Probabilistic Inference for Solving Markov Decision Processes

Toussaint, Storkey, ICML '06

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Computer Science M.Sc. Seminar
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Outline

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   - Motivation & Prior work
   - Main contribution
2. Research Problem
   - Solving Markov Decision Processes
3. Research Plan
   - Mixture of MDPs and likelihood
   - An EM-algorithm for computing the optimal policy
   - Relation to Policy Iteration
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   - Stochastic optimal control
5. Conclusion

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Motivation & Prior work
Main contribution

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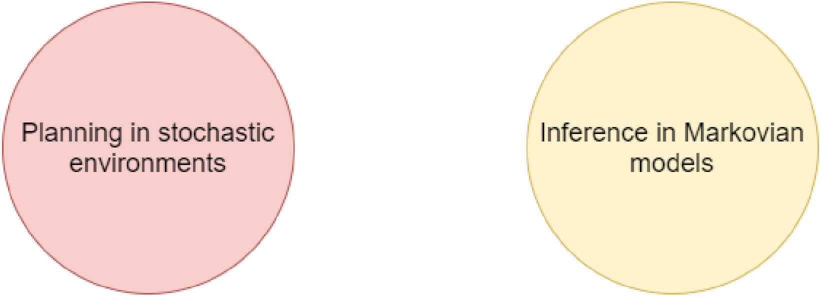
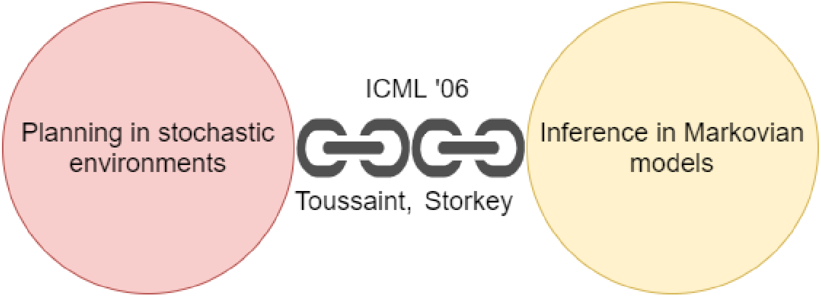
Motivation


Planning in stochastic environments
Inference in Markovian models

Motivation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Prior work

1. *Bui et al. (2002)* used inference on Abstract Hidden Markov Models for policy recognition, *but not for computing an optimal policy*.
2. *Attias (2003)* got close to translating the problem of planning to a problem of inference. However, *the total time T had to be fixed* and the MAP action sequence that is proposed as a solution *is not optimal*.
3. *Verma and Rao (2006)* used inference to compute plans, but again T has to be fixed and the plan is not optimal.

Main contribution

Contribution

1. Translate the problem of *maximizing the expected future return* exactly into a problem of *likelihood maximization in a latent variable model*, for arbitrary reward functions and episode lengths.

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1. Translate the problem of *maximizing the expected future return* exactly into a problem of *likelihood maximization in a latent variable model*, for arbitrary reward functions and episode lengths.
2. Demonstrate the approach on discrete & continuous stochastic optimal control problems,

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Examples 1: Discrete maze

(a)

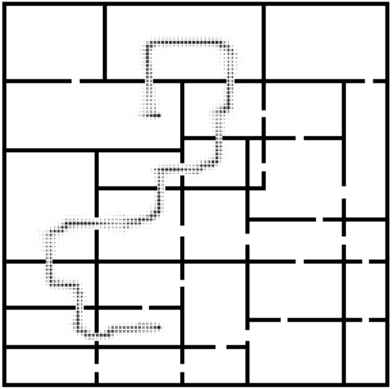

Figure 1: Posterior state-visiting-probabilities generated by our Probabilistic Infernce Planner (PIP)

Markov Decision Processes

Definition (MDP)

$$
\begin{aligned}
\text{state transition probability } & P(x_{t+1} \mid a_t, x_t) \\
\text{reward probability } & P(r_t \mid a_t; x_t), \qquad r_t \in \{0, 1\} \\
\text{action probability } & P(a_t \mid x_t; \pi), \qquad \pi \text{ a parameter}
\end{aligned}
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action probability $P(a_t \mid x_t; \pi), \qquad \pi$ a parameter

Definition (Policy π)

The action probabilities are parameterized by a policy:

$$P(a_t \mid x_t = i; \pi) = \pi_{ai} \qquad s.t. \sum_a \pi_{ai} = 1$$

Markov Decision Processes

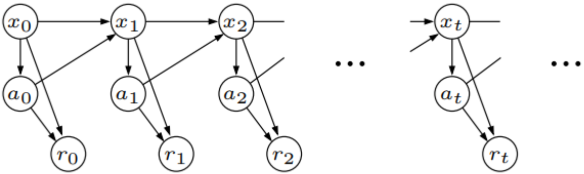

Figure 1. Dynamic Bayesian Network for a MDP. The x states denote the state variables, a the actions and r the rewards.

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Research problem

Definition (solving an MDP)

Solving an MDP means to find a parameter π of the graphical model in Figure 1 that maximizes the expected future return $V^{\pi}(i) = E\{\sum_{t=0}^{\infty} \gamma^t r_t \mid x_0 = i; \pi\}$, where $\gamma \in [0, 1]$ is a discount factor.

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research problem

The problem is to *solve the MDP*, i.e. to find a policy that maximizes the expected future return.

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Mixture of MDPs

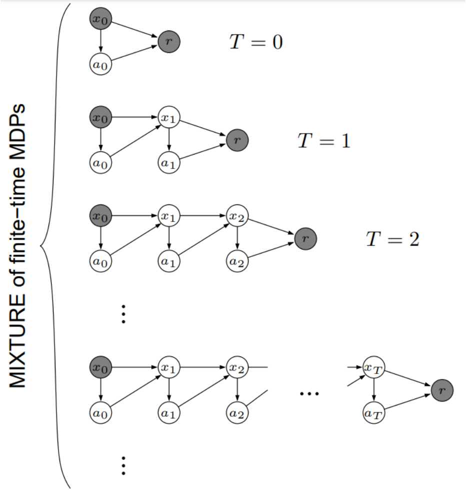

Figure 2. Mixture of finite-time MDPs.

Representing the joint distribution $P(\mathcal{X})$

full joint for finite time MDP

$$P(r, x_{0:T}, a_{0:T} \mid T; \pi) =$$

$$P(r \mid a_T, x_T) P(a_0 \mid x_0; \pi) P(x_0) \cdot \prod_{t=1}^{T} P(a_t \mid x_t; \pi) P(x_t \mid a_{t-1}, x_{t-1})$$

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full joint for mixture of finite-time MDPs

$$P(r, x_{0:T}, a_{0:T}, T; \pi) = P(r, x_{0:T}, a_{0:T} \mid T; \pi) P(T)$$

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$$P(r, x_{0:T}, a_{0:T}, T; \pi) = P(r, x_{0:T}, a_{0:T} \mid T; \pi) P(T)$$

prior over the total time

$$P(T) = \gamma^T (1 - \gamma)$$

Defining the likelihood

Defining the likelihood

Definition (likelihood for a finite-time MDP)

$$L_T^\pi(i) = P(r = 1 \mid x_0 = i, T; \pi) = E\{r \mid x_0 = i, T; \pi\}$$

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$$L^\pi(i) = P(r = 1 \mid x_0 = i; \pi) = \sum_T P(T) E\{r \mid x_0 = i, T; \pi\}$$

Corollary

$$L_T^\pi(i) = (1 - \gamma) V^\pi(i)$$

Theoretical Guarantee

Reminders

1. *Solving an MDP* means to find a parameter π of the graphical model in Figure 1 that maximizes the expected future return $V^{\pi}(i) = E\{\sum_{t=0}^{\infty} \gamma^{t} r_t \mid x_0 = i; \pi\}$.
2. The *likelihood for a mixture of MDPs* is given by $L^{\pi}(i) = P(r = 1 \mid x_0 = i; \pi) = \sum_{T} P(T) E\{r \mid x_0 = i, T; \pi\}$
3. This implies that $L_T^{\pi}(i) = (1 - \gamma) V^{\pi}(i)$

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Theorem

(proved) Maximizing the likelihood in the mixture of finite-time MDPs is equivalent to solving the MDP.

An EM-algorithm for computing the optimal policy

What are EM algorithms?

A class of algorithms consisting of two modes:

1. E-step: estimates missing variables.
2. M-step: optimizes parameters of model to best explain the data.

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E-step

For a given π, compute posteriors $P(T \mid x_0 = A, r = 1; \pi)$ and $P(x_{1:T}, a_{1:T} \mid x_0 = A, r = 1, T; \pi)$

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M-step

Adapt parameters π to optimize $V^\pi(A)$

E-step: forward-backward in all MDPs synchronously

Simplifying notation

1. $p(j \mid a, i) = P(x_{t+1} = j \mid a_t = a, x_t = i)$
2. $p(j \mid i; \pi) = P(x_{t+1} = j \mid x_t = i; \pi) = \sum_a p(j \mid a, i)\pi_{ai}$

E-step: forward-backward in all MDPs synchronously

Forward Propagation

$$\alpha_0(i) = \delta_{i=A}$$

$$\begin{aligned} \alpha_t(i) &= P(x_t = i \mid x_0 = A; \pi) \\ &= \sum_j p(i \mid j; \pi)\alpha_{t-1}(j) \end{aligned}$$

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Backward Propagation (temp.)

$$\tilde{\beta}_T(i) = \hat{\beta}(i)$$

$$\begin{aligned}\tilde{\beta}_t(i) &= P(r = 1 \mid x_t = i; \pi) \\ &= \sum_j p(j \mid i; \pi)\tilde{\beta}_{t+1}(j)\end{aligned}$$

Where

$$\hat{\beta}(i) = P(r = 1 \mid x_T = i; \pi) = \sum_a P(r = 1 \mid a_T = a, x_T = i)\pi_{ai}$$

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Backward Propagation (corrected)

$$\beta_0(i) = \hat{\beta}(i)$$

$$\begin{aligned}\beta_\tau(i) &= P(r = 1 \mid x_{T-\tau} = i; \pi) \\ &= \sum_j p(j \mid i; \pi)\beta_{\tau-1}(j)\end{aligned}$$

M-step: the policy update

Definition (expected complete log-likelihood)

$Q(\pi^*, \pi) = \sum_T \sum_{x_{0:T}, a_{0:T}} P(x_{0:T}, a_{0:T}, T \mid r = 1; \pi)$

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Fact

Maximizing $Q(\pi^*, \pi)$ w.r.t. π^* is achieved by setting $\pi^*_{ai} = P(a_t = a \mid x_t = i, r = 1; \boldsymbol{\pi})$

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However

exploiting the structure of the MDP, we can also write the likelihood as

$$P(r = 1 \mid x_0 = i; \boldsymbol{\pi}) = \sum_{aj} P(r = 1; a_t = a, x_t = j; \boldsymbol{\pi}^*)\pi^*_{aj} \cdot P(x_t = j \mid x_0 = i; \boldsymbol{\pi}^*)$$

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$$\pi^*_{ai} = \delta_{a=a^*(i)} \qquad \alpha^*(i) = \arg\max_a P(r = 1 \mid a_t = a, x_t = i; \pi)$$

Relation to Policy Iteration

E-step: Policy Evaluation

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M-step: Policy Update

1. Maximizing the Q-function w.r.t. the action *a* and state *i*.

Relation to Policy Iteration

Thus,

the EM-algorithm using exact inference and belief representation is effectively equivalent to Policy Iteration but computes the necessary quantities in a different way.

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the EM-algorithm using exact inference and belief representation is effectively equivalent to Policy Iteration but computes the necessary quantities in a different way.

However,

when using approximate inference or belief representations

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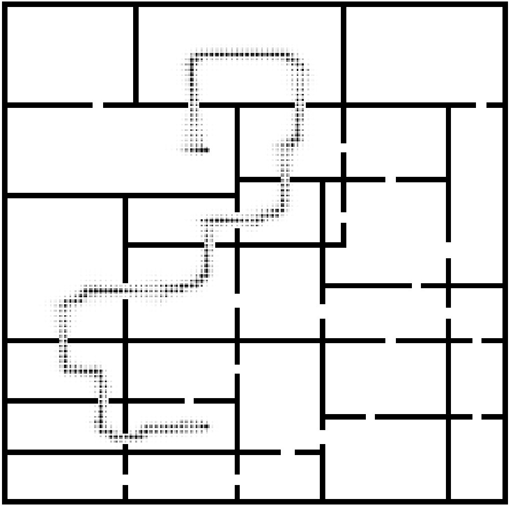
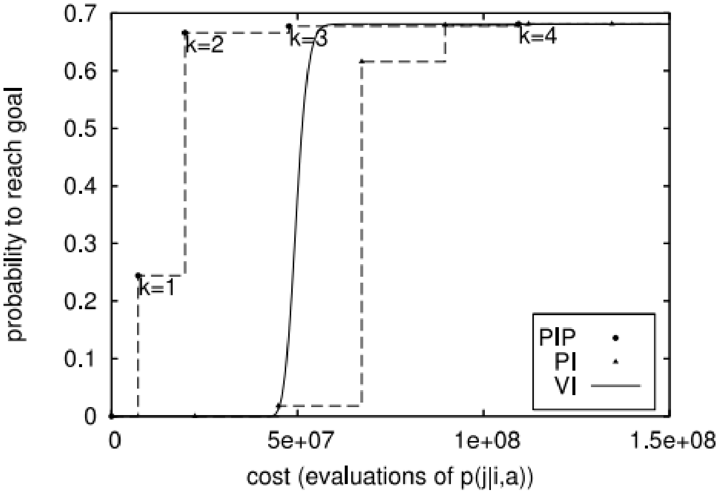
Discrete maze

(a)

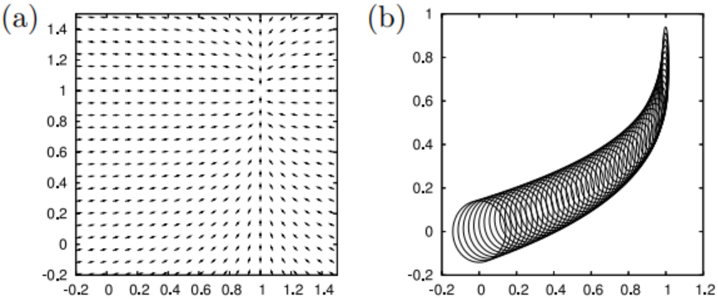


Discrete maze

(b)

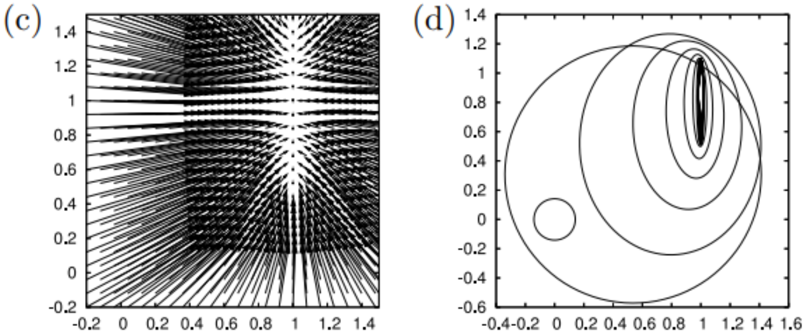


Stochastic optimal control - 'walker'

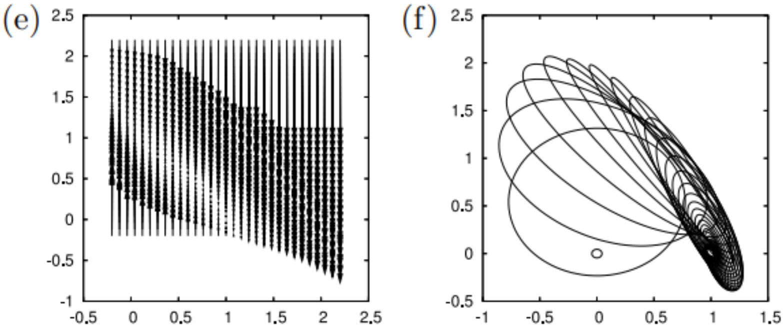


- $P(x' \mid u, x) = \mathcal{N}(x', \phi(u,x), Q + (|u|/\mu)^2 I)$ is *transitions*.
- $\alpha_0(x) = \mathcal{N}(x, (0,0), .01 I)$ is *start-state*.
- $P(r = 1 \mid x) = \mathcal{N}(x, (1,1), diag(.0001, .1))$ is *goal*.
- $\phi(u,x) = x + .1u$ is *control-law*.

Stochastic optimal control - 'golfer'



Stochastic optimal control - 'phase space'



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We can compute posteriors over actions, states, and the total time.

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We can compute posteriors over actions, states, and the total time.

The full variety of existing inference techniques can be applied to solving MDPs.

End

Thank You