Module 5 Homework - Noboru Hayashi

3.7.42

3.7.42

$$f_{X,Y}(x,y) = \frac{2}{3}(x+2y), \quad x \in [0,1], \ y \in [0,1]$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{2}{3}(x+2y)\,dy$$

$$= \frac{2}{3}\left(xy\Big|_0^1 + y^2\Big|_0^1 \right) = \frac{2}{3}(x+1)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 \frac{2}{3}(x+2y)\,dx$$

$$= \frac{2}{3}\left(\frac{1}{2}x^2\Big|_0^1 + 2xy\Big|_0^1 \right) = \frac{2}{3}\left(\frac{1}{2}+2y\right)$$

$$f_X(x) \cdot f_Y(y) = \frac{2}{3}(x+1) \cdot \frac{2}{3}\left(\frac{1}{2}+2y\right)$$

$$= \frac{4}{9}(x+1)\left(\frac{1}{2}+2y\right) \neq f_{X,Y}(x,y)$$

$$\therefore f_{X,Y}(x,y) \neq f_X(x) \cdot f_Y(y)$$

$\therefore$ X and Y are not independant

3.8.10

3.8.10 $\quad f_Y(y) = ay^2 e^{-by^2}, \quad y \geq 0$

$$W = \frac{m}{2} Y^2$$

$$F_W(w) = P(W \leq w) = P\left(\frac{m}{2} Y^2 \leq w\right)$$

$$= P\left(Y^2 \leq \frac{2w}{m}\right)$$

$$= P\left(-\sqrt{\frac{2w}{m}} \leq Y \leq \sqrt{\frac{2w}{m}}\right)$$

$$= P\left(Y \leq \sqrt{\frac{2w}{m}}\right) - P\left(Y \leq -\sqrt{\frac{2w}{m}}\right)$$

Since for $y < 0$, $f_Y(y) = 0$. So $F_Y(y) = 0$

$$\Rightarrow F_W(w) = P\left(Y \leq \sqrt{\frac{2w}{m}}\right) - 0$$

$$= F_Y\left(\sqrt{\frac{2w}{m}}\right)$$

Taking derivative wrt w :

$$\frac{d}{dw} F_W(w) = \frac{d}{dw} F_Y\left(\sqrt{\frac{2w}{m}}\right)$$

$$\Rightarrow f_W(w) = f_Y\left(\sqrt{\frac{2w}{m}}\right) \cdot \sqrt{\frac{2}{m}} \cdot \frac{1}{2\sqrt{w}} = f_Y\left(\sqrt{\frac{2w}{m}}\right) \cdot \frac{1}{\sqrt{2mw}}$$

$$\Rightarrow f_W(w) = \frac{1}{\sqrt{2mw}} \cdot a\left(\sqrt{\frac{2w}{m}}\right)^2 e^{-b\left(\sqrt{\frac{2w}{m}}\right)^2}$$

$$= \frac{a}{\sqrt{2mw}} \cdot \frac{2w}{m} \cdot e^{-\frac{\cancel{2}bw}{m}}$$

$$= \frac{a\sqrt{2w}}{m\sqrt{m}} \cdot e^{-\frac{2bw}{m}}, \quad \text{for } w \geq 0$$

3.8.12

3.8.~~10~~ 12 Find $F_W(w)$

a) $f_X(x) = 1 \quad x \in [0,1]$

$f_Y(y) = 1 \quad y \in [0,1]$

Let $W = Y/X$, according to theorem:

$$f_W(w) = \int_{-\infty}^{\infty} |x| f_X(x) f_Y(wx)\, dx$$

~~Since $x \in [0,1]$, $|x| = x$,~~

Since $f_X(x) = 0$ for $x \notin [0,1]$ and $f_Y(y) = 0$ for $y \notin [0,1]$

So, $0 \le x \le 1$ and $0 \le wx \le 1 \Rightarrow 0 \le x \le 1$

~~$\Rightarrow 0 \le x \le 1$,~~ and $0 \le x \le \frac{1}{w}$

if $0 \le w \le 1$, $\frac{1}{w} > 1$, so the upperbound of the integral is 1

$$\Rightarrow f_W(w) = \int_0^1 x \cdot 1 \cdot 1\, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$$

if $w > 1$, $\frac{1}{w} < 1$, the upperbound will be $\frac{1}{w}$

$$\Rightarrow f_W(w) = \int_0^{\frac{1}{w}} x \cdot 1 \cdot 1\, dx = \frac{1}{2} x^2 \Big|_0^{\frac{1}{w}} = \frac{1}{2w^2}$$

therefore
$$f_W(w) = \begin{cases} \frac{1}{2}, & w \in [0,1] \\ \frac{1}{2w^2}, & w > 1 \end{cases}$$

$$F_W(w) = \int_{-\infty}^{w} f_W(w)\, dw$$

if $0 \le w \le 1$:

$$= \int_0^w \frac{1}{2}\, dw = \frac{1}{2} w$$

if $w > 1$

$$= \int_0^1 \frac{1}{2}\, dw + \int_1^w \frac{1}{2w^2}\, dw$$

$$= \frac{1}{2} + \left(-\frac{1}{2w}\right)\Big|_1^w$$

$$= \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2w}\right) = 1 - \frac{1}{2w}$$

$$\Rightarrow F_W(w) = \begin{cases} \frac{1}{2} w, & w \in [0,1] \\ 1 - \frac{1}{2w}, & w > 1 \end{cases}$$

b) $f_X(x) = 2x, \quad x \in [0,1]$

$f_Y(y) = 2y, \quad y \in [0,1]$

$$f_W(w) = \int_{-\infty}^{\infty} |x| \, f_X(x) \, f_Y(wx) \, dx$$

$\because f_X(x) = f_Y(y) = 0$ if $x \notin [0,1]$, $y \notin [0,1]$

$\therefore$ For the integral, $x \in [0,1]$ & $wx \in [0,1]$

$\Rightarrow x \in [0,1]$ & $x \in [0, \frac{1}{w}]$

if $w \in [0,1]$, the upperbound is 1

$$\Rightarrow f_W(w) = \int_0^1 x \cdot 2x \cdot 2wx \, dx$$

$$= \int_0^1 4wx^3 \, dx = wx^4 \Big|_0^1 = w$$

if $x \in (1, \infty)$, the upperbound is $\frac{1}{w}$

$$\Rightarrow f_W(w) = \int_0^{\frac{1}{w}} 4wx^3 \, dx = wx^4 \Big|_0^{\frac{1}{w}} = \frac{1}{w^3}$$

$$\Rightarrow f_W(w) = \begin{cases} w, & w \in [0,1] \\ \frac{1}{w^3}, & w \in (1, \infty) \end{cases}$$

$$F_W(w) = \int_{-\infty}^{w} f_W(w) \, dw$$

For $w \in [0,1]$,

$$F_W(w) = \int_0^w w \, dw = \frac{1}{2}w^2 \Big|_0^w = \frac{1}{2}w^2$$

For $w \in (1, \infty)$

$$F_W(w) = \int_0^1 w \, dw + \int_1^w \frac{1}{w^3} \, dw$$

$$= \frac{1}{2}w^2 \Big|_0^1 + \left(-\frac{1}{2}w^{-2}\right) \Big|_1^w$$

$$= \frac{1}{2} + \left(\frac{1}{2} - \frac{1}{2w^2}\right)$$

$$= 1 - \frac{1}{2w^2}$$

$$\therefore F_W(w) = \begin{cases} \frac{1}{2}w^2, & w \in [0,1] \\ 1 - \frac{1}{2w^2}, & w \in (1, \infty) \end{cases}$$

3.9.8

3.9.8 Toss 2 fair dice : X, Y

Expected value of the product is $E(XY)$

According to theorem 3.9.3:

$$E(XY) = E(X)E(Y) \quad \text{for independent } X \,\&\, Y$$

$$E(X) = E(Y) = \sum_{n=1}^{6} \frac{1}{6} n$$

$$= \frac{1}{6}(1+2+\cdots+6) = \frac{21}{6} = \frac{7}{2} = 3.5$$

Therefore $E(XY) = 3.5 \times 3.5 = 12.25$

3.11.16

3.11.16 $\quad f_{X,Y}(x,y) = \frac{2}{5}(2x+3y)\,, \quad x,y \in [0,1]$

a) $$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^1 \frac{2}{5}(2x+3y)\,dy$$

$$= \frac{2}{5}\left(2xy + \frac{3}{2}y^2\right)\Big|_0^1 = \frac{2}{5}\left(2x + \frac{3}{2}\right)$$

$$= \frac{4}{5}x + \frac{3}{5}\,, \quad x \in [0,1]$$

b) $$f_{Y|x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{\frac{2}{5}(2x+3y)}{\frac{4}{5}x + \frac{3}{5}}$$

$$= \frac{2(2x+3y)}{4x+3} = \frac{4x+6y}{4x+3}\,, \quad y \in [0,1]$$

c) $$P\left(\frac{1}{4} \le Y \le \frac{3}{4} \,\middle|\, X = \frac{1}{2}\right) = \int_{\frac{1}{4}}^{\frac{3}{4}} f_{Y|\frac{1}{2}}(y)\,dy$$

$$= \int_{\frac{1}{4}}^{\frac{3}{4}} \frac{2+6y}{5} = \frac{2}{5}y + \frac{3}{5}y^2 \Big|_{\frac{1}{4}}^{\frac{3}{4}} \cancel{= \frac{2}{5} + \frac{3}{5}\left(\frac{9}{16} - \frac{1}{16}\right)}$$

$$= \frac{2}{5}\left(\frac{3}{4} - \frac{1}{4}\right) + \frac{3}{5}\left(\frac{9}{16} - \frac{1}{16}\right)$$

$$= \frac{2}{10} + \frac{3}{10} = \frac{1}{2}$$

d) $$E(Y|x) = \int_{-\infty}^{\infty} y \cdot f_{Y|x}(y)\,dy$$

$$= \int_0^1 y \cdot \frac{4x+6y}{4x+3}\,dy = \int_0^1 \frac{4x}{4x+3} \cdot y + \frac{6y^2}{4x+3}\,dy$$

$$= \left(\frac{2x}{4x+3}y^2 + \frac{2y^3}{4x+3}\right)\Big|_0^1$$

$$= \frac{2x}{4x+3} + \frac{2}{4x+3} = \frac{2x+2}{4x+3}$$