Module 6 Homework - Noboru Hayashi

4.2.12

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Let X be the number of bags lost by Midwestern during a typical week.

Given the table for the number of ~~bag~~ lost bags during each of the first 40 weeks in 2009.

the estimated value $\lambda = \dfrac{1\times 13+2\times 10+3\times 5+4\times 2+5\times 1}{40}$

$= \dfrac{61}{40} = 1.525$.

The distribution of X can be showned as:

No. of bags lost	frequency	proportion	$p_X(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$
0	9	$\frac{9}{40}$	$e^{-1.525}\dfrac{(1.525)^0}{0!} = \cancel{0.216}$ 0.2176
1	13	$\frac{13}{40}$	$e^{-1.525}\dfrac{(1.525)^1}{1!} = 0.3318$
2	10	$\frac{10}{40}$	$e^{-1.525}\dfrac{(1.525)^2}{2!} = 0.2531$
3	5	$\frac{5}{40}$	$e^{-1.525}\dfrac{(1.525)^3}{3!} = 0.1286$
4	2	$\frac{2}{40}$	$e^{-1.525}\dfrac{(1.525)^4}{4!} = 0.0490$
5	1	$\frac{1}{40}$	$e^{-1.525}\dfrac{(1.525)^5}{5!} = 0.0150$

total : $0.9951 \approx 1.00$.

Since $\Sigma p_X(k) \approx 1.00$, X can be ~~is~~ considered as a Poisson random variable.

4.2.28

4.2.28: $N = 50$ $\lambda = \frac{1.1}{100} = 0.011$

Let T be a life time of a bulb in hours:

$$T \sim \exp(\lambda = 0.011)$$

Therefore $P(T \le t) = 1 - e^{-\lambda t}$

$$= 1 - e^{-0.011t}$$

$\Rightarrow P($a spotlight will fail before 75 hours$)$

$$= P(T < 75) = 1 - e^{-0.011 \times 75}$$

$$= 0.5618$$

For $N=50$ independent spotlights, each has 0.5618 probability to fail before 75 hours.

$\Rightarrow$ the expected number of bulbs will fail to last for at least 75 hours is: $50 \times 0.5618 = 28.09$

≈ 28

4.3.14

4.3.14

Let X be the # of orders occurs in a game (demand) and m be the # of units concession ~~mago~~ manager placed.

$\Rightarrow P(\text{Demands exceeds supply})$

$= P(X > m)$

Given $n = 42200$ and $p = 0.38$, $X \sim Norm$.

$$\Rightarrow P(X > m) = P\left(\frac{X - np}{\sqrt{np(1-p)}} > \frac{m - np}{\sqrt{np(1-p)}}\right)$$

$$= P\left(Z > \frac{m - np}{\sqrt{np(1-p)}}\right)$$

$$= 1 - P\left(Z < \frac{m - np}{\sqrt{np(1-p)}}\right)$$

Since the manger wants $P(\text{Demand} > \text{Supply}) \leq 20\%$

$$\Rightarrow 1 - P\left(Z < \frac{m - np}{\sqrt{np(1-p)}}\right) \leq 0.2$$

$$P\left(Z < \frac{m - np}{\sqrt{np(1-p)}}\right) \geq 0.8$$

According to ~~table~~ standard normal table.

if $z = 0.85$, $P(Z < 0.85) = 0.80234$.

if $z = 0.84$, $P(Z < 0.84) = 0.79955$.

$\Rightarrow$

$$\frac{m - np}{\sqrt{np(1-p)}} = 0.85$$

$$\frac{m - 42200 \times 0.038}{\sqrt{42200 \times 0.38(1-0.38)}} = 0.85 \Rightarrow \frac{m - \overset{16036}{\cancel{16067.44}}}{99.71} = 0.85$$

$m = 16120.75$

≈ 16120

$\therefore$ The manger should place 16120 orders

4.4.6

4.4.6

For three dice, the combinations of faces having a sum of 4 are [1,1,2], [1,2,1], [2,1,1]

$\Rightarrow$ P(sum of 4 appears by rolling 3 dice)

$$= \frac{3}{6^3} = \frac{3}{216} = \frac{1}{72}$$

Given the CDF for a geometric random variable is: $F_X(t) = 1 - (1-p)^{[t]}$

$$\Rightarrow P(65 \leq X \leq 75) = F_X(75) - F_X(65)$$

$$= \cancel{1} - (1-p)^{75} + (1-p)^{65} - \cancel{1}$$

$$= (1-p)^{65} - (1-p)^{75}$$

$$= \left(\frac{71}{72}\right)^{65} - \left(\frac{71}{72}\right)^{75}$$

$$= 0.40288 - 0.35030$$

$$= 0.0526$$

4.6.6

4.6.6 Prove $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Let $Z \sim N(0,1)$, $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$

Then $E(Z^2) = \int_{-\infty}^{\infty} z^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz$

Since $f_Z(Z)$ and Z^2 is symmetric,

$$E(Z^2) = \int_0^{\infty} z^2 \frac{2}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}z^2} dz$$

$$= \frac{2}{\sqrt{2\pi}} \int_0^{\infty} z^2 \cdot e^{-\frac{1}{2}z^2} dz$$

Let $y = \frac{1}{2}z^2$, then $z = \sqrt{2y}$, $dz = \frac{1}{\sqrt{2y}} dy$

$$\Rightarrow E(Z^2) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} 2y \cdot e^{-y} \cdot \frac{1}{\sqrt{2y}} dy$$

$$= \frac{2}{\sqrt{\pi}} \int_0^{\infty} y^{\frac{1}{2}} \cdot e^{-y} dy = \frac{2}{\sqrt{\pi}} \Gamma(\frac{3}{2}) = \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\sqrt{\pi}} \Gamma(\frac{1}{2})$$

$$= \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2})$$

Since $E(Z^2) = Var(Z) + [E(Z)]^2 = 1 + 0^2 = 1$

$$\Rightarrow E(Z^2) = 1 = \frac{1}{\sqrt{\pi}} \Gamma(\frac{1}{2})$$

$$\Rightarrow \Gamma(\frac{1}{2}) = \sqrt{\pi}$$