The Recursion Theorem

A useful item to have in one's toolbox while working in ZFC set theory[1] is the ability to create inductive definitions. My goal in this post is to prove a theorem that allows us to do exactly that whenever the proper conditions are present.

1 Preliminaries

Definition 1.1. $\mathbb{N} = \{1, 2, ...\}$ and n^+ is the image of $n \in \mathbb{N}$ under the successor function[2].

Axiom 1.1. *For all $n \in \mathbb{N}$, we have that $n^+ \neq 1$.*

Axiom 1.2. *If $n, m \in \mathbb{N}$ and if $n^+ = m^+$, then $n = m$.*

2 Statement and Theorem

Theorem 2.1. *For any set X, if $a \in X$ and $f : X \to X$, then there exists a function $u : \mathbb{N} \to X$ such that $u(1) = a$ and $u(n^+) = f(u(n))$ for all other natural numbers n.*

Proof. Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N} \times X)$ be defined such that $(1, a) \in c$ and $(n^+, f(x)) \in c$ whenever $(n, x) \in c$ for all $c \in \mathcal{C}$. It's clear to see that $\mathcal{C}$ is nonempty since $\mathbb{N} \times X \in C$ and so we can form the intersection of all sets in $\mathcal{C}$ which we'll call u. Let S be the set of all natural numbers such that if $n \in S$ then there exists $(n, x) \in u$ and if $(n, w), (n, y) \in u$ then $w = y$. We will prove inductively that $S = \mathbb{N}$ which establishes u as a function. Furthermore, given how u is constructed, such a proof gives us exactly the kind of function we are looking for.

Suppose that $(1, b) \in u$, $a \neq b$, and consider the set $M = u \setminus \{(1, b)\}$. We claim that $(n^+, f(x)) \in M$ whenever $(n, x) \in M$. If that weren't the case then some $(n^+, f(x))$ would not be present in M for some $(n, x) \in M$. Well by the definition of M, for all α we have that $(\alpha \in u \wedge \alpha \neq (1, b)) \to \alpha \in M$ which implies that if $\alpha \notin M$, then $\alpha \notin u \vee \alpha = (1, b)$. Well it can't be the case that

[1] Short for Zermelo–Fraenkel set theory. See https://en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory for more information.

[2] If this doesn't make any sense, then a great review of this topic can be found in [1]. The proof of the main theorem for this post can be found in that text as well.

$(1, b) = (n^+, f(x))$ by Axiom 1.1. So then we must conclude that $(n^+, f(x)) \notin u$ which contradicts u's initial construction since $(n, x) \in u$ by the assumption that $(n, x) \in M$. Therefore, since $(1, a) \in M$ and for any other $(1, b) \in M$ we know that $a = b$, we conclude that $1 \in S$.

Now suppose that $n \in S$ which implies that there is an $(n, x) \in u$ for at most one x. It follows from the definition of u that $(n^+, f(x)) \in u$. Now if n^+ isn't in S, then there exists $(n^+, y) \in u$ such that $f(x) \neq y$. Consider the set $J = u \setminus \{(n^+, y)\}$ and some $(m, t) \in J$. If $m = n$, then $t = x$ since $n \in S$ and so $(n^+, f(x)) = (m^+, f(t)) \in J$. By construction of J, we note that $\alpha \notin J \to (\alpha \notin u \vee \alpha = (n^+, y))$. So if $m \neq n$, then we know that $m^+ \neq n^+$ by the Axiom 1.2 and so if $(m^+, f(t)) \notin J$ then we must conclude that $(m^+, f(t)) \notin u$ which is a contradiction to the construction of u. So $n^+ \in S$ and by the principal of mathematical induction we conclude that $\mathbb{N} = S$. $\square$

Corollary 2.1. *The function defined in Theorem 2.1 is unique.*

Proof. Let $a \in X$ and $f : X \to X$ for some set X. Suppose that there are two functions $F : \mathbb{N} \to \mathbb{X}$ and $G : \mathbb{N} \to \mathbb{X}$ where $F(1) = G(1) = a$, $F(n^+) = f(F(n))$, and $G(n^+) = f(G(n))$ for all natural numbers n. Let $F(1) = G(1)$ be the base case for an inductive proof and suppose $F(n) = G(n)$ for some natural number n. Well then $F(n^+) = f(F(n)) = f(G(n)) = G(n^+)$. So by the principle of mathematical induction, F and G are the same function. $\square$

References

[1] Paul R. Halmos. *Naive Set Theory*. Springer New York, NY, 1998.