Distributivity, Associativity, and Commutativity

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In [1], we created some fundamental tools for working with the natural numbers. In this post, we'll establish some of the core ideas that make working with addition and multiplication of the natural numbers such a pleasure.

1 Preliminaries

We start by recalling the theorems from [1] that restated the definitions of addition and multiplication of natural numbers in algebraic terms.

Theorem 1.1. *$n+1 = n^+$ where n^+ is the image of $n \in \mathbb{N}$ under the successor function.*

Theorem 1.2. $n + (m + 1) = (n + m) + 1$

Lemma 1.1. $n \cdot 1 = n$

Theorem 1.3. $n \cdot (m + 1) = n + (n \cdot m)$

1.1 Induction

Recall the following theorem from [1].

Theorem 1.4 (Principle of Mathematical Induction)**.** *If S is an inductive subset of $\mathbb{N}$, then $S = \mathbb{N}$.*

Preliminary Remark: It's often easier to work with Theorem 1.4 by restating it in an algebraically useful form.

Theorem 1.5. *If $S \subseteq \mathbb{N}$, $1 \in S$, and for all $n \in S$ it's the case that $n + 1 \in S$, then $S = \mathbb{N}$.*

Proof. Considering the equality between $n+1$ and n^+ as stated in Theorem 1.1, we can see that S is an inductive subset of $\mathbb{N}$ and so the hypothesis of Theorem 1.4 is fulfilled which leads us to conclude that $S = \mathbb{N}$. $\square$

2 A Few "Nice to Have" Theorems

2.1 Addition

Theorem 2.1 (Associativity of Addition). *$x + (y + z) = (x + y) + z$ for all $x, y, z \in \mathbb{N}$.*

Proof. We proceed by induction on z. Let S be the set of all natural numbers such that for $n \in S$ we have $x + (y + n) = (x + y) + n$. Well by Theorem 1.2, we know that $x + (y + 1) = (x + y) + 1$ and so $1 \in S$. So suppose that $n \in S$. Using Theorem 1.2 and our inductive hypothesis that $n \in S$, we can see that

$$\begin{aligned} x + (y + (n + 1)) &= x + ((y + n) + 1) \\ &= (x + (y + n)) + 1 \\ &= ((x + y) + n) + 1 \\ &= (x + y) + (n + 1) \end{aligned}$$

So by Theorem 1.5, $S = \mathbb{N}$. $\square$

Lemma 2.1. *$1 + n = n + 1$ for all $n \in \mathbb{N}$.*

Proof. We proceed by induction on n and let S be the set of all natural numbers such that $1+n = n+1$. Since $1+1 = 1+1$ we can see that $1 \in S$. Now let $k \in S$ and consider $1 + (k + 1)$. Well by Theorem 1.2 and our inductive hypothesis, $1 + (k + 1) = (1 + k) + 1 = (k + 1) + 1$ and so we conclude that $S = \mathbb{N}$. $\square$

Theorem 2.2 (Commutativity of Addition). *$x + y = y + x$ for all $x, y \in \mathbb{N}$*

Proof. We proceed by induction on y and let S be the set of all natural numbers such that $x + y = y + x$. Our base case of $1 \in S$ is established by Lemma 2.1 and so let's assume that $n \in S$ and consider $x + (n + 1)$. Well then using Theorem 1.2, Lemma 2.1, and our inductive hypothesis, we can see that

$$\begin{aligned} x + (n + 1) &= (x + n) + 1 \\ &= (n + x) + 1 \\ &= n + (x + 1) \\ &= n + (1 + x) \\ &= (n + 1) + x \end{aligned}$$

and so $S = \mathbb{N}$. $\square$

2.2 Multiplication

Preliminary Remark: We'll be referring to $x \cdot y$ as just xy as well as use Theorem 2.2, Theorem 2.1, and Theorem 1.5 without reference throughout these proofs.

2.3 Distributivity

Lemma 2.2 (Left Distributivity of Multiplication)**.** *$x(y+z) = xy + xz$ for all $x, y, z \in \mathbb{N}$.*

Proof. We proceed by induction on z. We use Theorem 1.3 to see that

$$\begin{aligned} x(y+1) &= x + xy \\ &= xy + x \\ &= xy + x \cdot 1 \end{aligned}$$

Now suppose that $x(y+n) = xy + xn$ for some $n \in \mathbb{N}$. We can see from using our inductive hypothesis and Theorem 1.3 that

$$\begin{aligned} x(y+(n+1)) &= x((y+n)+1) \\ &= x + x(y+n) \\ &= x + xy + xn \\ &= xy + x + xn \\ &= xy + x(n+1) \end{aligned}$$

□

Lemma 2.3 (Right Distributivity of Multiplication)**.** *$(y+z)x = yx + zx$ for all $x, y, z \in \mathbb{N}$.*

Proof. We proceed by induction on x. First note that $(y+z)\cdot 1 = y+z = y\cdot 1+z\cdot 1$. Now assume that $(y+z)n = yn + zn$ for some $n \in \mathbb{N}$. We can see from using our inductive hypothesis and Theorem 1.3 that

$$\begin{aligned} (y+z)(n+1) &= (y+z) + (y+z)n \\ &= y + z + yn + zn \\ &= y + yn + z + zn \\ &= y(n+1) + z(n+1) \end{aligned}$$

□

Theorem 2.3 (Distributivity of Multiplication)**.** *$x(y+z) = xy + xz$ and $(y+z)x = yx + zx$ for all $x, y, z \in \mathbb{N}$.*

Proof. This is established by Lemma 2.3 and Lemma 2.2. □

2.4 Associativity and Commutativity

Theorem 2.4 (Associativity of Multiplication)**.** *$x(yz) = (xy)z$ for all $x, y, z \in \mathbb{N}$.*

Proof. We proceed by induction on z. Note that $x(y \cdot 1) = xy = (xy) \cdot 1$ and suppose that $x(yn) = (xy)n$ for some $n \in \mathbb{N}$. Well then by Theorem 2.3 and our inductive hypothesis we have that

$$\begin{aligned} x(y(n+1)) &= x(yn + y) \\ &= x(yn) + xy \\ &= (xy)n + xy \\ &= (xy)(n+1) \end{aligned}$$

□

Lemma 2.4. *$1 \cdot n = n \cdot 1$ for all $n \in \mathbb{N}$*

Proof. We proceed by induction on n. Our base case is established by $1 \cdot 1 = 1 \cdot 1$ and so suppose that $1 \cdot k = k \cdot 1$ for some $k \in \mathbb{N}$ and consider $1 \cdot (k+1)$. Well by Theorem 1.3 and our inductive hypothesis we can see that

$$1 \cdot (k+1) = 1 + k \cdot 1 = 1 + k = k + 1 = (k+1) \cdot 1$$

□

Theorem 2.5 (Commutativity of Multiplication)**.** *$xy = yx$ for all $x, y \in \mathbb{N}$*

Proof. We proceed by induction on y and use Lemma 2.4 as our base case. So suppose that $xn = ny$ for some $n \in \mathbb{N}$ and observe that by our inductive hypothesis, Theorem 2.3, and Lemma 2.4 we have

$$\begin{aligned} x(n+1) &= xn + x \cdot 1 \\ &= nx + 1 \cdot x \\ &= (n+1)x \end{aligned}$$

□

References

[1] Noatmeal. The Recursion Theorem. https://noatmeal.github.io/recursion_theorem/, 2024.