# roller coaster tycoon 2

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# Contents

# 1 Formulation

## 1.1 Assumptions

- The terminology 'smooth' was used to describe the transition between pieces of track. It is assumed that this means they are both at the same location in space, meaning there will be no gaps, and that their gradient will be the same at the point of intersection, so that there is no sudden change gradient.
- The task sheet specified "The beginning and end sections... have been erected and are in a perfectly straight alignment". While this did clarify that the first, and end pieces of track were parallel, it did not state whether they had a slope. Therefore it was assumed that the start and end pieces of track had a gradient of 0.

## 1.2 Observations

- The track was divided into 3 segments. The first segment's start was partially undefined, say $(x, 80)$, and ended at $(0, 80)$. This was referred to as the "first" segment. The "middle" segment started at $(0, 80)$, and continued to $(150, 30)$. The final section stared at $(150, 30)$, and continued for an undefined length, say until $(x, 30)$. This final section was referred to as the "end".

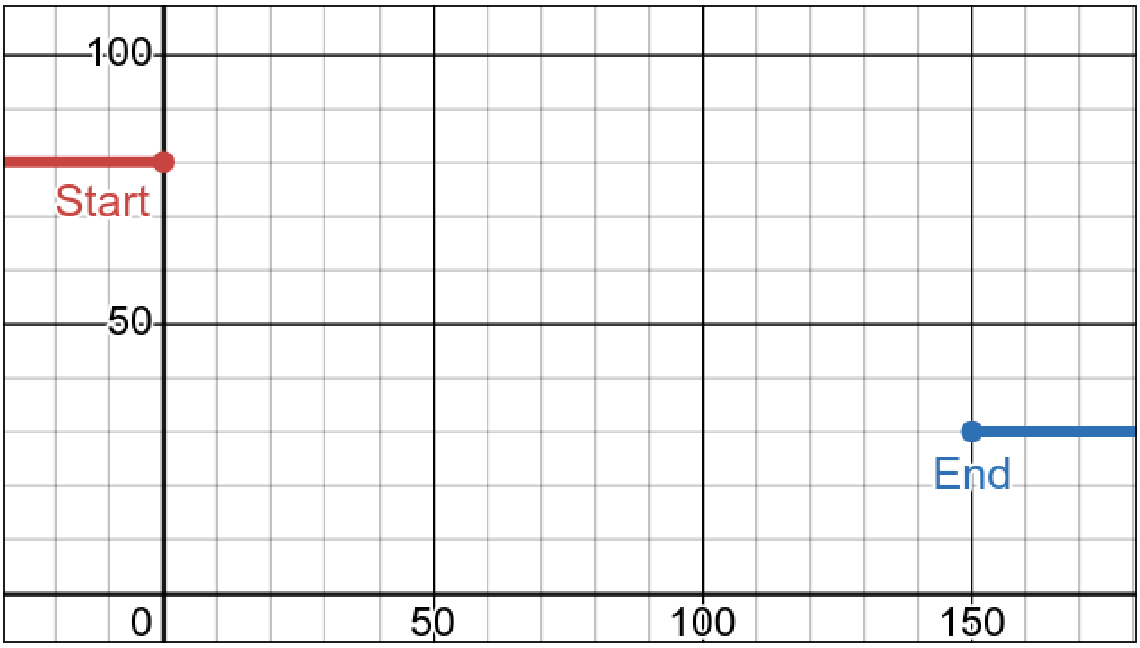


Figure 1: The proposed start and end sections of the track

- The roller coaster could be faithfully modelled in 2D to simplify calculations as the width of the track and much of its geometry were not relevant in this case and we were not provided with specifications regarding the width available.
- The task sheet defined the success criteria as causing the maximum amount of exhilaration, caused by "swift changes in direction, height and steepness".
- It was required that the track be constructed of at least 3 or more types of functions, including at least two types that are covered in Unit 3 Topic 2. The functions covered were exponential, logarithms, and trigonometric functions, like sine, cosine, and tangent.
- It was decided that any calculations involving trigonometric functions would use radians, rather than degrees, in order for them to naturally oscillate more with smaller change in $x$.

- In order for the roller coaster to have sufficient speed to overcome any peaks or hills in the track, each successive peak should be at a lower height than any preceding it.

### 1.3 Translation of aspects to Mathematical concepts and techniques

- Since the roller coaster had been assumed to be 2D, it's track could be represented on the Cartesian plane. This allowed us to use desmos to graph its track and perform calculations by letting 1 unit be 1 meter.
- The derivative function of modelled section of track could be used to determine the gradient at that point and therefore be used to determine if the track exceeded the specified "Maximum Slope for safety" requirement provided by the task sheet. In this case, it was $-2$.
- "Swift changes in direction, height and steepness" could be translated to swift changes in the $y-$axis, and gradient. However, there was a maximum gradient specified that must be considered.
- In order to achieve as much of a thrill as possible, the maximum slope should be reached whenever possible and appropriate.
- Cubic in general form

## 2 Solve

### 2.1 Modelling in Desmos

#### 2.1.1 The First Function

- Considering that the starting points for the middle section were already 80m in the air, it was determined that climbing further was futile as there was already sufficient height for the maximum gradient to be reached for a reasonable duration and the anticipation had already been built during the climb to the beginning of the middle section.
- By defining the cubic generated as $f(x)$, desmos could generate the derivative function, $f'(x)$, automatically. Therefore by checking $f'(x)$ for intercepts with y=-2, it could be determined whether the generated cubic exceeded the maximum gradient.
- Clearly the first point will be $(80, 0)$ to connect with the start of the first segment. However we required for $f'(0) = 0$ so the gradient so that the middle segment could smoothly connect to the first segment. By differentiating the general form of a cubic, it was found that $f'(x) = 3ax^2 + 2bx + c$. The only part of the equation that did not have a coefficient of $x^n$ is $c$, therefore $c$ must be equal to 0 for $f'(0) = 0$.
- It was considered that during the first drop, a trigonometric function could be added so that more variability in the gradient could occur, therefore causing more of a thrill.
- The sine function was chosen as it could be easily differentiated to the cosine function.
- This meant that the second turning point of the cubic had to be placed so that it would coincide with one of the lower turning points of the sine function.
- However the cubic was yet to be defined. We required maximum gradient during the drop. Since the derivative function was a quadratic, its lowest point could be considered the maximum gradient reached by the original.
- The $x$-coordinates of the turning points were found using the *vertex* formula, which stated that for turning point $(x, y)$, $x = \frac{-b}{2a}$.
- Considering the derivative function, $f'(x) = 3ax^2 + 2bx$, by substituting the corresponding coefficients of $x^2$ and $x$ as $a$, and $b$ into the vertex formula respectively, it was found that the $x$-coordinates of the turning point were $\frac{-2b}{2\times 3a} = \frac{-b}{3a}$.

- Therefore by finding $f'(\frac{-b}{3a}) = -2$, a generic function could be found with minimum gradient -2, located at turning point. It was found that $f'(\frac{-b}{3a}) = -2$ simplified to $b = \pm\sqrt{6a}$. Therefore the cubic $f(x) = ax^3 \pm \sqrt{6a}x + d, a \geq 0$ is found.

- It was decided that the function inserted would be $f(x) = ax^3 - \sqrt{6a}x + 80, a \geq 0$ as it placed the "drop" on the right hand side of the start of the coaster, and aligned the point on both tracks where their gradients were the same.

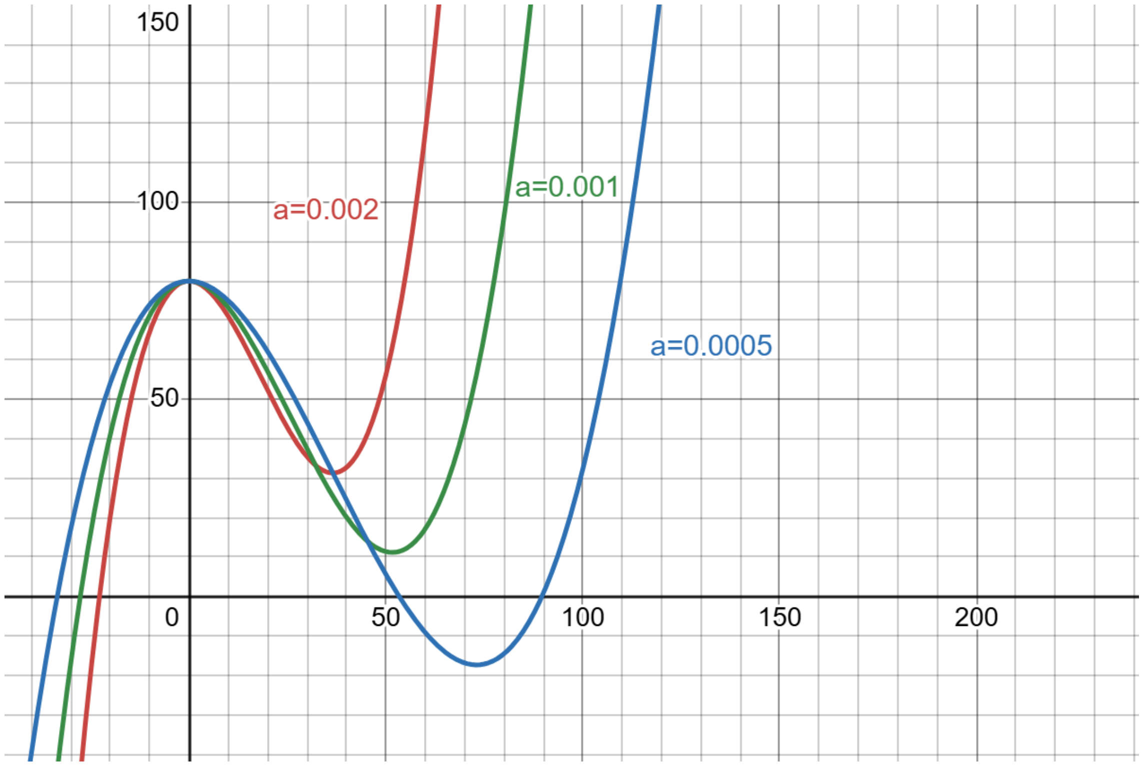


Figure 2: Graphs of $f(x) = ax^3 - \sqrt{6a}x^2 + 80$ with various **a** values

- The end of the first function was chosen to be $f(30)$ as it was observed that letting it proceed further may have resulted in the end of the track being too low to the ground for a thrill to be achieved.

- It was required that the turning point be at $x = 30$ so $f'(30) = 0$ was considered. Solving for $a$ gave $a = \frac{2}{675}$

- Substituting this into $f(x)$ gave $f(x) = \frac{2}{675}x^3 - \frac{2}{15}x^2 + 80$.

#### 2.1.2 The Second Function

- To connect the first and second functions, a point must be found where their gradients match.

- Consider $s(x) = a\sin(b(x + c)) + d$. Therefore $s'(x) = ab\cos(b(x + c))$. Clearly the minimum gradient is defined by $ab \times -1$, where $a, b \epsilon \mathbf{N}$, as $a$ and $b$ both multiply with $\cos(b(x + c))$ which naturally oscillates between 1, and -1.

- However, if the function was allowed to continue for a period or more, the peak of the next oscillation would be smaller than the previous one. Therefore if the function was to continue, it must be modified in order for each successive peak to be lower. This was achieved though the creation of a new function, $c(x) = a\sin(b(x + c)) + d + gx$, so that $c'(x) = ab\cos(b(x + c)) + g$.

- Clearly now when $g < 0$, the function will slope downwards throughout its oscillation.

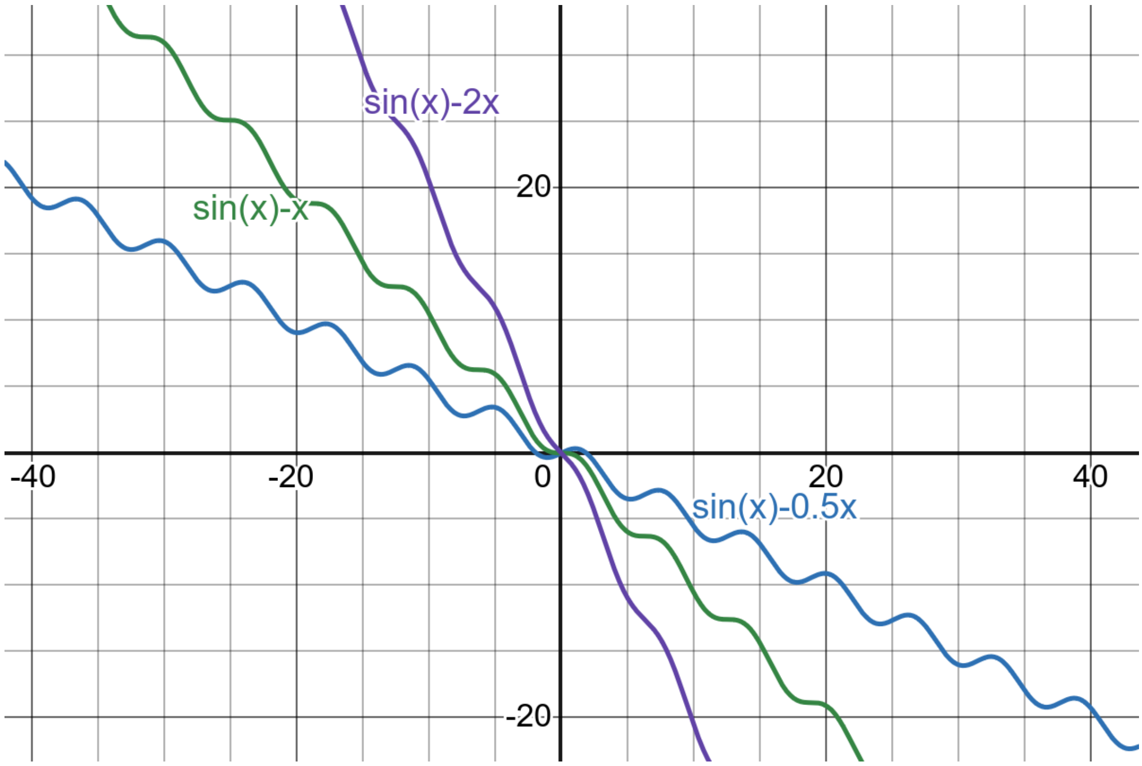


Figure 3: The function $y = sin(x)$ with various slopes applied

- By letting $g = -0.1$, suitable values for $a$ and $b$ can be resolved. Clearly $ab \times -1 - 0.1 = -2$ for minimum gradient to be achieved. Therefore through trial and error in Desmos it was found that for $c'(x) = -2$, $a = 9.5, b = 0.2$ were appropriate. Therefore all that was left to do was to let $f'(x) = c'(x)$, shifting horizontally until the minimums were at the same $x$ co-ordinates, then shift vertically shift using $d$ until the functions met.

- It was found that $9.5 \times 0.2\cos(0.2(30 + c)) - 0.1 = 0 \Rightarrow c = -6.174775557$, and therefore $9.5sin(0.2(30 - 6.174775557)) + d + 3 = 40 \Rightarrow d = 52.48683298$

### 2.1.3 The Third Function

- It was decided that the third function would be an exponential that travelled downwards towards the ground, becoming more steep in the process.

- Considering the general form of an exponential, $g(x) = ae^{b(x+c)} + d$, it was found that solving for variables $a, b, c$ and $d$ would require significant computation.

- Therefore it was decided that only $a$ and $b$ would need to be solved for, while $c = 0$, and $d = 0$.

- The point $(80, 0)$ was considered the origin for $g(x)$ during computation, so that initial requirements such as $g(80) = c(80)$ became $g(0) = c(80)$.

- Now that $g(x) = ae^{bx}$, $g'(x) = abe^{bx}$, we could solve for $a$ and $b$ using simultaneous equations.

- It was found that $a = \frac{c'(80)}{be^{80b}}$, therefore $g(80) = \frac{c'(80)}{be^{80b}} \times e^{80b}$. This gives $b = \frac{c'(80)}{c(80)} \approx -0.023308398873$.

- Since $a$ is a function of $b$, this also acts as a solution for $a$.

- However this solution causes the exponential to intersect with the $x-$axis, or the ground in this scenario, prematurely.

- Attempting to insert another condition such as $g(50) = 10$ causes the equation to become over constrained, therefore another technique must be used.

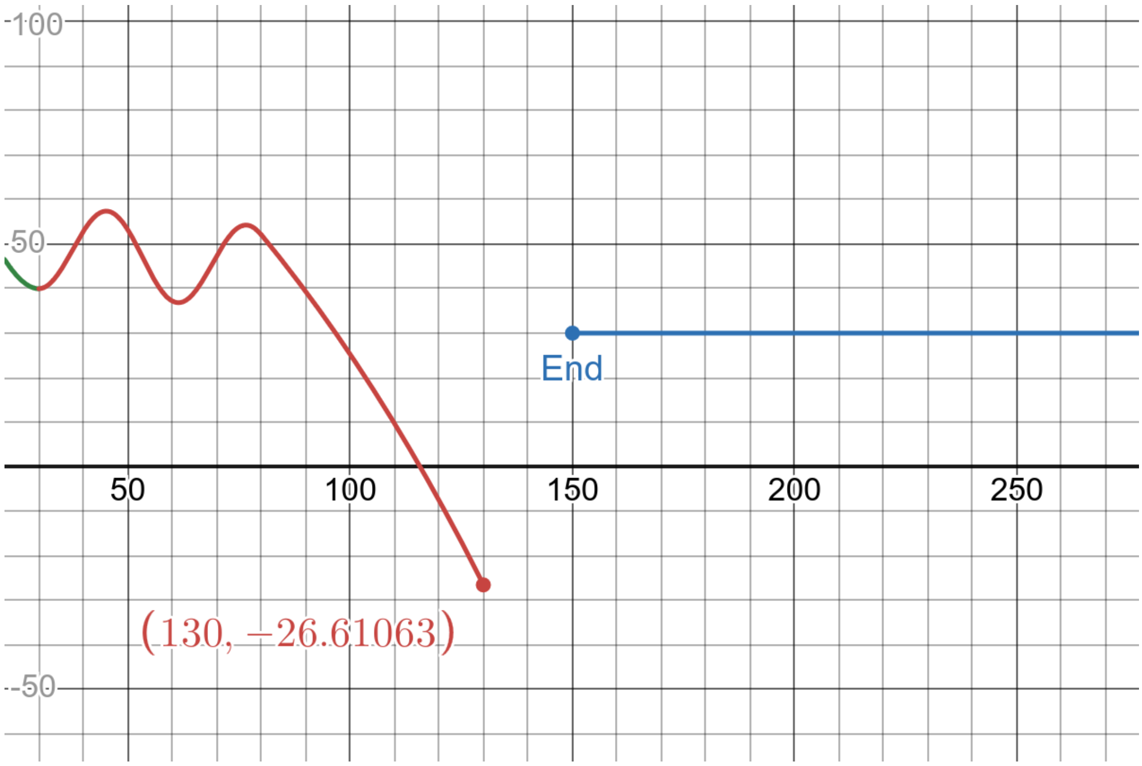


Figure 4: Graph of $g(x)$ prematurely intersects with the $x-$axis

- By considering $s'(\lambda) = g'(0)$, a suitable $x$ value for the starting point of the function was defined, which did not cause over constraint when later conditions were established.

- This was rearranged to give $a = \frac{s'(\lambda)}{b}$, which can be substituted into $g(\gamma) = -2$, which gave $g'(\gamma) = s'(\lambda)e^{\gamma b}$.

- Solving for $b$ gave $b = \frac{1}{\gamma}\ln(\frac{-2}{s'(\lambda)})$, which was then substituted back into $g(x) = ae^{bx}$ to give $g(x) = \frac{s'(\lambda)}{\frac{1}{\gamma}\ln(\frac{-2}{s'(\lambda)})} \cdot e^{\frac{1}{\gamma}\cdot\ln(\frac{-2}{s'(\lambda)})\cdot x}$, where $\gamma$ is the distance that the function takes to reach $g'(x) = -2$, and $\lambda$ is the x value that the function starts at, and joins with the previous function, $s(x)$.

- $x$ was replaced by $(x - \lambda)$ to horizontally shift, and $s(\lambda) - a$ was added to vertically shift so that $g(\lambda) = s(\lambda)$.

- After inserting the equation into desmos, suitable values for $\lambda$ and $\gamma$ were found. It was required that the equation terminate at $y = 10$ so that there would be suitable room for the final function to revert direction and climb. Therefore $g(\lambda + \gamma) = 10$, where $\gamma = 130 - \lambda$ was considered, as this would ensure that $g'(\lambda - \lambda - 130) = g'(130) = -2$, while still allowing for $\lambda$ to be altered until a suitable answer was found.

- It was found that for $g'(130) = -2$ and $g(130) = 10$, $\lambda = 77.2487110854776$. Therefore $g(x) = \frac{s'(\lambda)}{\frac{1}{\gamma}\ln(\frac{-2}{s'(\lambda)})} \cdot e^{\frac{1}{\gamma}\cdot\ln(\frac{-2}{s'(\lambda)})\cdot(x-\lambda)} + s(\lambda) - \frac{s'(\lambda)}{\frac{1}{\gamma}}\ln\left(\frac{-2}{s'(\lambda)}\right)$ where $\lambda = 77.2487110854776$, and $\gamma = 130 - \lambda$.

#### 2.1.4 The Fourth Function

- The final function was decided to be a cubic function, as multiple turning points were required, and the process for defining a cubic with constraints had already been established when creating the first function.

- It was required that for the cubic function, say $f_2(x) = a_2x^3 + b_2x^2 + C_2x + d_2$, $f_2'(130) = g'(130)$, $f_2(130) = g(130)$, $f_2'(150) = 0$, and $f_2(150) = 30$.

- However to simplify calculations, it was considered that $f_2'(0) = g'(130)$, $f_2(0) = g(130)$, $f_2'(20) = 0$, and $f_2(20) = 30$. Horizontal shift was found once

# 3 Evaluate and Verify