Chapter 12

The second derivative and applications

Syllabus

The second derivative and applications of differentiation (9 hours)

In this sub-topic, students will:

- ☐ understand the concept of the second derivative as the rate of change of the first derivative function
- ☐ recognise acceleration as the second derivative of displacement position with respect to time
- ☐ understand the concepts of concavity and points of inflection and their relationship with the second derivative
- ☐ understand and use the second derivative test for finding local maxima and minima
- ☐ sketch the graph of a function using first and second derivatives to locate stationary points and points of inflection
- ☐ solve optimisation problems from a wide variety of fields using first and second derivatives, where the function to be optimised is both given and developed.

12A The second derivative and acceleration

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 12.1:

Find the second derivative of each of the following with respect to x:

(a) $f(x) = 6x^4 - 4x^3 + 4x$

(b) $y = e^x \sin x$

Solution:

Example 12.2:

If $f(x) = e^{2x}$, find $f''(0)$.

Solution:

Example 12.3:
Consider $f(x) = x^3 - 2x^2 + 4x - 6$.

(a) Find $f''(x)$.

(b) Solve the equation $f''(x) = 0$ for x.

Solution:

Example 12.4:
Consider $y = x^2 e^x$.

(a) Find $\frac{d^2y}{dx^2}$.

(b) Solve the equation $\frac{d^2y}{dx^2} = 0$ for x.

Solution:

Example 12.5:

A particle moves along a straight line such that its position, x m, relative to a point O at time t seconds is given by $x = 5 + \sin(t\pi t)$ for $0 \leq t \leq 2$. Find:

(a) at what times and in what positions the particle will have zero velocity

(b) its acceleration at those instants.

Solution:

12B Using the second derivative in graph sketching

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 12.6:

For each of the following functions, find the coordinates of the points of inflection of the curve and state the intervals where the curve is concave up:

(a) $f(x) = x^3$ **(b)** $f(x) = -x^3$ **(c)** $f(x) = x^3 - 3x^2 + 1$

Solution:

Example 12.7:
Consider the graph of $y = f(x)$, where $f(x) = x^2(10 - x)$.

(a) Find the coordinates of the stationary points and determine their nature using the second derivative test.

(b) Find the coordinates of the point of inflection and find the gradient at this point.

(c) On the one set of axes, sketch the graphs of $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ for $x \in [0, 10]$.

Solution:

Example 12.8:

Using a graphics calculator, find approximate coordinates for the stationary points and the points of inflection on the graph of the function

$$f(x) = e^x \sin x, \quad x \in [0, 2\pi]$$

Solution:

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Example 12.9:

Sketch the graph of $f(x) = x^4 - 8x^3 + 18x^2 + 4$, locating the stationary points and the points of inflection.

Solution:

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12C Absolute maximum and minimum values

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 12.10:

Let $f(x) = x^2 + 2$ for $x \in [-2, 4]$. Find the absolute maximum value and the absolute minimum value of the function.

Solution:

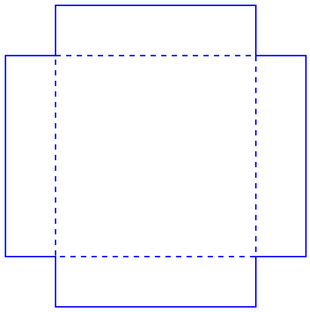
Example 12.11:

Let $f(x) = x^3 + 2$ for $x \in [-2, 1]$. Find the maximum and minimum values of the function.

Solution:

Example 12.12:

From a square piece of metal of side length 2 m, four squares are removed as shown in the diagram. The metal is then folded along the dashed lines to form an open box with height x m.



- **(a)** Show that the volume of the box, V m^3, is given by $V = 4x^3 - 8x^2 + 4x$.
- **(b)** Find the value of x that gives the box its maximum volume and show that the volume is a maximum for this value.
- **(c)** Sketch the graph of V against x for a suitable domain.
- **(d)** If the height of the box must be less than 0.3 m, i.e. $x \leq 0.3$, what will be the maximum volume of the box?

Solution:

12D Optimisation problems

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 12.13:

A farmer has sufficient fencing to make a rectangular pen of perimeter 200 metres. What dimensions will give an enclosure of maximum area?

Solution:

Example 12.14:

Two variables x and y are such that $x^4y = 8$. A third variable z is defined by $z = x + y$. Find the values of x and y that give z a stationary value. Use the second derivative test to show that this value of z is a minimum.

Solution:

Example 12.15:

A cylindrical tin canister closed at both ends has a surface area of 100 cm^2. Find, correct to two decimal places, the greatest volume it can have. If the radius of the canister can be at most 2 cm, find the greatest volume it can have.

Solution:

Example 12.16:

A TV cable company has 1000 subscribers who are paying \$5 per month. It can get 100 more subscribers for each \$0.10 decrease in the monthly fee. What monthly fee will yield the maximum revenue and what will this revenue be?

Solution:

Example 12.17:

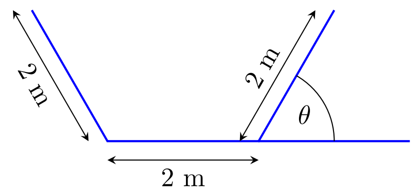
A manufacturer annually produces and sells 10 000 shirts. Sales are uniformly distributed throughout the year. The production cost of each shirt is \$23 and the carrying costs (storage, insurance, interest) depend on the total number of shirts in a production run. (A production run is the number, x of shirts which are under production at a given time.)

The set-up costs for a production run are \$40. The annual carrying costs are $\$x^{\frac{3}{2}}$. Find the size of a production run that minimises the total set-up and carrying costs for a year.

Solution:

Example 12.18:

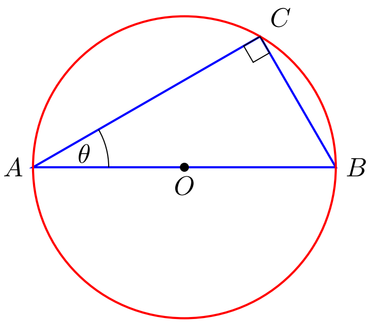
The cross-section of a drain is to be an isosceles trapezium, with three sides of length 2 metres, as shown. Find the angle θ that maximises the cross-sectional area, and find this maximum area.



Solution:

Example 12.19:

The figure shows a circular lake, centre O, of radius 2 km. A man swims across the lake from A to C at 3 km/h and then walks around the edge of the lake from C to B at 4 km/h.



(a) If $\angle BAC = \theta$ radians and the total time taken is T hours, show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta)$$

(b) Find the value of θ for which $\dfrac{dT}{d\theta} = 0$ and determine whether this gives a maximum or minimum value of $T (0° < \theta° < 90°)$.

Solution:

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Example 12.20:

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-0.1t}$. At what time will the population be at a maximum? Find the maximum population.

Solution:

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Example 12.21:

Assume that the number of bacteria present in a culture at time t is given by $N(t)$, where $N(t) = 36te^{-.1t}$.

(a) Sketch the graphs of $N(t)$ against t and $N'(t)$ against t.

(b) Find the maximum rates of increase and decrease of the population and the times at which these occur.

Solution:

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Chapter 13

Trigonometry using the sine and cosine rules

Syllabus

Cosine and sine rules (9 hours)

In this sub-topic, students will:

- ☐ recall sine, cosine and tangent as ratios of side lengths in right-angled triangles
- ☐ understand the unit circle definition of $\cos(\theta)$, $\sin(\theta)$ and $\tan(\theta)$ and periodicity using degrees and radians
- ☐ establish and use the sine (ambiguous case is required) and cosine rules and the formula area $= \dfrac{1}{2}bc\sin(A)$ for the area of a triangle
- ☐ construct mathematical models using the sine and cosine rules in two- and three-dimensional contexts (including bearings in two-dimensional context) and use the model to solve problems; verify and evaluate the usefulness of the model using qualitative statements and quantitative analysis.

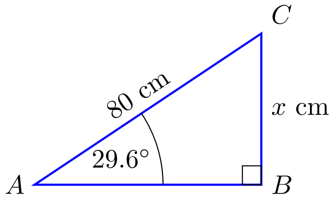
13A Reviewing trigonometry

Foundational Concepts	Vocabulary & Definitions
Big Ideas & Concepts	Procedures

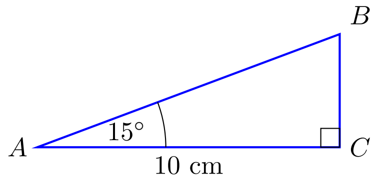
Example 13.1:

Complete the following.

(a) Find the value of x correct to two decimal places.



(b) Find the length of the hypotenuse correct to two decimal places.



Solution:

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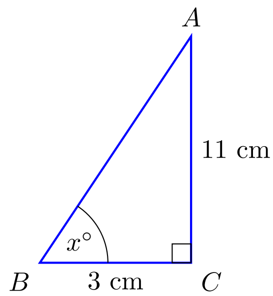
Example 13.2:

Find the magnitude of $\angle ABC$.



Solution:

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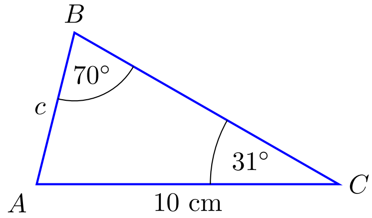
13B The sine rule

Foundational Concepts	Vocabulary & Definitions
Big Ideas & Concepts	Procedures

Example 13.3:

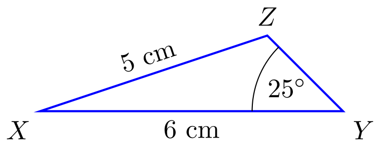
Use the sine rule to find the length of AB.



Solution:

Example 13.4:

Use the sine rule to find the magnitude of angle XZY in the triangle, given that $Y = 25°$, $y = 5$ cm and $z = 6$ cm.



Solution:

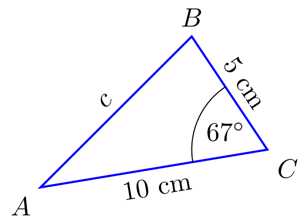
13C The cosine rule

Foundational Concepts Vocabulary & Definitions

Big Ideas & Concepts Procedures

Example 13.5:

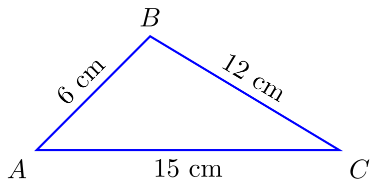
For triangle ABC, find the length of AB in centimetres correct to two decimal places.



Solution:

Example 13.6:

Find the magnitude of angle ABC.



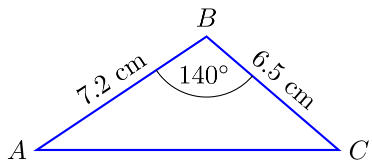
Solution:

13D The area of a triangle

Foundational Concepts	Vocabulary & Definitions
Big Ideas & Concepts	Procedures

Example 13.7:

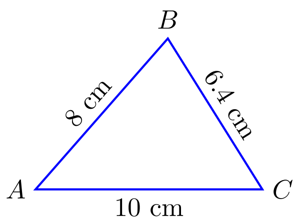
Find the area of triangle ABC shown in the diagram.



Solution:

Example 13.8:

Find the area of the triangle, correct to three decimal places.

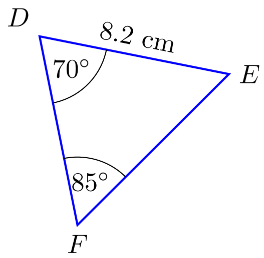


Solution:

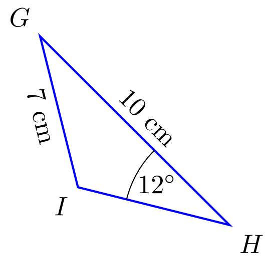
Example 13.9:

Find the area of each of the following triangles, correct to three decimal places:

(a)



(b)



Solution:

13E Angles of elevation, angles of depression and bearings

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 13.10:

The pilot of a helicopter flying at 400 m observes a small boat at an angle of depression of 1.2°. Calculate the horizontal distance of the boat to the helicopter.

Solution:

Example 13.11:

The light on a cliff-top lighthouse, known to be 75 m above sea level, is observed from a boat at an angle of elevation of 7.1°. Calculate the distance of the boat from the lighthouse.

Solution:

Example 13.12:

From the point A, a man observes that the angle of elevation of the summit of a hill is $10°$. He then walks towards the hill for 500 m along flat ground. The summit of the hill is now at an angle of elevation of $14°$. Find the height of the hill above the level of A.

Solution:

Example 13.13:

The road from town A runs due west for 14 km to town B. A television mast is located due south of B at a distance of 23 km. Calculate the distance and bearing of the mast from the centre of town A.

Solution:

Example 13.14:

A yacht starts from a point A and sails on a bearing of 038° for 3000 m. It then alters its course to a bearing of 318° and after sailing for a further 3300 m reaches a point B. Find:

(a) the distance AB

(b) the bearing of B from A.

Solution:

13F Problems in three dimensions

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 13.15:

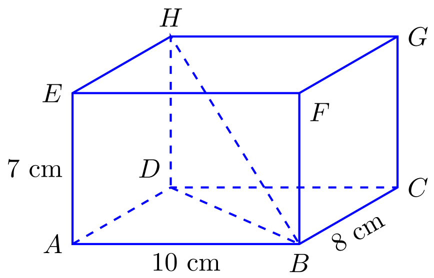
$ABCDEFGH$ is a cuboid. Find:

(a) the distance DB

(b) the distance HB

(c) the magnitude of angle HBD

(d) the magnitude of angle HBA.



Solution:

Example 13.16:

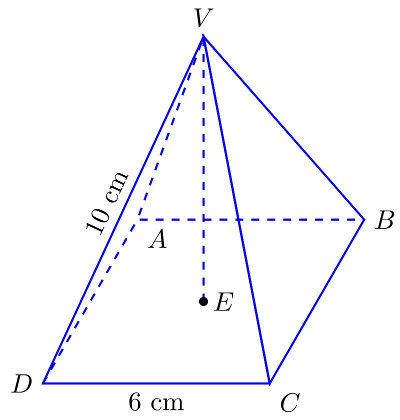
The figure shows a pyramid with a square base. The base has sides 6 cm long and the edges VA, VB, VC and VD are each 10 cm long.

(a) Find the length of DB.

(b) Find the length of BE.

(c) Find the length of VE.

(d) Find the magnitude of angle VBE.



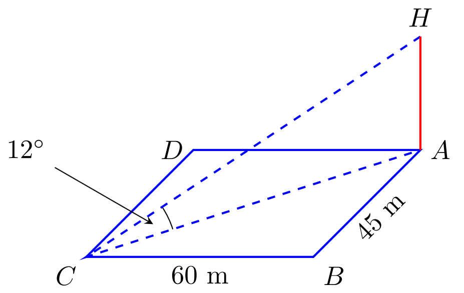
Solution:

Example 13.17:

A communications mast is erected at corner A of a rectangular courtyard $ABCD$ with side lengths 60 m and 45 m as shown. If the angle of elevation of the top of the mast from C is $12°$, find:

(a) the height of the mast

(b) the angle of elevation of the top of the mast from B.



Solution:

13G Angles between planes and more complex 3D problems

Foundational Concepts Vocabulary & Definitions

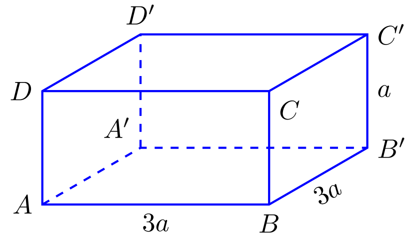
Big Ideas & Concepts Procedures

Example 13.18:

For the cuboid shown in the diagram, find:

(a) the angle between AC' and the plane $ABB'A'$

(b) the angle between the planes ACD' and DCD'.



Solution:

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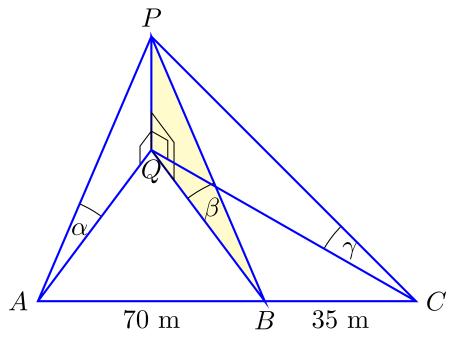
Example 13.19:

Three points A, B and C are on a horizontal line such that $AB = 70$ m and $BC = 35$ m. The angles of elevation of the top of a tower are α, β and γ, where

$$\tan\alpha = \frac{1}{13}, \qquad \tan\beta = \frac{1}{15}, \qquad \tan\gamma = \frac{1}{20}$$

as shown in the diagram.

The base of the tower is at the same level as A, B and C. Find the height of the tower.



Solution:

Example 13.20:

A sphere rests on the top of a vertical cylinder which is open at the top. The inside diameter of the cylinder is 8 cm. The sphere projects 8 cm above the top of the cylinder. Find the radius length of the sphere.

Solution:

Chapter 14

Refresher on probability and discrete random variables

14A Sample spaces and probability

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 14.1:

Find the sample space when three coins are tossed and the results noted.

Solution:

Example 14.2:

If one card is chosen at random from a well-shuffled deck of 52 cards, what is the probability that the card is:

(a) an ace

(b) not a heart

(c) an ace or a heart

(d) either a king or an ace?

Solution:

Example 14.3:

500 people were questioned and classified according to age and whether or not they regularly use social media. The results are shown in the table.

Do you regularly use social media?			
	Age < 25	**Age ≥ 25**	**Total**
Yes	200	100	300
No	40	160	200
Total	240	260	500

One person is selected from these 500. Find the probability that:

(a) the person regularly uses social media

(b) the person is less than 25 years of age

(c) the person is less than 25 years of age and does not regularly use social media.

Solution:

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Example 14.4:

Suppose that a die is tossed 1000 times and the following outcomes observed:

Outcome	1	2	3	4	5	6
Frequency	135	159	280	199	133	97

(a) Use this information to estimate the probability of observing a 6 when this die is rolled.

(b) What outcome would you predict to be most likely the next time the die is rolled?

Solution:

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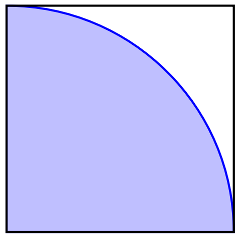
Example 14.5:

A dartboard consists of a square of side length 2 metres containing a blue one-quarter of a circular disc centred at the bottom-left vertex of the square, as shown.

If a dart thrown at the square is equally likely to hit any part of the square, and it hits the square every time, find the probability of it hitting the blue region.



Solution:

Example 14.6:

Simone visits the dentist every 6 months for a checkup. The probability that she will need her teeth cleaned is 0.35, the probability that she will need a filling is 0.1 and the probability that she will need both is 0.05.

(a) What is the probability that she will not need her teeth cleaned on a visit, but will need a filling?

(b) What is the probability that she will not need either of these treatments?

Solution:

14B Conditional probability and independence

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 14.7:

In a certain town, the probability that it rains on any Monday is 0.21. If it rains on Monday, then the probability that it rains on Tuesday is 0.83. If it does not rain on Monday, then the probability of rain on Tuesday is 0.3. For a given week, find the probability that it rains:

(a) on both Monday and Tuesday

(b) on Tuesday.

Solution:

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Example 14.8:

Adrienne, Regan and Michael are doing the dishes. Since Adrienne is the oldest, she washes the dishes 40% of the time. Regan and Michael each wash 30% of the time. When Adrienne washes the probability of at least one dish being broken is 0.01, when Regan washes the probability is 0.02, and when Michael washes the probability is 0.03. Their parents don't know who is washing the dishes one particular night.

(a) What is the probability that at least one dish will be broken?

(b) Given that at least one dish is broken, what is the probability that the person washing was Michael?

Solution:

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Example 14.9:
As part of an evaluation of the school tuck shop, all students at a Senior Secondary College (Years 10 – 12) were asked to rate the tuck shop as poor, good or excellent. The results are shown in the table.
What is the probability that a student chosen at random from this college:

(a) is in Year 12

(b) is in Year 12 and rates the tuck shop as excellent

(c) is in Year 12, given that they rate the tuck shop as excellent

(d) rates the tuck shop as excellent, given that they are in Year 12?

	Year			
Rating	**10**	**11**	**12**	**Total**
Poor	30	20	10	60
Good	80	65	35	180
Excellent	60	65	35	160
Total	170	150	80	400

Solution:

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Example 14.10:
The probability that Monica remembers to do her homework is 0.7, while the probability that Patrick remembers to do his homework is 0.4. If these events are independent, then what is the probability that:

(a) both will do their homework

(b) Monica will do her homework but Patrick forgets?

Solution:

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14C Discrete random variables

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 14.11:

A jar contains four white and six black balls. What is the probability that, if three balls are drawn at random from the jar, with replacement, a white ball will be drawn exactly once?

Solution:

Example 14.12:

Consider the table shown.

x	0	1	2	3
$p(x)$	0.2	0.3	0.1	0.4

(a) Does this meet the conditions to be a discrete probability distribution?

(b) Use the table to find $\Pr(X \leq 2)$.

Solution:

Example 14.13:

Let X be the number of heads showing when a fair coin is tossed three times.

(a) Find the probability distribution of X and show that all the probabilities sum to 1.

(b) Find the probability that one or more heads show.

(c) Find the probability that more than one head shows.

Solution:

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Example 14.14:

The random variable X represents the number of chocolate chips in a certain brand of biscuit, and is known to have the following probability distribution.

x	2	3	4	5	6	7
$p(x)$	0.01	0.25	0.40	0.30	0.02	0.02

Find:

(a) $\Pr(X \geq 4)$

(b) $\Pr(X \geq 4 | X > 2)$

(c) $\Pr(X < 5 | X > 2)$

Solution:

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14D Expected value, variance and standard deviation

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 14.15:

A person may buy a lucky ticket for \$1. They have a 20% chance of winning \$2, a 5% chance of winning \$11, and otherwise they lose. Is this a good game to play?

Solution:

Example 14.16:

A coin is biased in favour of heads such that the probability of obtaining a head on any single toss is 0.6. The coin is tossed three times and the results noted. If X is the number of heads obtained on the three tosses, find $E(X)$, the expected value of X.

Solution:

Example 14.17:

For the random variable X defined in Example 16, find:

(a) $E(3X+1)$

(b) $E(X^2)$

Solution:

Example 14.18:

For a \$5 monthly fee, a TV repair company guarantees customers a complete service. The company estimates the probability that a customer will require one service call in a month as 0.05, the probability of two calls as 0.01 and the probability of three or more calls as 0.00. Each call costs the repair company \$40. What is the TV repair company's expected monthly gain from such a contract?

Solution:

Example 14.19:

Suppose that a discrete random variable X has the probability distribution shown in the following table, where $c > 0$.

$\boldsymbol{x}$	$-c$	c
$\mathbf{Pr}(\boldsymbol{X = x})$	0.5	0.5

Find the standard deviation of X.

Solution:

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Example 14.20:

For the probability distribution shown, find $E(X^2)$ and $[E(X)]^2$ and hence find the variance of X.

$\boldsymbol{x}$	0	1	2	3
$\mathbf{Pr}(\boldsymbol{X = x})$	0.08	0.18	0.4	0.34

Solution:

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Example 14.21:

If X is a random variable such that $Var(X) = 9$, find:

(a) $Var(3X + 2)$

(b) $Var(-X)$

Solution:

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Example 14.22:

The number of chocolate bars, X, sold by a manufacturer in any month has the following distribution:

x	100	150	200	250	300	400
$p(x)$	0.05	0.15	0.35	0.25	0.15	0.05

What is the probability that X takes a value in the interval $\mu - 2\sigma$ to $\mu + 2\sigma$?

Solution:

Example 14.23:

A manufacturer knows that the mean number of faulty light bulbs in a batch of 10 000 is 12 with a standard deviation of 3. He wishes to claim to his clients that 95% of batches will contain between c_1 and c_2 faulty light bulbs (where c_1 and c_2 are symmetric about the mean). What are two possible values of c_1 and c_2?

Solution:

Chapter 15

Bernoulli sequences and the binomial distribution

Syllabus

Bernoulli distributions (3 hours)

In this sub-topic, students will:

- ☐ use a Bernoulli random variable as a model for two-outcome situations
- ☐ identify contexts suitable for modelling by Bernoulli random variables
- ☐ recognise and determine the mean p and variance $p(1-p)$ of the Bernoulli distribution with parameter p
- ☐ use Bernoulli random variables and associated probabilities to model data and solve practical problems.

Binomial distributions (5 hours)

In this sub-topic, students will:

- ☐ understand the concepts of Bernoulli trials and the concept of a binomial random variable as the number of 'successes' in n independent Bernoulli trials, with the same probability of success p in each trial
- ☐ identify contexts suitable for modelling by binomial random variables
- ☐ determine and use the probabilities $P(X=r) = \binom{n}{r}p^r(1-p)^{n-r}$ associated with the binomial distribution with parameters n and p
- ☐ calculate the mean np and variance $np(1-p)$ of a binomial distribution using technology and algebraic methods
- ☐ identify contexts suitable to model binomial distributions and associated probabilities to solve practical problems, including the language of 'at most' and 'at least'.

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15A Introduction to Bernoulli sequences and the binomial distribution

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 15.1:

Suppose that a netball player has a probability of $\frac{1}{3}$ of scoring a goal each time she attempts to goal. She repeatedly has shots for goal. Is this a Bernoulli sequence?

Solution:

Example 15.2:

Find the probability of obtaining exactly three heads when a fair coin is tossed seven times, correct to four decimal places.

Solution:

Example 15.3:

The probability that a person currently in prison has ever been imprisoned before is 0.72. Find the probability that of five prisoners chosen at random at least three have been imprisoned before, correct to four decimal places.

Solution:

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Example 15.4:

The probability of a netballer scoring a goal is 0.3. Find the probability that out of six attempts the netballer scores a goal:

(a) four times

(b) four times, given that she scores at least one goal.

Solution:

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15B The graph, expectation and variance of a binomial distribution

Foundational Concepts	Vocabulary & Definitions
Big Ideas & Concepts	Procedures

Example 15.5:

Construct and compare the graph of the binomial probability distribution for 20 trials ($n = 20$) with probability of success:

(a) $p = 0.2$ **(b)** $p = 0.5$ **(c)** $p = 0.8$

Solution:

Example 15.6:

An examination consists of 30 multiple-choice questions, each question having three possible answers. A student guesses the answer to every question. Let X be the number of correct answers.

(a) How many will she expect to get correct? That is, find $E(X) = \mu$.

(b) Find $Var(X)$.

Solution:

15C Finding the sample size

Foundational Concepts

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Procedures

Example 15.7:

The probability of winning a prize in a game of chance is 0.48.

(a) What is the least number of games that must be played to ensure that the probability of winning at least once is more than 0.95?

(b) What is the least number of games that must be played to ensure that the probability of winning at least twice is more than 0.95?

Solution:

15D Proofs for the expectation and variance

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Chapter 16

Continuous random variables

Syllabus

General continuous random variables (6 hours)

In this sub-topic, students will:

- ☐ use relative frequencies and histograms obtained from data to estimate probabilities associated with a continuous random variable
- ☐ understand the concepts of a probability density function, cumulative distribution function, and probabilities associated with a continuous random variable given by integrals; examine simple types of continuous random variables and use them in appropriate contexts
- ☐ calculate the expected value, variance and standard deviation of a continuous random variable in simple cases
- ☐ understand standardised normal variables (z-values, z-scores) and use these to compare samples.

16A Introduction to continuous random variables

Foundational Concepts

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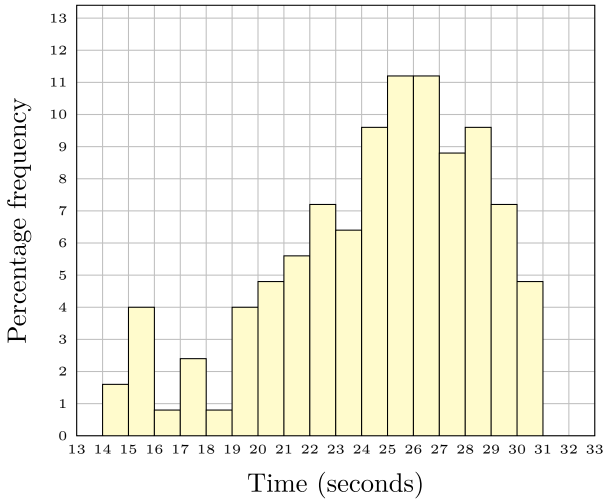
Procedures

Example 16.1:

Let T represent the time (in seconds) that it takes a student to complete a particular puzzle. The following percentage frequency histogram was obtained by recording the times taken to complete the puzzle by 500 students, with each recorded time rounded down to a whole number of seconds.



Use the histogram to estimate:

(a) $\Pr(19 \leq T < 22)$

(b) $\Pr(T \geq 28)$

Solution:

Example 16.2:

Consider the function f with the rule:

$$f(x) = \begin{cases} 1.5\left(1 - x^2\right) & \text{if } 0 \le x \le 1 \\ 0 & \text{if } x > 1 \text{ or } x < 0 \end{cases}$$

(a) Sketch the graph of f.

(b) Show that f is a probability density function.

(c) Find $\Pr(X > 0.5)$, where the random variable X has probability density function f.

Solution:

Example 16.3:

Suppose that the random variable X has the probability density function with rule:

$$f(x) = \begin{cases} cx & \text{if } 0 \leq x \leq 2 \\ 0 & \text{if } x > 2 \text{ or } x < 0 \end{cases}$$

(a) Find the value of c that makes f a probability density function.

(b) Find $\Pr(X > 1.5)$.

(c) Find $\Pr(1 \leq X \leq 1.5)$.

Solution:

Example 16.4:

Consider the exponential probability density function f with the rule:

$$f(x) = \begin{cases} 2e^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

(a) Sketch the graph of f.

(b) Show that f is a probability density function.

(c) Find $\Pr(X > 1)$, where the random variable X has probability density function f.

Solution:

Example 16.5:

The time (in seconds) that it takes a student to complete a puzzle is a random variable X with a density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \geq 5 \\ 0 & x < 5 \end{cases}$$

(a) Find the probability that a student takes less than 12 seconds to complete the puzzle.

(b) Find the probability that a student takes between 8 and 10 seconds to complete the puzzle, given that he takes less than 12 seconds.

Solution:

16B Mean and median for a continuous random variable

Foundational Concepts

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Procedures

Example 16.6:

Find the expected value of the random variable X which has probability density function with rule:

$$f(x) = \begin{cases} 0.5x & 0 \leq x \leq 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Solution:

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Example 16.7:

Let X be a random variable with probability density function f given by

$$f(x) = \begin{cases} 0.5x & 0 \leq x \leq 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Find:

(a) the expected value of X^2

(b) the expected value of e^X.

Solution:

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Example 16.8:

The duration of telephone calls to the order department of a large company is a random variable, X minutes, with probability density function:

$$f(x) = \begin{cases} \dfrac{1}{3}e^{-\frac{x}{3}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Find the value of a such that 90% of phone calls last less than a minutes.

Solution:

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Example 16.9:

Suppose the probability density function of weekly sales of topsoil, X (in tonnes), is given by the rule:

$$f(x) = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Find the median value of X, and interpret.

Solution:

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16C Measures of spread

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 16.10:

Find the variance and standard deviation of the random variable X which has the probability density function f with rule:

$$f(x) = \begin{cases} 0.5x & 0 \leq x \leq 2 \\ 0 & x < 0 \text{ or } x > 2 \end{cases}$$

Solution:

Example 16.11:

The life of a certain brand of battery, X hours, is a continuous random variable with mean 50 and variance 16. Find an (approximate) interval for the time period for which 95% of the batteries would be expected to last.

Solution:

Example 16.12:

Determine the interquartile range of the random variable X which has the probability density function:

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & x < 0 \text{ or } x > 1 \end{cases}$$

Solution:

16D Properties of mean and variance

Foundational Concepts

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Procedures

Example 16.13:

Suppose that X is a continuous random variable with mean $\mu = 10$ and variance $\sigma^2 = 2$.

(a) Find $E(2X + 1)$.

(b) Find $Var(1 - 3E)$.

(c) If X has probability density function f, describe the rule of a probability density function f for $2X + 1$.

Solution:

16E Cumulative distribution functions

Foundational Concepts

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Procedures

Example 16.14:

The time, X seconds, that it takes a student to complete a puzzle is a random variable with density function given by

$$f(x) = \begin{cases} \dfrac{5}{x^2} & x \geq 5 \\ 0 & X < 5 \end{cases}$$

(a) Find $F(x)$, the cumulative distribution function of X.

(b) Use the cumulative distribution function to find:

(i) $\Pr(X \leq 7)$ **(ii)** $\Pr(X \geq 6)$ **(iii)** $\Pr(10 \leq X \leq 20)$

Solution:

Example 16.15:

The time to failure (in hundreds of hours) for a certain electronic component is a random variable X with cumulative distribution function F given by

$$F(x) = \begin{cases} 1 - e^{-x^2} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find the rule for a probability density function f for X.

Solution:

Chapter 17

The normal distribution

Syllabus

Normal distributions (6 hours)

In this sub-topic, students will:

- ☐ identify contexts, such as naturally occurring variations, that are suitable for modelling by normal random variables
- ☐ recognise features of the graph of the probability density function of the normal distribution with mean μ and standard deviation σ and the use of the standard normal distribution
- ☐ calculate probabilities and quantiles associated with a given normal distribution using technology and use these to solve practical problems.

17A The normal distribution

Foundational Concepts | Vocabulary & Definitions

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Example 17.1:

On the same set of axes, sketch the graphs of the probability density functions of the standard normal distribution and the normal distribution with:

(a) mean 1 and standard deviation 1

(b) mean 1 and standard deviation 2.

(A calculator can be used to help.)

Solution:

17B Standardisation

Foundational Concepts

Vocabulary & Definitions

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Procedures

Example 17.2:

Experience has shown that the scores obtained on a commonly used IQ test can be assumed to be normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 15$.

Approximately what percentage of the distribution lies within one, two or three standard deviations of the mean?

Solution:

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Example 17.3:

From Example 2 we know that 95% of the scores in the IQ distribution lie between 70 and 130 (that is, within two standard deviations of the mean). What percentage of the scores are more than two standard deviations above or below the mean (in this instance, less than 70 or greater than 130)?

Solution:

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17C Determining normal probabilities

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 17.4:

Suppose that Z is a standard normal random variable (that is, it has mean $\mu = 0$ and standard deviation $\sigma = 1$). Find:

(a) $\Pr(-1 < Z < 2.5)$

(b) $\Pr(Z > 1)$

Solution:

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Example 17.5:

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.
Find k such that $\Pr(X \leq k) = 0.95$.

Solution:

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Example 17.6:

Suppose X is normally distributed with mean $\mu = 100$ and standard deviation $\sigma = 6$.
Find values of c_1 and c_2 (symmetric about the mean) such that $\Pr(c_1 < X < c_2) = 0.95$.

Solution:

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17D Solving problems using the normal distribution

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 17.7:

The time taken to complete a logical reasoning task is normally distributed with a mean of 55 seconds and a standard deviation of 8 seconds.

(a) Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task.

(b) Find the probability, correct to four decimal places, that a randomly chosen person will take less than 50 seconds to complete the task, if it is known that this person took less than 60 seconds to complete the task.

Solution:

Example 17.8:

Limits of acceptability imposed on the lengths of a certain batch of metal rods are 1.925 cm and 2.075 cm. It is observed that, on average, 5% are rejected as undersized and 5% are rejected as oversized.
Assuming that the lengths are normally distributed, find the mean and standard deviation of the distribution.

Solution:

17E The normal approximation to the binomial distribution

Foundational Concepts | Vocabulary & Definitions

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Example 17.9:

A sample of 1000 people from a certain city were asked to indicate whether or not they were in favour of the construction of a new freeway. It is known that 30% of people in this city are in favour of the new freeway. Find the approximate probability that between 270 and 330 people in the sample were in favour of the new freeway.

Solution:

Chapter 18

Sampling and estimation

Syllabus

Random sampling (3 hours)

In this sub-topic, students will:

- ☐ understand the concept of a random sample
- ☐ discuss sources of bias in samples, and procedures to ensure randomness
- ☐ investigate the variability of random samples from various types of distributions, including uniform, normal and Bernoulli, using graphical displays of real and simulated data.

Sample proportions (6 hours)

In this sub-topic, students will:

- ☐ understand the concept of the sample proportion $\hat{p}$ as a random variable whose value varies between samples, and the formulas for the mean p and standard deviation $\sqrt{p(1-p)/n}$ of the sample proportion $\hat{p}$
- ☐ consider the approximate normality of the distribution of $\hat{p}$ for large samples
- ☐ simulate repeated random sampling, for a variety of values of p and a range of sample sizes, to illustrate the distribution of $\hat{p}$ and the approximate standard normality of $\dfrac{\hat{p}-p}{\sqrt{(\hat{p}(1-\hat{p})/n)}}$ where the closeness of the approximation depends on both n and p.

Confidence intervals for proportions (8 hours)

In this sub-topic, students will:

- ☐ understand the concept of an interval estimate for a parameter associated with a random variable
- ☐ use the approximate confidence interval $\left(\hat{p}-z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}, \hat{p}+z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}\right)$, as an interval estimate for p, where z is the appropriate quantile for the standard normal distribution
- ☐ define the approximate margin of error $E = z\sqrt{\dfrac{\hat{p}(1-\hat{p})}{n}}$ and understand the trade-off between margin of error and level of confidence
- ☐ use simulation to illustrate variations in confidence intervals between samples and to show that most but not all confidence intervals contain p.

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18A Populations and samples

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 18.1:

A researcher wishes to evaluate how well the local library is catering to the needs of a town's residents. To do this she hands out a questionnaire to each person entering the library over the course of a week. Will this method result in a random sample?

Solution:

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Example 18.2:

Use a random number generator to select a group of six students from the following class:

- Denise
- Matt
- Teresa
- Sue
- Sharyn
- Mark
- Peter
- Nick
- Miller
- William
- Anne
- Darren
- Tom
- David
- Sally
- Janelle
- Steven
- Jane
- Georgia
- Jaimie

Solution:

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Example 18.3:

Use a random number generator to select another group of six students from the same class, and determine the proportion of females in the sample.

Solution:

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18B The exact distribution of the sample proportion

Foundational Concepts | Vocabulary & Definitions

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Example 18.4:

A bag contains six blue balls and four red balls. If we take a random sample of size 4, what is the probability that there is one blue ball in the sample $\left(\hat{p} = \frac{1}{4}\right)$?

Solution:

Example 18.5:

A bag contains six blue balls and four red balls. Use the sampling distribution in the previous table to determine the probability that the proportion of blue balls in a sample of size 4 is more than $\frac{1}{4}$.

Solution:

Example 18.6:

Use the sampling distribution in the previous table to determine the probability that, in a random sample of four Australian 17-year-olds, the proportion attending school is less than 50%.

Solution:

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Example 18.7:

Use the probability distribution to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6.

$\hat{p}$	0	0.25	0.5	0.75	1
$\Pr(\hat{P} = \hat{p})$	0.0081	0.0756	0.2646	0.4116	0.2401

Solution:

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Example 18.8:

Use these rules to determine the mean and standard deviation of the sample proportion $\hat{P}$ from Example 6. Are they the same as those found in Example 7?

Solution:

Example 18.9:

Suppose that 70% of 17-year-olds in Australia attend school. If a random sample of size 20 is chosen from this population, find:

(a) the probability that the sample proportion is equal to the population proportion (0.7)

(b) the probability that the sample proportion lies within one standard deviation of the population proportion

(c) the probability that the sample proportion lies within two standard deviations of the population proportion.

Solution:

18C Approximating the distribution of the sample proportion

Foundational Concepts

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Example 18.10:

Assume that 55% of people in Australia have blue eyes. Use your calculator to illustrate a possible distribution of sample proportions $\hat{p}$ that may be obtained when 200 different samples (each of size 100) are selected from the population.

Solution:

Example 18.11:

Assume that 60% of people have a driver's licence. Using the normal approximation, find the approximate probability that, in a randomly selected sample of size 200, more than 65% of people have a driver's licence.

Solution:

18D Confidence intervals for the population proportion

Foundational Concepts	Vocabulary & Definitions
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Example 18.12:

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 20 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution:

Example 18.13:

A survey found that 237 out of 500 undergraduate university students questioned intended to take a postgraduate course in the future. Find a 95% confidence interval for the proportion of undergraduates intending to take a postgraduate course.

Solution:

Example 18.14:

Find an approximate 95% confidence interval for the proportion p of primary school children in Australia who regularly use social media, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

Solution:

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Example 18.15:

Determine the sample size required to achieve a margin of error of 2% in an approximate 95% confidence interval for the proportion p of primary school children in Australia who use social media, if the sample proportion $\hat{p}$ is found to be 0.7.

Solution:

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Example 18.16:

Consider the proportion p of primary school children in Australia who use social media.

(a) Calculate and compare 90%, 95% and 99% confidence intervals for p, if we select a random sample of 200 children and find the sample proportion $\hat{p}$ to be 0.7.

(b) Determine the sample size required to achieve a margin of error of 2% in an approximate 99% confidence interval for p.

Solution:

Example 18.17:

Suppose that we toss a fair coin 20 times, and determine the proportion of heads observed in this sample. Suppose further that this is repeated 10 times.

(a) Use your calculator to generate 10 values of the sample proportion $\hat{p}$ of heads in 20 coin tosses.

(b) Use your calculator to find an approximate 90% confidence interval for the population proportion p from each of these values of the sample proportion $\hat{p}$.

(c) How many of these intervals contain the value of the population proportion p?

(d) How many of these intervals would you expect to contain the value of the population proportion p?

Solution: