Module 11 Homework - Noboru Hayashi

11.2.18

11.2.18

a) $y = bx$

From Q.8.2.5: $n=9$

x	y
180	11.0
26	1.4
68	3.0
88	6.0
12	0.9
67	4.0
56	3.5
18	1.4
30	2.7

From Q.11.2.14: $b = \dfrac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2}$

$$\Rightarrow b = \frac{\sum_{i=1}^{9} x_i y_i}{\sum_{i=1}^{9} x_i^2} = \frac{3329.4}{54437} = 0.06116$$

$$\Rightarrow y = 0.06116x$$

b) $x = 120$

$$\Rightarrow y = 0.06116 \times 120$$

$$= 7.34$$

$\Rightarrow$ The projected revenue is 7.34 million

11.2.26

11.2.26 Fit $y = ax^b$ model

For logarithmic regression ($y = ax^b$):

$$b = \frac{n\sum_{i=1}^{n} \log x_i \cdot \log y_i - \left(\sum_{i=1}^{n} \log x_i\right)\left(\sum_{i=1}^{n} \log y_i\right)}{n\sum_{i=1}^{n} (\log x_i)^2 - \left(\sum_{i=1}^{n} \log x_i\right)^2}$$

$$\log a = \frac{\sum_{i=1}^{n} \log y_i - b\sum_{i=1}^{n} \log x_i}{n}$$

From the Question: $n = 11$, $\sum_{i=1}^{11} \log x_i \cdot \log y_i = 32.01$

$\sum_{i=1}^{11} \log x_i = 17.75$, $\sum_{i=1}^{11} \log y_i = 18.97$

$\sum_{i=1}^{11} (\log x_i)^2 = 31.07$

$$\Rightarrow b = \frac{11 \times 32.01 - 17.75 \times 18.97}{11 \times 31.07 - (17.75)^2} = 0.576$$

$$\log a = \frac{18.97 - 0.576 \times 17.75}{11} = 0.7951$$

$$\Rightarrow a = 10^{0.7951} = 6.239$$

Therefore $y = 6.24 x^{0.58}$

11.3.2

a)

11.3.2 $y = 81.088 + 0.412x$, $S = 11.78848$

a) $\hat{\beta}_0 = 81.088$, $\hat{\beta}_1 = 0.412$

From Theorem 11.3.6:

$100(1-\alpha)\%$ CI for β_1 is:

$$\left[\hat{\beta}_1 - t_{\alpha/2, n-2} \cdot \frac{S}{\sqrt{\Sigma(x_i - \bar{x})^2}} , \hat{\beta}_1 + t_{\alpha/2, n-2} \cdot \frac{S}{\sqrt{\Sigma(x_i - \bar{x})^2}}\right]$$

$$\Sigma(x_i - \bar{x})^2 = \Sigma(x_i)^2 + n\bar{x}^2 - 2\bar{x}\Sigma x_i = \Sigma(x_i)^2 + \frac{1}{n}\left((\Sigma x_i)^2 - 2(\Sigma x_i)^2\right)$$

$$= \Sigma(x_i)^2 - \frac{1}{n}(\Sigma x_i)^2$$

Since from Q 11.2.7: $\sum_{i=1}^{26} x_i = 360$, $\sum_{i=1}^{26} y_i = 2256.6$

$\sum_{i=1}^{26} x_i^2 = 5365.08$, $\sum_{i=1}^{26} x_i y_i = 31402$

$$\Rightarrow \Sigma(x_i - \bar{x})^2 = 5365.08 - \frac{1}{26} \cdot 360^2$$

$$= 380.46$$

plus, $t_{0.025, 26-2} = 2.0639$

$\Rightarrow$ CI for β_1:

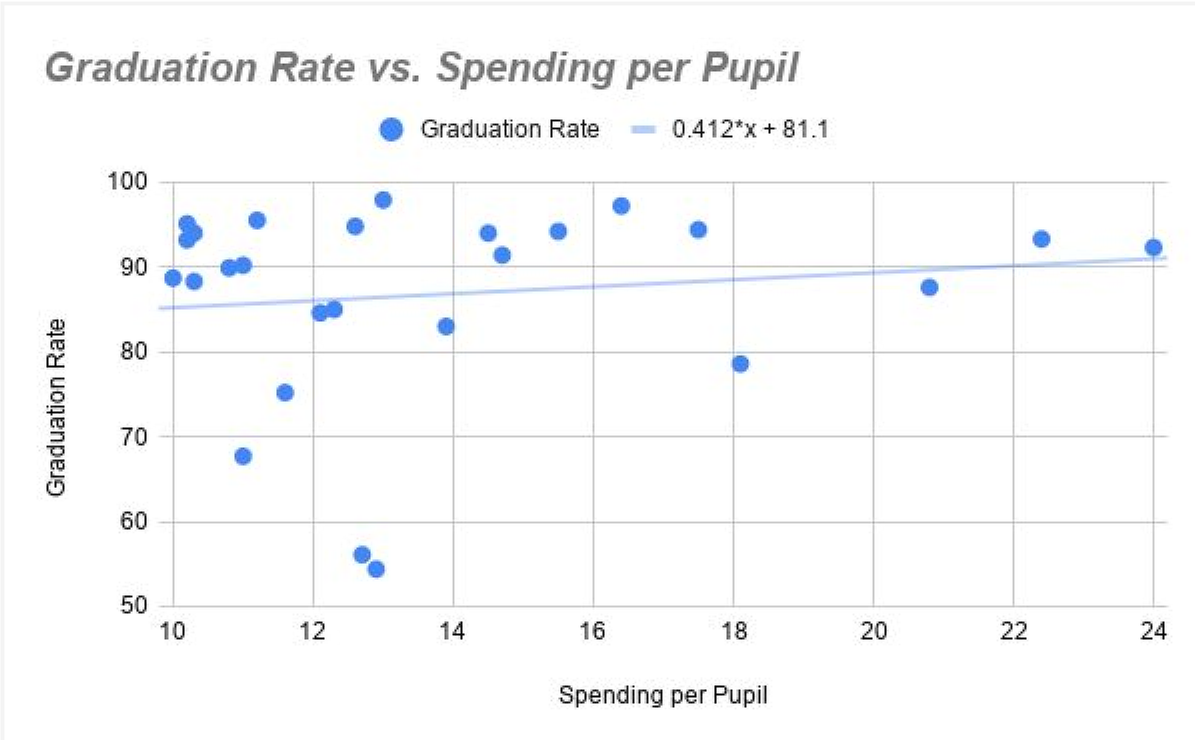
$$\left[0.412 - 2.0639 \cdot \frac{11.78848}{\sqrt{380.46}} , 0.412 + 2.0639 \cdot \frac{11.78848}{\sqrt{380.46}}\right]$$

$$= [0.412 - 1.247 , 0.412 + 1.247] = [-0.835 , 1.659]$$

b)

b) Since 0 is contained in CI, we are not able to reject $H_0: \beta_1 = 0$

c)



Data points from Scatter plot seem to be randomly distributed, and don't show a strong linear relationship between Graduation Rate vs. Spending per Pupil. Therefore, we cannot conclude that two variables are independent(beta-1 = 0).

11.4.2

11.4.2 $f_{X,Y}(x,y) = x+y$, $0<x<1$, $0<y<1$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_{-\infty}^{\infty} x \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_{0}^{1} x \int_0^1 x+y\,dy\,dx = \int_0^1 x\left[xy+\tfrac{1}{2}y^2\right]_0^1 dx$$

$$= \int_0^1 x^2+\tfrac{1}{2}x\,dx = \left[\tfrac{1}{3}x^3+\tfrac{1}{4}x^2\right]_0^1 = \tfrac{1}{3}+\tfrac{1}{4} = \tfrac{7}{12}$$

$$E(X^2) = \int_0^1 x^2\left(x+\tfrac{1}{2}\right)dx = \int_0^1 x^3+\tfrac{x^2}{2}\,dx = \left[\tfrac{x^4}{4}+\tfrac{x^3}{6}\right]_0^1 = \tfrac{5}{12}$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 y \int_0^1 x+y\,dx\,dy = \int_0^1 y\left[\tfrac{1}{2}x^2+xy\right]_0^1 dy = \int_0^1 y\left(\tfrac{1}{2}+y\right)dy$$

$$= \int_0^1 y^2+\tfrac{1}{2}y\,dy = \left(\tfrac{1}{3}y^3+\tfrac{1}{4}y^2\right)_0^1 = \tfrac{7}{12}$$

$$E(Y^2) = \int_0^1 y^2\cdot\left(\tfrac{1}{2}+y\right)dy = \int_0^1 y^3+\tfrac{1}{2}y^2\,dy = \tfrac{1}{4}y^4+\tfrac{1}{6}y^3\Big|_0^1 = \tfrac{5}{12}$$

$$\Rightarrow Var(X) = E(X^2)-[E(X)]^2 = \tfrac{5}{12}-\left(\tfrac{7}{12}\right)^2 = \tfrac{11}{144}$$

$$Var(Y) = \tfrac{11}{144}$$

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$

$$= \int_0^1\int_0^1 x^2y+xy^2\,dx\,dy = \int_0^1 \tfrac{1}{3}x^3y+\tfrac{1}{2}x^2y^2\Big|_0^1 dy$$

$$= \int_0^1 \tfrac{1}{3}y+\tfrac{1}{2}y^2\,dy = \tfrac{1}{6}y^2+\tfrac{1}{6}y^3\Big|_0^1 = \tfrac{1}{3}$$

$$\Rightarrow Cov(X,Y) = E(XY)-E(X)E(Y) = \tfrac{1}{3}-\tfrac{7}{12}\times\tfrac{7}{12} = -\tfrac{1}{144}$$

Therefore $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X\sigma_Y} = \dfrac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} = \dfrac{-\frac{1}{144}}{\sqrt{11/144}\cdot\sqrt{11/144}}$

$$= -\tfrac{1}{144} \div \tfrac{11}{144} = -\tfrac{1}{11} = -0.0909$$

11.5.4

11.5.4 $\mu_X = 56$, $\mu_Y = 11$, $\sigma_X^2 = 1.2$, $\sigma_Y^2 = 2.6$, $\rho = 0.6$

$P(10 < Y < 10.5 \mid X = 55)$:

Since $E(Y|x) = \mu_Y + \frac{\rho\sigma_Y}{\sigma_X}(x - \mu_X)$ & $Var(Y|x) = (1-\rho^2)\sigma_Y^2$

$\Rightarrow E(Y|x=55) = 11 + \frac{0.6\cdot\sqrt{2.6}}{\sqrt{1.2}}(55-56) = 10.1168$

$Var(Y|x=55) = (1-0.6^2)\,2.6 = 1.664.$

$$P(10 < Y < 10.5 \mid x=55) = P\left(\frac{10-E(Y|x=55)}{\sqrt{Var(Y|x=55)}} < \frac{Y-E(Y|x=55)}{\sqrt{Var(Y|x=55)}} < \frac{10.5-E(Y|x=55)}{\sqrt{Var(Y|x=55)}}\right)$$

$$= P\left(\frac{10-10.1168}{\sqrt{1.664}} < \frac{Y-E(Y|x=55)}{\sqrt{Var(Y|x=55)}} < \frac{10.5-10.1168}{\sqrt{1.664}}\right) = P(-0.09 < Z < 0.297)$$

$= P(Z < 0.297) - P(Z < -0.09)$ $\cancel{= P(Z < 0.297)}$

$= 0.6179 - 0.4641$ (According $N(0,1)$ table)

$= \underline{0.1538}$

$P(10.5 < \bar{Y} < \overset{11}{\cancel{10.5}} \mid x=55)$ given $n=4$:

$$P(10.5 < \bar{Y} < \overset{11}{\cancel{10.5}} \mid x=55) = P\left(\frac{10.5 - 10.1168}{\sqrt{1.664/4}} < \frac{\bar{Y}-E(Y|x=55)}{\sqrt{Var(Y|x=55)/n}} < \frac{\overset{11}{\cancel{10.5}} - 10.1168}{\sqrt{1.664/4}}\right)$$

$= P(0.59 < Z < 1.369) = P(Z < 1.369) - P(Z < 0.59)$

$= 0.9147 - 0.7224 = \underline{0.1923}$