

# The Recursion Theorem

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## 1 Preliminaries

**Definition 1.1.**  $\mathbb{N} = \{1, 2, \dots\}$  and  $n^+$  is the image of  $n \in \mathbb{N}$  under the successor function.

## 2 Statement and Theorem

**Theorem 2.1.** *For any set  $X$ , if  $a \in X$  and  $f : X \rightarrow X$ , then there exists a function  $u : \mathbb{N} \rightarrow X$  such that  $u(1) = a$  and  $u(n^+) = f(u(n))$  for all other natural numbers  $n$ .*

*Proof.* Let  $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N} \times X)$  be defined such that  $(1, a) \in c$  and  $(n^+, f(x)) \in c$  whenever  $(n, x) \in c$  for all  $c \in \mathcal{C}$ . It's clear to see that  $\mathcal{C}$  is nonempty since  $\mathbb{N} \times X \in \mathcal{C}$  and so we can form the intersection of all sets in  $\mathcal{C}$  which we'll call  $u$ . Let  $S$  be the set of all natural numbers such that if  $n \in S$  then there exists  $(n, x) \in u$  and if  $(n, w), (n, y) \in u$  then  $w = y$ . We will prove inductively that  $S = \mathbb{N}$  which establishes  $u$  as a function. Furthermore, given how  $u$  is constructed, such a proof gives us exactly the kind of function we are looking for.

Suppose that  $(1, b) \in u$ ,  $a \neq b$ , and consider the set  $M = u \setminus \{(1, b)\}$ . We claim that  $(n^+, f(x)) \in M$  whenever  $(n, x) \in M$ . If that weren't the case then some  $(n^+, f(x))$  would not be present in  $M$  for some  $(n, x) \in M$ . Well by the definition of  $M$ , for all  $\alpha$  we have that  $(\alpha \in u \wedge \alpha \neq (1, b)) \rightarrow \alpha \in M$  which implies that if  $\alpha \notin M$ , then  $\alpha \notin u \vee \alpha = (1, b)$ . Well it can't be the case that  $(1, b) = (n^+, f(x))$  since  $n^+ \neq 1$  for any natural number  $n$ . So then we must conclude that  $(n^+, f(x)) \notin u$  which contradicts  $u$ 's initial construction since  $(n, x) \in u$  by the assumption that  $(n, x) \in M$ . Therefore, since  $(1, a) \in M$  and for any other  $(1, b) \in M$  we know that  $a = b$ , we conclude that  $1 \in S$ .

Now suppose that  $n \in S$  which implies that there is an  $(n, x) \in u$  for at most one  $x$ . It follows from the definition of  $u$  that  $(n^+, f(x)) \in u$ . Now if  $n^+$  isn't in  $S$ , then there exists  $(n^+, y) \in u$  such that  $f(x) \neq y$ . Consider the set  $J = u \setminus \{(n^+, y)\}$  and some  $(m, t) \in J$ . If  $m = n$ , then  $t = x$  since  $n \in S$  and so  $(n^+, f(x)) = (m^+, f(t)) \in J$ . By construction of  $J$ , we note that  $\alpha \notin J \rightarrow (\alpha \notin u \vee \alpha = (n^+, y))$ . So if  $m \neq n$ , then we know that  $m^+ \neq n^+$  by the fourth Peano axiom and so if  $(m^+, f(t)) \notin J$  then we must conclude that  $(m^+, f(t)) \notin u$  which is a contradiction to the construction of  $u$ . So  $n^+ \in S$  and by the principle of mathematical induction we conclude that  $\mathbb{N} = S$ .  $\square$

**Corollary 2.1.** *The function defined in [Theorem 2.1](#) is unique.*

*Proof.* Let  $a \in X$  and  $f : X \rightarrow X$  for some set  $X$ . Suppose that there are two functions  $F : \mathbb{N} \rightarrow X$  and  $G : \mathbb{N} \rightarrow X$  where  $F(1) = G(1) = a$ ,  $F(n^+) = f(F(n))$ , and  $G(n^+) = f(G(n))$  for all natural numbers  $n$ . Let  $F(1) = G(1)$  be the base case for an inductive proof and suppose  $F(n) = G(n)$  for some natural number  $n$ . Well then  $F(n^+) = f(F(n)) = f(G(n)) = G(n^+)$ . So by the principle of mathematical induction,  $F$  and  $G$  are the same function.  $\square$