf  $\Psi$ .** To define  $\Psi$ , we observe that the contributions of an individual of size  $x$  to the population in the next time step involves the sum of two independent random variables: the contribution due to survival and growth and the contribution due to reproduction. We will identify

two pgfs,  $\Psi_g$  and  $\Psi_f$ , for each of these processes separately. Then, we make use of a fundamental property of pgfs

**Fundamental Property 1:** The pgf for a sum of independent random variables is the product of the pgfs of these random variables.

to get that

$$\Psi = \Psi_g \times \Psi_f.$$

To write down each of these pgfs, we make use of another fundamental property of pgfs:

**Fundamental Property 2:** The pgf for a sum of  $N$  independent, identically distributed random variables  $X_i$  is the composition of the pgf for  $N$  with the pgf for the  $X_i$ .

For survival and growth,  $\Psi_g(h)(x)$  corresponds to integrating  $h^{\mathbf{s}}$  over all possible contributions  $\mathbf{s}$  from an individual of size  $x$  surviving and growing. These contributions are of two types:  $\mathbf{s} = 0$  when the individual dies, and  $\mathbf{s} = (1; y)$  when the individual survives and grows to size  $y$ . The first event occurs with probability  $1 - s(x)$  and the infinitesimal probability of the second event is  $s(x)G(y, x)dy$ . As  $h^{\mathbf{s}} = 1$  when  $\mathbf{s} = 0$  and  $h^{\mathbf{s}} = h(y)$  when  $\mathbf{s} = (1; y)$ , integrating over all possible contributions due to survival and growth yields

$$\Psi_g(h)(x) = (1 - s(x)) \times 1 + s(x) \int h(y)G(y, x)dy.$$

For fecundity,  $\Psi_f(h)(x)$  is given by integrating  $h^{\mathbf{s}}$  over all possible states  $\mathbf{s}$  corresponding to the offspring produced by an individual of size  $x$ . To write this down, we begin by conditioning on the event that an individual of size  $x$  flowers. On this event, the individual produces a Poisson number  $N$  of offspring with mean  $f(x)$ . The pgf for  $N$  is given by  $\phi(x, \xi) = \exp(-f(x)(\xi - 1))$  where  $\xi$  is a dummy variable. The size of each of these offspring is drawn interdependently from the common offspring distribution  $e(y)dy$ . Hence, the contribution of a flowering individual of size  $x$  is the sum of  $N$  independent random variables with distribution  $e(y)dy$ . By **Fundamental Property 2** of pgfs, we need to take the composition of the pgf  $\phi$  for  $N$  with the pgf  $\Psi_e$  for a single offspring,

212 namely

$$\Psi_e(h)(x) = \int h(y)e(y)dy.$$

213 Thus, we get the pgf associated with a flowering individual is the following composition of pgfs

$$\Psi_{flowering}(h)(x) = \phi(x, \int h(y)e(y)dy)$$

214 To get the pgf for flowering and non-flowering contributions to fecundity, we observe that fecundity  
 215 contributions of an individual of size  $x$  is given by the sum of a Bernoulli number of flowering  
 216 individuals where the probability of success is  $p(x)$ . By the **Fundamental Property 2**, we need  
 217 to compose the pgf of a Bernoulli, which is  $\theta(\xi) = 1 - p(x) + p(x)\xi$  where  $\xi$  is the dummy variable,  
 218 with the pgf  $\Psi_{flowering}$  of a flowering individual. This composition yields

$$\Psi_f(h)(x) = \theta(\Psi_{flowering}(h)(x)) = 1 - p(x) + p(x)\phi(x, \int h(y)e(y)dy).$$

219 Using **Fundamental Property 1**, we multiply  $\Psi_g$  and  $\Psi_f$  to get  $\Psi$ , the pgf for the individual-  
 220 based IPM:

$$\Psi(h)(x) = \left( (1 - s(x)) \times 1 + s(x) \int h(y)G(y, x)dy \right) \left( 1 - p(x) + p(x)\phi(x, \int h(y)e(y)dy) \right).$$

221 **Numerically implementing and using the pgf  $\Psi$ .** To approximate  $\Psi$  numerically, we discretize  
 222 the size interval  $[0, 2b]$  using  $n = 500$  equal sized intervals of width  $\Delta x = 2b/n$ . We used the larger  
 223 size interval of  $[0, 2b]$ , where  $b$  is the size of the largest observed individual, to minimize the effects  
 224 of eviction (see below). We created a vector  $\vec{x}$  corresponding to the midpoints of these intervals.  
 225 Using this vector we discretized the survival function as a vector  $\vec{s} = s(\vec{x})$ , the growth kernel as  
 226 a matrix using the outer product  $G$  of the growth kernel  $g$  with  $\vec{x}$ , the probability of flowering  
 227 function as a vector  $\vec{p} = p(\vec{x})$ , the fecundity function as a vector  $\vec{f} = f(\vec{x})$ , and the offspring size  
 228 distribution as a vector  $\vec{e} = e(\vec{x})$ .

For the previously described methods to work, it is critical that column sums for  $G$  and the sum of  $\vec{e}$  equal one. For most IPMs, this will not occur automatically due to individuals being evicted from the size interval  $[0, 2b]$ . There are a variety of ways to handle this issue [Williams et al., 2012]. As the offspring size vector nearly summed to one, we simply re-normalized it so that it summed to one. For the growth matrix  $G$ , we treated eviction as mortality. To do this, we took one minus the column sums of  $G$ , subtracted these sums from the survival vector, and then normalized the column sums of  $G$  so that they added to one. When taking the product of survival and growth, the resulting mean-field IPM is unaffected by these changes.

Using these discretized demographic components and the pgf  $\phi$  for fecundity, we get the discretized pgf  $\Psi$ , given by

$$\Psi_{discrete}(\vec{h}) = (1 - \vec{s} + \vec{s} \circ (G^T \vec{h})) \circ (1 - \vec{p} + \vec{p} \circ \phi(\vec{x}, \vec{e}^T \vec{h}))$$

where  $\vec{h}$  corresponds to a discretized function i.e. a vector of length  $n$ ,  $^T$  denotes the transpose of a matrix or vector, and  $\circ$  denotes element by element multiplication.

Iterating  $\Psi$  yields how extinction probabilities  $E_t(x)$  vary with time for a population initiated with a single individual (Figure 2). Intuitively, this figure illustrates that the probability of extinction decreases with the size of founding individual, and that extinction probabilities increase over time. Furthermore,  $E_t(x)$  as  $t$  increases are approaching limiting extinction probabilities  $E_\infty(x)$  which always equals one (Figure 2A). This stems from the fact that the dominant eigenvalue of the mean-field IPM is less than one, consistent with the results of Salguero-Gómez et al. [2012]. By increasing seed survivorship by a factor of three, the dominant eigenvalue of the mean-field IPM becomes greater than one and the asymptotic extinction probabilities become less than one (Figure 2B). We approximated these asymptotic extinction probabilities by iterating  $\Psi$  until the difference between  $E_t$  and  $E_{t+1}$  were below a specified error tolerance.

[Figure 2 about here.]

252 To scale things up to an entire population initially in state  $\mathbf{s}_0$ , recall that the extinction probability  
 253 at time  $t$  is  $E_t^{\mathbf{s}_0}$ . Figure 3 illustrates how the extinction probabilities over time vary for a population  
 254 with initially 100 individuals of the smallest size  $x = 1$  (i.e.  $\mathbf{s}_0 = (1; 100)$ ) versus a population with  
 255 5 or 8 individuals of the largest size  $x = 60$  typically observed in the field (i.e.,  $\mathbf{s}_0 = (60; 5)$  or  
 256  $(60; 8)$ ). Figure 3 suggests that, from the extinction risk perspective, about 6 or 7 larger individuals  
 257 are equivalent to 100 of the smallest individuals. These types of comparisons may be particularly  
 258 useful when trying to assess whether planting small or larger individuals are more effective for  
 259 establishment success.

260 [Figure 3 about here.]

261 We also used the data to see how extinction risk over different time frames depends on the size  
 262 and composition of the population. Specifically, for different founding population abundances  $N$ ,  
 263 we randomly sampled  $N$  individuals from the data and computed extinction risk of this sampled  
 264 population over 5 and 10 year periods (Figure 4). For each founding population abundance  $N$ , we  
 265 considered 500 random samples of size  $N$ . Figure 4A illustrates that, on average, log extinction risk  
 266 decreases with the founding population size  $N$  and is greater for the 10 year period than the 5 year  
 267 period. For smaller founding population sizes  $N$ , there is substantial overlap in the distributions of  
 268 extinction times for the 5 and 10 year time frames. This overlap stems from founding populations  
 269 of mostly large individuals being more likely to persist at least 10 years than founding populations  
 270 of mostly small individuals persisting at least 5 years. Consistent with this explanation, Figure 4B  
 271 illustrates that mean size of an individual within a founding population has a strong negative  
 272 correlation with (log) extinction probability, and the slope of this correlation is steeper over shorter  
 273 time frames than longer time frames.

274 [Figure 4 about here.]

## RECOMMENDATIONS, EXTENSIONS, AND FUTURE CHALLENGES

To implement the methods presented here, there are two main steps. First, one needs to identify the main demographic processes of the population, the order in which these processes occur relative to the censuses used for data collection, and develop the statistical models for the each of the demographic processes. Rees et al. [2014] and Merow et al. [2014] provide excellent reviews on the philosophical and methodological issues associated with this step. Typically, whenever a study has sufficient data for constructing the mean-field IPM, there is no need to collect any additional data to build the individual-based IPM. Unlike the mean-field IPMs, though, the individual-based IPMs make use of the complete distributional information associated with fecundity. As with other areas of stochastic demography, the shapes, not just the means, of these distributions may have significant effects on the likelihood of extinction or establishment success [Lloyd-Smith et al., 2005]. Hence, it is best to examine several options (e.g. Poisson versus negative Binomial versus zero-inflated distributions) to identify which distribution does a better job of describing the fecundity data. As in all areas of modeling, if there is significant uncertainty about the “best” choice of the model, one should perform the analyses with each of the alternative fecundity distributions to identify the sensitivity of predictions of extinction risk to these alternatives.

The second step, the focus of this paper, involves constructing the probability-generating functional  $\Psi$ . For the uninitiated, this step may be intimidating. However, there are a three basic principles that simplify this construction. First, while this pgf  $\Psi$  takes functions to functions, one should focus on writing down  $\Psi(h)(x)$ , which involves understanding the contributions of a single individual of size  $x$  to the next census. Second, one can often break up these contributions into a sum of independent contributions, find the pgfs associated with these simpler contributions, and then use the two fundamental properties of pgfs to “stitch” together  $\Psi$ . Third, the distributions used to describe the number of offspring produced by an individual typically involve random

299 variables (e.g. Poisson, negative binomial) for which the associated pgfs are well-known. Finally,  
 300 whenever in doubt, find a collaborator that you trust to help put the pieces together correctly.

301 For the individual-based models considered here, we assumed the environment remains constant  
 302 over time. However, IPMs have and continue to be used to study the effects of fluctuating en-  
 303 vironmental conditions on population demography and life history evolution [Childs et al., 2004,  
 304 Dahlgren and Ehrlén, 2011, Rees and Ellner, 2009]. The methods described here easily extend to  
 305 fluctuating environments. Specifically, if  $\Psi_t$  is the pgf of the individual-based IPM associated with  
 306 year  $t$ , then the probability of extinction by year  $t + 1$  for a population initially in state  $\mathbf{s}$  is

$$\Psi_0(\Psi_1(\dots \Psi_t(h_0)))^{\mathbf{s}}$$

307 where  $h_0$ , as before, is the zero function (see Appendix B for justification). Note that the com-  
 308 position here is in the reverse order of what one does when iterating the mean-field IPM model  
 309 forward in time. While we know of no formal proof, in the case of a stationary and ergodic envi-  
 310 ronment with suitable technical assumptions, we conjecture the following limit theorem holds: the  
 311 asymptotic extinction probabilities are strictly less than one if and only if the stochastic growth  
 312 rate (aka dominant Lyapunov exponent) of the mean-field IPM is positive. For multi-type branch-  
 313 ing processes, Athreya and Karlin [1971, Theorem 12] proved this result under suitable technical  
 314 hypotheses.

315 Despite this relatively straightforward extension to temporally variable environments, many chal-  
 316 lenges remain. From the computational perspective, finding the efficient methods to deal with  
 317 multi-dimensional states variables (e.g. size and location, or multi-dimensional traits) continues to  
 318 be a challenge, as it is for mean-field IPMs. While the analytical methods presented here cover  
 319 multi-dimensional state variables, their numerical implementation involves approximating multi-  
 320 dimensional integrals which can be computationally expensive. From the analytical perspective,  
 321 accounting for temporal correlations in individual growth or reproductive rates (e.g. individuals

that grew larger than expected in one year being more likely to grow larger than expected in the next year) or correlations among individuals are particularly important challenges as strong correlations likely have large effects on extinction risk. Finally, and perhaps most importantly, the manners in which size-structured demography may shape extinction risk for real-world population remains to be understood. One might hope that by applying these methods to the many data sets for which IPMs have been developed, as well as future data sets, might provide a computationally efficient means to explore how size-structure for populations differing in their evolutionary history and environmental context influences their vulnerability to extinction.

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## APPENDIX A: THE PGF FOR MULTI-TYPE BRANCHING PROCESSES

To see how this definition of  $\Psi$  recovers the pgf for the classical multi-type branching processes (see, for example, Caswell [2001]), assume there are  $k$  individual states with  $\mathcal{S} = \{1, 2, \dots, k\}$ . In this special case, functions  $h : \mathcal{S} \rightarrow \mathbb{R}$  can be viewed as vectors:  $h = (h(1), h(2), \dots, h(k))$ . The probability update rules,  $m(i, d\mathbf{s})$ , are determined by the probabilities  $p(i; n_1, n_2, \dots, n_k)$  that an individual in state  $i$  is “replaced” in the next time step by  $n_1$  individuals of type 1,  $n_2$  individuals of type 2,  $\dots$ ,  $n_k$  individuals of type  $k$ . Then the pgf  $\Psi$  is defined by

$$\begin{aligned} \Psi(h)(i) &= \int h^{\mathbf{s}} m(x, d\mathbf{s}) \\ &= \sum_{n_1, \dots, n_k} \prod_{i=1}^k h(i)^{n_i} p(i; n_1, n_2, \dots, n_k). \end{aligned}$$

where the sum is taken over all  $k$ -tuples  $n_1, n_2, \dots, n_k$  of non-negative integers.

## APPENDIX B: EXTINCTION PROBABILITIES IN A TIME VARYING ENVIRONMENT

Let  $m_0(x, d\mathbf{s}), m_1(x, d\mathbf{s}), m_2(x, d\mathbf{s}), \dots$  be a sequence of probability measures defining the population update rules at times  $0, 1, 2, \dots$  i.e.  $\mathbb{P}[\mathbf{s}_{t+1} \in A | \mathbf{s}_t = (1; x)] = \int_A m_t(x, d\mathbf{s})$  for any  $A \subset \mathcal{S}$ . For each of these probability measures, define the pgf

$$\Psi_t(h)(x) = \int h^{\mathbf{s}} m_t(x, d\mathbf{s})$$

for any continuous function  $h : X \rightarrow \mathbb{R}$ . We claim that for  $t \geq 0$  and  $\tau \geq 0$

$$(5) \quad \Psi_t(\Psi_{t+1}(\dots \Psi_{t+\tau}(h_0)))(x)$$

is the probability of extinction by time  $t + \tau + 1$  for a population consisting of one individual in state  $x$  at time  $t$ . As before, we argue by induction on  $\tau$ . For  $\tau = 0$  and any  $t \geq 0$ , the definition of  $\Psi_t$  implies

$$(6) \quad \Psi_t(h_0)(x) = \int_{\{0\}} 1 \times m_t(x, d\mathbf{s}) + \int_{\mathcal{S} \setminus \{0\}} 0 \times m_t(x, d\mathbf{s}) = \mathbb{P}[\mathbf{s}_{t+1} = 0 | \mathbf{s}_t = (1; x)].$$

Now suppose that equation (5) holds for some  $\tau \geq 0$  and all  $t$ ; we will show it holds  $\tau + 1$  and all  $t$ . Fix  $t \geq 0$  and assume  $\mathbf{s}_t = (1; x)$ . On the event that  $\mathbf{s}_{t+1} = (n_1, \dots, n_k; x_1, \dots, x_k)$  is the population state at time  $t + 1$ , extinction occurs by time  $t + \tau + 2$  only if each of the lineages of the  $n_1 + \dots + n_k$  individuals go extinct in the next  $\tau + 1$  time steps. As the fates of these lineages are independent of one another, it follows that

$$\begin{aligned} \mathbb{P}[\mathbf{s}_{t+\tau+2} = 0 | \mathbf{s}_t = (1; x), \mathbf{s}_{t+1} = (n_1, \dots, n_k; x_1, \dots, x_k)] &= \prod_{i=1}^k \mathbb{P}[\mathbf{s}_{t+\tau+2} = 0 | \mathbf{s}_{t+1} = (1; x_i)]^{n_i} \\ &= \prod_{i=1}^k (\Psi_{t+1}(\Psi_{t+2}(\dots \Psi_{t+1+\tau}(h_0)))(x_i))^{n_i} \\ &= \Psi_{t+1}(\Psi_{t+2}(\dots \Psi_{t+\tau+1}(h_0)))^{(n_1, \dots, n_k; x_1, \dots, x_k)} \end{aligned}$$

423 where the second equality follows from our inductive hypothesis that (5) holds for  $\tau$  and any  $t$ . By  
 424 the law of total probability and conditioning over all possible states at time  $t + 1$ , we get

$$\begin{aligned}\mathbb{P}[s_{t+\tau+2} = 0 | \mathbf{s}_t = (1; x)] &= \int \Psi_{t+1}(\Psi_{t+2}(\dots \Psi_{t+\tau+1}(h_0)))^{\mathbf{s}} m_t(x, d\mathbf{s}) \\ &= \Psi_t(\Psi_{t+1}(\dots \Psi_{t+\tau+1}(h_0)))(x).\end{aligned}$$

425 As  $t \geq 0$  was arbitrary, this completes the inductive proof.

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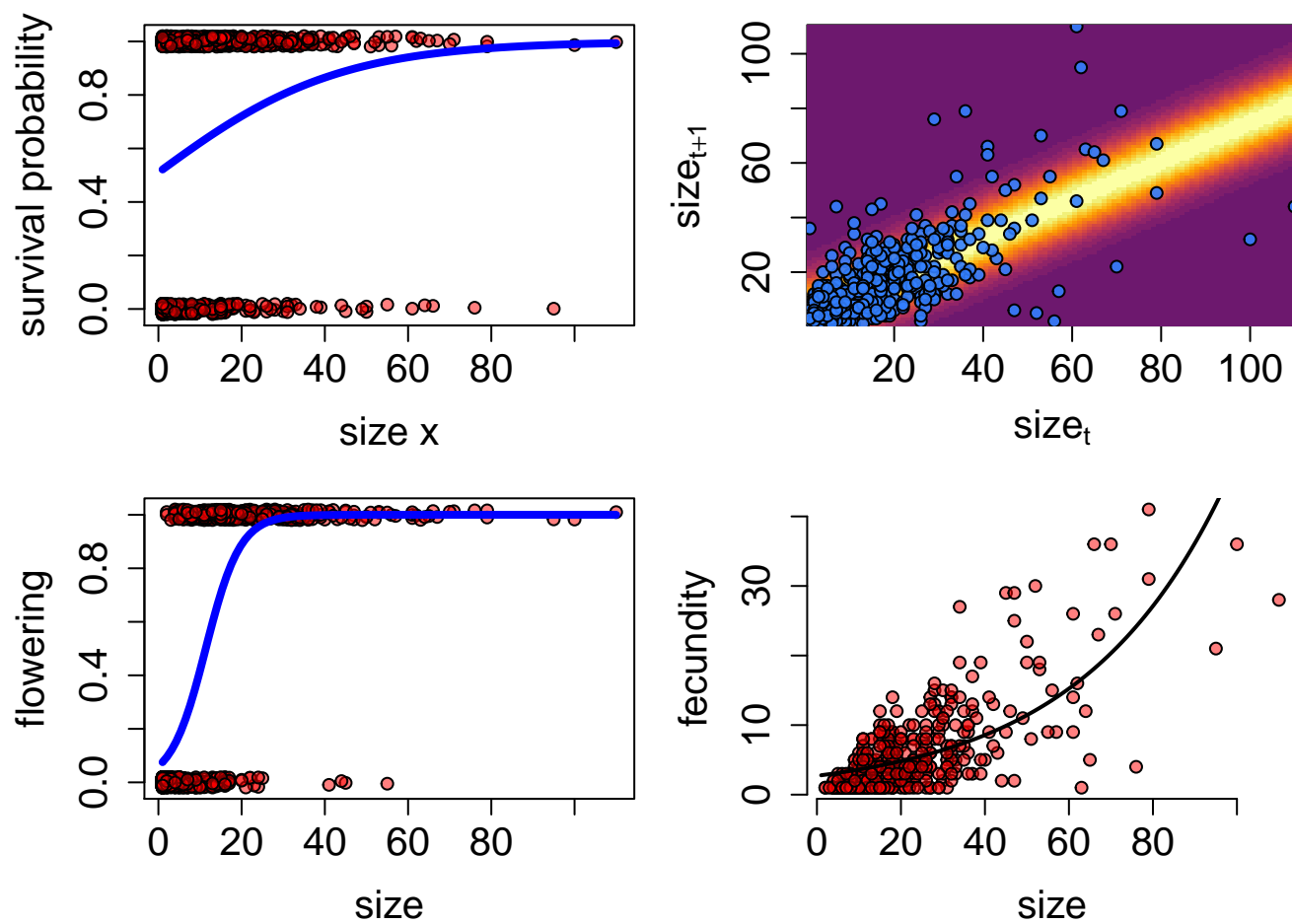


FIGURE 1. The demographic kernels for Plateau Yellow Miner's Candle with the corresponding data from Salguero et al. 2012.

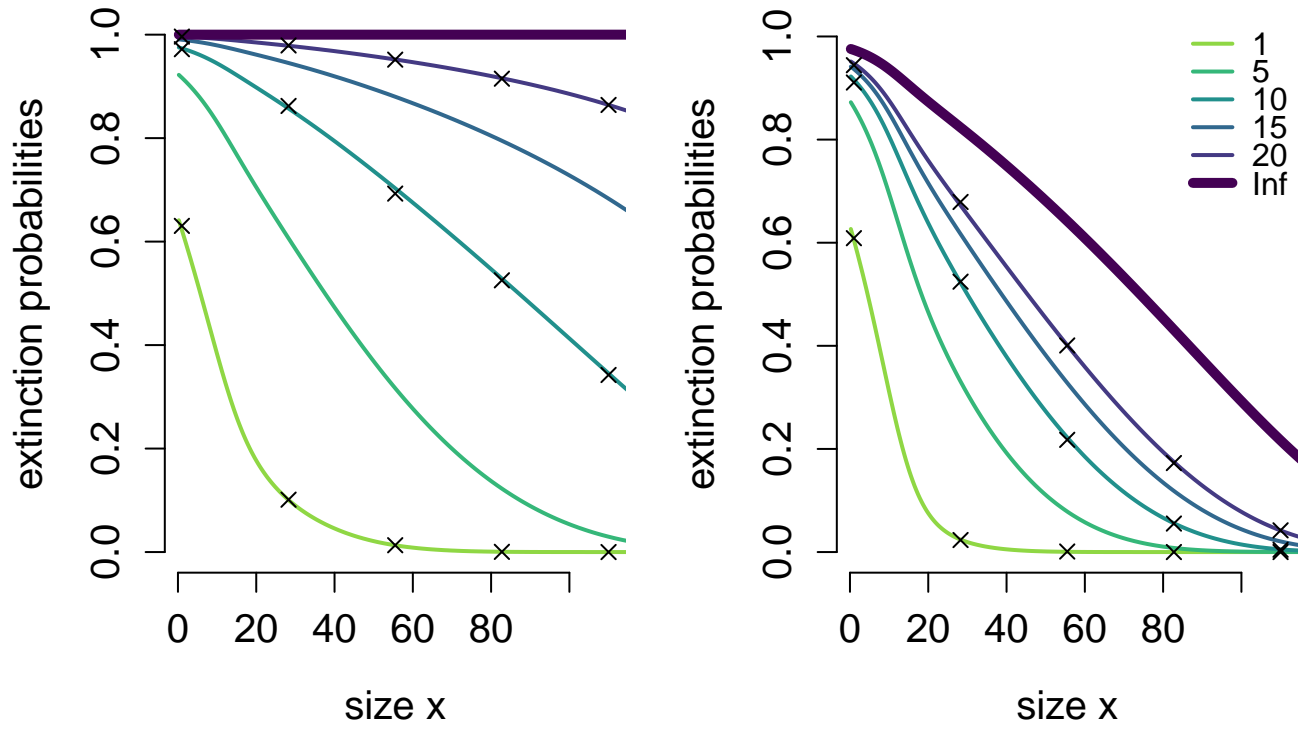


FIGURE 2. Extinction probabilities for a population initially consisting of a single individual of size  $x$ . Different curves correspond to extinction occurring in 1, 5, 10, 15 or 20 years. Asymptotic extinction probabilities are shown by the thicker curve. In A, the extinction curves for the baseline individual-based IPM. In B, extinction curves for the case when seed survival is increased by a factor of three. Crosses indicate estimates of extinction probabilities from 10,000 simulations of a full individual-based model (see code in [Schreiber and Ross, 2015]).



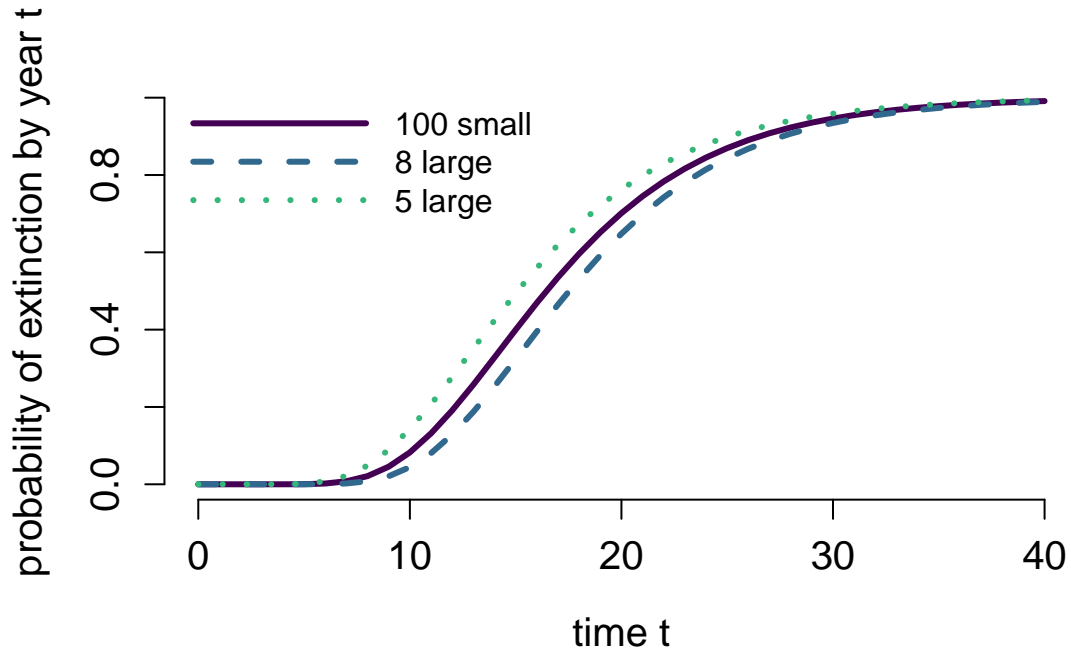


FIGURE 3. Extinction probabilities as a function of time for populations with 100 individuals of size  $x = 1$ , and populations with 5 or 8 individuals of size  $x = 60$ .

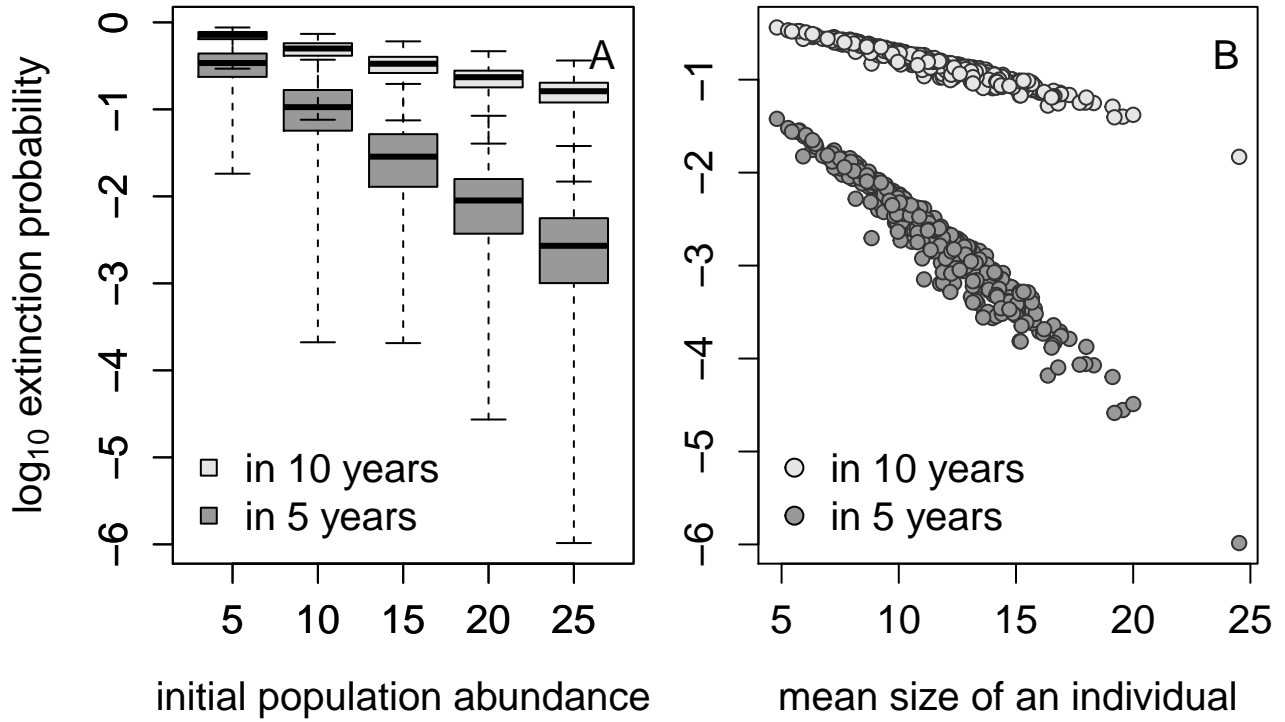


FIGURE 4. Extinction probabilities for founding populations of different abundance. For each founding population abundance  $N$ , 500 samples consisting of  $N$  randomly chosen individuals from the data set were used to create a founding population of  $N$  individuals. Extinction probabilities by year 5 and year 10 were calculated for each of these sample populations. In A, extinction probability is plotted as a box plot for different  $N$  values. In B, extinction probability is plotted against the mean size of an individual for a population abundance  $N = 25$ .