Final Exam - Noboru Hayashi

1

1. $\frac{y}{x} = ae^{-x/b}$

$y = axe^{-x/b}$

Taking log: $\log y = \log a + \log x - \frac{x}{b}$

Let $\log a = A$ , $-\frac{1}{b} = B$ , $\log y = Y$

$\Rightarrow Y = A + \log x - Bx$

$S^2 = \Sigma (Y_i - \hat{Y}_i)^2 = \sum_{i=1}^{n} (Y_i - A - \log x_i - Bx_i)^2$

To minimize $S^2$, $\frac{\partial S^2}{\partial A} = 0$ & $\frac{\partial S^2}{\partial B} = 0$

$$\Rightarrow \begin{cases} \Sigma\ 2(Y_i - A - \log x_i - Bx_i)(-1) = 0 \\ \Sigma\ 2(Y_i - A - \log x_i - Bx_i)(-x_i) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \Sigma(Y_i - A - \log x_i - Bx_i) = 0 \\ \Sigma(x_iY_i - Ax_i - x_i\log x_i - Bx_i^2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \Sigma Y_i - nA - \Sigma\log x_i - B\Sigma x_i = 0 \\ \Sigma x_iY_i - A\Sigma x_i - \Sigma x_i \log x_i - B\Sigma x_i^2 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \Sigma x_i \cdot \Sigma Y_i - nA\Sigma x_i - \Sigma x_i \cdot \Sigma \log x_i = B(\Sigma x_i)^2 \\ n\Sigma x_iY_i - nA\Sigma x_i - n\Sigma x_i \log x_i = nB\Sigma x_i^2 \end{cases}$$

$\Rightarrow B((\Sigma x_i)^2 - n\Sigma x_i^2) = \Sigma x_i \Sigma Y_i - n\Sigma x_iY_i - \Sigma x_i \Sigma \log x_i + n\Sigma x_i \log x_i$

$\Rightarrow B = \frac{\Sigma x_i \Sigma Y_i - n\Sigma x_iY_i - \Sigma x_i \Sigma \log x_i + n\Sigma x_i \log x_i}{(\Sigma x_i)^2 - n\Sigma x_i^2}$

$A = \frac{\Sigma Y_i - \Sigma \log x_i - B\Sigma x_i}{n}$

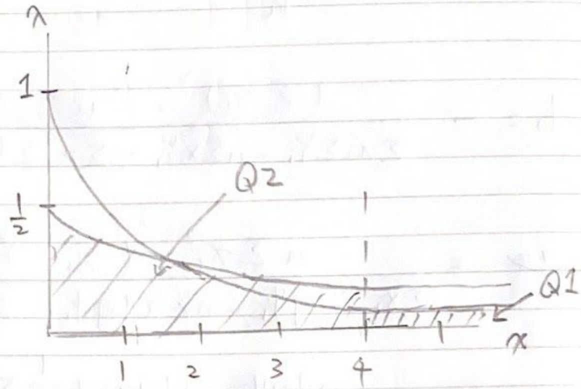
$\Rightarrow b = -\frac{(\Sigma x_i)^2 - n\Sigma x_i^2}{\Sigma x_i \Sigma Y_i - n\Sigma x_iY_i - \Sigma x_i \Sigma \log x_i + n\Sigma x_i \log x_i}$

$= \frac{n\Sigma x_i^2 - (\Sigma x_i)^2}{\Sigma x_i \Sigma \log y_i - n\Sigma x_i \log y_i - \Sigma x_i \Sigma \log x_i + n\Sigma x_i \log x_i}$

$a = e^{\frac{\Sigma \log y_i - \Sigma \log x_i + \frac{1}{b}\Sigma x_i}{n}}$

2. $f(x) = \frac{1}{\alpha} \exp(-x/\alpha)$

a) $Exp(\lambda = \frac{1}{\alpha})$




P(Type I Error)

= P(Reject $H_0$ | $H_0$ true)

$= P(x > 4 \mid \alpha = 1)$

$= \int_4^\infty \frac{1}{\alpha} \exp(-x/\alpha) dx \Big| \alpha = 1$

$= \int_4^\infty e^{-x} dx = -e^{-x} \Big|_4^\infty = 0 + e^{-4} = 0.0183$

b) P(Type II Error) = P(Fail to reject $H_0$ | $H_0$ False)

$= P(x \leq 4 \mid \alpha = 2)$

$= \int_0^4 f(x) dx \mid \alpha = 2 = \int_0^4 \frac{1}{2} e^{-\frac{x}{2}} dx$

$= -e^{-\frac{x}{2}} \Big|_0^4 = -e^{-2} + 1 = 0.86466$

c) $P(\text{Type I}) + P(\text{Type II}) = \int_x^\infty \cancel{\tfrac{1}{\alpha}} f(x) dx \Big| \alpha = 1 + \int_0^x f(x) dx \Big| \alpha \neq 1$

$= \int_x^\infty e^{-x} dx + \int_0^x \frac{1}{\alpha} e^{-\frac{x}{\alpha}} dx$

$= e^{-x} - e^{-\frac{x}{\alpha}} + 1 = 1 - e^{-\frac{x}{\alpha}} + e^{-x}$

Take $\frac{\partial}{\partial x} = \frac{1}{\alpha} e^{-\frac{x}{\alpha}} - e^{-x} = 0$

$\frac{1}{\alpha} e^{-\frac{x}{\alpha}} = e^{-x}$ log $\Rightarrow \log \frac{1}{\alpha} - \frac{x}{\alpha} = -x \Rightarrow x = \frac{\log 1/\alpha}{1/\alpha - 1}$

3. $f(x) = 2x$ $\quad x \in (0,1)$ , $\quad$ 15 for $(0, 0.5)$
25 for $(0.5, 1)$

$$\int_0^{0.5} 2x\,dx = x^2 \Big|_0^{0.5} = 0.25$$

$\Rightarrow$ Expected frequency ~~of~~ $(0, 0.5)$ is $40 \times 0.25 = 10$
$(0.5, 1)$ is 30

$\Rightarrow$ Test Statistic : $\chi^2 = \frac{(15-10)^2}{10} + \frac{(25-30)^2}{30} = 3.33$.

Since $\chi^2_{0.05}$ with df : $2-1 = 1$ is $3.841 > \chi^2_{Test}$

$\Rightarrow$ The $f(x)$ fits to the data.

4. $g(\theta) = \frac{\pi}{2} \sin(\pi\theta)$

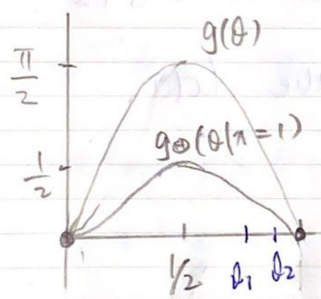
a) $$g_\Theta(\theta \mid X=1) = \frac{P_X(1 \mid \theta)\, \cancel{f_\Theta(\theta)}\, g(\theta)}{\int_{-\infty}^{\infty} P_X(1 \mid \theta)\, \underset{g(\theta)}{\cancel{f_\Theta(\theta)}}\, d\theta}$$

$$= \frac{\binom{1}{1} \theta^1 (1-\theta)^0 \cdot \frac{\pi}{2} \sin(\pi\theta)}{\int_{-\infty}^{\infty} \binom{1}{1} \theta^1 (1-\theta)^0 \cdot \frac{\pi}{2} \sin(\pi\theta)\, d\theta}$$

$$= \frac{\theta \cdot \frac{\pi}{2} \sin(\pi\theta)}{\int_0^1 \theta \cdot \frac{\pi}{2} \sin(\pi\theta)\, d\theta} = \cancel{\frac{\theta \cdot \frac{\pi}{2} \sin(\pi\theta)}{-\frac{\pi}{2}\theta \cos(\pi\theta)\theta - \frac{1}{2}\sin(\pi\theta)}}$$

$$= \cancel{\frac{1}{-\tan(\pi\theta) - \frac{1}{\theta\pi}}}$$

b




$$= \frac{\frac{\pi}{2}\, \theta \sin(\pi\theta)}{\pi/2} = \theta \sin(\pi\theta)$$

c) $\theta_1$ has higher posterior prob.

5) $f(x) = \cancel{\frac{1}{\Gamma(\alpha)\beta^\alpha}} \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta}$

a) $L(\beta) = \prod_{i=1}^{n} f(x_i)$

$= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-\frac{x_i}{\beta}}$

$= \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^n \left(\prod_{i=1}^{n} x_i\right)^{\alpha-1} e^{-\frac{\Sigma x_i}{\beta}}$

Take log : $\log(L(\beta)) = -n[\log(\Gamma(\alpha)) + \alpha\log(\beta)]$

$+ (\alpha-1)\log\prod_{i=1}^{n} x_i - \frac{\Sigma x_i}{\beta}$

$= -n\log(\Gamma(\alpha)) - \alpha n\log(\beta) + (\alpha-1)\log\prod_{i=1}^{n} x_i - \frac{1}{\beta}\Sigma x_i$

Take $\frac{\partial}{\partial\beta}$ : $-\frac{\alpha n}{\beta} + \frac{1}{\beta^2}\Sigma x_i = 0$

$\Rightarrow \alpha n\beta = \Sigma x_i$

$\Rightarrow \beta = \frac{\Sigma x_i}{n\alpha}$

b) Take $\frac{\partial^2}{\partial\beta^2}$ of $\log(L(\beta))$:

$\frac{\alpha n}{\beta^2} - \frac{2}{\beta^3}\Sigma x_i$

Apply $\beta = \frac{\Sigma x_i}{n\alpha}$ :

$\Rightarrow \frac{\partial^2(\log(L(\beta)))}{\partial\beta^2} = \frac{\alpha^3 n^3}{(\Sigma x_i)^2} - \frac{2\alpha^3 n^3}{(\Sigma x_i)^{\cancel{3}2}} \cancel{\Sigma x_i} = -\frac{\alpha^3 n^3}{(\Sigma x_i)^2} < 0$

$\Longrightarrow$ Concave down, the $\beta$-value corresponds to maximum likelihood

6) Let $n$ be the minimum number of coin tosses.
Let $X$ be the number of heads.

$E(X) = np = 0.52n \qquad S_x = \sqrt{np(1-p)} = \sqrt{0.2496n}$

To find $n$ that

$$P(X > \frac{n}{2}) \geq 0.99 \Rightarrow 1 - P(X \leq \frac{n}{2}) \geq 0.99$$

$$\Rightarrow P(X \leq \frac{n}{2}) \leq 0.01$$

$\Rightarrow$ ~~P~~ Since $n$ will be large enough, and assume

$$X \sim Norm(0.52n, \sqrt{0.2496n})$$

From Normal dist table.

$$P(Z \leq -2.33) \leq 0.01$$

$$\Rightarrow \frac{\frac{n}{2} - 0.52n}{\sqrt{0.2496n}} \leq \cancel{0.01} -2.33$$

$$-0.02n \leq -2.33\sqrt{0.2496n}$$

$$0.02n \geq 2.33\sqrt{0.2496n}$$

$$0.0004n^2 \geq 1.35505n$$

$$0.0004n \geq 1.35505$$

$$n \geq 3387.6$$

$$\Rightarrow n = 3388$$

7) $\mu_X = 1$ , $\sigma_X = 2$ , $\mu_Y = 4$ , $\sigma_Y = 3$ , $\rho = 0.4$

a) Since ~~X ~ Normal~~ there exists $Z \sim N(0,1)$ that

$$X = \sigma_X Z + \mu_X$$

$$\Rightarrow P(X > 1) = P(\sigma_X Z + \mu_X > 1)$$

$$= P\left(Z > \frac{1-\mu_X}{\sigma_X}\right) = P\left(Z > \frac{1-1}{2}\right) = P(Z > 0)$$

Since $Z \sim \text{Norm}(0,1)$

$$\Rightarrow P(Z>0) = 0.5$$

b) $E(X|y) = \mu_X + \frac{\rho \sigma_X}{\sigma_Y}(y - \mu_Y)$

$$\Rightarrow E(X|y=8) = 1 + \frac{0.4 \times 2}{3}(8-4)$$

$$= \frac{31}{15} = 2.0667$$

$$Var(X|y) = (1-\rho^2)\sigma_X^2 = (1-0.4^2)\cdot 2^2 = 3.36$$

$$P(X>1 \mid y=8) = P\left(\frac{X - E(X|Y=8)}{\sqrt{Var(X|y=8)}} > \frac{1 - E(X|Y=8)}{\sqrt{Var(X|y=8)}}\right)$$

$$= P\left(Z > \frac{1-2.0667}{\sqrt{3.36}}\right) = P(Z > -0.582)$$

$$= 1 - P(Z < -0.582)$$

From $N(0,1)$ table: $= 1 - 0.2810 = 0.719$