Determining the relationship between masses in equilibrium and the angle of a frictionless plane

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May 2025

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1 Introduction

1.1 Research Question

When the mass of an on a frictionless plane is altered, and the mass of a hanging object adjusted so equilibrium is achieved, what is the precision and uncertainty of this method in comparison to conventional measuring techniques?

1.2 Rationale

The original experiment was conducted to determine the relationship between the mass of a carriage (C_m) and a hanging mass (H_m) when a frictionless plane was inclined at different angles. The results confirmed the theoretical relationship $H_m = C_m \sin(\theta)$.

It was noticed during the experiment that the angle measurement device, an 'angle gun', had a large uncertainty ($\pm 0.5\,\text{deg}$) compared to the scale used to measure masses (± 0.005 grams). It was questioned whether the relationship between the masses in equilibrium could be used to determine the angle of the plane with improved precision and uncertainty.

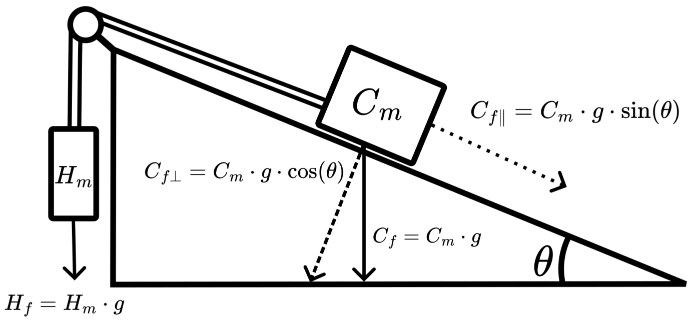


Figure 1: Diagram of experimental setup

Considering Newtons first law, under equilibrium the net force is 0 (Britannica, 2023). This implies that under equilibrium, $C_{f\parallel} = H_f$.

By plotting the experimental results of the experiment in Excel, a line of best fit can be determined. Since the relationship between the masses is established to be linear, we can use the gradient as a parameter to calculate the angle.

$$
\begin{aligned}
H_m &= C_m \cdot \sin(\theta) \\
C_m &= H_m \cdot \frac{1}{\sin(\theta)} && \because \text{Make dependent variable subject} \\
\frac{C_m}{H_m} &= \text{gradient} && \because \text{gradient} = \frac{\text{rise}}{\text{run}} \\
\frac{H_m}{C_m} &= \frac{1}{\text{gradient}} = \sin(\theta) \because \sin(\theta) = \frac{O}{H} \\
\therefore\ \theta &= \sin^{-1}\left(\frac{1}{\text{gradient}}\right)
\end{aligned}
$$

Since the angle is now expressed in terms of the gradient, the maximum and minimum slopes derived from experimental data can be used to determine the uncertainty in the angle by calculating the angle using each one of these extremes, then dividing the difference between them by 2.

1.3 Methodology

1.3.1 Modifications

The following modifications to the method were implemented

- The plane was kept at a constant angle throughout the entire duration of the experiment. This was done to isolate it from the independent and dependent variables and ensure that the results of all trials would point to the same relationship between them and the angle.
- The independent variable became the hanging mass (H_m). Since this mass could be directly measured and processed without the use of trigonometric functions, the only uncertainty in it's value should be random error from the scale. By conducting multiple trials with the same parameters, random error can be negated significantly.
- The dependent variable became the carriage mass (C_m) as the large area inside each carriage allowed for fine adjustment of its mass via the addition of brass weights.

1.3.2 Materials

- Angle gun
- Measuring tape
- Frictionless plane
- Pulley
- Blower fan
- Brass weights
- Blue tack
- Scale
- Carriage

1.3.3 Method

1. Set up slope at a constant angle as shown in Figure 1. It will remain at this angle for the entire duration of the experiment.
2. Perform and record the following measurements
   - Along the length of the plane from the ground, to very edge of where the string pivots downwards. Record as "Hypotenuse"
   - From the point on the ground used as the starting point for the first measurement to the point in which the tensioned string that H_m is connected to touches the ground. Record as "Adjacent"
   - From the edge of the point where the string pivots downwards to the ground. Record as "Opposite"
3. Set the hanging mass (H_m) to its minimum value initially.
4. Turn on the blower fan
5. Alter the mass of the carriage (C_m) until equilibrium with the H_m is achieved, i.e. The carriage remains stationary.
6. If the carriage does not have sufficient space for more weight, replace with a larger carriage or link an additional carriage to the chain.

7. Turn off blower fan
8. Measure and record masses.
9. Repeat for 3 trials with current H_m value.
10. Increase H_m by 50 grams.
11. Repeat until $H_m \geq\approx 300$g or until equilibrium cannot be achieved with equipment.

1.3.4 Risk Assessment

Frictionless plane

- Mishandling of heavy masses on the frictionless plane could result in them sliding down the slope at high speed. This could damage equipment of cause injury. The blower fan will be turned off not required, and one person will always be supporting the carriage whenever possible to prevent this.
- Using too low fan speed on the frictionless plane may not create enough of an air pocket to support heavy weights. This could create friction between the surfaces, which could damage equipment and introduce inaccuracies. The fan will be set to the highest possible speed throughout the experiment to negate the possibility of this occurring.

Blower

- Leaving the blower fan on for extended periods may cause overheating, as clearly stated in its instructions. This further implies the need for the blower to be turned off when not required.

Masses

- Heavy masses or items containing many brass weights may cause injury if dropped or mishandled. participants will wear enclosed footwear to negate injury if this occurs.

2 Results and Evaluation

2.1 Results

2.1.1 Raw Data

Carriage mass					Hanging mass			
1	2	3	Uncertainty	Average	1	2	3	Average
138.550	139.200	138.540	0.330	138.763	50.180	50.160	50.140	50.160
277.270	278.870	278.010	0.800	278.050	100.240	101.230	100.220	100.563
418.050	418.480	418.500	0.225	418.343	150.160	150.390	150.270	150.273
554.870	555.460	554.890	0.295	555.073	200.180	200.180	200.170	200.177
696.860	696.990	697.860	0.500	697.237	250.180	250.210	250.200	250.197
848.190	849.060	849.060	0.435	848.770	300.300	300.200	300.240	300.247

Figure 2: Raw results with additional calculations

2.1.2 Sample Calculations

Absolute uncertainty for C_m when $H_m = 50.160$

$$
\begin{aligned}
\sigma(C_m) &= \pm\frac{\max - \min}{2} \\
&= \pm\frac{139.20 - 138.55}{2} \\
&= \pm 0.325
\end{aligned}
$$

Average mass of C_m when $H_m = 50.16$

$$\begin{aligned}\bar{C_m} &= \frac{\Sigma_{i=1}^{n} C_m}{n} \\ &= \frac{138.55 + 139.20 + 138.54}{3} \\ &= 137.76\end{aligned}$$

2.1.3 Prerequisite trigonometric measurements

Hypotenuse	**Length**	**Height**
2.65	2.50	0.98

Figure 3: Table showing the side lengths of triangle formed by incline plane

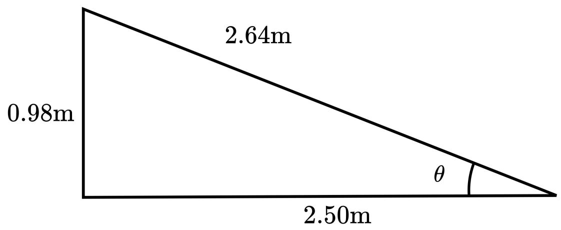


Figure 4: Diagram showing the side lengths of triangle formed by incline plane

2.1.4 Plotting

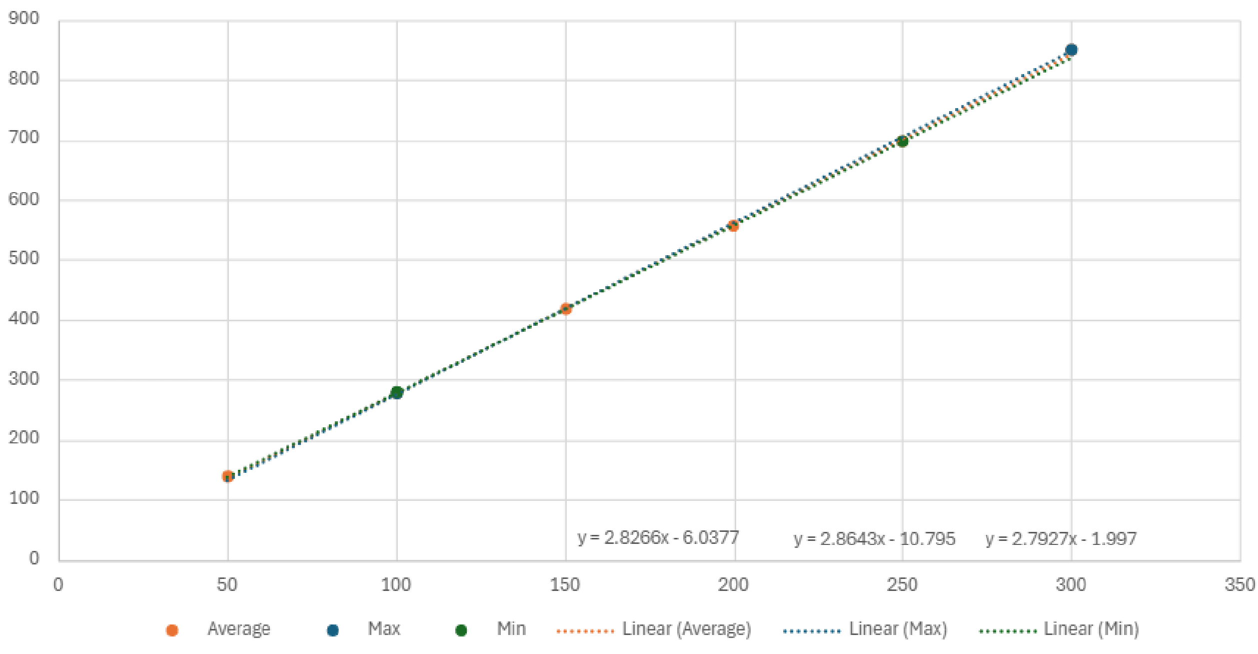


Figure 5: Average H_m (x-axis) and C_m (y-axis) extrapolated from experimental data

3 Discussion

3.1 Analysis of evidence

3.1.1 Identification of trends

Figure 5 depicts a clear linear relationship formed by the data plotted. This supports the previously investigated relationship $H_m = C_m \cdot \sin(\theta)$. However, clearly this graph represents the rearranged equation $C_m = H_m \cdot \frac{1}{\sin(\theta)}$.
This can be confirmed by determining the ratio between each of its points:

As H_m doubles from 50.14 to 100.06, C_m increases by a factor of $\frac{278.05}{138.76} = 2.0038 \approx 2.00$
As H_m doubles from 50.14 to 100.06, C_m increases by a factor of $\frac{555.07}{278.05} = 1.9963 \approx 2.00$

The presence of vertical shift in the line of best fit implies there is some inaccuracy in the data. In theory the relationship should be directly proportional To find the angle, the gradient of the slope of average values for each trial can be extrapolated from the line of best fit, and substituted into the equation found for angle in terms of gradient.

$$\begin{aligned}\theta &= \sin^{-1}\left(\frac{1}{\text{gradient}}\right)\\ &= \sin^{-1}\left(\frac{1}{2.8266}\right)\\ &= 20.7188^\circ\end{aligned}$$

3.1.2 Uncertainty

Finding uncertainty

Since the value of H_m was deliberately set, any uncertainty was due to the inherent random error in the scale used to measure it. By averaging H_m across trials, a more reliable value can be found, however it should be noted that although this negates the effect of random error, the systematic uncertainty ($\pm 0.005g$) is not affected as it is determined by the scale's precision.
To find the absolute uncertainty in the gradient, the maximum and minimum slopes were required. By considering the two average carriage masses with the largest uncertainties, say m_1, m_2, then plotting the line between $m_1 + \sigma_1$ and $m_2 - \sigma_2$, then $m_1 - \sigma_1$ and $m_2 + \sigma_2$, the equations of the maximum and minimum slopes could be solved as a line of best fit using Excel.

Absolute uncertainty in the gradient

$$\begin{aligned}\sigma(\text{gradient}) &= \pm\,\frac{\max - \min}{2}\\ \therefore\ \sigma(\text{gradient}) &= \pm\,\frac{2.8643 - 2.7927}{2}\\ &= \pm\,\frac{0.0716}{2}\\ &= \pm\,0.0358\end{aligned}$$

Absolute uncertainty in the angle

$$\sigma(\text{angle}) = \pm \frac{\max - \min}{2}$$

$$\sigma(\text{angle}) = \pm \frac{\sin^{-1}(\frac{1}{2.8643}) - \sin^{-1}(\frac{1}{2.7927})}{2}$$

$$= \pm\, 0.274138198466°$$

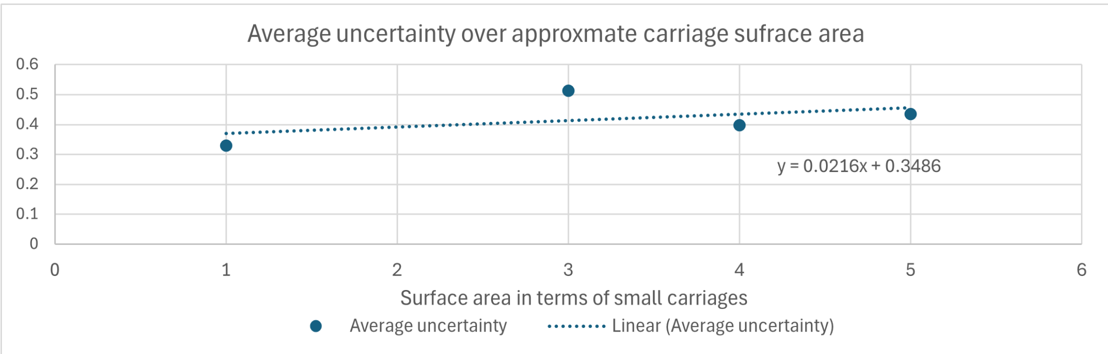
Possible sources of uncertainty

Angle consistency

Since the experiment was conducted over multiple class periods, the experimental setup was dismantled and reassembled part way through. Additionally the plane was bumped on multiple occasions, requiring it to be returned back to its original location and angle. Despite the angle of the plane measuring the same throughout the experiment, the uncertainty in the tool used to set the angle of the plane ($\pm 0.5°$) meant that the angle was likely inconsistent. This was not accounted for in any way.

Friction

It was assumed that the frictionless plane was completely frictionless. This was not the case, and multiple factors likely altered the degree of friction across it throughout the experiment. It was noticed that when masses were in equilibrium, pushing the cart would cause it to move for a while, then slow down and stop. If the plane was truly frictionless then the plane would not have slowed down, but continued until it reached the end stop. It was theorised that the surface area of the carriage on the plane determined how much of an air pocket could form underneath it. This was briefly investigated as the type of carriage used was recorded for each trial. Considering one big carriage having the approximate equivalent surface area of two small carriages, the average uncertainty for each carriage surface area was graphed. However, rather than a reduction in uncertainty as surface area increased, an increase was observed. Another source of friction is the pulley and string system that connects the masses. Although the pulley moves freely, there is likely play or static fiction in the system, as well as elasticity in the string.



4 Evaluation

4.1 Reliability and validity

Comparison to trigonometric calculations

Using the lengths of each side of the right angle triangle formed by the incline plane and the ground, the angle between then can be found. It was found that the angle differed based on which trigonometric function was used.

$$\sin^{-1}\left(\frac{0.98}{2.64}\right) = 21.79°$$
$$\cos^{-1}\left(\frac{2.50}{2.64}\right) = 18.74°$$
$$\tan^{-1}\left(\frac{0.98}{2.50}\right) = 21.41°$$

This indicates error in the measurement process. While assuming these values are equally reliable, then averaging them, would give a valid angle, this process introduces uncertainty. It was decided that the most reliable values were the hypotenuse since it was the length of the incline plane and physically could not have changed, and the vertical measurement taken from the top of the plane to the ground, as it was the easiest to confirm as parallel during the measurement process.
Considering the ruler only measured in half centimetre increments, its uncertainty was ± 0.25cm, or ± 0.0025m. By considering the sine value of $\frac{2.64-\sigma}{0.98+\sigma}$, and $\frac{2.64+\sigma}{0.98-\sigma}$, the absolute uncertainty in this measurement can be found.

$$\sin^{-1}\left(\frac{0.98+0.0025}{2.64-0.0025}\right) = 21.87$$
$$\sin^{-1}\left(\frac{0.98-0.0025}{2.64+0.0025}\right) = 21.71$$

$$\begin{aligned}\sigma &= \frac{\max - \min}{2} \\ &= \frac{21.87 - 21.71}{2} \\ &= \pm 0.08\end{aligned}$$

Therefore the angle of the slope as determined via trigonometry was $\sin\left(\frac{0.98}{2.64}\right) = 21.79° \pm 0.08$.
While the uncertainty is very low, the error when compared to the measured angle of 20° is

$$\frac{21.79 - 20}{20} = 8.95\%$$

This is a very large error in the context of using this value as a reference for comparison and therefore this measurement was deemed unusable.
The possibility was considered that during the measurement process, the hypotenuse was measured as the length of the plane, rather than where the plane would intersect with the ground, to the point in which the string begins travelling downwards. Assuming 7cm of unaccounted for length on either side of the slope, the total length of the hypotenuse would be 2.78m. Assuming 10cm of unaccounted for horizontal length due to poor procedure, the total length of the "Adjacent" would be 2.60m. These changes would result in an average angle across sine, cosine, and tangent calculations of $20.68° \pm 0.045$, which falls in line with the expected results.

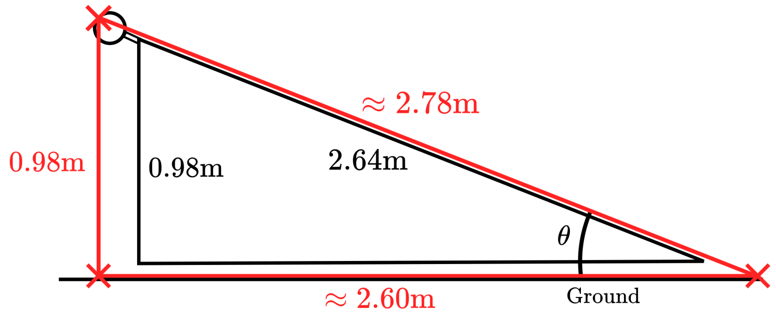


Figure 6: Recorded measurements (black) and theorised measurements (red)

However since these measurements are purely theoretical, they cannot be used to quantify error or any other results.

Comparison to angle gun As previously mentioned, the line of best fit constructed by the data did not cross through the origin, which implies error in the experimental process. This was likely due to the variables previously discussed, such as friction and procedural error. Considering the unreliability of data derived from length measurementation, a final error analysis was unable to be conducted. The only option available is a direct comparison in terms of uncertainty and prevision in comparison to the angle gun.

Uncertainty comparison

$$\sigma(\text{Angle gun}) = \pm 0.5^\circ > \sigma(\text{Mass method}) = \pm 0.27^\circ$$

Clearly the uncertainty of the angle found using masses in equilibrium is smaller than the uncertainty of the angle gun by nearly a factor of half.

4.2 Improvements and Extensions

Improvements

The experimental process could be improved through the implementation of the following:

- Conducting measurements using a rigid "square" (tool, not shape) to ensure that measurements are taken from the correct position and at angles parallel to the ground or other features.
- A stronger blower fan could reduce friction, therefore improving accuracy.
- More rigid string, or steel wire, could be utilised to remove the elastic component from the linkage between masses.
- A digital angle meter could be used to find the expected angle with more precision so that a reliable error analysis could be conducted.

Extensions

The following extensions could extend data on a specific aspect of the experiment or pivot to investigation to a tangentially related topic.

- A pivot to the investigation could investigate masses not in equilibrium and how their masses affect their acceleration.
- Investigation into deceleration of masses when force is applied to quantify the friction of the supposedly "frictionless" plane.

5 Conclusion

This experiment quantified the angle of a frictionless plane using the relationship between a mass on the plane, and a hanging mass linked to it. Despite systematic and procedural errors such as friction, unintentional equipment reconfiguration, and incorrect measurementation, the angle was found with increased prevision and reduced uncertainty ($20.72^\circ \pm 0.27^\circ$) in comparison to conventional equipment such as an angle gun ($20^\circ \pm 0.5^\circ$). It is evident that this is a viable method for determining angle when performed correctly, however, further investigation is required to determine the exact error of this method.

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