Midterm Exam - Noboru Hayashi

1.

1. P(a child has blue eyes) $= \frac{1}{4}$

a)

$$P(X \geq 2 \mid X \geq 1)$$

$$= \frac{P(X \geq 2,\ X \geq 1)}{P(X \geq 1)} = \frac{P(X \geq 2)}{P(X \geq 1)}$$

$$P(X \geq 2) = P(X=3) + P(X=2)$$

$$= \binom{3}{3} \frac{1}{4}^3 + \cancel{\binom{2}{2}} \binom{3}{2} \frac{1}{4}^2 \cdot \left(1 - \frac{1}{4}\right)^1$$

$$= \frac{1}{64} + 3 \cdot \frac{1}{16} \cdot \frac{3}{4}$$

$$= \frac{1}{64} + \frac{9}{64} = \frac{5}{32} = 0.15625$$

$$P(X \geq 1) = 1 - P(X=0) = 1 - \binom{3}{0}\left(\frac{3}{4}\right)^3$$

$$= 1 - \frac{27}{64} = \frac{37}{64} = 0.578125$$

$$\Rightarrow P(X \geq 2 \mid X \geq 1) = \frac{P(X \geq 2)}{P(X \geq 1)}$$

$$= \frac{5}{\cancel{32}_1} \cdot \frac{\cancel{64}^2}{37} = \frac{10}{37}$$

$$= 0.2703,$$

b) P(at least 2 children have blue eyes | the youngest has blue eyes)

= P(one of the other two has blue eyes)

+ P(both of the other two have blue eyes)

= 1 - P(~~None~~ Neither of the other two has blue eyes)

$$= 1 - \left(1 - \frac{1}{4}\right)^2$$

$$1 - \frac{9}{16} = \frac{7}{16} = 0.4375$$

2.

2 Let D be a voter is decided, ND for undecided

A be a voter is affiliated, NA for unaffiliated

	D	ND
A	70% × 94%	20% × 6%
NA	30% × 94%	80% × 6%
Total	94%	6%

$\Longrightarrow$

	D	ND	Total
A	65.8%	1.2%	67%
NA	28.2%	4.8%	33%
Total	94%	6%	100%

a) P(Undecided | Unaffiliated)

$$= P(ND \mid NA) = \frac{P(ND, NA)}{P(NA)} = \frac{4.8\%}{33\%}$$

$$= 0.1455 = 14.55\%$$

b) P(Decided | Affiliated)

$$= P(D \mid A) = \frac{P(D, A)}{P(A)} = \frac{65.8\%}{67\%}$$

$$= 0.9821$$

$$= 98.21\%$$

3

3.

P(a coin is heads) = 0.5

pdf for X when $X \sim U(0,1)$

$$f_X(x) = P(\text{a coin is heads}) \cdot \frac{1}{1-0}$$

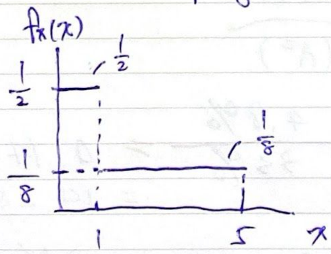
$$= \frac{1}{2} \cdot 1 = \frac{1}{2}$$

pdf for X when $X \sim U(1,5)$

$$f_X(x) = P(\text{a coin is tails}) \cdot \frac{1}{5-1}$$

$$= \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

$$\Rightarrow f_X(x) = \begin{cases} \frac{1}{2}, & x \in [0,1] \\ \frac{1}{8} & x \in (1,5) \end{cases}$$



b) $$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_0^1 \frac{1}{2} x \, dx + \int_1^5 \frac{1}{8} x \, dx$$

$$= \frac{x^2}{4} \Big|_0^1 + \frac{x^2}{16} \Big|_1^5$$

$$= \frac{1}{4} + \frac{25}{16} - \frac{1}{16} = \frac{1}{4} + \frac{\cancel{24}^{\,3}}{\cancel{16}_{\,2}} = \frac{7}{4} = 1.75$$

4) P(At least 1 of individuals having a cancer)

$= \cancel{P} \; 1 - P$(None of the having a cancer)

$= 1 - P$(a person not having a cancer)^{n}

$= 1 - (99\%)^{n} \geq 95\%$. (Since $1-0.99^{x}$ is an increasing function from 1%)

$\Rightarrow \quad 0.99^{n} \leq 0.05$

if $n = 298, \quad 0.99^{n} = 0.05003$

$n = 299, \quad 0.99^{n} = 0.04953$

$\Rightarrow$ the minimum is 299

5) $f_X(x) = 2x \quad 0 \le x \le 1$

a) Mean radius $= E(X)$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx$$

$$= \frac{2}{3} x^3 \Big|_0^1 = \frac{2}{3}$$

Mean radius is $\frac{2}{3}$

b) Median radius R ~~satf~~ satisfies $F_X(R) = \frac{1}{2}$

$$\Rightarrow \int_{-\infty}^{R} f_X(x)\, dx = 0.5$$

$$\int_0^R 2x\, dx = 0.5$$

$$x^2 \Big|_0^R = 0.5$$

$$R^2 = 0.5$$

$$R = \frac{\sqrt{2}}{2}$$

Median radius is $\frac{\sqrt{2}}{2}$

c) $E(\text{Area}) = E(\pi X^2)$

$$= \int_{-\infty}^{\infty} \pi x^2 \cdot f_X(x)\, dx = \int_0^1 2\pi x^3\, dx$$

$$= \frac{1}{2} \pi X^4 \Big|_0^1 = \frac{1}{2}\pi$$

Mean area is $\frac{1}{2}\pi$

6) $f_T(t) = e^{-t}$, $t \geq 0$

a) From the pdf, ~~each~~ life span of each light bulb is exponetially distrubuted by $\lambda = 1$.

So ~~E(T)~~ $E(T)$ for each bulb is $\frac{1}{\lambda} = 1$.

Since each light bulb is independent, ~~therefo~~ and the length of at least 1 working bulb in the room is equal to both light bulb's failure time.

Hence, we expect at least one bulb working until $t = 1$

b) Most likely time for room to go dark is $E(T) = 1$.

7. $f_{X,Y}(x,y) = x+y \quad , \quad 0<x<1 \ , \ 0<y<1$

$$f_Y(y) = \int_0^1 x+y \; dx = \frac{1}{2}x^2 + xy \Big|_0^1$$

$$= \frac{1}{2} + y \ ,$$

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{x+y}{\frac{1}{2}+y}$$

$$\Rightarrow P(X>0.4 \mid y=0.6) = 1 - P(X \le 0.4 \mid y=0.6)$$

$$= 1 - \int_0^{0.4} f_{X|Y=0.6}(x,y)\, dx$$

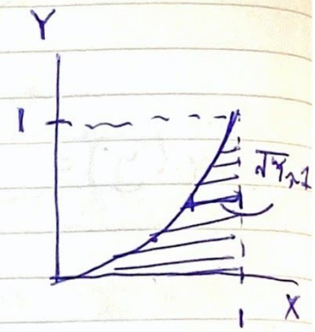
$$= 1 - \int_0^{0.4} \frac{x+0.6}{0.5+0.6}\, dx = 1 - \int_0^{0.4} \frac{10}{11}x + \frac{6}{11}\, dx$$

$$= 1 - \left(\frac{5}{11}x^2 + \frac{6}{11}x\right)\Big|_0^{0.4}$$

$$= 1 - \left(\frac{\cancel{5}}{11}\cdot\frac{4}{\cancel{25}_{5}} + \frac{6}{11}\cdot\frac{2}{5}\right)$$

$$= 1 - \left(\frac{4}{55} + \frac{12}{55}\right) = \frac{39}{55} = 0.709$$

8) $f(x,y) = x+y$, $0<x<1$, $0<x<1$



$$P(X > \sqrt{Y}) = \int_0^1 \int_{\sqrt{y}}^1 f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_{\sqrt{y}}^1 x+y \quad dx\,dy$$

$$= \int_0^1 \frac{1}{2}x^2 + xy \Big|_{\sqrt{y}}^1 \quad dy = \int_0^1 \frac{1}{2} + y - \frac{1}{2}y - y^{\frac{3}{2}}\,dy$$

$$= \int_0^1 \frac{1}{2} + \frac{1}{2}y - y^{\frac{3}{2}}\,dy = \frac{1}{2}y + \frac{1}{4}y^2 - \frac{2}{5}y^{\frac{5}{2}} \Big|_0^1$$

$$= \frac{1}{2} + \frac{1}{4} - \frac{2}{5} = \frac{10+5-8}{20} = \frac{7}{20}$$

9 $g(t) = \frac{1}{2} f(t; \mu=-2, \sigma=1) + \frac{1}{2} f(t; \mu=2, \sigma=1)$

Let $f_1 = f(t; \mu=-2, \sigma=1)$, $f_2 = f(t; \mu=2, \sigma=1)$

$E(g(t)) = E[\frac{1}{2} f_1 + \frac{1}{2} f_2]$

According to the linearity of expectation:

$$E[\frac{1}{2} f_1 + \frac{1}{2} f_2] = E(\frac{1}{2} f_1) + E(\frac{1}{2} f_2)$$
$$= \frac{1}{2} E(f_1) + \frac{1}{2} E(f_2)$$
$$= \frac{1}{2} \cdot -2 + \frac{1}{2} \cdot 2$$
$$= 0$$

Mean is 0

$Var[g(t)] = Var[\frac{1}{2} f_1 + \frac{1}{2} f_2]$

according the property of the variance

$$V[\frac{1}{2} f_1 + \frac{1}{2} f_2] = \frac{1}{2}^2 Var[f_1] + \frac{1}{2}^2 Var[f_2] + 2(\frac{1}{2})^2 Cov[f_1, f_2]$$

Since two distribution are identical, $Cov[f_1, f_2] = Var[f_1] = Var[f_2]$

$$\Rightarrow Var[g(t)] = (\frac{1}{2})^2 \cdot \sigma_1^2 + (\frac{1}{2})^2 \cdot \sigma_2^2 + \frac{1}{2} \cdot \sigma_1^2$$
$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{2} = 1$$

$\Rightarrow$ Standard Deviation is $\sqrt{1} = 1$