Chapter 4

Vectors

Syllabus

Geometric proofs using vectors (8 hours)

In this sub-topic, students will:

- ☐ prove the diagonals of a parallelogram meet at right angles if and only if it is a rhombus
- ☐ prove midpoints of the sides of a quadrilateral join to form a parallelogram
- ☐ prove the sum of the squares of the lengths of a parallelograms diagonals is equal to the sum of the squares of the lengths of the sides
- ☐ prove an angle in a semicircle is a right angle
- ☐ prove geometric results in the plane and construct simple proofs in three dimensions.

4G Geometric proofs

Foundational Concepts	Vocabulary & Definitions
Big Ideas & Concepts	**Procedures**

Example 4.1:

Prove that the diagonals of a rhombus are perpendicular.

Solution:

Example 4.2:

Prove that the angle subtended by a diameter in a circle is a right angle.

Solution:

Example 4.3:

Prove that the medians of a triangle are concurrent.

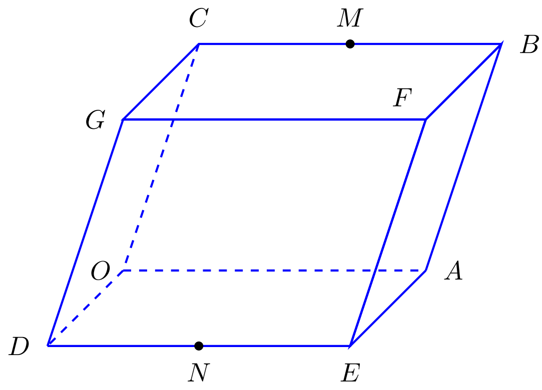
Solution:

Example 4.4:

Consider a parallelepiped $OABCDEFG$ as shown.

(a) Prove that the diagonals OF and CE bisect each other.

(b) Let M be the midpoint of CB, and let N be the midpoint of DE. Prove that the midpoint of MN is the point where the diagonals OF and CE intersect.



Solution:

Example 4.5:

Prove the diagonals of a prallelogram meet at right angles if and only if it is a rhombus.

Solution:

Example 4.6:

Prove midpoints of the sides of a quadrilateral join to form a parallelogram.

Solution:

Example 4.7:
Prove the sum of the squares of the lengths of a parallelograms diagonals is equal to the sum of the squares of the lengths of the sides.

Solution:

Chapter 7

Trigonometry

Syllabus

The basic trigonometric functions (2 hours)

In this sub-topic, students will:

- ☐ find all solutions of $f(a(x-b))$ where $f(\theta)$ is one of $\sin(\theta)$, $\cos(\theta)$ or $\tan(\theta)$
- ☐ sketch and graph functions with rules of the form $y = f(a(x-b))$ where $f(\theta)$ is one of $\sin(\theta)$, $\cos(\theta)$ or $\tan(\theta)$.

The reciprocal trigonometric functions, secant, cosecant and cotangent (3 hours)

In this sub-topic, students will:

- ☐ define the reciprocal trigonometric functions, sketch their graphs, and graph simple transformations of them.

Trigonometric identities (8 hours)

In this sub-topic, students will:

- ☐ prove and apply the Pythagorean identities
- ☐ prove and apply the angle sum, difference and double-angle identities for sines and cosines
- ☐ prove and apply the identities for products of sines and cosines expressed as sums and differences
- ☐ convert sums $a\cos(x) + b\sin(x)$ to $R\cos(x \pm a)$ or $R\sin(x \pm a)$ and apply these to sketch graphs, solve equations of the form $a\cos(x) + b\sin(x) = c$ and solve real-world problems
- ☐ use the binomial theorem to prove and apply multi-angle trigonometric identities up to $\sin(4x)$ and $\cos(4x)$.

Applications of trigonometric functions to model periodic phenomena (5 hours)

In this sub-topic, students will:

- ☐ model periodic motion using sine and cosine functions and understand the relevance of the period and amplitude of these functions in the model.

7.1 Solving trigonometric equations

7.1.1 Solving trigonometric equations

Foundational Concepts

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Example 7.1 – Solving an equation of the type $\sin(x) = c$:

Solve the following equations for the given domain.

(a) $\sin(x) = \frac{1}{2}$, $0 \le x \le 2\pi$

(b) $\sin(x) = -\frac{\sqrt{2}}{2}$, $0 \le x \le 2\pi$

Solution:

Example 7.2 – Solving an equation of the type $\cos(x) = c$:

Solve the equation $2\cos(x) + \sqrt{3} = 0$, $0 \le x \le 2\pi$.

Solution:

Example 7.3 – Solving an equation of the type $\tan(x) = c$**:**

Solve the equation $\tan(x) - 1 = 0$, $0° \leq x \leq 360°$.

Solution:

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Example 7.4 – Solving equations with different domains**:**

Solve the following equations for the given domain.

(a) $\tan(x) = -\sqrt{3}$, $-\pi \leq x \leq \pi$

(b) $\sin(x) = \frac{1}{2}$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

Solution:

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Example 7.5 – Solving equations of the form $\sin(ax) = c$, $\cos(ax) = c$ or $\tan(ax) = c$**:**
Solve the following equations.

(a) $\cos(2x) = \frac{1}{2}$, $0 \leq x \leq 2\pi$

(b) $\sin\left(\frac{x}{3}\right) = 0.6$, $0 \leq x \leq 6\pi$

Solution:

Example 7.6 – Solving equations of the form $\sin(x-b)=c$, $\cos(x-b)=c$ or $\tan(x-b)=c$**:**

Solve the following equations.

(a) $\cos\left(x-30^\circ\right)=\dfrac{\sqrt{2}}{2}$, $0^\circ \leq x \leq 360^\circ$

(b) $\sin\left(x+\dfrac{5\pi}{3}\right)=\dfrac{1}{2}$, $0 \leq x \leq 2\pi$

Solution:

Example 7.7 – Solving more complex equations**:**

Solve the equation $\tan\left(3\left(x - \frac{\pi}{4}\right)\right) = -1$, $0 \leq x \leq 2\pi$.

Solution:

7.1.2 General solutions

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Example 7.8 – Solving equations with an unrestricted domain**:**

Solve these equations, leaving your answer in general form.

(a) $\sin(x) = \dfrac{\sqrt{3}}{2}$

(b) $\tan(x) = \dfrac{1}{\sqrt{2}}$

Solution:

Example 7.9 – Solving equations with an unrestricted domain along the vertical axis**:**

Solve $\cos(2x) = 0$.

Solution:

7.2 Graphing and modelling with trigonometric functions

7.2.1 Graphs of sine and cosine functions

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 7.10 – Graphing functions of the form $f(x) = A\cos(ax)$**:**

Consider the function $f(x) = 3\cos(2x)$, $0 \leq x \leq 2\pi$

(a) Write the amplitude and period of $f(x)$

(b) Calculate the x-intercepts, by solving the equation $f(x) = 0$.

(c) Determine the equation of the median line.

(d) Sketch the graph of the function, labelling all key features, including the coordinates of the maximum and minimum points and the median line.

Solution:

Example 7.11 – Graphing functions of the form $f(x) = A\sin(ax)$ for $A < 0$**:**

Consider the function $f(x) = -3\sin(2x)$, $0 \leq x \leq 2\pi$

(a) Determine the period, amplitude and range of the function.

(b) Determine the coordinates of the x-intercepts for the graph of $y = f(x)$.

(c) Determine the equation of the median line.

(d) Sketch the graph of the function, labelling all major points in coordinate form, including the maximum and minimum points and showing the median line.

Solution:

Example 7.12 – Graphing a trigonometric function with a vertical translation**:**

Consider the function $f(x) = 3\sin(2x) + 4$, $0 \leq x \leq 2\pi$.

(a) Determine the amplitude, period and range of the function.

(b) Calculate the intercepts of $f(x)$ with the median line.

(c) Sketch the graph of the function, labelling all key points in coordinate form and showing the median line.

Solution:

Example 7.13 – Graphing a trigonometric function with a horizontal translation**:**
Consider the function $f(x) = 3\sin\left(2x - \frac{\pi}{2}\right)$, $0 \leq x \leq 2\pi$.

(a) Determine the amplitude, period and range of the function.

(b) Calculate the coordinates of the x-intercepts for the graph of $y = f(x)$.

(c) Sketch the graph of the function, labelling all key features.

Solution:

Example 7.14 – Determining transformations of trigonometric functions**:**

Determine the transformations that have taken place to transform either $f(x) = \sin(x)$ or $f(x) = \cos(x)$ into the given function. Also list the period and amplitude of each function.

(a) $f(x) = \cos\left(3x + \frac{\pi}{6}\right)$ **(b)** $f(x) = -\sin(\pi x) + 2$ **(c)** $f(x) = 6\cos\left(\frac{1}{2}x - \frac{\pi}{4}\right)$

Solution:

7.2.2 Graphs of the tangent function

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Example 7.15 – Graphing functions of the form $f(x) = \tan(ax)$**:**

Consider the function $f(x) = \tan(2x)$, $0 \leq x \leq 2\pi$.

(a) Determine the period, equations of the asymptotes and x-intercepts of the function.

(b) Sketch the graph of $y = f(x)$.

(c) List the transformations that have taken place to transform $y = \tan(x)$ into $y = \tan(2x)$.

Solution:

Example 7.16 – Graphing functions of the form $f(x) = \tan(a(x - b))$**:**
Consider $h(x) = \tan\left(2\left(x - \frac{\pi}{3}\right)\right)$, $0 \le x \le \pi$.

(a) Sketch the graph of $y = h(x)$, showing the asymptotes and the coordinates of key points.

(b) List the transformations in sequential order to produce the graph of $y = h(x)$ from the graph of $f(x) = \tan(x)$.

Solution:

7.2.3 Modelling with trigonometric functions

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Example 7.17 – Modelling with a trigonometric function**:**

The height of a carriage on a Ferris wheel, h metres above a platform from which people will enter and exit the ride, t minutes after the ride starts, is modelled by the function $h(t) = -10\cos\left(\frac{\pi}{9}t\right) + 10$.

(a) Demonstrate that one complete cycle of the ride takes 18 minutes.

(b) Calculate the maximum height of the ride above the platform.

(c) Sketch the graph of $h(t)$ for $0 \leq t \leq 18$.

(d) In a single ride, determine when the carriage is 5 metres above the platform.

Solution:

7.3 The reciprocal trigonometric functions

7.3.1 Calculating values of reciprocal functions

Foundational Concepts

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Example 7.18 – Calculating reciprocal trigonometric functions for exact values**:**
Calculate the exact value of each of the following.

(a) $\sec(\pi)$ **(b)** $\csc\left(\frac{\pi}{2}\right)$ **(c)** $\csc(\pi)$ **(d)** $\cot\left(\frac{\pi}{3}\right)$

Solution:

Example 7.19 – Solving equations involving reciprocal trigonometric functions**:**

Solve the equation $\sec(x) = 2$, for $0 \leq x \leq 2\pi$.

Solution:

7.3.2 Graphs of reciprocal trigonometric functions

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Procedures

Example 7.20 – Sketching graphs of the reciprocal trigonometric functions of the form $f(ax)$:

Sketch the graph of $f(x) = \sec(2x)$ for $0 \leq x \leq 2\pi$, stating the domain, range and period. Also show the asymptotes and any intercepts with the axes.

Solution:

Example 7.21 – Sketching reciprocal trigonometric functions of the form $Af(x-b)$**:**

Sketch the graph of $f(x) = 2\cot\left(x - \dfrac{\pi}{4}\right)$ for $0 \leq x \leq 2\pi$, stating the domain, range and period. Also show the asymptotes and any intercepts with the axes.

Solution:

Example 7.22 – Sketching reciprocal functions of the form $f(a(x-b))+c$:

Sketch the graph of $f(x) = \csc\left(x + \dfrac{\pi}{4}\right) + 1$ for $0 \le x \le 2\pi$, stating the domain, range and period. Also show the asymptotes and any intercepts with the axes.

Solution:

7.4 The Pythagorean trigonometric identities

Foundational Concepts

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Example 7.23 – Calculating the values of trigonometric expressions using identities**:**

Calculate the values of the following.

(a) $\sin(\theta)$ for $\dfrac{\pi}{2} \leq \theta \leq \pi$ if $\cos(\theta) = -\dfrac{7}{25}$.

(b) $\cos(\theta)$ for $\pi \leq \theta \leq 2\pi$ if $\tan(\theta) = 3$.

Solution:

Example 7.24 – Simplifying trigonometric expressions**:**

Simplify the following.

(a) $\cos^2(A) + \sin^2(A) + \dfrac{1}{\cot^2(A)}$.

(b) $\dfrac{\sin(\theta)}{1-\cos(\theta)} + \dfrac{\sin(\theta)}{1+\cos(\theta)}$, where $\cos(\theta) \neq \pm 1$.

Solution:

Example 7.25 – Using trigonometric identities to solve equations**:**

Solve the following equations in the given domain.

(a) $\cos^2(A) - \sin^2(A) = \frac{1}{2}$, $0 \leq A \leq 2\pi$.

(b) $\csc^2(x) - \sqrt{3}\cot(x) = 1$, $0 \leq x \leq \pi$.

Solution:

Example 7.26 – Proving trigonometric identities**:**
Prove that $\dfrac{\csc(\theta)}{\tan(\theta)+\cot(\theta)} = \cos(\theta)$.

Solution:

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Example 7.27 – Proving a more complex trigonometric identity**:**
Prove that $\dfrac{\cos(x)}{1-\sin(x)} = \sec(x)+\tan(x)$.

Solution:

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7.5 Compound and double-angle identities

7.5.1 Sums and differences

Foundational Concepts

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Procedures

Example 7.28 – Using identities to simplify trigonometric expressions**:**
Demonstrate that $\sin\left(\dfrac{3\pi}{2} + x\right) = -\cos(x)$

Solution:

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Example 7.29 – Using a compound angle identity to determine a trigonometric expression**:**
If x and y are acute angles such that $\sin(x) = \dfrac{1}{4}$ and $\cos(y) = \dfrac{3}{4}$, determine $\sin(x + y)$ without calculating x or y.

Solution:

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Example 7.30 – Determining the exact value of a trigonometric expression using a compound angle identity**:**
Determine the exact value of $\cos(15°)$.

Solution:

Example 7.31 – Simplifying trigonometric expressions with compound angles**:**
Simplify $\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)+\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$.

Solution:

7.5.2 Double-angle identities

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Example 7.32 – Simplifying trigonometric expressions using double-angle identities**:**
Simplify $(\sin(3x) - \cos(3x))^2$.

Solution:

Example 7.33 – Using a double-angle formula to calculate the value of trigonometric expressions**:**
Given that $\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$, use a double-angle identity to demonstrate that $\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$.

Solution:

Example 7.34 – Using half-angle identities**:**

Complete the following.

(a) Use the identity $\cos(2x) = 1 - 2\sin^2(x)$, demonstrate that $\sin\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{2}}$.

(b) If $\cos(A) = \frac{3}{5}$ for $\frac{3\pi}{2} \leq A \leq 2\pi$, determine the exact value of $\sin\left(\frac{A}{2}\right)$ without calculating A.

Solution:

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Example 7.35 – Using double-angle identities in a proof**:**

Complete the following.

(a) Prove that $(\cos(\theta) - \sin(\theta))^2 = 1 - \sin(2\theta)$.

(b) Prove that $\cos(3x) = 4\cos^3(x) - 3\cos(x)$

Solution:

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Example 7.36 – Solving equations using double-angle identities**:**

Complete the following.

(a) Use the expansion found in Worked Example 35 for $\cos(3x)$ to solve the equation $4\cos^3(x) - 3\cos(x) = \dfrac{1}{2}$ for $0 \leq x \leq 2\pi$.

(b) Determine the value of θ such that $\sqrt{2}\sin(2\theta) = 2\cos(\theta)$, $0 \leq \theta \leq 2\pi$.

Solution:

7.5.3 Changing products to sums and differences

Foundational Concepts

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Procedures

Example 7.37 – Converting products to sums or differences of trigonometric expressions**:**
Express $\sin(\theta)\cos(3\theta)$ as a sum or difference.

Solution:

7.6 Transformations of trigonometric expressions

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Example 7.38 – Expressing a sum or difference in the form $R\sin(x \pm \alpha)$**:**

Express $\sin(x) - \cos(x)$ in the form $R\sin(x - \alpha)$, where R is a positive constant and $0 \leq \alpha \leq \dfrac{\pi}{2}$

Solution:

Example 7.39 – Expressing a sum or difference in the form $R\cos(x \pm \alpha)$ to solve an equation**:**
Consider the expression $\sqrt{3}\cos(x) - \sin(x)$.

(a) Express $\sqrt{3}\cos(x) - \sin(x)$ in the form $R\cos(x + \alpha)$ where $R > 0$ and $0 \leq \alpha \leq \dfrac{\pi}{2}$.

(b) Solve the equation $\sqrt{3}\cos(x) - \sin(x) = 1$, $0 \leq x \leq 2\pi$.

Solution:

Example 7.40 – Expressing a sum or difference in the form $R\sin(x \pm \alpha)$**:**
In Worked Example 38, the expression $\sin(x) - \cos(x)$ was found to be equal to $\sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$.
Use this relationship to sketch the graph of $y = \sin(x) - \cos(x)$.

Solution:

Example 7.41 – Solving a problem using the transformation of an expression to the form $R\sin(x+\alpha)$**:**
A rectangle $OABC$ is formed with one vertex from the origin O, point A at $0, 3\cos(\theta))$ and point C at $(4\sin(\theta), 0)$.

(a) Express $4\sin(\theta) + 3\cos(\theta)$ in the form $R\sin(\theta+\alpha)$ where $R > 0$ and $0 \leq \theta \leq \dfrac{\pi}{2}$.

(b) State the coordinates of B in terms of θ.

(c) Express the perimeter of the rectangle in terms of $f(\theta)$.

(d) Calculate the maximum value of the perimeter of the rectangle.

Solution:

Chapter 6
Graphing functions

Syllabus

Sketching graphs (5 hours)

In this sub-topic, students will:

- ☐ use and apply the notation $|x|$ for the absolute value for the real number x and the graph of $y = |x|$
- ☐ examine the relationship between the graph of $y = f(x)$ and the graphs of $y = \dfrac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$
- ☐ sketch the graphs of simple rational functions where the numerator and denominator are polynomials to degree 3 without technology.

6.1 Absolute value functions

6.1.1 Absolute values functions

Foundational Concepts

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Procedures

Example 6.1 – Determining the values of numerical expressions involving absolute values**:**
Determine each of the following.

(a) $|-5|$ **(b)** $-3\,|-12 + 2 \times 5|$ **(c)** $-5 \times |-6| - |5 \times -6|$

Solution:

Example 6.2 – Solving simple equations involving absolute values**:**
Solve the following equations.

(a) $|x + 3| = 5$ **(b)** $|10 - 2x| = 4$

Solution:

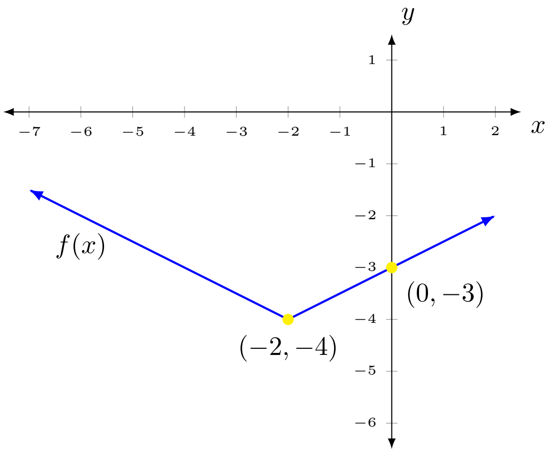
Example 6.3 – Sketching graphs of simple transformations of $y = |x|$**:**

Sketch the graph of the function $y = -2|x-3| + 4$, identifying the coordinates of the vertex, and any intercepts with the axes.

Solution:

Example 6.4 – Determining the equation of a simple absolute graph**:**

Determine the equation of the function drawn below.



Solution:

6.1.2 Graphs of the form $y = |f(x)|$ and $y = f(|x|)$

Foundational Concepts

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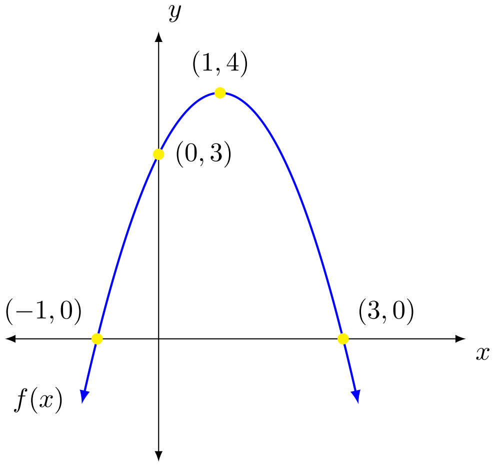
Procedures

Example 6.5 – Sketching graphs of the form $y = |f(x)|$ and $y = f(|x|)$**:**

For the function $f(x)$ graphed on the right, sketch the following graphs, clearly indicate the location of all intercepts and turning points.

(a) $y = |f(x)|$

(b) $y = f(|x|)$



Solution:

Example 6.6 – Solving more complex equations involving absolute values**:**

Solve for x.

$$\left|\frac{1}{2}x - 6\right| = x + 3$$

Solution:

6.2 Reciprocal functions

Foundational Concepts | Vocabulary & Definitions

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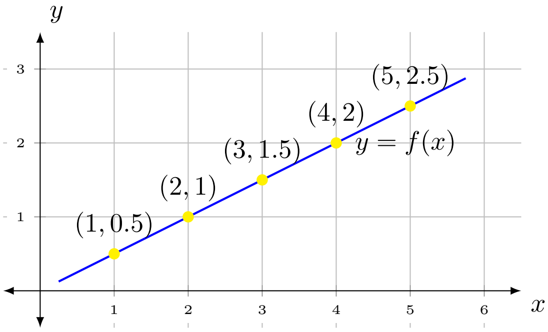
Example 6.7 – Sketching reciprocal functions from coordinates**:**

A section of the graph of a function $f(x)$ is shown with the coordinates of several points displayed.

Replicate this diagram.

Then determine the reciprocal of the y-coordinate of each point given.

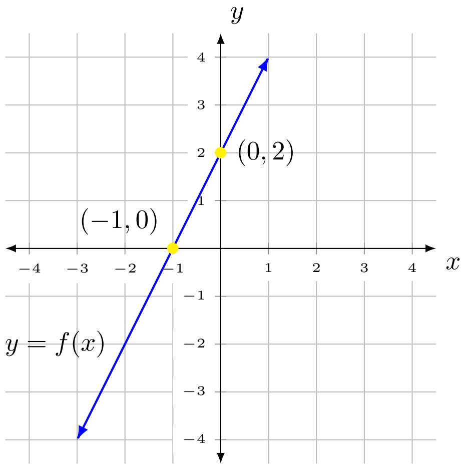
Hence, sketch the corresponding section of $y = \dfrac{1}{f(x)}$ on the same set of axes.



Solution:

Example 6.8 – Sketching $y = \dfrac{1}{f(x)}$ from the graph of $y = f(x)$ when $f(x)$ is linear**:**

The graph of the function $f(x)$ is shown on the right. Sketch the graph of $y = \dfrac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes and intercepts.



Solution:

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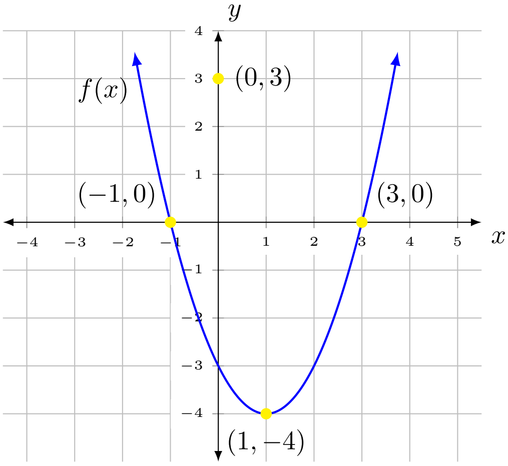
Example 6.9 – Sketching $y = \dfrac{1}{f(x)}$ from the graph of $y = f(x)$ when $f(x)$ contains local extrema**:**



The graph of the function $f(x)$ is shown on the right. Sketch the graph of $y = \dfrac{1}{f(x)}$ on the same set of axes, clearly identifying all asymptotes, intercepts, and local extrema.

Solution:

6.3 Rational functions

6.3.1 Sketching rational functions with linear denominators

Foundational Concepts

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Big Ideas & Concepts

Procedures

Example 6.10 – Sketching a rational function by first simplifying**:**

Simplify the rational function $y = \dfrac{2x^2 - 8}{x + 2}$ by identifying common factors in the numerator and denominator. Hence, sketch the graph of this function, clearly identifying any intercepts and discontinuous points.

Solution:

Example 6.11 – Sketching functions of the form $y = \dfrac{ax+b}{cx+d}$**:**

Sketch the function $y = \dfrac{6-3x}{2x+2}$, clearly indicating all intercepts and asymptotes.

Solution:

6.3.2 Sketching rational functions with quadratic denominators

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 6.12 – Sketching functions of the form $y = \dfrac{ax+b}{cx^2+dx+e}$**:**

Sketch the function $y = \dfrac{-2x+2}{x^2-9}$, clearly indicating all intercepts and asymptotes.

Solution:

Example 6.13 – Sketching functions of the form $y = \dfrac{ax^2 + bx + c}{dx^2 + ex + f}$:

Sketch the function $y = \dfrac{2x^2 + 2}{x^2 + 2x - 3}$, clearly indicating all intercepts and asymptotes.

Solution:

6.3.3 Sketching rational functions with an oblique asymptote

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 6.14 – Sketching functions of the form $y = \dfrac{ax^2+bx+c}{dx+e}$**:**

Sketch the function $y = \dfrac{x^2+3x}{x-1}$, clearly indicating all intercepts and asymptotes.

Solution:

Chapter 11

Techniques of integration

Syllabus

Integration techniques (10 hours)

In this sub-topic, students will:

- ☐ integrate using the trigonometric identities $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$, $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and $1 + \tan^2(x) = \sec^2(x)$
- ☐ use substitution $u = g(x)$ to integrate expressions of the form $f\left(g(x)\right) g'(x)$
- ☐ establish and use the formula $\int \frac{1}{x}\, dx = \ln|x| + c$, for $x \neq 0$
- ☐ find and use the inverse trigonometric functions: arcsine, arccosine and arctangent
- ☐ find and use the derivative of the inverse trigonometric functions: arcsine, arccosine and arctangent
- ☐ integrate expressions of the form $\frac{\pm 1}{\sqrt{a^2 - x^2}}$ and $\frac{a}{a^2 + x^2}$
- ☐ use partial fractions where necessary for integration in simple cases
- ☐ integrate by parts.

11A Finding definite integrals and using the modulus function

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 11.1:

Evaluate each of the following integrals:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \cos(3x)dx$ **(b)** $\displaystyle\int_0^1 e^{2x} - e^x dx$ **(c)** $\displaystyle\int_0^{\frac{\pi}{8}} \sec^2(2x)dx$ **(d)** $\displaystyle\int_0^1 \sqrt{2x+1}dx$

Solution:

Example 11.2:

Evaluate each of the following:

(a) $|-3 \times 2|$

(b) $|-3| \times |2|$

(c) $\left|\dfrac{-4}{2}\right|$

(d) $\dfrac{|-4|}{|2|}$

(e) $|-6 + 2|$

(f) $|-6| + |2|$

Solution:

Example 11.3:

Evaluate each of the following:

(a) $\dfrac{d}{dx}(\ln|x|)$ for $x \neq 0$.

(b) $\dfrac{d}{dx}(\ln|\sec x|)$ for $x \notin \{\dfrac{(2k+1)\pi}{2} : k \in \mathbb{Z}\}$.

Solution:

Example 11.4:

Calculate the following:

(a) An anti-derivative of $\dfrac{1}{4x+2}$ for $x \neq -\dfrac{1}{2}$.

(b) $\displaystyle\int_0^1 \frac{1}{4x+2}dx.$

(c) $\displaystyle\int_{-2}^{-1} \frac{1}{4x+2}dx.$]

Solution:

11B Derivatives of inverse trigonometric functions

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 11.5:

Differentiate each of the following with respect to x:

(a) $\sin^{-1}\left(\dfrac{x}{3}\right)$ **(b)** $\cos^{-1}(4x)$ **(c)** $\tan^{-1}\left(\dfrac{2x}{3}\right)$ **(d)** $\sin^{-1}\left(x^2-1\right)$

Solution:

11C Anti-derivatives involving inverse trigonometric functions

Foundational Concepts

Vocabulary & Definitions

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Example 11.6:

Calculate an anti-derivative of each of the following:

(a) $\dfrac{1}{\sqrt{9-x^2}}$ **(b)** $\dfrac{1}{\sqrt{9-4x^2}}$ **(c)** $\dfrac{1}{9+4x^2}$

Solution:

Example 11.7:

Evaluate each of the following definite integrals:

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}dx$ **(b)** $\displaystyle\int_0^2 \frac{1}{4+x^2}dx$ **(c)** $\displaystyle\int_0^1 \frac{3}{\sqrt{9-4x^2}}dx$

Solution:

11D Integration by substitution

Foundational Concepts

Vocabulary & Definitions

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Procedures

Example 11.8:

Differentiating each of the following with respect to x:

(a) $\left(2x^2+1\right)^5$ **(b)** $\cos^3 x$ **(c)** e^{3x^2}

we have:

(a) Let $y=\left(2x^2+1\right)^5$ and $u=2x^2+1$.
Then $y=u^5$, $\dfrac{dy}{du}=5u^4$ and $\dfrac{du}{dx}=4x$.
Then:

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dx}\\&=5u^4\cdot 4x\\&=20u^4x\\&=20x\left(2x^2+1\right)^4\end{aligned}$$

(b) Let $y=\cos^3 x$ and $u=\cos x$.
Then $y=u^3$, $\dfrac{dy}{du}=3u^2$ and $\dfrac{du}{dx}=-\sin x$.
Then:

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\cdot\frac{du}{dx}\\&=3u^2\cdot(-\sin x)\\&=3\cos^2\cdot(-\sin x)\\&=-3\cos^2 x\sin x\end{aligned}$$

(c) Let $y=e^{3x^2}$ and $u=3x^2$.
Then $y=e^u$, $\dfrac{dy}{du}=e^u$ and $\dfrac{du}{dx}=6x$.
Then:

$$\begin{aligned}\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}&=e^u\cdot 6x\\&=6xe^{3x^2}\end{aligned}$$

Note:

(a) $\displaystyle\int 20x\left(2x^2+1\right)^4dx=\left(2x^2+1\right)^5+c$, then letting $h(x)=x^2+1$, we have:
$\displaystyle\int 5h'(x)\left(h(x)\right)^4dx=\left(h(x)\right)^5+c$

(b) $\displaystyle\int -3\cos^2 x\sin x dx=\cos^3 x+c$, then letting $h(x)=\cos x$, we have:
$\displaystyle\int 3h'(x)\left(h(x)\right)^2dx=\left(h(x)\right)^3+c$

(c) $\displaystyle\int 6xe^{3x^2}dx=e^{3x2}+c$, then letting $h(x)=3x^2$, we have:
$\displaystyle\int h'(x)e^{(h(x)}dx=e^{h(x)}+c$

Example 11.9:

Determine an anti-derivative of each of the following:

(a) $\sin x \cos^2 x$ **(b)** $5x^2\left(x^3-1\right)^{\frac{1}{2}}$ **(c)** $3xe^{x^2}$

Solution:

Example 11.10:

Determine an anti-derivative of each of the following:

(a) $\dfrac{2}{x^2 + 2x + 6}$

(b) $\dfrac{3}{\sqrt{9 - 4x - x^2}}$

Solution:

Example 11.11:

Determine an anti-derivative of each of the following:

(a) $(2x+1)\sqrt{x+4}$

(b) $\dfrac{2x+1}{(1-2x)^2}$

(c) $x^2\sqrt{3x-1}$

Solution:

11E Definite integrals by substitution

Foundational Concepts	Vocabulary & Definitions
Big Ideas & Concepts	Procedures

Example 11.12:

Evaluate $\int_0^4 3x\sqrt{x^2+9}$

Solution:

Example 11.13:

Evaluate the following:

(a) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^3 x dx$

(b) $\displaystyle\int_0^1 2x^2 e^{x^3} dx$

Solution:

11F Using trigonometric identities for integration

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 11.14:

Evaluate the following:

(a) $\displaystyle\int \cos^2 x\, dx$

(b) $\displaystyle\int \tan^2 x\, dx$

(c) $\displaystyle\int \sin(2x)\cos(2x)\, dx$

(d) $\displaystyle\int \cos^4 x\, dx$

(e) $\displaystyle\int \sin^3 x \cos^2 x\, dx$

Solution:

11G Partial fractions

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 11.15:
Resolve $\dfrac{3x+5}{(x-1)(x+3)}$ into partial fractions.

Solution:

Example 11.16:
Resolve $\dfrac{2x+10}{(x+1)(x-1)^2}$ into partial fractions.

Solution:

Example 11.17:

Resolve $\dfrac{x^2+6x+5}{(x-2)(x^2+x+1)}$ into partial fractions.

Solution:

Example 11.18:
Express $\dfrac{x^5+2}{x^2-1}$ as partial fractions.

Solution:

Example 11.19:

Evaluate $\displaystyle\int \frac{3x+5}{(x-1)(x+3)}\,dx$.

Solution:

Example 11.20:

Evaluate $\displaystyle\int \frac{x^5+2}{x^2-1}\,dx$.

Solution:

Example 11.21:
Express $\dfrac{3x+1}{(x+2)^2}$ in partial fractions and hence evaluate $\displaystyle\int \frac{3x+1}{(x+2)^2}\,dx$.

Solution:

Example 11.22:

Determine an anti-derivative of $\dfrac{4}{(x+1)(x^2+1)}$ by first expressing it as partial fractions.

Solution:

11H Integration by parts

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 11.23:

Calculate an anti-derivative of each of the following:

(a) $x \cos x$ **(b)** xe^x **(c)** $\arcsin x$

Solution:

Example 11.24:
Determine $\int x^2 e^x \, dx$.

Solution:

Example 11.25:

Evaluate $\int e^x \cos x \, dx$.

Solution:

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Example 11.26:

Evaluate $\int_1^2 \ln x \, dx$.

Solution:

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11I Further techniques and miscellaneous exercises

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 11.27:

Determine the derivative of $\sin^{-1}(x) + x\sqrt{1-x^2}$, hence evaluate $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx$.

Solution:

Chapter 12

Applications of integration

Syllabus

Applications of integral calculus (9 hours)

In this sub-topic, students will:

- ☐ calculate areas between curves determined by functions
- ☐ determine volumes of solids of revolution about either axis
- ☐ use the numerical integration method of Simpson's rule, using technology
- ☐ use and apply the probability density function, $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$, of the exponential random variable with parameter $\lambda > 0$, and use the exponential random variables and associated probabilities and quantiles to model data and solve practical problems.

12A The fundamental theorem of calculus

Foundational Concepts

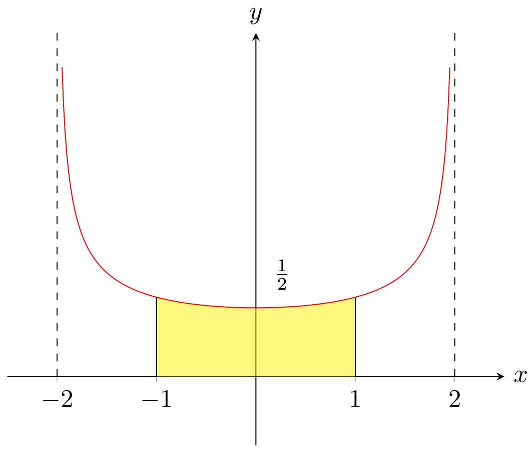
Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 12.1:

The graph of $y = \dfrac{1}{\sqrt{4-x^2}}$ is shown. Determine the area of the shaded region.



Solution:

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Example 12.2:

Determine the area under the graph of $y = \dfrac{6}{4+x^2}$ between $x = -2$ and $x = 2$.

Solution:

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Example 12.3:

Sketch the graph of $f(x) = \sin^{-1}(2x), x \in [-\frac{1}{2}, \frac{1}{2}]$. Shade the region defined by the inequalities $0 \leq x \leq \frac{1}{2}$ and $0 \leq y \leq f(x)$. Determine the area of this region.

Solution:

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Example 12.4:

Sketch the graph of $y = \dfrac{1}{4 - x^2}$. Shade the region for the area determined by $\displaystyle\int_{-1}^{1} \frac{1}{4 - x^2} dx$ and calculate this area.

Solution:

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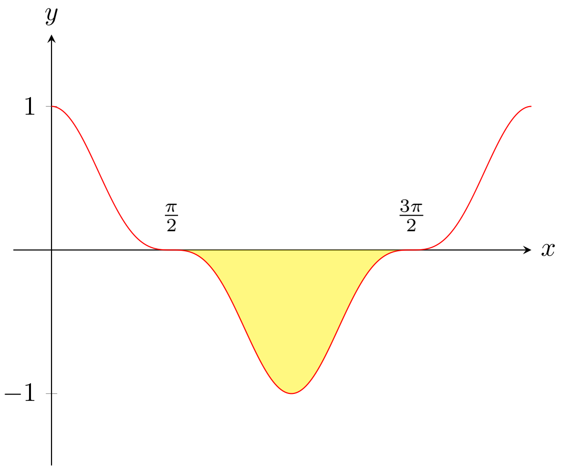
Example 12.5:

The graph of $y = \cos^3 x$ is shown. Determine the area of the shaded region.



Solution:

12B Area of a region between two curves

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 12.6:

Calculate the area of the region bounded by the parabola $y = x^2$ and the line $y = 2x$.

Solution:

Example 12.7:

Calculate the area of the region enclosed by the curves with equations $y = x^2 + 1$ and $y = 4 - x^2$ and the lines $x = -1$ and $x = 1$.

Solution:

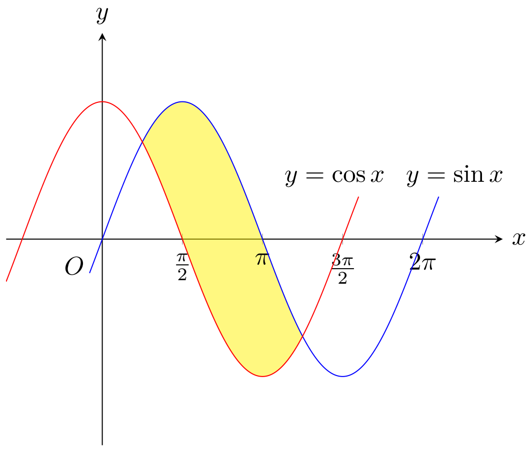
Example 12.8:

Determine the area of the region enclosed by the graphs of $f(x) = x^3$ and $g(x) = x$.

Solution:

Example 12.9:

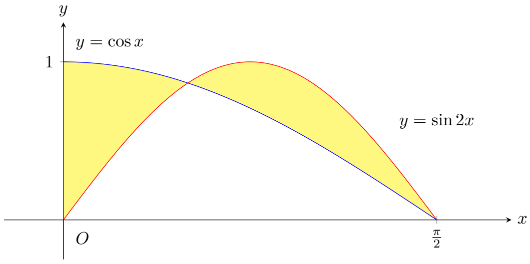
Calculate the area of the shaded region.



Solution:

Example 12.10:

Calculate the area of the shaded region.



Solution:

12C Integration using a graphics calculator

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

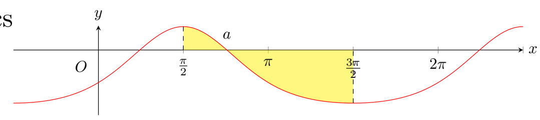
Example 12.11:

Using a graphics calculator, evaluate $\int_1^2 \ln x \, dx$.

Solution:

Example 12.12:

The graph of $y = e^{\sin x} - 2$ is as shown. Using a graphics calculator, determine the area of the shaded regions.



Solution:

Example 12.13:

Plot the graph of $F(x) = \int_1^x \frac{1}{t}\, dt$ for $x > 1$.

Solution:

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Example 12.14:

Use a graphics calculator to calculate an approximate value of $\int_0^{\frac{\pi}{3}} \cos\left(x^2\right)\, dx$ and to plot the graph of $f(x) = \int_0^x \cos\left(t^2\right)\, dt$ for $-\frac{\pi}{4} \leq x \leq \pi$.

Solution:

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12D Volumes of solids of revolution

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 12.15:

Calculate the volume of the solid of revolution formed by rotating the curve $y = x^3$ about:

(a) the x-axis for $0 \leq x \leq 1$

(b) the y-axis for $0 \leq y \leq 1$

Solution:

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Example 12.16:

Determine the volume of the solid of revolution when the region bounded by the graphs of $y = 2e^{2x}$, $y = 1$, $x = 0$ and $x = 1$.

Solution:

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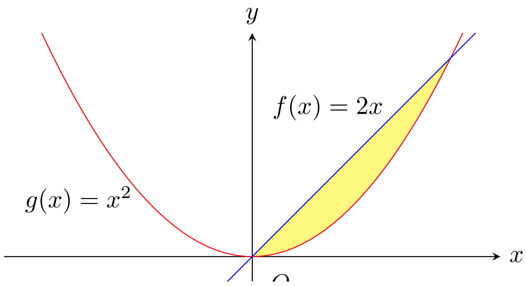
Example 12.17:

The shaded region is rotated around the x-axis.
Calculate the volume of the resulting solid.



Solution:

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Example 12.18:

A solid is formed when the region bounded by the x-axis and the graph of $y = 3\sin(2x)$, $0 \le x \le \frac{\pi}{2}$ is rotated around the x-axis. Determine the volume of this solid.

Solution:

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Example 12.19:

The curve $y = 2\sin^{-1} x$, $0 \le x \le 1$, is rotated around the y-axis to form a solid of revolution. Calculate the volume of this solid.

Solution:

12E The exponential probability distribution

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 12.20:

The time, X minutes, that a shop assistant waits before the next customer arrives is known to be exponentially distributed, with probability density function given by

$$f(x) = \begin{cases} 0.2e^{-0.2x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Calculate the probability that he will wait more than 8 minutes for the next customer to arrive.

Solution:

Example 12.21:

For the situation in Example 20, calculate the mean adn standard deviation of the time that the shop assistant waits for a customer.

Solution:

Example 12.22:

The time, T minutes, that it takes a librarian to locate a book is exponentially distributed with a mean of 3 minutes. Determine:

(a) the probability density function of T

(b) the probability that it takes her less than 2 minutes to find a book.

Solution:

Example 12.23:

For the situation in Example 20, determine the cumulative distribution function of X, and hence calculate the probability that the shop assistant waits between 5 and 10 minutes for a customer.

Solution:

12F Simpson's rule

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 12.24:

Use Simpson's rule to estimate the integral $\int_2^5 x^3\,dx$.

Solution:

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Example 12.25:

Use Simpson's rule to estimate the integral $\int_0^1 e^x\,dx$.

Solution:

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Example 12.26:

Use Simpson's rule with 10 subintervals to estimate the integral $\displaystyle\int_0^1 e^x \, dx$.

Solution:

Chapter 8

Vector calculus

Syllabus

Vector calculus (5 hours)

In this sub-topic, students will:

- ☐ consider position of vectors as a function of time
- ☐ derive the Cartesian equation of a path given as a vector equation in two dimensions, including circles, ellipses and hyperbolas
- ☐ differentiate and integrate a vector function with respect to time
- ☐ determine equations of motion of a particle travelling in a straight line with both constant and variable acceleration
- ☐ apply vector calculus to motion in a plane, including projectile and circular motion.

8A Summary of differentiation and anti-differentiation

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 8.1:

Differentiate each of the following with respect to x:

(a) $\sqrt{x}\sin x$ **(b)** $\dfrac{x^2}{\sin x}$ **(c)** $\cos\left(x^2+1\right)$

Solution:

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Example 8.2:

Differentiate each of the following w.r.t. x:

(a) $\tan\left(5x^2+3\right)$ **(b)** $\tan^3 x$ **(c)** $\sec^2(3x)$

Solution:

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Example 8.3:

Determine the second derivative of each fo the following w.r.t. x:

(a) $f(x) = 6x^4 - 4x^3 + 4x$ **(b)** $y = e^x \sin x$

Solution:

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Example 8.4:

Determine an anti-derivative of each of the following:

(a) $\sin\left(3x - \frac{\pi}{4}\right)$ **(b)** e^{3x+4} **(c)** $6x^3 - \frac{2}{x^2}$

Solution:

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Example 8.5:

Determine $f(x)$ for each of the following.

(a) $f'(x) = 3x^2 + 1$, given $f(0) = 1$.

(b) $f''(x) = -9.8$, given $f(0) = 0$ and $f'(0) = 3$.

Solution:

8B Position vectors as a function of time

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 8.6:

Each of the following vector functions describes the motion of a particle by giving its position vector, $\boldsymbol{r}(t)$, at time t. For each vector function:

i. Determine the Cartesian relation that represents the path of the particle, determine its domain and range, and sketch its graph.

ii. Give the starting point of the particle's motion, the direction of motion and the period of motion (if applicable).

(a) $\boldsymbol{r}(t) = -5\cos\left(\frac{\pi t}{2}\right)\boldsymbol{i} + 5\sin\left(\frac{\pi t}{2}\right)\boldsymbol{j},\ t \geq 0$

(b) $\boldsymbol{r}(t) = -5\cos\left(\frac{\pi t}{2}\right)\boldsymbol{i} + 12\sin\left(\frac{\pi t}{2}\right)\boldsymbol{j},\ t \geq 0$

(c) $\boldsymbol{r}(t) = 4\tan\left(\frac{\pi t}{2}\right)\boldsymbol{i} - 3\sec\left(\frac{\pi t}{2}\right)\boldsymbol{j},\ t \in (1, 3)$

Solution:

8C Vector calculus

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 8.7:
Let $\boldsymbol{r}(t) = 20t\boldsymbol{i} + \left(15t - 5t^2\right)\boldsymbol{j}$. Determine $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$.

Solution:

Example 8.8:
Let $\boldsymbol{r}(t) = \cos t\boldsymbol{i} - \sin t\boldsymbol{j} + 5t\boldsymbol{k}$. Determine $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$.

Solution:

Example 8.9:

Let $\boldsymbol{r}(t) = t\boldsymbol{i} + \left((t-1)^3 + 1\right)\boldsymbol{j}$. Determine $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$ where $\boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j}$.

Solution:

Example 8.10:

Let $\boldsymbol{r}(t) = e^t\boldsymbol{i} + \left((e^t-1)^3 + 1\right)\boldsymbol{j}$. Determine $\dot{\boldsymbol{r}}(\alpha)$ and $\ddot{\boldsymbol{r}}(\alpha)$ where $\boldsymbol{r}(\alpha) = \boldsymbol{i} + \boldsymbol{j}$.

Solution:

Example 8.11:

A curve is described by the vector equation $\boldsymbol{r}(t) = 2\cos t\boldsymbol{i} + 2\sin t\boldsymbol{j}$.

(a) Determine $\dot{\boldsymbol{r}}(t)$ and $\ddot{\boldsymbol{r}}(t)$.

(b) Determine the gradient of the curve at the point (x, y), where $x = 2\cos t$ and $y = 3\sin t$.

Solution:

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Example 8.12:

A curve is described by the vector equation $\boldsymbol{r}(t) = \sec(t)\boldsymbol{i} + \tan(t)\boldsymbol{j}$, with $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(a) Determine the gradient of the curve at the point (x, y), where $x = \sec(t)$ and $y = \tan(t)$.

(b) Determine the gradient of the curve where $t = \frac{\pi}{4}$.

Solution:

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Example 8.13:

Let $\ddot{\boldsymbol{r}}(t) = 10\boldsymbol{i} - 12\boldsymbol{k}$.

(a) Determine $\dot{\boldsymbol{r}}(t)$ if $\dot{\boldsymbol{r}}(0) = 30\boldsymbol{i} - 20\boldsymbol{j} + 10\boldsymbol{k}$.

(b) Determine $\ddot{\boldsymbol{r}}(t)$ if additionally $\boldsymbol{r}(0) = 0\boldsymbol{i} + 0\boldsymbol{j} + 2\boldsymbol{k}$.

Solution:

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Example 8.14:

Given $\ddot{\boldsymbol{r}}(t) = -9.8\boldsymbol{j}$ with $\boldsymbol{r}(0) = \boldsymbol{0}$ and $\dot{\boldsymbol{r}} = 30\boldsymbol{i} + 40\boldsymbol{j}$, determine $\boldsymbol{r}(t)$.

Solution:

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8D Velocity and acceleration for motion along a curve

Foundational Concepts | Vocabulary & Definitions

Big Ideas & Concepts | Procedures

Example 8.15:

The position of an object is $\boldsymbol{r}(t)$ metres at time t seconds, where $\boldsymbol{r}(t) = e^t\boldsymbol{i} + \frac{2}{9}e^{2t}\boldsymbol{j}, t \geq 0$.

(a) Determine the velocity vector for time t.

(b) Determine the acceleration vector for time t.

(c) Determine the speed at time t.

Solution:

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Example 8.16:

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = \left(2t - t^2\right)\boldsymbol{i} + \left(t^2 - 3t\right)\boldsymbol{j} + 2t\boldsymbol{k}$, where $t \geq 0$.

(a) Determine the velocity of the particle at time t.

(b) Determine the speed of the particle at time t.

(c) Determine the minimum speed of the particle.

Solution:

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Example 8.17:

The position of a projectile at time t is given by $\boldsymbol{r}(t) = 400t\boldsymbol{i} + \left(500t - 5t^2\right)\boldsymbol{j}$, for $t \geq 0$, where $\boldsymbol{i}$ is the unit vector in a horizontal direction and $\boldsymbol{j}$ is a unit vector vertically up. The projectile is fired from a point on the ground.

(a) Determine the time taken to reach the ground again.

(b) Determine the speed at which the projectile hits the ground.

(c) Determine the maximum height of the projectile.

(d) Determine the initial speed of the projectile.

Solution:

Example 8.18:

The position vector of a particle at time t is given by $\boldsymbol{r}(t) = 2\sin(2t)\,\boldsymbol{i} + \cos(2t)\,\boldsymbol{j} + 2t\boldsymbol{k}$, where $t \geq 0$.

(a) Determine the velocity at time t.

(b) Determine the speed of the particle at time t.

(c) Determine the maximum speed.

(d) Determine the minimum speed.

Solution:

Example 8.19:

The position vectors at time $t \geq 0$, of particles A and B are given by

$$\boldsymbol{r}_A(t) = \left(t^3 - 9t + 8\right)\boldsymbol{i} + t^2\boldsymbol{j}$$
$$\boldsymbol{r}_B(t) = \left(2 - t^2\right)\boldsymbol{i} + (3t - 2)\boldsymbol{j}$$

Prove that A and B collide while travelling at the same speed but at right angles to each other.

Solution:

8E Motion in a straight line

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 8.20:

A particle moves along a straight line such that its position vector, $\boldsymbol{r}(t)$ cm, at time t seconds is given by $\boldsymbol{r}(t) = \left(3t - t^3\right) \boldsymbol{i}$, for $t \geq 0$.

(a) Determine its initial position.

(b) Determine its position when $t = 2$.

(c) Determine its initial velocity.

(d) Determine its velocity when $t = 2$.

(e) Determine its speed when $t = 2$.

(f) Determine when and where the velocity is zero.

Solution:

Example 8.21:

A cricket ball is projected vertically upwards from ground level with an initial speed of 15 m/s. Take the origin O to be the point of projection and take $\boldsymbol{j}$ to be the unit vector vertically up, where the unit of distance is metres.

Relative to this frame of reference, let $\boldsymbol{r}(t)$ m be the position of the ball at time t seconds. Then $\ddot{\boldsymbol{r}} = -g\boldsymbol{j}$, where g m/s^2 is the magnitude of acceleration due to gravity ($g \approx 9.8$).

(a) Determine an expression for $\dot{\boldsymbol{r}}(t)$.

(b) Determine an expression for $\boldsymbol{r}(t)$.

(c) Determine the max height reached by the ball.

(d) Determine when the ball returns to ground level.

Solution:

8F Projectile motion

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 8.22:

A particle is projected from a point on horizontal ground with speed 50 m/s at an angle of 30° to the horizontal. Let $\boldsymbol{i}$ and $\boldsymbol{j}$ be unit vectors in the horizontal (x) and vertical (y) directions respectively. Neglect air resistance.

(a) Determine the initial velocity vector.

(b) Determine the velocity vector at time t seconds.

(c) Determine the position vector at time t seconds.

(d) Determine the Cartesian equation of the path.

Solution:

8G Circular motion

Foundational Concepts

Vocabulary & Definitions

Big Ideas & Concepts

Procedures

Example 8.23:

A particle is moving around a circle of radius 3 m with a constant speed of 2 m/s. It is known that $\theta = 0$ at time $t = 0$.

(a) Determine the angular velocity of the particle.

(b) Determine the position of the particle at time $t = \pi$ seconds.

(c) Determine the velocity of the particle at time $t = \pi$ seconds.

(d) Determine the acceleration of the particle at time $t = \pi$ seconds.

Solution:

Example 8.24:

A particle moves at a constant speed of 8 m/s around a circle with radius of 4 m. Assume that $\theta = 0$ when $t = 0$.

(a) Determine the position of the particle, relative to the centre of the circle, at time t seconds.

(b) Determine the veloicty of the particle at time t seconds.

(c) Determine the acceleration of the particle at time t seconds.

Solution: