Module 8 Homework - Noboru Hayashi

5.2.12

5.2.12

$$f_Y(y;\theta) = \frac{2y}{\theta^2}, \quad 0 \le y \le \theta$$

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta) = \frac{2^n \prod_{i=1}^{n} y}{\theta^{2n}} = 2^n(\theta^{-2})^n \prod_{i=1}^{n} y$$

$$\Rightarrow \ln(L(\theta)) = n\ln 2 - 2n\ln\theta + \ln(\prod y)$$

$$\Rightarrow \frac{\partial}{\partial\theta}\left(\ln(L(\theta))\right) = -\frac{2n}{\theta}$$

Let $\frac{\partial}{\partial\theta}(\ln(L(\theta))) = 0$

$$\Rightarrow \frac{2n}{\theta} = 0$$

Since $\frac{2n}{\theta} = 0$ is not possible with $n > 0$,

$L(\theta)$ continuosly decreases as θ increases.

Also $\theta \ge y_i$ for all i, so the minimum value of θ will be y_{max}

$$\Rightarrow \hat{\theta} = y_{MAX}$$

5.3.10

5.3.10 $\quad k = 192 \quad n = 540 \quad \frac{k}{n} = 0.356$

According to theorem, ~~CI for~~ 95% CI is

$$\left[\frac{k}{n} - z_{0.025}\sqrt{\frac{(k/n)(1-k/n)}{n}},\ \frac{k}{n} + z_{0.025}\sqrt{\frac{(k/n)(1-k/n)}{n}} \right]$$

$$\Rightarrow \left[0.356 - 1.96\sqrt{\frac{0.356 \times 0.644}{540}},\ 0.356 + 1.96\sqrt{\frac{0.356 \times 0.644}{540}} \right]$$

$$= [\,0.356 - 1.96 \times 0.0206,\ 0.356 + 1.96 \times 0.0206\,]$$

$$= [\,0.356 - 0.0404,\ 0.356 + 0.0404\,]$$

$$= [\,0.3156,\ 0.3964\,]$$

Thus, 95% CI is $[0.3156, 0.3964]$

5.3.14

5.3.14

50% CI: $(0.57, 0.63)$

Since CI is $\left[\frac{k}{n} - z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}, \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}}\right]$

$$\Rightarrow \frac{k}{n} = \frac{\text{upper} + \text{lower}}{2} = \frac{0.63 + 0.57}{2} = 0.60$$

So, $\begin{cases} \frac{k}{n} - z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}} = 0.57 \\ \frac{k}{n} + z_{\alpha/2}\sqrt{\frac{(k/n)(1-k/n)}{n}} = 0.63 \end{cases} \Rightarrow z_{\alpha/2}\sqrt{\frac{0.6 \times 0.4}{n}} = 0.03$

$$\Rightarrow z_{0.25}\sqrt{\frac{0.24}{n}} = 0.03$$

$$0.67\sqrt{\frac{0.24}{n}} = 0.03$$

$$\sqrt{\frac{0.24}{n}} \approx 0.04478$$

$$\frac{0.24}{n} \approx 0.0020$$

$$0.0020n \approx 0.24$$

$$n \approx 120$$

There are 120 observations

5.4.20

5.4.20

$X \sim Pois(\lambda)$, $E(X) = \lambda$, $Var(X) = \lambda$

$\hat{\lambda}_1 = X_1$, $\hat{\lambda}_2 = \bar{X}$

$E(\hat{\lambda}_1) = E(X_1) = \lambda \Rightarrow$ unbiased

$$E(\hat{\lambda}_2) = E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{\cancel{n}} \cdot \cancel{n}\lambda = \lambda \Rightarrow \text{unbiased}$$

$Var(\hat{\lambda}_1) = Var(X_1) = \lambda$

$$Var(\hat{\lambda}_2) = Var(\bar{X}) = Var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)$$

$$= \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} Var(X_i)$$

$$= \frac{1}{n^2} \cdot \sum_{i=1}^{n} \lambda = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n}\lambda = \frac{\lambda}{n}$$

$\Rightarrow$ The relative efficiency of $\hat{\lambda}_1$ to $\hat{\lambda}_2$ is:

$$Var(\hat{\lambda}_2) / Var(\hat{\lambda}_1) = \cancel{\lambda / \frac{\lambda}{n}}$$

$$\frac{\lambda}{n} / \lambda = \frac{1}{n}$$

So $\hat{\lambda}_2$ is more efficient

5.6.6

5.6.6: $Y_1, Y_2, \cdots, Y_n$. $f_Y(y;\theta) = \theta y^{\theta-1}$, $y \in [0,1]$

$$W = h(Y_1, Y_2, \cdots, Y_n) = \prod_{i=1}^{n} Y_i$$

$$L(\theta) = \prod_{i=1}^{n} f_Y(y_i;\theta) = \theta^n \prod_{i=1}^{n} (y_i^{\theta-1}) = \theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta-1}$$

Set $g(h;\theta) = \theta^n h^{\theta-1}$, and $b(Y_1, Y_2, \cdots Y_n) = 1$

$$\Rightarrow L(\theta) = \theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta-1} = g(h(Y_1, Y_2, \cdots, Y_n)) \cdot b(Y_1, Y_2, \cdots, Y_n)$$

Thus $W = h(Y_1, Y_2, \cdots, Y_n) = \prod_{i=1}^{n} Y_i$ is a sufficient statistics

For $L(\theta)$, apply logarithm to both sides:

$$\ln(L(\theta)) = \ln\left(\theta^n \left(\prod_{i=1}^{n} y_i\right)^{\theta-1}\right) = n \ln\theta + (\theta-1) \ln\left(\prod_{i=1}^{n} y_i\right)$$

Apply $\frac{d}{d\theta}$:

$$\frac{d}{d\theta}\left(\ln(L(\theta))\right) = \frac{n}{\theta} + \ln\left(\prod_{i=1}^{n} y_i\right)$$

Set the $\frac{d}{d\theta} = 0$: $\frac{n}{\theta} + \ln\left(\prod_{i=1}^{n} y_i\right) = 0$

$$\frac{n}{\theta} = -\ln\left(\prod_{i=1}^{n} y_i\right)$$

$$\hat{\theta} = -\frac{n}{\ln\left(\prod_{i=1}^{n} y_i\right)}$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\ln W}$$

$\Rightarrow$ Hence, $\hat{\theta}$ is a function of W