Module 9 Homework - Noboru Hayashi

6.2.10

6.2.10 Set μ as true average blood pressure (mmHg)

$H_0: \mu = 120$

$H_a: \mu > 120$

From information: $\sigma = 12$, $N = 50$, $\bar{x} = 125.2$

$$Z = \frac{\bar{x} - \mu}{\sigma / \sqrt{n}} = \frac{125.2 - 120}{12/\sqrt{50}} = 3.064$$

$$P\text{-value} = P(Z \geq 3.064) = 1 - P(Z < 3.06)$$

According to cdf for Normal Distribution:

$$1 - P(Z < 3.06) = 1 - (0.5 + 0.49889)$$

$$= 0.00111 < 0.05.$$

$\Rightarrow$ At $\alpha = 0.05$ level of significance, reject H_0.

So the stress elevates blood pressure

6.3.6

6.3.6 If people don't postpone their deaths to wait birthdays the probablity at a person's death month for each month p_0 will be $\frac{1}{12}$

$\Rightarrow H_0: \ p = \frac{1}{12}$

$H_a: \ p < \frac{1}{12}$

Given $k = 16, \quad n = 348, \quad \alpha = 0.05$

$\Rightarrow Z_\alpha = -1.645$

Test Statistic:

$$Z = \frac{k - np_0}{\sqrt{np_0(1-p_0)}} = \frac{16 - 348 \cdot \frac{1}{12}}{\sqrt{348 \cdot \frac{1}{12}\left(1 - \frac{1}{12}\right)}}$$

$$= \frac{16 - 29}{\sqrt{29 \cdot \frac{11}{12}}} = -2.52 < Z_\alpha$$

$\Rightarrow$ Reject H_0 at $\alpha = 0.05$ level

6.4.18

6.4.18

$H_0 : \lambda = 6$

$H_a : \lambda < 6$

Rejecting H_0 if $k \leq 2$, $p_X(k) = e^{-\lambda}\lambda^k / k!$ $k = 0, 1, 2 \ldots$

a) $P(\text{Type 1 error}) = P(\text{Reject } H_0 \mid H_0 \text{ is true})$

$= P(k \leq 2 \mid \lambda = 6)$

$= P_X(1) + P_X(2) + P_X(0)$ given $\lambda = 6$

$= \frac{e^{-6}6^1}{1!} + \frac{e^{-6}6^2}{2!} + \frac{e^{-6}6^0}{0!}$

$= e^{-6}(6 + 18 + 1) = 25e^{-6} = 0.0620$

b) $P(\text{Type 2 error}) = P(\text{Fail to reject } H_0 \mid H_0 \text{ is false})$

$= P(k > 2 \mid \lambda = 4)$

$= 1 - P(k \leq 2 \mid \lambda = 4)$

$= 1 - \left(\frac{e^{-4}4^0}{0!} + \frac{e^{-4}4^1}{1!} + \frac{e^{-4}4^2}{2!}\right)$

$= 1 - e^{-4}(1 + 4 + 8) = 1 - 13e^{-4}$

$= 1 - 0.2381 = 0.7619$

6.5.2

6.5.2

Let parameter spaces $\omega = \{\lambda : \lambda = \lambda_0\}$, $\Omega = \{\lambda : \lambda \neq \lambda_0, \lambda > 0\}$

The likelihood function is :

$$L = L(\lambda) = \prod_{i=1}^{10} \lambda e^{-\lambda y_i} = \lambda^{10} e^{-\lambda \sum_{i=1}^{10} y_i}$$

Take logarithm and differentiating w.r.t λ:

$$\ln(L(\lambda)) = 10 \ln \lambda - \lambda \sum_{i=1}^{10} y_i$$

$$\frac{\partial}{\partial \lambda}(L(\lambda)) = \frac{10}{\lambda} - \sum_{i=1}^{10} y_i$$

Let $\frac{\partial}{\partial \lambda} = 0$:

$$\Rightarrow \frac{10}{\lambda} = \sum_{i=1}^{10} y_i$$

$$\hat{\lambda} = \frac{10}{\sum_{i=1}^{10} y_i}$$

$\Rightarrow$ Generalized likelihood ratio is :

$$\Lambda = \frac{\max_{\omega} L(\lambda)}{\max_{\Omega} L(\lambda)} = \frac{L(\lambda_0)}{L(\hat{\lambda})} = \frac{\lambda_0^{10} e^{-\lambda_0 \sum y_i}}{\left(\frac{10}{\sum y_i}\right)^{10} e^{-\frac{10}{\sum y_i} \sum y_i}}$$

$$= \left(\frac{e\lambda_0}{10}\right)^{10} \cdot e^{-\lambda_0 \sum_{i=1}^{10} y_i} \left(\sum_{i=1}^{10} y_i\right)^{10}$$

$\Rightarrow$ GLRT reject H_0 whenever λ^* is chosen for $0 < \Lambda < \lambda^*$

so that $P(0 < \Lambda \leq \lambda^* \mid H_0 \text{ is true}) = \alpha = 0.05$

Integral would have to be evaluated to determine the critical value is :

$$\int_0^{\lambda^*} f_\Lambda(t \mid H_0)\, dt = 0.05$$

7.3.2

7.3.2 Let X be chi-square variate $\chi^2(k)$

$$M_X(t) = E(e^{tX})$$

$$= \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$$

pdf for $\chi^2(k)$ is : $\frac{1}{2^{\frac{k}{2}}\Gamma(k/2)} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}$

Let $c = \frac{1}{2^{k/2}\Gamma(k/2)}$

$$\Rightarrow M_X(t) = \int_{-\infty}^{\infty} c e^{tx} x^{\frac{k}{2}-1} e^{-\frac{x}{2}}\,dx$$

$$= c\int_{-\infty}^{\infty} e^{-(\frac{1}{2}-t)x} x^{\frac{k}{2}-1}\,dx$$

Let $y = (\frac{1}{2}-t)x$ $\quad \frac{dy}{dx} = \frac{1}{2}-t \Rightarrow dx = \frac{2}{1-2t}dy$

$x = \frac{2}{1-2t}y$

$$\Rightarrow M_X(t) = c\int_{-\infty}^{\infty} e^{-y} \left(\frac{2}{1-2t}y\right)^{\frac{k}{2}-1} \frac{2}{1-2t}\,dy$$

$$= c\left(\frac{2}{1-2t}\right)^{\frac{k}{2}} \int_{-\infty}^{\infty} e^{-y}\cdot y^{\frac{k}{2}-1}\,dy$$

$$= c\left(\frac{2}{1-2t}\right)^{\frac{k}{2}}\cdot\Gamma\left(\frac{k}{2}\right)$$

$$= \frac{1}{\cancel{2^{\frac{k}{2}}}\cancel{\Gamma(\frac{k}{2})}}\cdot\left(\frac{\cancel{2}}{1-2t}\right)^{\frac{k}{2}}\cdot\cancel{\Gamma\left(\frac{k}{2}\right)} = (1-2t)^{-\frac{k}{2}}$$

$$E(X) = \left[\frac{d}{dt}M_X(t)\right]_{t=0}$$

$$= \left(\frac{d}{dt}(1-2t)^{-\frac{k}{2}}\right)_{t=0}$$

$$= \left[-\frac{k}{2}(1-2t)^{-\frac{k}{2}-1}\cdot -2\right]_{t=0}$$

$$= \left[k(1-2t)^{-\frac{k}{2}-1}\right]_{t=0}$$

$$= k \quad \Rightarrow \quad E(\chi^2_{(n)}) = n$$

$$E(X^2) = \frac{d^2}{dt^2}[M_X(t)]_{t=0}$$

$$= \frac{d}{dt}\left[k(1-2t)^{-\frac{k}{2}-1}\right]$$

$$= \left[k(1-2t)^{-\frac{k}{2}-2}\left(-\frac{k}{2}-1\right)\cdot -2\right]_{t=0}$$

$$= k(k+2)$$

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= k(k+2) - k^2$$

$$= \cancel{k^2} + 2k - \cancel{k^2} = 2k$$

$$\Rightarrow Var(\chi^2_{(n)}) = 2n$$

7.4.2

7.4.2

a) $P(-x \le T_{22} \le x) = 0.98$

$P(T_{22} \le x) - P(T_{22} \ge -x) = 0.98$

$1 - P(T_{22} > x) - P(T_{22} > x) = 0.98$

$1 - 2P(T_{22} > x) = 0.98$

$P(T_{22} > x) = 0.01$

According to t table (A.2), $\cancel{x = 2.508}$ $t_{0.01,22} = 2.508$

$\Rightarrow x = 2.508$

b) $\cancel{P(T_{13} \ge}$ $P(T_{13} \ge x) = 0.85$

$1 - P(T_{13} < x) = 0.85$

$P(T_{13} < x) = 0.15$

$P(T_{13} \ge -x) = 0.15$

According to t table with $df = 13$, $\alpha = 0.15$.

$t_{0.15,13} = 1.079$

Then $-x = 1.079 \Rightarrow x = -1.079$

c) $P(T_{26} < x) = 0.95$

$1 - P(T_{26} \ge x) = 0.95$

$P(T_{26} \ge x) = \cancel{0.95}\ 0.05$

$df = 26$, $\alpha = 0.05 \Rightarrow x = 1.7056$

d) $P(T_2 \ge x) = 0.025$

$df = 2$, $\alpha = 0.025$, $t_{0.025,2} = 4.3027$

$\Rightarrow x = 4.3027$

7.5.16

7.5.16 Test:

$H_0: \sigma^2 = 1$

$H_a: \sigma^2 > 1$ with $\alpha = 0.05$

$\sum_{i=1}^{30} y_i = 758.62$, $\sum_{i=1}^{30} y_i^2 = 19195.7938$, $n = 30$.

$$S^2 = \frac{n \sum^{30} y_i^2 - (\sum y_i)^2}{n(n-1)} = \frac{30 \cdot 19195.7938 - (758.62)^2}{30(30-1)}$$

$$= \frac{369.5096}{870} = 0.4247$$

From Chi-Square table, critical value for test statistic

is: $\chi^2_{0.95, 29} = 42.557$

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{(30-1) \cdot 0.4247}{1} = 12.3163 < \chi^2_{0.95, 29}$$

$\Rightarrow$ Fail to reject H_0