

The Recursion Theorem

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A useful item to have in one's toolbox while working in ZFC set theory¹ is the ability to create inductive definitions. My goal in this post is to prove a theorem that allows us to do exactly that whenever the proper conditions are present. We'll also work out a couple of fun examples at the end.

1 Preliminaries

If the following material doesn't make much sense, then a great review can be found in the first half of [1]. The proof of the main theorem for this post can be found in that text as well.

1.1 Natural Numbers

We use the following two axioms of ZFC to define the natural numbers.

Axiom 1.1 (Infinity). $\exists x(\emptyset \in x \wedge \forall y \in x(y \cup \{y\} \in x))$

Axiom 1.2 (Comprehension). $\forall v \exists y \forall x(x \in y \Leftrightarrow x \in v \wedge \phi(x))$ where ϕ is a formula that y is not free in.

Definition 1.1. A set I is inductive if and only if $\emptyset \in I$ and $\forall y \in I(y \cup \{y\} \in I)$.

Example 1.1. Some examples of elements of an inductive set are

- \emptyset
- $\emptyset \cup \{\emptyset\} = \{\emptyset\}$
- $\{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$
- $\{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Definition 1.2. Let J be the set whose existence is guaranteed by [Axiom 1.1](#). We use [Axiom 1.2](#) to form the set

$$\mathbb{N} = \{y \in J : \forall I(I \text{ is inductive} \Rightarrow y \in I)\}$$

One should think of \mathbb{N} as the “smallest” inductive set. An element of \mathbb{N} is called a *natural number*.

¹Short for Zermelo–Fraenkel set theory. See https://en.wikipedia.org/wiki/Zermelo-Fraenkel_set_theory for more information.

Example 1.2. Figure 1 shows a few compact names for natural numbers that are often used in place of the set theory notation².

| Element of \mathbb{N} | Compact Name |
|--|--------------|
| \emptyset | 1 |
| $\{\emptyset\}$ | 2 |
| $\{\emptyset, \{\emptyset\}\}$ | 3 |
| $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ | 4 |

Figure 1: Compact names for elements of \mathbb{N} .

Definition 1.3. The function $S : \mathbb{N} \rightarrow \mathbb{N}$ defined by $S(n) = n \cup \{n\}$ for all $n \in \mathbb{N}$ is called the *successor function on \mathbb{N}* . For $n \in \mathbb{N}$, we will refer $S(n)$ as n^+ .

Example 1.3. Recall Example 1.1 and Example 1.2. One can deduce that

- $1^+ = 2$
- $2^+ = 3$
- $3^+ = 4$
- $4^+ = 5$
- $5^+ = 6$

1.2 Peano Axioms

While we prove and refer to the following as theorems, it is common nomenclature to refer to them as *The Five Peano Axioms*.

Theorem 1.1. $1 \in \mathbb{N}$

Theorem 1.2. If $n \in \mathbb{N}$, then $n^+ \in \mathbb{N}$.

Theorem 1.3 (The Principle of Mathematical Induction). If $S \subseteq \mathbb{N}$, $1 \in S$, and $n \in S \Rightarrow n^+ \in S$, then $S = \mathbb{N}$.

Theorem 1.4. For all $n \in \mathbb{N}$, we have that $n^+ \neq 1$.

Theorem 1.5. If $n, m \in \mathbb{N}$ and $n^+ = m^+$, then $n = m$.

²Note that 1 is considered the first natural number in this post.

2 Existence and Uniqueness

Theorem 2.1. *For any set X , if $a \in X$ and $f : X \rightarrow X$, then there exists a function $u : \mathbb{N} \rightarrow X$ such that $u(1) = a$ and $u(n^+) = f(u(n))$ for all other natural numbers n .*

Proof. Let $\mathcal{C} \subseteq \mathcal{P}(\mathbb{N} \times X)$ be defined such that $(1, a) \in c$ and $(n^+, f(x)) \in c$ whenever $(n, x) \in c$ for all $c \in \mathcal{C}$. It's clear to see that \mathcal{C} is nonempty since $\mathbb{N} \times X \in \mathcal{C}$ and so we can form the intersection of all sets in \mathcal{C} which we'll call u .

Let S be the set of all natural numbers such that if $n \in S$ then there exists $(n, x) \in u$ and if $(n, w), (n, y) \in u$ then $w = y$. We will prove inductively that $S = \mathbb{N}$ which establishes that u is a function. Furthermore, given how u is constructed, such a proof gives us exactly the kind of function we are looking for.

Suppose that $(1, b) \in u$, $a \neq b$, and consider the set $M = u \setminus \{(1, b)\}$. We claim that $(n^+, f(x)) \in M$ whenever $(n, x) \in M$. If that weren't the case then some $(n^+, f(x))$ would not be present in M for some $(n, x) \in M$. Well by the definition of M , for all α we have that $(\alpha \in u \wedge \alpha \neq (1, b)) \rightarrow \alpha \in M$ which implies that if $\alpha \notin M$, then $\alpha \notin u \vee \alpha = (1, b)$. Well it can't be the case that $(1, b) = (n^+, f(x))$ by [Theorem 1.4](#). So then we must conclude that $(n^+, f(x)) \notin u$ which contradicts u 's initial construction since $(n, x) \in u$ by the assumption that $(n, x) \in M$. Therefore, since $(1, a) \in M$ and for any other $(1, b) \in M$ we know that $a = b$, we conclude that $1 \in S$.

Now suppose that $n \in S$ which implies that there is an $(n, x) \in u$ for at most one x . It follows from the definition of u that $(n^+, f(x)) \in u$. Now if n^+ isn't in S , then there exists $(n^+, y) \in u$ such that $f(x) \neq y$. Consider the set $J = u \setminus \{(n^+, y)\}$ and some $(m, t) \in J$. If $m = n$, then $t = x$ since $n \in S$ and so $(n^+, f(x)) = (m^+, f(t)) \in J$. By construction of J , we note that $\alpha \notin J \rightarrow (\alpha \notin u \vee \alpha = (n^+, y))$. So if $m \neq n$, then we know that $m^+ \neq n^+$ by the [Theorem 1.5](#) and so if $(m^+, f(t)) \notin J$ then we must conclude that $(m^+, f(t)) \notin u$ which is a contradiction to the construction of u . So $n^+ \in S$ and by the principle of mathematical induction we conclude that $\mathbb{N} = S$. \square

Corollary 2.1. *The function defined in [Theorem 2.1](#) is unique.*

Proof. Let $a \in X$ and $f : X \rightarrow X$ for some set X . Suppose that there are two functions $F : \mathbb{N} \rightarrow X$ and $G : \mathbb{N} \rightarrow X$ where $F(1) = G(1) = a$, $F(n^+) = f(F(n))$, and $G(n^+) = f(G(n))$ for all natural numbers n . Let $F(1) = G(1)$ be the base case for an inductive proof and suppose that $F(n) = G(n)$ for some natural number n . Well then $F(n^+) = f(F(n)) = f(G(n)) = G(n^+)$. So by the principle of mathematical induction, F and G are the same function. \square

3 Examples

Let's define a couple of familiar concepts by induction.

Definition 3.1 (Addition). For any natural number $n \in \mathbb{N}$, we have $n^+ \in \mathbb{N}$ by [Theorem 1.2](#). If we apply n^+ and S to [Theorem 2.1](#) and [Corollary 2.1](#), we can define a unique function $s_n : \mathbb{N} \rightarrow \mathbb{N}$ where $s_n(1) = n^+$ and $s_n(m^+) = s_n(m)^+$ for all $m \in \mathbb{N}$. For any such function s_n and $m \in \mathbb{N}$, we refer to $s_n(m)$ as $n + m$ and call it the *sum of n and m* .

Example 3.1. Recall [Example 1.3](#). We can see that

$$2 + 2 = s_2(2) = s_2(1)^+ = (2^+)^+ = 3^+ = 4$$

Definition 3.2 (Multiplication). For every $n \in \mathbb{N}$, we take $s_n : \mathbb{N} \rightarrow \mathbb{N}$ from [Definition 3.1](#) and we define $p_n : \mathbb{N} \rightarrow \mathbb{N}$ where $p_n(1) = n$ and $p_n(m^+) = s_n(p_n(m)) = n + p_n(m)$. We refer to $p_n(m)$ as $n \cdot m$. We call this the *product of n and m* or *n multiplied by m* .

Example 3.2. Recall [Example 3.1](#). We can see that

$$\begin{aligned} 2 \cdot 3 &= 2 + (2 \cdot 2) \\ &= 2 + (2 + (2 \cdot 1)) \\ &= 2 + (2 + 2) \\ &= 2 + 4 \text{ by } \text{Example 3.1} \\ &= (2 + 3)^+ \\ &= ((2 + 2)^+)^+ \\ &= (4^+)^+ \text{ by } \text{Example 3.1} \\ &= 6 \end{aligned}$$

References

- [1] Paul R. Halmos. *Naive Set Theory*. Springer New York, NY, 1998.