Module 4 Homework - Noboru Hayashi
3.10.4,

3.10.4 $\quad N = 5 \quad f_Y(y) = 2y \quad 0 \le y \le 1$

$P(Y_1' < 0.6 < Y_5')^c = P(Y_1' > 0.6) \cup P(Y_5' < 0.6)$

$\because P(Y_1' > 0.6) = P(Y_1', Y_2', Y_3', Y_4', Y_5' > 0.6)$

$P(Y_5' < 0.6) = P(Y_1', Y_2', Y_3', Y_4', Y_5' < 0.6)$

$\therefore P(Y_1' > 0.6)$ and $P(Y_5' < 0.6)$ are disjoint

$\therefore P(Y_1' > 0.6) \cup P(Y_5' < 0.6) = P(Y_1' > 0.6) + P(Y_5' < 0.6)$

$\because F_Y(y) = \int_0^y 2y\,dy = y^2 \Big|_0^y = y^2$

$\therefore f_{Y_{min}}(y) = f_{Y_1'}(y) = n[1 - F_Y(y)]^{n-1} f_Y(y)$

$= 5(1-y^2)^4\, 2y$

$f_{Y_{max}}(y) = f_{Y_5'}(y) = n[F_Y(y)]^{n-1} f_Y(y)$

$= 5(y^2)^4\, 2y$

$\therefore P(Y_1' > 0.6) = \int_{0.6}^1 5(1-y^2)^4 \cdot 2y\,dy = -(1-y^2)^5 \Big|_{0.6}^1$

$= 0 + 0.1073 \qquad = 0.1073$

$P(Y_5' < 0.6) = \int_0^{0.6} 5(y^2)^4 \cdot 2y\,dy = y^{10} \Big|_0^{0.6} = 0.0060$

$\therefore P(Y_1' < 0.6 < Y_5') = 1 - (0.1073 + 0.0060) = 0.8867$

MC 3.10.4

```
> # MC 3.10.4
>
> # init count
> cnt = 0
>
> # Since F(y) = y^2
> # inverse function of F(y), f(u) = u^0.5
> inv = function(u){return (u^0.5) }
>
> # Simulation for 1000 times
> for (i in 1:1000){
+ samples = sort(runif(5))
+ y = lapply(samples, inv)
+ if (y[1] <0.6 && y[5] > 0.6) {cnt = cnt+1}
+ }
>
>
> cnt/1000
[1] 0.884
```

⇒ MC Simulated number is close to the calculated number 0.8867

3.10.6,

3.10.6 $\quad Y_1, Y_2, \ldots, Y_n \quad f_Y(y) = e^{-y}, \; y \geq 0$

$$F_Y(y) = \int_0^y e^{-y}\,dy = -e^{-y}\Big|_0^y = 1 - e^{-y}$$

$$f_{Y_{min}}(y) = n[1 - F_Y(y)]^{n-1} f_Y(y)$$

$$= n[1-(1-e^{-y})]^{n-1} \cdot e^{-y} = n(e^{-y})^{n-1} \cdot e^{-y}$$

$$= n \cdot e^{-y(n-1)-y} = ne^{-ny}$$

$$P(Y_{min} < 0.2) = \int_0^{0.2} ne^{-ny}\,dy = -e^{-ny}\Big|_0^{0.2}$$

$$= -e^{-0.2n} + 1$$

$$P(Y_{min} < 0.2) > 0.9 \Rightarrow -e^{-0.2n} + 1 > 0.9$$

$$e^{-0.2n} < 0.1$$

$$-0.2n < \ln(0.1) \approx -2.3026$$

$$n > 11.51$$

So, the smallest n for which $P(Y_{min} < 0.2) > 0.9$

is $\underline{12}$

MC 3.10.6

```
> # MC3.10.6
>
> # init cnt
> cnt = 0
>
> # Calculated n
> n=12
>
> # Since CDF is 1-exp(-y)
> # its inverse is f(u) = -log(1-u)
> inv = function(u) {return ( -log(1-u) )}
>
> # simulation for n = 12
> for (i in 1:1000){
+ samples = runif(n)
+ y = sapply(samples, inv)
+ if (min(y) < 0.2){cnt = cnt+1}
+ }
>
> cnt/1000
[1] 0.916
>
>
> # Simulation for n = 11
>
> cnt = 0
> for (i in 1:1000){
+ samples = runif(n-1)
+ y = sapply(samples, inv)
+ if (min(y) < 0.2){cnt = cnt+1}
+ }
>
> cnt/1000
[1] 0.888
```

If n is 12, simulated P(Ymin<0.2) is slightly over 0.9. While with n = 11, P(Ymin<0.2) is below 0.9

3.10.16,

if all 3 components will fall within two months of one another means the time spans of between Y'1 & Y'2 and Y'2 & Y'3 are smaller than 2:

3.10.16 $\quad n=3 \quad f_Y(y) = e^{-y}$, ~~$y \geq 0$~~ $y>0$

P(all 3 components will fail within two months of one another)

$= P(\overset{Y_2'-Y_1'}{\cancel{R}} \leq 2) \cap P(Y_3' - Y_2' \leq 2)$

$= P(Y_2' \leq 2 + Y_1') \cdot P(Y_3' \leq Y_2' + 2)$

$\leq 2 \quad \leq 2$

$Y_1' \quad Y_2' \quad Y_3'$

$\because f_{Y_i' Y_j'}(u,v) = \dfrac{n!}{(i-1)!(j-i-1)!(n-j)!}[F_Y(u)]^{i-1}[F_Y(v)-F_Y(u)]^{j-i-1}$

$[1-F_Y(v)]^{n-j} \cdot f_Y(u) \cdot f_Y(v)$

$F_Y(y) = \int_0^y e^{-y}\,dy = -e^{-y}\Big|_0^y = 1-e^{-y}$

$\therefore f_{Y_1'Y_2'}(u,v) = \dfrac{3!}{(1-1)!(0)!(3-2)!}[F_Y(u)]^{1-1}[F_Y(v)-F_Y(u)]^{2-1-1} \cdot$

$[1-F_Y(v)]^{3-2} \cdot f_Y(u) \cdot f_Y(v)$

$= 3!\,(1-F_Y(v)) \cdot f_Y(u) \cdot f_Y(v) = 6(e^{-v})e^{-u} \cdot e^{-v}$

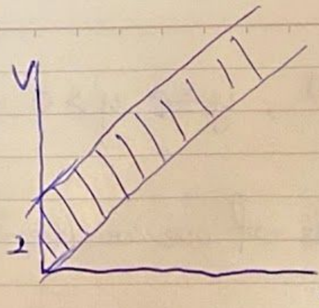
$= 6e^{-2v} \cdot e^{-u}$

$f_{Y_2'Y_3'}(u,v) = \dfrac{3!}{(2-1)!(3-2-1)!(3-3)!}[F_Y(u)]^{2-1}[F_Y(v)-F_Y(u)]^{3-2-1}[1-F_Y(v)]^{3-3} \cdot$

$f_Y(u) \cdot f_Y(v)$

$= 3!\,[F_Y(u)] \cdot f_Y(u) \cdot f_Y(v) = 6(1-e^{-u}) \cdot e^{-u} \cdot e^{-v}$

$= 6(e^{-u} \cdot e^{-v} - e^{-2u} \cdot e^{-v})$



$u \in (0, \infty)$

$v \in (u, u+2)$

$$P(Y_2^1 \le Y_1^2 + 2) = \int_0^\infty \int_u^{u+2} f_{Y_1 Y_2^1}(u,v)\, dv\, du$$

$$= \int_0^\infty \int_u^{u+2} 6\, e^{-2v} \cdot e^{-u}\, dv\, du$$

$$= \int_0^\infty 6e^{-u} \left[-\frac{1}{2} e^{-2v} \right]_u^{u+2} du = \int_0^\infty 6e^{-u} \left(\frac{1}{2} e^{-2u} - \frac{1}{2} e^{-2u-4} \right) du$$

$$= \int_0^\infty 3e^{-3u} - 3e^{-3u-4}\, du = -e^{-3u} + e^{-3u-4} \Big|_0^\infty$$

$$= e^{-3u}(e^{-4} - 1) \Big|_0^\infty = \cancel{e^{-4}} = (0-1)(e^{-4}-1)$$

$$= 1 - e^{-4}$$

$$P(Y_3^1 \le Y_2^2 + 2) = \int_0^\infty \int_u^{u+2} f_{Y_2^1 Y_3^1}(u,v)\, dv\, du = \int_0^\infty \int_u^{u+2} 6(e^{-u} e^{-v} - e^{-2u} \cdot e^{-v})\, dv\, du$$

$$= 6\int_0^\infty -e^{-u}e^{-v} + e^{-2u}e^{-v} \Big|_u^{u+2} du = 6\int_0^\infty -e^{-2u-2} + e^{-3u-2} + e^{-2u} - e^{-3u}\, du$$

$$= 6\int_0^\infty e^{-2u}(1 - e^{-2}) - e^{-3u}(1 - e^{-2})\, du = 6(1-e^{-2}) \int_0^\infty e^{-2u} - e^{-3u}\, du$$

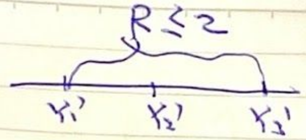
$$= 6(1-e^{-2}) \left(-\frac{1}{2} e^{-2u} + \frac{1}{3} e^{-3u} \right) \Big|_0^\infty = 6(1-e^{-2}) \left(\frac{1}{2} - \frac{1}{3} \right)$$

$$= 1 - e^{-2}$$

$$P(Y_2^1 \le Y_1^1 + 2) \wedge P(Y_3^1 \le Y_2^1 + 2) = (1 - e^{-4}) \times (1 - e^{-2}) \approx 0.8488$$

if the question is asking the range (timespan between Y’1 and Y’3) :

If $P(Y_3' \le Y_1' + 2)$:



$$f_{Y_1'Y_3'}(u,v) = \frac{3!}{(1-1)!(3-1-1)!(3-3)!}[F_Y(u)]^{1-1}\cdot[F_Y(v)-F_Y(u)]^{3-1-1}\cdot[1-F_Y(v)]^{3-3}\cdot f_Y(u)\cdot f_Y(v)$$

$$= 6[F_Y(v)-F_Y(u)]\cdot f_Y(u)\, f_Y(v)$$

$$= 6[\cancel{1}-e^{-v}\cancel{-1}+e^{-u}]\cdot e^{-u}\cdot e^{-v} = 6(e^{-u}-e^{-v})\cdot e^{-u}\cdot e^{-v}$$

$$P(Y_3' \le Y_1' + 2) = \int_0^{\infty}\int_u^{u+2} f_{Y_1'Y_3'}(u,v)\, dv\, du$$

$$= 6\int_0^{\infty}\int_u^{u+2} e^{-2u-v} - e^{-u-2v}\, dv\, du$$

$$= 6\int_0^{\infty} -e^{-2u-v} + \frac{1}{2}e^{-u-2v}\Big|_u^{u+2}\, du$$

$$= 6\int_0^{\infty} -e^{-3u-2} + \frac{1}{2}e^{-3u-4} + e^{-3u} - \frac{1}{2}e^{-3u}\, du$$

$$= 6\int_0^{\infty} -e^{-3u}\left(e^{-2} - \frac{1}{2}e^{-4} - \frac{1}{2}\right) du$$

$$= 6\left(e^{-2} - \frac{1}{2}e^{-4} - \frac{1}{2}\right)\cdot \frac{1}{3}e^{-3u}\Big|_0^{\infty}$$

$$= 2\left(e^{-2} - \frac{1}{2}e^{-4} - \frac{1}{2}\right)\cdot(0-1) = 1 - 2e^{2} + e^{-4}$$

$$= 0.7476$$

3.12.6,

3.12.6

$$f_Y(y) = \begin{cases} y, & 0 \le y \le 1 \\ 2-y, & 1 \le y \le 2 \\ 0, & \text{elsewhere} \end{cases}$$

$$M_Y(t) = E(e^{tY}) = \int_{-\infty}^{\infty} e^{ty} f_Y(y)\, dy$$

$$= \int_0^1 e^{ty} \cdot y\, dy + \int_1^2 e^{ty}(2-y)\, dy$$

$$= \frac{1}{t}\left(e^{ty} \cdot y - \int e^{ty} dy\right)_0^1 + \frac{1}{t}\left(e^{ty}(2-y) + \int e^{ty} dy\right)_1^2$$

$$= \frac{1}{t}\left(e^{ty} \cdot y - \frac{1}{t} e^{ty}\right)_0^1 + \frac{1}{t}\left(e^{ty}(2-y) + \frac{1}{t} e^{ty}\right)_1^2$$

$$= \frac{1}{t}\left(\cancel{e^t} - \frac{1}{t} e^t + \frac{1}{t}\right) + \frac{1}{t}\left(\frac{1}{t} e^{2t} - \cancel{e^t} - \frac{1}{t} e^t\right)$$

$$= \frac{1}{t}\left(\frac{1}{t} e^{2t} - \frac{2}{t} e^t + \frac{1}{t}\right) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

$$\Longrightarrow M_Y(t) = \frac{e^{2t} - 2e^t + 1}{t^2}$$

3.12.8

3.12.8

$$f_Y(y) = ye^{-y} \quad , \quad 0 \le y$$

$$M_Y(t) = \int_{-\infty}^{\infty} e^{ty} f_Y(y)\,dy$$

$$= \int_0^{\infty} e^{ty} \cdot ye^{-y}\,dy = \int_0^{\infty} y \cdot e^{(t-1)y}\,dy$$

let $1-t = \lambda$

$$\Rightarrow M_Y(t) = \int_0^{\infty} y \cdot e^{-\lambda y} = \frac{1}{\lambda}\int_0^{\infty} y \cdot \lambda e^{-\lambda y}\,dy$$

Since $\int_0^{\infty} y \cdot \lambda e^{-\lambda y}\,dy$ is the expected value of exponential distribution with $\lambda = 1-t$,

So the mean is $\frac{1}{\lambda} = \frac{1}{1-t}$

$$\Rightarrow M_Y(t) = \frac{1}{\lambda} \cdot \frac{1}{\lambda} = \frac{1}{(1-t)^2}$$