MATH 470 final paper *

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1 Introduction

In this paper, we give a new, categorical definition of a σ-algebra. We define measures and integration on those σ-categories, and we extend measures and integrals to pre-σ-categories, a more general setting than σ-algebras.

In section 2, we introduce the notion of a categorical complement based on the universal property of set complements. In section 3, we define σ-categories and show that every small σ-category has a representation as a σ-algebra of sets. The standard definition of a σ-algebra starts with a power set and pares it down to only include chosen measurable sets. Similarly, we discuss how σ-categories are subcategories of pre-σ-categories; however, pre-σ-categories are quite unlike power sets, making this a qualitatively different definition. In section 4, we define analogs of measures, measurable functions, and integrals on σ-categories. We then extend these notions to pre-σ-categories and calculate a very simple integral on a pre-σ-category that is not a σ-algebra. We conclude in section 5 with a list of further questions and directions for exploration.

2 Complemented Categories

2.1 Precomplemented Categories

Given a subset A of a set X, its complement A^{c} in X satisfies the following universal property: a set B is a subset of A^{c} if and only if A and B are disjoint, or in other words, if and only if there exists a function $A \cap B \to \emptyset$. We take this universal property as inspiration for the following categorical definition of a complement. As we'll see, this definition is much broader, and

*Thank you to Professor Pablo Boixeda Alvarez for supervising my independent study in category theory!

it will take several additional assumptions until we recover all the familiar behaviors of complements in section 3.2.

Definition 2.1 (precomplemented category, complement)**.** Let C be a category with a symmetric bifunctor $- \cap - : C \times C \to C$ and an object $\emptyset$. The triple $(C, \cap, \emptyset)$ is *precomplemented* if for each object $B \in C$, there exists a *complement* object B^{c} representing the functor $C(- \cap B, \emptyset)$. This means there is an isomorphism

$$\rho_{A,B} : C(A, B^{\mathsf{c}}) \cong C(A \cap B, \emptyset)$$

for all objects $A, B \in C$, and for a fixed B, the collection $(\rho_{A,B})_{A \in C}$ is natural in A.

Since B^{c} is an object representing a functor, it is unique (up to a canonical isomorphism).

Example 2.2. In a closed, symmetric monoidal category C, the tensor product $\otimes$ has an adjoint, the internal hom functor. Then $(C, \otimes, \emptyset)$ is complementary for any object $\emptyset$. The complement of an object $B \in C$ is $\hom(B, \emptyset)$, since for all objects $A \in C$, we have the natural isomorphism

$$C(A \otimes B, \emptyset) \cong C(A, \hom(B, \emptyset)).$$

Example 2.3. Let C be the following preorder with $\emptyset < A < X$:

$$\begin{tikzcd} X \\ A \arrow[u] \\ \emptyset \arrow[u] \end{tikzcd}$$

$(C, \inf, \emptyset)$ is a precomplemented category: For any object $B \in C$, we have natural isomorphisms

$$\begin{aligned} C(\inf\{B, \emptyset\}, \emptyset) &\cong C(B, X) \\ C(\inf\{B, X\}, \emptyset) &\cong C(B, \emptyset) \\ C(\inf\{B, A\}, \emptyset) &\cong C(B, \emptyset). \end{aligned}$$

So $\emptyset^{\mathsf{c}} \cong X$, and $X^{\mathsf{c}} \cong A^{\mathsf{c}} \cong \emptyset$. Note that taking a complement is not an involution in this category, since $(A^{\mathsf{c}})^{\mathsf{c}} \cong X \ncong A$.

A precomplemented category C comes with an assignment $(-)^{\mathsf{c}} : A \mapsto A^{\mathsf{c}}$ on its objects. We can extend this to a functor $(-)^{\mathsf{c}} : C^{\mathrm{op}} \to C$ in a canonical way using the Yoneda lemma.

Lemma 2.4. *In a precomplemented category $(C, \cap, \emptyset)$, there is a unique way to extend the assignment $(-)^{\mathsf{c}} : B \mapsto B^{\mathsf{c}}$ of objects of C to a functor $(-)^{\mathsf{c}} : C^{\mathrm{op}} \to C$ such that the collection $\rho = (\rho_{A,B})_{A,B \in C}$ is natural in both A and B. We call $(-)^{\mathsf{c}}$ the "complement functor".*

Proof. Given a morphism $f : B \to B'$, we will construct a natural transformation between the represented functors $C(-, B'^{\mathsf{c}}) \Rightarrow C(-, B^{\mathsf{c}})$. Then, we will apply the Yoneda lemma to obtain the desired morphism $f^{\mathsf{c}} : B'^{\mathsf{c}} \to B^{\mathsf{c}}$.

First, bifunctoriality of $\cap$ gives a natural transformation $1_{-} \cap f : - \cap B \Rightarrow - \cap B'$. Whiskering that natural transformation with the functor $C(-, \emptyset)$ returns a natural transformation $C(- \cap B', \emptyset) \Rightarrow C(- \cap B, \emptyset)$. By definition, B'^{c} and B^{c} represent those functors, yielding a natural transformation

$$C(-, B'^{\mathsf{c}}) \cong C(- \cap B', \emptyset) \;\Rightarrow\; C(- \cap B, \emptyset) \cong C(-, B^{\mathsf{c}}).$$

Explicitly, its component at A is $\rho^{-1}_{A,B} \circ (1_A \cap f)^* \circ \rho_{A,B'}$.

By the Yoneda lemma, there exists a unique morphism $f^{\mathsf{c}} : B'^{\mathsf{c}} \to B^{\mathsf{c}}$ such that this natural transformation equals f^{c}_*. Therefore, the component of f^{c}_* at A makes the following diagram commute:

$$\begin{tikzcd}
C(A, B'^{\mathsf{c}}) \arrow[r, "\sim", "\rho_{A,B'}"'] \arrow[d, "f^{\mathsf{c}}_*"'] & C(A \cap B', \emptyset) \arrow[d, "(1_A \cap f)^*"] \\
C(A, B^{\mathsf{c}}) & C(A \cap B, \emptyset) \arrow[l, "\sim"', "\rho^{-1}_{A,B}"]
\end{tikzcd}$$

This is the naturality square for ρ in the variable B. ρ is also natural in A by definition. Hence there exists a unique assignment of morphisms $f \mapsto f^{\mathsf{c}}$ making ρ natural in both variables. Finally, note that the assignment $f \mapsto f^{\mathsf{c}}$ is functorial: The steps we took to construct the natural transformation $C(-, B'^{\mathsf{c}}) \Rightarrow C(-, B^{\mathsf{c}})$ were functorial, as is the Yoneda embedding. $\square$

Lemma 2.5. *The functor $(-)^{\mathsf{c}}$ in a precomplemented category $(C, \cap, \emptyset)$ is its own mutual right adjoint. In other words, we have bijections*

$$\rho^{-1}_{B,A} \circ \rho_{A,B} : C(A, B^{\mathsf{c}}) \cong C(B, A^{\mathsf{c}})$$

that are natural in A and B. The adjunction's unit is $\eta : 1_C \Rightarrow ((-)^{\mathsf{c}})^{\mathsf{c}}$ with $\eta_A = (\rho^{-1}_{A,A^{\mathsf{c}}} \circ \rho_{A^{\mathsf{c}},A})(1_{A^{\mathsf{c}}})$. Its counit is $\epsilon = \eta^{\mathrm{op}}$, the image of η in C^{op}.

Proof. The left hand square in the diagram below commutes by the naturality of ρ in its first variable, and the right hand square commutes by the naturality of ρ in its second variable. The fact that $\cap$ is symmetric, so $A \cap B \cong B \cap A$, is what allows us to put the two squares together:

$$
\begin{tikzcd}
C(A, B^{\mathsf{c}}) \arrow[r, "\sim", "\rho_{A,B}"'] \arrow[d, "f^*"'] & C(A \cap B, \emptyset) \arrow[r, "\sim", "\rho^{-1}_{B,A}"'] \arrow[d, "(f \cap 1_B)^*"'] & C(B, A^{\mathsf{c}}) \arrow[d, "f^{\mathsf{c}}_*"'] \\
C(A', B^{\mathsf{c}}) \arrow[r, "\sim", "\rho_{A',B}"'] & C(A' \cap B, \emptyset) \arrow[r, "\sim", "\rho^{-1}_{B,A'}"'] & C(B, A'^{\mathsf{c}})
\end{tikzcd}
$$

The commuting rectangle shows naturality of the bijection $C(A, B^{\mathsf{c}}) \cong C(B, A^{\mathsf{c}})$ in its "A" slot. A similar argument using ρ's naturality in both its variables shows the bijection $C(A, B^{\mathsf{c}}) \cong C(B, A^{\mathsf{c}})$ to be natural in its "B" slot. Hence these bijections form an adjunction

$$(-)^{\mathsf{c}} \,:\, C^{\mathrm{op}} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} C \,:\, (-)^{\mathsf{c}}.$$

By definition, the component of the unit at A is the image of $1_{A^{\mathsf{c}}}$ under the adjunction, $\eta_A = (\rho^{-1}_{A,A^{\mathsf{c}}} \circ \rho_{A^{\mathsf{c}},A})(1_{A^{\mathsf{c}}})$.

When discussing the counit, it will be helpful to distinguish between C and C^{op}. To that end, we temporarily adopt the following notation:

$$
\begin{aligned}
D &:= C^{\mathrm{op}} \\
(-)^{\mathsf{c}} &: D \to C \\
(-)^{\mathsf{d}} &:= ((-)^{\mathsf{c}})^{\mathrm{op}} : C \to D.
\end{aligned}
$$

In this notation, the adjunction becomes $C(A, B^{\mathsf{c}}) \cong D(A^{\mathsf{d}}, B)$. Then the components of the unit and counit are

$$
\begin{aligned}
C(A^{\mathsf{c}}, A^{\mathsf{c}}) &\cong D(A, (A^{\mathsf{c}})^{\mathsf{d}}) \\
1_{A^{\mathsf{c}}} &\mapsto \eta_A,
\end{aligned}
$$

$$
\begin{aligned}
D(A^{\mathsf{d}}, A^{\mathsf{d}}) &\cong C((A^{\mathsf{d}})^{\mathsf{c}}, A) \\
1_{A^{\mathsf{d}}} &\mapsto \epsilon_A.
\end{aligned}
$$

By definition, $1_{A^{\mathsf{d}}} = (1_A)^{\mathsf{d}} = ((1_A)^{\mathsf{c}})^{\mathrm{op}} = (1_{A^{\mathsf{c}}})^{\mathrm{op}}$ is the image of $1_{A^{\mathsf{c}}}$ in $D = C^{\mathrm{op}}$. This relationship holds after applying the adjunction; ϵ_A is the image of η_A in C^{op}. $\square$

2.2 Complemented Categories

As seen in example 2.3, the unit η in a precomplemented category is not necessarily an isomorphism. When η is an isomorphism, $(-)^{\mathsf{c}}$ becomes an involution, $(A^{\mathsf{c}})^{\mathsf{c}} \cong A$, as the notation suggests. We also recover some other familiar properties of set-theoretic complements.

Definition 2.6 (complemented category)**.** A precomplemented category $(C, \cap, \emptyset)$ is *complemented* if its unit $\eta : 1_C \Rightarrow ((-)^\mathsf{c})^\mathsf{c}$ is a natural isomorphism.

Definition 2.7 ($\cup$, X)**.** Let $(C, \cap, \emptyset)$ be a precomplemented category. We define $X = \emptyset^\mathsf{c}$, and we define $\cup : C \times C \to C$ to be the symmetric bifunctor

$$- \cup - := ((-)^\mathsf{c} \cap (-)^\mathsf{c})^\mathsf{c}.$$

Proposition 2.8. *In a complemented category $(C, \cap, \emptyset)$:*

1. *$(-)^\mathsf{c} : C^{\mathrm{op}} \to C$ is an duality, i.e. a self-adjoint equivalence $C^{\mathrm{op}} \xrightarrow{\sim} C$.*
2. *$\emptyset^\mathsf{c} \cong X$ and $X^\mathsf{c} \cong \emptyset$.*
3. *There are "De Morgan" isomorphisms $(- \cap -)^\mathsf{c} \cong (-)^\mathsf{c} \cup (-)^\mathsf{c}$ and $(- \cup -)^\mathsf{c} \cong (-)^\mathsf{c} \cap (-)^\mathsf{c}$ which are natural in both variables.*
4. *$(C^{\mathrm{op}}, \cup, X)$ is a complemented category, with B^c representing the functor $C^{\mathrm{op}}(- \cup B, X) = C(X, - \cup B)$. (This is in addition to B^c representing the functor $C(- \cap B, \emptyset)$).*

Proof. **1.** When η is an isomorphism, so is $\epsilon = \eta^{\mathrm{op}}$. Any pair of adjoint functors whose unit and counit are isomorphisms is an equivalence of categories. We already know $(-)^\mathsf{c} : C^{\mathrm{op}} \to C$ is self-adjoint, so it is a duality.

2. $X = \emptyset^\mathsf{c}$ by definition, and we have $\eta_\emptyset : \emptyset \cong (\emptyset^\mathsf{c})^\mathsf{c} = X^\mathsf{c}$.

3. Whiskering preserves natural isomorphisms. The first natural isomorphism is $(\eta_A \cap \eta_B)^\mathsf{c} : (- \cap -)^\mathsf{c} \cong (-)^\mathsf{c} \cup (-)^\mathsf{c}$. The second natural isomorphism is $\eta_{A^\mathsf{c} \cap B^\mathsf{c}} : (-)^\mathsf{c} \cap (-)^\mathsf{c} \cong (- \cup -)^\mathsf{c}$.

4. For all objects $A, B^\mathsf{c} \in C^{\mathrm{op}}$, we have

$$\begin{aligned}
C^{\mathrm{op}}(A, B^\mathsf{c}) &\cong C(B^\mathsf{c}, A) & \\
&\cong C(B^\mathsf{c}, (A^\mathsf{c})^\mathsf{c}) & \text{since } \eta_A : A \cong (A^\mathsf{c})^\mathsf{c} \\
&\cong C(B^\mathsf{c} \cap A^\mathsf{c}, \emptyset) & \text{via } \rho_{B^\mathsf{c}, A^\mathsf{c}} \\
&\cong C^{\mathrm{op}}((B^\mathsf{c} \cap A^\mathsf{c})^\mathsf{c}, (\emptyset)^\mathsf{c}) & \text{by full faithfulness of } (-)^\mathsf{c} \\
&\cong C^{\mathrm{op}}(A \cup B, X) & \text{by definition of } X \text{ and } \cup.
\end{aligned}$$

This composition of natural isomorphisms is natural in A. So B^c represents $C^{\mathrm{op}}(- \cup B, X)$, and $(C^{\mathrm{op}}, \cup, X)$ is precomplemented. Both $(C^{\mathrm{op}}, \cup, X)$ and $(C, \cap, \emptyset)$ share the same representing objects B^c. That means $(C^{\mathrm{op}}, \cup, X)$'s complement functor is the opposite of $(C, \cap, \emptyset)$'s complement functor. Therefore, $((-)^\mathsf{c})^{\mathrm{op}}$ has unit η^{op}, which is an isomorphism because η is, and $(C^{\mathrm{op}}, \cup, X)$ is complemented. $\square$

3 σ-Categories

So far, we have established a way of defining complements in a category through a universal property. Now, we use this notion of complement to model σ-algebras as categories.

3.1 Definition

Definition 3.1 (pre-σ-category)**.** A category C is a *pre-σ-category* if

- C has all countable products, which we denote $\cap$ rather than the usual $\times$. In particular, it has an initial object $\emptyset$.
- $\emptyset$ is *strict*, menaing $|C(A, \emptyset)| \leq 1$ for all objects $A \in C$.
- $(C, \cap, \emptyset)$ is a precomplemented category.
- $\cap$ naturally distributes over $\cup$, $(- \cap -) \cup \bullet \cong (- \cup \bullet) \cap (- \cup \bullet)$ (here, both bullets on the right are the same, but the two dashes may be different).

Definition 3.2 (σ-category)**.** A pre-σ-category C for which $(C, \cap, \emptyset)$ is complemented (not just precomplemented) is a *σ-category.*

Example 3.3. Let $\mathcal{A}$ be a σ-algebra on a set X. Then we can view $\mathcal{A}$ as a category whose objects are measurable subsets, and with a single morphism $A \to B$ if and only if $A \subset B$. This makes $\mathcal{A}$ a σ-category.

Although it is not obvious from the definition, this example is paradigmatic: we will see in section 3.2 that every small σ-category is equivalent to a σ-algebra.

Lemma 3.4. *Let C be a σ-category. Then C is a preorder, and C has countable coproducts, given by $\cup$.*

Proof. For any objects $A, B \in C$, we have $|C(A, B^{\mathsf{c}})| = |C(A \cap B, \emptyset)| \leq 1$ by $\emptyset$'s strictness. Since C is a complemented category, the object B^{c} is generic, so every hom set is of the form $C(A, B^{\mathsf{c}})$. Hence C is a preorder.

Next, we know the countable product $\bigcap_i A_i$ is the inverse limit of

$$\cdots \to \bigcap_{i=1}^{k} A_i \to \cdots \to \bigcap_{i=1}^{2} A_i \to A_1,$$

where the morphisms are projections. Since $(-)^{\mathsf{c}} : C^{\mathrm{op}} \to C$ is a equivalence, it brings limits to colimits. This means for each k, the product $\bigcap_{i=1}^{k} A_i$

becomes the coproduct $(\bigcap_{i=1}^k A_i)^\mathsf{c} \cong \bigcup_{i=1}^k A_i^\mathsf{c}$. Also, the inverse limit $\bigcap_{i\in\mathbb{N}} A_i$ becomes the colimit of

$$\cdots \leftarrow \bigcup_{i=1}^{k} A_i^\mathsf{c} \leftarrow \cdots \leftarrow \bigcup_{i=1}^{2} A_i^\mathsf{c} \leftarrow A_1^\mathsf{c}.$$

Since each object in the diagram is a finite coproduct with an inclusion morphism into the next, the colimit of this diagram is the same thing as the countable coproduct of $\{A_i^\mathsf{c}\}_{i\in\mathbb{N}}$. We denote this coproduct $(\bigcap_{i\in\mathbb{N}} A_i)^\mathsf{c}$ as $\bigcup_{i\in\mathbb{N}} A_i^\mathsf{c}$, so De Morgan's laws hold in the countable case. Since A_i^c is generic, we conclude that C has all countable coproducts, given by $\cup$. □

3.2 Representing σ-Categories in Sets

In this subsection, we prove that every small σ-category has a representation as a σ-algebra on some set. This means σ-categories provide an alternate definition of σ-algebras. In light of this result, we will be able to use every familiar set-theoretic property—such as $A \leq B$ implying $A \cap B = A$—when working with small σ-categories.

Definition 3.5 (Boolean algebra)**.** Recall that a *Boolean algebra* is a complemented lattice, meaning a set $\mathcal{B}$ along with operations $\cap, \cup : \mathcal{B} \times \mathcal{B} \to \mathcal{B}$ and $(-)^\mathsf{c} : \mathcal{B} \to \mathcal{B}$ satisfying

1. associativity of $\cap$ and $\cup$
2. commutativity of $\cap$ and $\cup$
3. both distributive laws for $\cap$ and $\cup$
4. existence of identities $\emptyset$ and X for $\cap$ and $\cup$
5. complementarity: $B \cap B^\mathsf{c} = \emptyset$ and $B \cup B^\mathsf{c} = X$ for all $B \in \mathcal{B}$.

Every Boolean algebra has a natural preorder structure given by $A \leq B \iff A \cap B = A$.

Lemma 3.6. *In a σ-category C, $A \leq B \iff A \cap B \cong A$.*

Proof. In a preorder, such as C, the product $A \cap B$ is $\inf\{A, B\}$. Since $\inf\{A, B\}$ is a lower bound for A and B, $\inf\{A, B\} \cong A$ implies $A \leq B$. Conversely, when $A \leq B$, an object is less than A if and only if it is less than A and B, meaning A satisfies the universal property of the product $A \cap B$ and $A \cong A \cap B$. □

Theorem 3.7. *Every small σ-category C has a representation as a σ-algebra. Specifically, there exists a σ-algebra $\mathcal{A}$ of sets such that, when viewed as a category, there is an equivalence $C \cong \mathcal{A}$.*

Proof. First, we'll show the set of C's objects form a Boolean algebra with operations $(-)^{\mathsf{c}}$, $- \cap -$, and $- \cup -$ (limit and colimit viewed as bifunctors):

1,2. The limit and colimit bifunctors are associative and commutative.

3. We have one distributive law by definition. The other follows dually from applying $(-)^{\mathsf{c}}$ to the one we have already.

4. Initial objects are identities for limits $\cap$: $C(\emptyset \cap A, B) \cong C(\emptyset, B) \times C(A, B) \cong C(A, B)$, so $\emptyset \cap A \cong A$. Dually, X is an identity for $\cup$.

5. Since $\emptyset$ is strict, the image of $1_{A^{\mathsf{c}}}$ under $C(A^{\mathsf{c}}, A^{\mathsf{c}}) \cong C(A \cap A^{\mathsf{c}}, \emptyset)$ is the unique morphism $A \cap A^{\mathsf{c}} \to \emptyset$. Also, $\emptyset$ is initial, so there is a unique morphism $\emptyset \to A \cap A^{\mathsf{c}}$. Because C is a preorder, the compositions of these morphisms are identities, meaning they are isomorphisms $\emptyset \cong A \cap A^{\mathsf{c}}$. Dually, $A \cup A^{\mathsf{c}} \cong X$.

By the Stone representation theorem for Boolean algebras[1], the Boolean algebra formed by the objects of C is isomorphic as a Boolean algebra to some field $\mathcal{A}$ of sets. Since C has not just finite but countable limits and colimits, $\mathcal{A}$ is specifically a σ-algebra. To show C is equivalent to $\mathcal{A}$ as categories, we need to show that their preorder structures are the same. This is the case since the preorder on $\mathcal{A}$ is given by $A \leq B$ if and only if $A \cap B = A$, while the preorder in C has the corresponding condition, $A \leq B$ if and only if $A \cap B \cong A$. $\square$

3.3 The σ-Category Associated to a Pre-σ-Category

We have seen that small σ-categories amount to σ-algebras. When defining a σ-algebra the usual way, we start with the power set $\mathcal{P}(X)$ of some set X and select a subset $\mathcal{A} \subset \mathcal{P}(X)$ to be our σ-algebra. In this subsection, we show that the categorical definition of σ-category also arises from selecting a subcategory of a larger category—even though the larger category, a pre-σ-algebra, does not closely resemble a power set.

Definition 3.8 (measurable functor)**.** Let C and D be pre-σ-categories. A functor $F : C \to D$ is *measurable* if it preserves $\cap$ and $(-)^{\mathsf{c}}$.

[1] M.H. Stone, The theory of representations of Boolean algebras, Trans. Amer. Math. Soc. 40 (1936) 37-111.

Note that the identity functor is measurable, as is the composite of two measurable functors, allowing the following definition.

Definition 3.9 ($\mathsf{PreSig}, \mathsf{Sig}$)**.** PreSig is the category of all pre-σ-categories with measurable functors for morphisms. Sig is the full subcategory of σ-categories in PreSig.

Lemma 3.10. *The image of the functor* $(-)^{\mathsf{c}}$ *in a pre-*σ*-category* C *is a* σ*-category. This gives rise to a functor* $\sigma : \mathsf{PreSig} \to \mathsf{Sig}$.

Proof. We'll denote the image of $(-)^{\mathsf{c}}$ by $\sigma(C)$. First, we have that $\sigma(C)$ is a preorder, since $|C(A^{\mathsf{c}}, B^{\mathsf{c}})| = |C(A^{\mathsf{c}} \cap B, \emptyset)| \leq 1$ by strictness of $\emptyset$. Next, for each object $A \in C$, we have opposing morphisms

$$\eta_A^{\mathsf{c}} : ((A^{\mathsf{c}})^{\mathsf{c}})^{\mathsf{c}} \to A^{\mathsf{c}}$$
$$\eta_{A^{\mathsf{c}}} : A^{\mathsf{c}} \to ((A^{\mathsf{c}})^{\mathsf{c}})^{\mathsf{c}}.$$

These opposing morphisms in the preorder $\sigma(C)$ must be isomorphisms. Restricting η and the fucntors 1_C and $((-)^{\mathsf{c}})^{\mathsf{c}}$ to $\sigma(C)$, we find η is a natural isomorphism between them. That means De Morgan's laws hold within $\sigma(C)$, which in turn means it has all countable limits and colimits—for example, $A^{\mathsf{c}} \cap B^{\mathsf{c}} \cong (A \cup B)^{\mathsf{c}}$ is in $\sigma(C)$. Finally, $\sigma(C)$ has all complements by definition, making it a σ-category.

Given a measurable functor $F : C \to D$ in PreSig, let $\sigma(F)$ be the restriction of F to $\sigma(C)$. The range of $\sigma(F)$ lies in $\sigma(D)$ because F preserves complements, $F(A^{\mathsf{c}}) = (FA)^{\mathsf{c}}$. Hence the assignment $C \mapsto \sigma(C), F \mapsto \sigma(F)$ yields a functor $\sigma : \mathsf{PreSig} \to \mathsf{Sig}$. $\square$

Theorem 3.11. Sig *is a reflective subcategory of* PreSig *with unit* σ^2.

Proof. Let C be a pre-σ-category, let D be a σ-category, and let $F : C \to D$ be measurable. We want to show that there exists a unique measurable functor $G : \sigma^2(C) \to D$ such that the following diagram commutes:

$$\begin{tikzcd} C \arrow[r, "\sigma_C^2"] \arrow[dr, "F"'] & \sigma^2(C) \arrow[d, dashed, "G"] \\ & D \end{tikzcd}$$

We know objects and morphisms in $\sigma^2(C)$ are of the form $(A^{\mathsf{c}})^{\mathsf{c}}$ and $(f^{\mathsf{c}})^{\mathsf{c}}$. For the triangle to commute, F and G must agree on objects and morphisms of that form; we need $G((A^{\mathsf{c}})^{\mathsf{c}}) := F((A^{\mathsf{c}})^{\mathsf{c}})$ and $G((f^{\mathsf{c}})^{\mathsf{c}}) := F((f^{\mathsf{c}})^{\mathsf{c}})$. This fully determines G, so if G satisfies the commutative diagram, it is the unique such functor. Note that G inherits measurability from F: for all objects $B, B' \in \sigma^2(C)$,

$$G(B^{\mathsf{c}}) = F(B^{\mathsf{c}}) \cong (FB)^{\mathsf{c}} = (GB)^{\mathsf{c}}$$
$$G(B \cap B') = F(B \cap B') \cong FB \cap FB' = GB \cap GB'$$

and similarly for morphisms. Next, because C is complemented, we have $FA \cong ((FA)^{\mathsf{c}})^{\mathsf{c}}$, and because F preserves complements, we find

$$FA \cong ((FA)^{\mathsf{c}})^{\mathsf{c}} \cong F((A^{\mathsf{c}})^{\mathsf{c}}) = G((A^{\mathsf{c}})^{\mathsf{c}}).$$

A similar argument holds on morphisms. Hence $F \cong G \circ \sigma^2$, as desired. $\square$

4 Measures and Integration on Sig and PreSig

Now that we have σ-categories, we turn to defining categorical analogs of measures and integrals on them. From now on, we only consider small σ-categories. The set-theoretic definition of a measure carries over directly:

Definition 4.1 (measure)**.** Let $[0, \infty]$ be a category with a morphism $x \to y$ if and only if $x \leq y$. A *measure* on a σ-category C is a functor $\mu : C \to [0, \infty]$ such that

- $\mu(\emptyset) = 0$.
- If a countable collection of objects $\{A_i\}$ in C are pairwise disjoint, meaning $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then $\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$.

This has all the usual properties of measures when the σ-category is small. Note that the functoriality of a measure does not affect the analogy with sets since set-theoretic measures are always monotonic. One novel feature of this definition of measure is that it can extend to a pre-σ-category by precomposing with σ^2.

Next, we give several definitons that build up to defining integrals, starting with measurable functors. With sets, a function $f : X \to Y$ between two sets equipped with σ-algebras $\mathcal{A}$ and $\mathcal{A}'$ is *measurable* if $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ sends measurable sets to measurable sets, or in other words, if f^{-1} restricts to $f^{-1}|_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}$. In our setup, we do not have access to the "elements" of X, so we cannot define a measurable function f the same way. However, what we do know is that the objects in the σ-categories C and D correspond to the elements of $\mathcal{A}$ and $\mathcal{A}'$. In analogy with $f^{-1}|_{\mathcal{A}'} : \mathcal{A}' \to \mathcal{A}$, we start with a functor $F : D \to C$. Then, we place additional restrictions on F to make it seem as though it arose from an underlying function f. Namely, for any function f of sets, f^{-1} preserves intersections and complements, so we require the same for our functor F:

Definition 4.2 (measurable functor)**.** Let C and D be pre-σ-categories. A functor $F : D \to C$ is *measurable* if it preserves $\cap$ and $(-)^{\mathsf{c}}$.

This is the same as the definition of measurable functor that we introduced in section 3.3.

Now we continue the analogy with sets by introducing the analogs of measurable functions needed to define Lebesgue integrals: integrable simple functions, an ordering of measurable functions, and f_+ and f_-. In each case, we take the set-theoretic function, find an explicit forumla for its inverse image, and use that formula to define our measurable function.

Definition 4.3 ($\mathcal{B}$, ISF)**.** Let $\mathcal{B}$ denote the Borel σ-algebra on the real numbers, viewed as a σ-category, and let C be any σ-category. Let I be a countable index set, let each a_i be a non-negative real number, and let each A_i be an object in C. An *integrable simple functor* $\sum_{i\in I} a_i \chi_{A_i} : \mathcal{B} \to C$ is the functor which acts as follows on objects $S \in \mathcal{B}$:

$$S \mapsto \bigcup_{J \in \mathcal{P}(I) \text{ s.t. } \sum_{j\in J} a_j \in S} \left(\bigcap_{j\in J} A_j \cap \bigcap_{i \in J^{\mathsf{c}}} A_i^{\mathsf{c}} \right).$$

Here, $\mathcal{P}(I)$ is the power set of I, and $J^{\mathsf{c}} = I \setminus J$.

Note that unlike with sets, our notion of ISF can support countably many "summands" of indicator functors, not just finitely many.

Next, we define an ordering on measurable functors. Given two functions $f, g : X \to \mathbb{R}$, we normally define an ordering in terms of elements, saying $f \leq g \iff f(x) \leq g(x)\ \forall x \in X$. But that is equivalent to the definition $f \leq g \iff f^{-1}(x,\infty) \subset g^{-1}(x,\infty)\ \forall x \in \mathbb{R}$. We can adapt the second definition to measurable functors:

Definition 4.4 ($F \leq G$)**.** Let F and G be measurable functors. We say $F \leq G$ if $F(x,\infty) \leq G(x,\infty)$ for all $x \in \mathbb{R}$.

The next definition is the analog of the non-negative functions f_+ and f_- associated to a measurable function f,

$$f_+(x) = \begin{cases} f(x), & f(x) \geq 0 \\ 0, & f(x) < 0 \end{cases}$$

$$f_-(x) = \begin{cases} -f(x), & f(x) \leq 0 \\ 0, & f(x) > 0. \end{cases}$$

Definition 4.5 (F_+, F_-)**.** A measurable functor $F : \mathcal{B} \to C$ gives rise to functors $F_+, F_- : \mathcal{B} \to C$ defined by

$$
\begin{aligned}
F_+ &: S \mapsto \begin{cases} F(S \cap [0,\infty)) \cup F(-\infty,0), & 0 \in S \\ F(S \cap [0,\infty)), & 0 \notin S \end{cases} \\
F_- &: S \mapsto F_+(-S).
\end{aligned}
$$

Finally, we define the analog of restricting a function to a measurable domain A.

Definition 4.6 ($F \cdot \chi_A$)**.** A measurable functor $F : \mathcal{B} \to C$ and an object $A \in C$ give rise to the functor $F \cdot \chi_A : \mathcal{B} \to C$ defined by

$$
F \cdot \chi_A : S \mapsto \begin{cases} (F(S) \cap A) \cup A^{\mathsf{c}}, & 0 \in S \\ F(S) \cap A, & 0 \notin S. \end{cases}
$$

Lemma 4.7. *Let C be a σ-category, $F : \mathcal{B} \to C$ be measurable, and A be an object in C.*

1. *Every ISF from $\mathcal{B} \to C$ is a measurable functor.*
2. *$F_+, F_- : \mathcal{B} \to C$ are measurable functors with $0 \leq F_+$ and $0 \leq F_-$.*
3. *$F \cdot \chi_A : \mathcal{B} \to C$ is a measurable functor.*

Proof. Any assignment of objects $F : \mathrm{ob}(\mathcal{B}) \to \mathrm{ob}(C)$ that preserves $\cap$ is a functor because the preorder relation in a σ-category only depends on $\cap$:

$$
S \leq S' \implies S \cong S \cap S' \implies F(S) \cong F(S) \cap F(S') \implies F(S) \leq F(S').
$$

By proving that these functors preserve $\cap$ and $(-)^{\mathsf{c}}$ on objects, we also confirm that they are indeed functors.

We will only present the proof for F_+ being measurable; the proofs for F_-, $F \cdot \chi_A$, and ISFs use the same principles. First, we'll show F_+ preserves $(-)^{\mathsf{c}}$. To start, note that any measurable functor sends $\mathbb{R}$ to X:

$$
X = \emptyset^{\mathsf{c}} \cong F(\emptyset)^{\mathsf{c}} \cong F(\emptyset^{\mathsf{c}}) \cong F(\mathbb{R}).
$$

Now if $0 \in S$, we find

$$
\begin{aligned}
F_+(S)^{\mathsf{c}} &= (F(S \cap [0,\infty))^{\mathsf{c}} \\
&\cong F(S^{\mathsf{c}} \cup (-\infty,0)) \\
&\cong (F(S^{\mathsf{c}}) \cup F(-\infty,0)) \cap X \\
&\cong (F(S^{\mathsf{c}}) \cup F(-\infty,0)) \cap (F[0,\infty) \cup F(-\infty,0)) \\
&\cong (F(S^{\mathsf{c}}) \cap F[0,\infty)) \cup F(-\infty,0) \\
&= F_+(S^{\mathsf{c}}).
\end{aligned}
$$

This uses F's measurability, De Morgan's laws, and distributivity of $\cup$ over $\cap$. For the other case, where $0 \notin S$, we have $0 \in S^{\mathsf{c}}$, so we can apply the above result to S^{c}:

$$\begin{aligned} F(S) &\cong F((S^{\mathsf{c}})^{\mathsf{c}}) \cong F(S^{\mathsf{c}})^{\mathsf{c}} \\ &\implies F(S)^{\mathsf{c}} \cong F(S^{\mathsf{c}}). \end{aligned}$$

Next, we'll show F_+ preserves $\cup$. Since C is complemented, De Morgan's laws then imply F_+ preserves $\cap$. Consider a countable collection of sets $\{S_j, T_i\} \in \mathcal{B}$, where each S_j does not contain 0 and each T_i does contain 0. Suppose there is at least one T_i, so the union over all S_j and T_i contains 0 (the case where it does not is similar). Using measurability of F, we find

$$\begin{aligned} F_+\left(\bigcup S_j \cup \bigcup T_i\right) &= F\left(\left(\bigcup S_j \cup \bigcup T_i\right) \cap [0,\infty)\right) \cup F(-\infty,0) \\ &\cong F\left(\bigcup(S_j \cap [0,\infty)) \cup \bigcup(T_i \cap [0,\infty))\right) \ \cup\ F\left(\bigcup(-\infty,0)\right) \\ &\cong F\left(\bigcup(S_j \cap [0,\infty))\right) \ \cup\ F\left(\bigcup(T_i \cap [0,\infty)) \cup \bigcup(-\infty,0)\right) \\ &\cong \bigcup F(S_j \cap [0,\infty)) \ \cup\ \bigcup F(T_i \cap [0,\infty)) \cup F(-\infty,0) \\ &= \bigcup F_+(S_j) \cup \bigcup F_+(T_i). \end{aligned}$$

Lastly, we'll show that $F_+ \geq 0$. Suppose $x > 0$. Then $0(x,\infty) = \emptyset$, which is initial, so $0(x,\infty) \cong \emptyset \leq F_+(x,\infty)$. Now suppose $x \leq 0$. In this case, $0 \in (x,\infty)$, so

$$F_+(x,\infty) = F((x,\infty) \cap [0,\infty)) \cup F(-\infty,0) \cong F([0,\infty) \cup (-\infty,0) \cong F(\mathbb{R}) \cong X$$

using the measurability of F. X is terminal, so $0(x,\infty) \leq X \cong F_+(x,\infty)$, completing the proof that $0 \leq F_+$. The proof that $0 \leq F_-$ is similar. ☐

We are now ready to define integrals on σ-categories.

Definition 4.8 (integration on Sig)**.** Let C be a σ-category and $\mu : C \to [0,\infty]$ be a measure. We define the integral $\int F \, \mathrm{d}\mu$ of a measurable functor $F : \mathcal{B} \to C$ in stages:

- If $F = \sum_{i \in I} a_i \chi_{A_i}$ is an ISF, then

$$\int F \, \mathrm{d}\mu = \sum_{i \in I} a_i \mu(A_i).$$

(If $a_i = 0$ and $\mu(A_i) = \infty$, we say $a_i \mu(A_i) = 0$.)

- If $F \geq 0$, then
$$\int F \,\mathrm{d}\mu = \sup_{G \text{ is an ISF and } G \leq F} \left\{ \int G \right\}.$$
- If F is any measurable functor,
$$\int F \,\mathrm{d}\mu = \int F_+ \,\mathrm{d}\mu - \int F_- \,\mathrm{d}\mu.$$
- If F is any measurable functor and A is any object in C,
$$\int_A F \,\mathrm{d}\mu = \int F \cdot \chi_A \,\mathrm{d}\mu.$$

Finally, we can extend this definition to PreSig.

Definition 4.9 (integration on PreSig)**.** Let C be a pre-σ-category and let $\mu : \sigma^2(C) \to [0, \infty]$ be a measure. We define the integral of a measurable functor $F : \mathcal{B} \to C$ by
$$\int F \,\mathrm{d}\mu = \int (\sigma^2 \circ F) \,\mathrm{d}\mu.$$

Example 4.10. Recall the following preorder C with $\emptyset < A < X$:

$$\begin{array}{c} X \\ \uparrow \\ A \\ \uparrow \\ \emptyset \end{array}$$

This is a pre-σ-category but not a σ-category; we have $A^{\mathsf{c}} = \emptyset$ and $(A^{\mathsf{c}})^{\mathsf{c}} = X$. We can equip $\sigma^2(C)$ with the measure $\mu(\emptyset) = 0, \mu(X) = 1$. Define the measurable functor
$$\begin{aligned} F : \mathcal{B} &\to C \\ S &\mapsto \begin{cases} X, & 1 \in S \\ \emptyset, & 1 \notin S. \end{cases} \end{aligned}$$

Then $\sigma^2 \circ F$ is the indicator functor χ_X, so we have
$$\int F \,\mathrm{d}\mu = \int \chi_X \,\mathrm{d}\mu = \mu(X) = 1.$$

5 Further Questions

- In a precomplemented category, the image of $1_{A^{\mathsf{c}}}$ under $C(A^{\mathsf{c}}, A^{\mathsf{c}}) \cong C(A \cap A^{\mathsf{c}}, \emptyset)$ is a morphism $\delta_A : A \cap A^{\mathsf{c}} \to \emptyset$. The collection $(\delta_A)_{A \in C}$ is a cone under a diagram whose objects are of the form $A \cap A^{\mathsf{c}}$ with nadir $\emptyset$. Under what conditions is this a colimit cone? What would be the consequences of $\emptyset$ being this colimit? What happens when δ is an isomorphism (which it is in pre-σ-categories)?

- The opposing morphisms

$$\eta_{A^{\mathsf{c}}} : A^{\mathsf{c}} \rightleftarrows ((A^{\mathsf{c}})^{\mathsf{c}})^{\mathsf{c}} : \eta_A^{\mathsf{c}}$$

  are present in any precomplemented category, not just in a pre-σ-algebra. What can we say about them in this setting? If we assemble the category of precomplemented categories, is its full subcategory of complemented categories a reflective subcategory?

- In a precomplemented category, the self-adjoint functor $(-)^{\mathsf{c}}$ gives rise to a monad with endofunctor $((-)^{\mathsf{c}})^{\mathsf{c}}$. What are the Eilenberg-Moore and Kleisli categories for this monad? Is the image of $((-)^{\mathsf{c}})^{\mathsf{c}}$ a complemented category?

- Can we characterize fibers of the functor $\sigma : \mathsf{PreSig} \to \mathsf{Sig}$? When do two pre-$\sigma$-categories have the same associated σ-algebra?

- All measurable functions between σ-algebras give rise to measurable functors between the corresponding σ-categories. Are there measurable functors that don't arise this way?

- Can we stretch notions of measures and integration to complemented or precomplemented categories?

- Is there a way to make the power set $\mathcal{P}(\mathbb{R})$ a pre-σ-category such that $\sigma^2(\mathcal{P}(\mathbb{R})) = \mathcal{B}$? That would let us integrate over sets that are not Lebesgue measurable.