The Recursion Theorem

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A useful item to have in one's toolbox while working in ZFC set theory[1] is the ability to create inductive definitions. My goal in this post is to prove a theorem that allows us to do exactly that whenever the proper conditions are present. We'll also work out a couple of fun examples at the end.

1 Preliminaries

If the following material doesn't make much sense, then a great review can be found in the first half of [1]. The proof of the main theorem for this post can be found in that text as well.

1.1 Natural Numbers

We use the following two axioms of ZFC to define the natural numbers.

Axiom 1.1 (Infinity)**.** $\exists x(\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x))$

Axiom 1.2 (Comprehension)**.** $\forall v \exists y \forall x(x \in y \Leftrightarrow x \in v \wedge \phi(x))$ *where* ϕ *is a formula that* y *is not free in.*

Definition 1.1. A set I is inductive if and only if $\emptyset \in I$ and $\forall y \in I(y \cup \{y\} \in I)$.

Example 1.1. Some examples of elements of an inductive set are

- $\emptyset$
- $\emptyset \cup \{\emptyset\} = \{\emptyset\}$
- $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$
- $\{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Definition 1.2. Let J be a set who's existence is guaranteed by Axiom 1.1. We use Axiom 1.2 to form the set

$$\mathbb{N} = \{y \in J : \forall I(I \text{ is inductive} \Rightarrow y \in I)\}$$

One should think of $\mathbb{N}$ as the "smallest" inductive set. An element of $\mathbb{N}$ is called a *natural number*.

[1]Short for Zermelo–Fraenkel set theory. See https://en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory for more information.

Example 1.2. Figure 1 shows a few compact names for natural numbers that are often used in place of the set theory notation[2].

Element of $\mathbb{N}$	Compact Name
$\emptyset$	1
$\{\emptyset\}$	2
$\{\emptyset, \{\emptyset\}\}$	3
$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$	4

Figure 1: Compact names for elements of $\mathbb{N}$.

Definition 1.3. The function $S : \mathbb{N} \to \mathbb{N}$ defined by $S(n) = n \cup \{n\}$ for all $n \in \mathbb{N}$ is called the *successor function on* $\mathbb{N}$. For $n \in \mathbb{N}$, we will refer $S(n)$ as n^+.

Example 1.3. Recall Example 1.1 and Example 1.2. One can deduce that

- $1^+ = 2$
- $2^+ = 3$
- $3^+ = 4$

and we'll define two more compact names to be used in an informative example at the end of the post:

- $4^+ = 5$
- $5^+ = 6$

1.2 Peano Axioms

While we prove and refer to the following as theorems, it is common nomenclature to refer to these as *The Five Peano Axioms*.

Theorem 1.1. $1 \in \mathbb{N}$

Proof. Since 1 is in every inductive set, then $1 \in \mathbb{N}$. □

Theorem 1.2. *If $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. By definition, $\forall I(I \text{ is inductive} \Rightarrow n \in I)$ and so for any inductive set K, we can then conclude that $n^+ \in K$ since $n \in K$. Recall that $n \in J$ from Definition 1.2. By defintion, $n^+ \in J$. So we conclude from these two facts that $n^+ \in \mathbb{N}$. □

Theorem 1.3 (The Principle of Mathematical Induction)**.** *If $S \subseteq \mathbb{N}, 1 \in S$, and $n \in S \Rightarrow n^+ \in S$, then $S = \mathbb{N}$.*

[2]Note that 1 is considered the first natural number in this post.

Proof. We will argue by subset inclusions that $S = \mathbb{N}$. By hypothesis, we are given that $n \in S \Rightarrow n \in \mathbb{N}$. Now suppose that $n \in \mathbb{N}$. We can see that S is an inductive set and so by definition of $\mathbb{N}$, $n \in S$. Therefore, we conclude that $S = \mathbb{N}$. $\square$

Definition 1.4. We refer to invoking Theorem 1.3 as an *argument by induction.*

Theorem 1.4. *For all $n \in \mathbb{N}$, we have that $n^+ \neq 1$.*

Proof. n^+ is nonempty and 1 is empty. So we conclude that $n^+ \neq 1$. $\square$

Definition 1.5. A set T is transitive if and only if $x \in T \Rightarrow x \subseteq T$.

Lemma 1.1. *Every natural number is transitive.*

Proof. We argue by induction. Let S be the set of all transitive natural numbers. $1 \in S$ trivially. Suppose $n \in S$. Since $n^+ = n \cup \{n\}$, then for all $x \in n^+$ either $x \in n$ or $x = n$. If $x \in n$, then by our inductive hypothesis that $n \in S$, we know that $x \subseteq n$ which allows us to conclude that $x \subseteq n^+$. If $x = n$, then we also conclude that $x \subseteq n^+$. Therefore, $n^+ \in S$ and so $S = \mathbb{N}$. $\square$

Theorem 1.5. *If $n, m \in \mathbb{N}$ and $n^+ = m^+$, then $n = m$.*

Proof. Suppose the hypothesis is true. Well then $n \in n^+ \Rightarrow n \in m^+$ and so either $n \in m$ or $n = m$. By the same type of observation, either $m \in n$ or $m = n$. If $n \neq m$, then $m \in n$ and $n \in m$. Well then Lemma 1.1 tells us that $m \subseteq n$ and $n \subseteq m$ and so $m = n$, a contradiction. $\square$

2 Existence and Uniqueness

Theorem 2.1. *For any set X, if $a \in X$ and $f : X \to X$, then there exists a function $u : \mathbb{N} \to X$ such that $u(1) = a$ and $u(n^+) = f(u(n))$ for all other natural numbers n.*

Proof. Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N} \times X)$ be defined such that $(1, a) \in c$ and $(n^+, f(x)) \in c$ whenever $(n, x) \in c$ for all $c \in \mathcal{C}$. It's clear to see that $\mathcal{C}$ is nonempty since $\mathbb{N} \times X \in C$ and so we can form the intersection of all sets in $\mathcal{C}$ which we'll call u.

Let S be the set of all natural numbers such that if $n \in S$ then there exists $(n, x) \in u$ and if $(n, w), (n, y) \in u$ then $w = y$. We will prove inductively that $S = \mathbb{N}$ which establishes that u is a function. Furthermore, given how u is constructed, such a proof gives us exactly the kind of function we are looking for.

Suppose that $(1, b) \in u$, $a \neq b$, and consider the set $M = u \setminus \{(1, b)\}$. We claim that $(n^+, f(x)) \in M$ whenever $(n, x) \in M$. If that weren't the case then some $(n^+, f(x))$ would not be present in M for some $(n, x) \in M$. Well by the definition of M, for all α we have that $(\alpha \in u \wedge \alpha \neq (1, b)) \to \alpha \in M$ which implies that if $\alpha \notin M$, then $\alpha \notin u \vee \alpha = (1, b)$. Well it can't be the case that $(1, b) = (n^+, f(x))$ by Theorem 1.4. So then we must conclude that $(n^+, f(x)) \notin u$ which contradicts u's initial construction since $(n, x) \in u$ by the assumption that $(n, x) \in M$. Therefore, since $(1, a) \in M$ and for any other $(1, b) \in M$ we know that $a = b$, we conclude that $1 \in S$.

Now suppose that $n \in S$ which implies that there is an $(n, x) \in u$ for at most one x. It follows from the definition of u that $(n^+, f(x)) \in u$. Now if n^+ isn't in S, then there exists $(n^+, y) \in u$ such that $f(x) \neq y$. Consider the set $J = u \setminus \{(n^+, y)\}$ and some $(m, t) \in J$. If $m = n$, then $t = x$ since $n \in S$ and so $(n^+, f(x)) = (m^+, f(t)) \in J$. By construction of J, we note that $\alpha \notin J \to (\alpha \notin u \vee \alpha = (n^+, y))$. So if $m \neq n$, then we know that $m^+ \neq n^+$ by the Theorem 1.5 and so if $(m^+, f(t)) \notin J$ then we must conclude that $(m^+, f(t)) \notin u$ which is a contradiction to the construction of u. So $n^+ \in S$ and by the principal of mathematical induction we conclude that $\mathbb{N} = S$. $\square$

Corollary 2.1. *The function defined in Theorem 2.1 is unique.*

Proof. Let $a \in X$ and $f : X \to X$ for some set X. Suppose that there are two functions $F : \mathbb{N} \to \mathbb{X}$ and $G : \mathbb{N} \to \mathbb{X}$ where $F(1) = G(1) = a$, $F(n^+) = f(F(n))$, and $G(n^+) = f(G(n))$ for all natural numbers n. Let $F(1) = G(1)$ be the base case for an inductive proof and suppose that $F(n) = G(n)$ for some natural number n. Well then $F(n^+) = f(F(n)) = f(G(n)) = G(n^+)$. So by the principle of mathematical induction, F and G are the same function. $\square$

3 Examples

Let's define a couple of familiar concepts using what we've shown.

Definition 3.1 (Addition)**.** For any natural number $n \in \mathbb{N}$, we have $n^+ \in \mathbb{N}$ by Theorem 1.2. If we apply n^+ and the successor function on $\mathbb{N}$ to Theorem 2.1 and Corollary 2.1, we can define a unique function $s_n : \mathbb{N} \to \mathbb{N}$ where $s_n(1) = n^+$ and $s_n(m^+) = s_n(m)^+$ for all $m \in \mathbb{N}$. For any such function s_n and $m \in \mathbb{N}$, we refer to $s_n(m)$ as $n + m$ and call it the *sum of* n *and* m.

Example 3.1. Recall Example 1.3. We can see that

$$2 + 2 = s_2(2) = s_2(1)^+ = (2^+)^+ = 3^+ = 4$$

Definition 3.2 (Multiplication)**.** For every $n \in \mathbb{N}$, we take $s_n : \mathbb{N} \to \mathbb{N}$ from Definition 3.1 and we define $p_n : \mathbb{N} \to \mathbb{N}$ where $p_n(1) = n$ and $p_n(m^+) = s_n(p_n(m)) = n + p_n(m)$. We refer to $p_n(m)$ as $n \cdot m$. We call this the *product of* n *and* m or n *multiplied* by m.

Example 3.2. Recall Example 3.1. We can see that

$$\begin{aligned} 2 \cdot 3 &= 2 + (2 \cdot 2) \\ &= 2 + (2 + (2 \cdot 1)) \\ &= 2 + (2 + 2) \\ &= 2 + 4 \text{ by Example 3.1} \\ &= (2 + 3)^+ \\ &= ((2 + 2)^+)^+ \\ &= (4^+)^+ \text{ by Example 3.1} \\ &= 6 \end{aligned}$$

References

[1] Paul R. Halmos. *Naive Set Theory.* Springer New York, NY, 1998.