

# The Recursion Theorem

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A useful item to have in one's toolbox while working in ZFC set theory<sup>1</sup> is the ability to create inductive definitions. My goal in this post is to prove a theorem that allows us to do exactly that whenever the proper conditions are present. We'll also work out a couple of fun examples at the end.

## 1 Preliminaries

If the following material doesn't make much sense, then a great review can be found in the first half of [1]. A proof for the main theorem of this post can be found in that text as well.

### 1.1 Natural Numbers

**Definition 1.1.** A set  $I$  is inductive if and only if  $\emptyset \in I$  and  $\forall y \in I (y \cup \{y\} \in I)$ .

**Example 1.1.** Here are some elements one would find in an inductive set

- $\emptyset$
- $\emptyset \cup \{\emptyset\} = \{\emptyset\}$
- $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$
- $\{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

**Axiom 1.1** (Infinity).  $\exists I (I \text{ is inductive})$

**Axiom 1.2** (Comprehension).  $\forall v \exists y \forall x (x \in y \Leftrightarrow x \in v \wedge \phi(x))$  where  $\phi$  is a formula that  $y$  is not free in.

**Definition 1.2.** Let  $I$  be a set whose existence is guaranteed by [Axiom 1.1](#). We use [Axiom 1.2](#) to form the set

$$\mathbb{N} = \{y \in I : \forall J (J \text{ is inductive} \Rightarrow y \in J)\}$$

One should think of  $\mathbb{N}$  as the “smallest” inductive set<sup>2</sup>. An element of  $\mathbb{N}$  is called a *natural number*.

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<sup>1</sup>Short for Zermelo–Fraenkel set theory. See [https://en.wikipedia.org/wiki/Zermelo-Fraenkel\\_set\\_theory](https://en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory) for more information.

<sup>2</sup>We will prove this in [Theorem 1.3](#)

**Example 1.2.** Figure 1 shows a few compact names for natural numbers that are often used in place of the set theory notation<sup>3</sup>.

Element of $\mathbb{N}$	Compact Name
$\emptyset$	1
$\{\emptyset\}$	2
$\{\emptyset, \{\emptyset\}\}$	3
$\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$	4

Figure 1: Compact names for elements of  $\mathbb{N}$ .

**Definition 1.3.** The function  $S : \mathbb{N} \rightarrow \mathbb{N}$  defined by  $S(n) = n \cup \{n\}$  for all  $n \in \mathbb{N}$  is called the *successor function on  $\mathbb{N}$* . For  $n \in \mathbb{N}$ , we will refer  $S(n)$  as  $n^+$ .

**Example 1.3.** Recall Example 1.1 and Example 1.2. One can deduce that

- $1^+ = 2$
- $2^+ = 3$
- $3^+ = 4$

and we'll define two more compact names here to be used in an informative example at the end of the post:

- $4^+ = 5$
- $5^+ = 6$

## 1.2 Peano Axioms

While we prove and refer to the following as theorems, it is common nomenclature to refer to these as *The Five Peano Axioms*.

**Theorem 1.1.**  $1 \in \mathbb{N}$

*Proof.* Since 1 is in every inductive set, then  $1 \in \mathbb{N}$ . □

**Theorem 1.2.** If  $n \in \mathbb{N}$ , then  $n^+ \in \mathbb{N}$ .

*Proof.* Let  $n \in \mathbb{N}$ . By definition  $\forall I(I \text{ is inductive} \Rightarrow n \in I)$ , and so for any inductive set  $K$  we can conclude that  $n^+ \in K$  since  $n \in K$ . Recall that  $n \in I$  from Definition 1.2. By definition,  $n^+ \in I$ . So we conclude from these two facts that  $n^+ \in \mathbb{N}$ . □

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<sup>3</sup>Note that 1 is considered the first natural number in this post.

**Theorem 1.3** (Principle of Mathematical Induction). *If  $S$  is an inductive subset of  $\mathbb{N}$ , then  $S = \mathbb{N}$ .*

*Proof.* We will argue by subset inclusions that  $S = \mathbb{N}$ . By hypothesis, we are given that  $S \subseteq \mathbb{N}$ . Now suppose that  $n \in \mathbb{N}$ . By hypothesis,  $S$  is an inductive set and so by definition of  $\mathbb{N}$ ,  $n \in S$ . Hence,  $S = \mathbb{N}$ .  $\square$

**Definition 1.4.** We refer to invoking [Theorem 1.3](#) as an *argument by induction*.

**Theorem 1.4.** *For all  $n \in \mathbb{N}$ , we have that  $n^+ \neq 1$ .*

*Proof.* Let  $n \in \mathbb{N}$ . Recall that  $n^+ = n \cup \{n\}$  and so  $n \in n^+$ . Well  $n \notin \emptyset$  and so it cannot be the case that  $n^+ = 1$ . Therefore  $n^+ \neq 1$ .  $\square$

**Definition 1.5.** A set  $T$  is transitive if and only if  $x \in T \Rightarrow x \subseteq T$ .

**Lemma 1.1.** *Every natural number is transitive.*

*Proof.* We argue by induction. Let  $S$  be the set of all transitive natural numbers.  $1 \in S$  trivially. Suppose  $n \in S$ . Since  $n^+ = n \cup \{n\}$ , then for all  $x \in n^+$  either  $x \in n$  or  $x = n$ . If  $x \in n$ , then by our inductive hypothesis that  $n \in S$ , we know that  $x \subseteq n$  which allows us to conclude that  $x \subseteq n^+$ . If  $x = n$ , then we also conclude that  $x \subseteq n^+$ . Therefore,  $n^+ \in S$  which completes the inductive argument.  $\square$

**Theorem 1.5.** *If  $n, m \in \mathbb{N}$  and  $n^+ = m^+$ , then  $n = m$ .*

*Proof.* Suppose the hypothesis is true. Well then  $n \in n^+ \Rightarrow n \in m^+$  and so either  $n \in m$  or  $n = m$ . By the same type of observation, either  $m \in n$  or  $m = n$ . If  $n \neq m$ , then  $m \in n$  and  $n \in m$ . Well then [Lemma 1.1](#) tells us that  $m \subseteq n$  and  $n \subseteq m$  and so  $n = m$ , a contradiction. Therefore it must be the case that  $n = m$ .  $\square$

## 2 Existence and Uniqueness

**Theorem 2.1** (Recursion). *For any set  $X$ , if  $a \in X$  and  $f : X \rightarrow X$ , then there exists a function  $u : \mathbb{N} \rightarrow X$  such that  $u(1) = a$  and  $u(n^+) = f(u(n))$  for all other natural numbers  $n$ .*

*Proof.* Let  $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N} \times X)$  be defined such that  $(1, a) \in c$  and  $(n^+, f(x)) \in c$  whenever  $(n, x) \in c$  for all  $c \in \mathcal{C}$ . It's clear to see that  $\mathcal{C}$  is nonempty since  $\mathbb{N} \times X \in \mathcal{C}$  and so we can form the intersection of all sets in  $\mathcal{C}$  which we'll call  $u$ .

Let  $S$  be the set of all natural numbers such that if  $n \in S$  then there exists  $(n, x) \in u$  and if  $(n, w), (n, y) \in u$  then  $w = y$ . We will prove inductively that  $S = \mathbb{N}$  which establishes that  $u$  is a function. Furthermore, given how  $u$  is constructed, such a proof gives us exactly the kind of function we are looking for.

Suppose that  $(1, b) \in u$ ,  $a \neq b$ , and consider the set  $M = u \setminus \{(1, b)\}$ . We claim that  $(n^+, f(x)) \in M$  whenever  $(n, x) \in M$ . Since  $(1, a) \in M$ , this will give us that  $M \in \mathcal{C}$  and  $u \not\subseteq M$  which is a contradiction to  $u$  being the intersection of all members of  $\mathcal{C}$ . That will then allow us to say that  $(1, a) \in u$  and for all  $(1, b) \in u$  it would be the case that  $a = b$ .

So suppose that  $(n^+, f(x))$  is not present in  $M$  for some  $(n, x) \in M$ . Well by the definition of  $M$ , for all  $\alpha$  we have that  $(\alpha \in u \wedge \alpha \neq (1, b)) \rightarrow \alpha \in M$  which implies by contraposition that if  $\alpha \notin M$ , then  $\alpha \notin u \vee \alpha = (1, b)$ . Well it can't be the case that  $(n^+, f(x)) = (1, b)$  by [Theorem 1.4](#). So then we must conclude that  $(n^+, f(x)) \notin u$  which contradicts  $u$ 's construction since  $(n, x) \in u$  by the assumption that  $(n, x) \in M$ . We now must conclude that  $M \in \mathcal{C}$  which gives us the base case for our inductive argument as discussed in the previous paragraph.

Now suppose that  $n \in S$  which implies that there is an  $(n, x) \in u$  for at most one  $x$ . It follows from the definition of  $u$  that  $(n^+, f(x)) \in u$ . Now if  $n^+$  isn't in  $S$ , then there exists  $(n^+, y) \in u$  such that  $f(x) \neq y$ . Suppose that this is the case and consider the set  $J = u \setminus \{(n^+, y)\}$  with some element  $(m, t) \in J$ .

If  $m = n$ , then  $t = x$  since  $n \in S$  and so  $(n^+, f(x)) = (m^+, f(t)) \in J$ . By construction of  $J$ , we note that  $\alpha \notin J \rightarrow (\alpha \notin u \vee \alpha = (n^+, y))$  for all  $\alpha$ . So if  $m \neq n$ , then we know that  $m^+ \neq n^+$  by [Theorem 1.5](#) which means that if  $(m^+, f(t)) \notin J$  then we must conclude that  $(m^+, f(t)) \notin u$  which is a contradiction to the construction of  $u$  and so  $(m^+, f(t)) \in J$ .

Therefore in every case,  $(m^+, f(t)) \in J$  whenever  $(m, t) \in J$ . Since it also the case that  $(1, a) \in J$  since  $n^+ \neq 1$  by [Theorem 1.4](#), we conclude that  $J \in \mathcal{C}$  and  $u \not\subseteq J$  which is a contradiction to  $u$ 's construction as being the intersection of all members of  $\mathcal{C}$ .

So  $n^+ \in S$  and by the principle of mathematical induction we conclude that  $\mathbb{N} = S$ .  $\square$

**Corollary 2.1.** *The function defined in [Theorem 2.1](#) is unique.*

*Proof.* Let  $a \in X$  and  $f : X \rightarrow X$  for some set  $X$ . Suppose that there are two functions  $F : \mathbb{N} \rightarrow X$  and  $G : \mathbb{N} \rightarrow X$  given by [Theorem 2.1](#)

where  $F(1) = G(1) = a$  and for all other natural numbers  $n$  we have that  $F(n^+) = f(F(n))$  and  $G(n^+) = f(G(n))$ . Let  $F(1) = G(1)$  be the base case for an inductive proof and suppose that  $F(n) = G(n)$  for some natural number  $n$ . Well then  $F(n^+) = f(F(n)) = f(G(n)) = G(n^+)$  by the inductive hypothesis. So by the principle of mathematical induction,  $F$  and  $G$  are the same function.  $\square$

### 3 Examples

Let's define and work with a couple of familiar concepts using what we've shown.

#### 3.1 Addition

**Definition 3.1.** For any natural number  $n \in \mathbb{N}$ , we have  $n^+ \in \mathbb{N}$  by [Theorem 1.2](#). If we apply  $n^+$  and the successor function on  $\mathbb{N}$  to [Theorem 2.1](#) and [Corollary 2.1](#), we can define a unique function  $s_n : \mathbb{N} \rightarrow \mathbb{N}$  where  $s_n(1) = n^+$  and  $s_n(m^+) = s_n(m)^+$  for all  $m \in \mathbb{N}$ .

**Definition 3.2** (Addition). Let  $+$  :  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be the function that maps  $+(n, m)$  to  $s_n(m)$  for all  $n, m \in \mathbb{N}$ . We almost always write this using inline notation as  $n + m$  and call it the *sum of  $n$  and  $m$* .

**Preliminary Remark:** In an effort to recover some familiar looking calculations we prove the following lemma and theorem.

**Lemma 3.1.**  $n + 1 = n^+$

*Proof.*  $n + 1 = s_n(1) = n^+$  □

**Theorem 3.1.**  $n + (m + 1) = (n + m) + 1$

*Proof.*

$$\begin{aligned} n + (m + 1) &= s_n(m + 1) \text{ by Definition 3.2} \\ &= s_n(m^+) \text{ by Lemma 3.1} \\ &= (s_n(m))^+ \text{ by Definition 3.1} \\ &= s_n(m) + 1 \text{ by Lemma 3.1} \\ &= (n + m) + 1 \text{ by Definition 3.2} \end{aligned}$$

□

**Example 3.1.** Recall [Example 1.3](#). One can deduce that

- $2 = 1^+ = 1 + 1$
- $3 = 2^+ = 2 + 1$
- $4 = 3^+ = 3 + 1$
- $5 = 4^+ = 4 + 1$
- $6 = 5^+ = 5 + 1$

**Example 3.2.** Recall [Example 3.1](#). We can see that

$$2 + 2 = 2 + (1 + 1) = (2 + 1) + 1 = 3 + 1 = 4$$

## 3.2 Multiplication

**Definition 3.3.** For every  $n \in \mathbb{N}$ , we take  $s_n : \mathbb{N} \rightarrow \mathbb{N}$  from [Definition 3.2](#) and define  $p_n : \mathbb{N} \rightarrow \mathbb{N}$  where  $p_n(1) = n$  and  $p_n(m^+) = n + p_n(m)$ .

**Definition 3.4** (Multiplication). Let  $\cdot : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  be the function that maps  $\cdot(n, m)$  to  $p_n(m)$  for all  $n, m \in \mathbb{N}$ . We almost always write this using inline notation as  $n \cdot m$  and call it the *product of  $n$  and  $m$* .

**Preliminary Remark:** In an effort to recover some familiar looking calculations we prove the following lemma and theorem.

**Lemma 3.2.**  $n \cdot 1 = n$

*Proof.*  $n \cdot 1 = p_n(1) = n$  □

**Theorem 3.2.**  $n \cdot (m + 1) = n + (n \cdot m)$

*Proof.*

$$\begin{aligned} n \cdot (m + 1) &= p_n(m + 1) \text{ by } \text{Definition 3.4} \\ &= p_n(m^+) \text{ by } \text{Lemma 3.1} \\ &= n + p_n(m) \text{ by } \text{Definition 3.3} \\ &= n + (n \cdot m) \text{ by } \text{Definition 3.4} \end{aligned}$$

□

**Example 3.3.** Recall [Example 3.1](#) once more. We can see that

$$\begin{aligned} 2 \cdot 3 &= 2 \cdot (2 + 1) \\ &= 2 + (2 \cdot 2) \\ &= 2 + (2 \cdot (1 + 1)) \\ &= 2 + (2 + (2 \cdot 1)) \\ &= 2 + (2 + 2) \\ &= 2 + 4 \text{ by } \text{Example 3.2} \\ &= 2 + (3 + 1) \\ &= (2 + 3) + 1 \\ &= (2 + (2 + 1)) + 1 \\ &= ((2 + 2) + 1) + 1 \\ &= (4 + 1) + 1 \\ &= 5 + 1 = 6 \end{aligned}$$

## References

- [1] Paul R. Halmos. *Naive Set Theory*. Springer New York, NY, 1998.