Failure of QM with special relativity(cont.)

In the region of light cone $x = \pm ct$ on the t v.s x diagram centered at x_0, there are three types of trajectories:

1. light like- follow the $x - ct = 0$. Trajectory. This is exclusively for massless particles.
2. time like- intervals "inside" the light cone $\to x - ct < 0$. This is the trajectory for all massive particles.
3. space like- intervals where the end points are outside of the light cone $\to x - ct > 0$. This is not allowed in our universe.

Therefore, we can conclude two rules using the concept of events in the space time coordinates

1. The future lightcone contains all possible events(a point at a certain (x, t)) that could be influenced by event at x_0.

2. The past lightcone contains all possible events that could have influenced the event at x_0.

In one word: **information cannot travel past the speed of light.**

Propergation of particle

Consider the following QM propergater:

$$\left\langle x \middle| e^{-i\widehat{H}t} \middle| y \right\rangle \tag{1}$$

where $\widehat{H} = E$, and $E^2 = \vec{\boldsymbol{p}}^2 + m^2$. This gives $\widehat{H} = \sqrt{\vec{\boldsymbol{p}}^2 + m^2}$.

We insert a complete set of momentum states(using completeness) between the $\langle x|$:

$$\begin{aligned} &\frac{1}{(2\pi)^3} \int \mathrm{d}p \mathrm{d}\vec{\boldsymbol{p}} \langle x|p\rangle \left\langle \vec{\boldsymbol{p}} \middle| e^{-i\sqrt{\vec{\boldsymbol{p}}^2+m^2}}t \middle| y \right\rangle \\ &= \frac{1}{(2\pi)^3} \int \mathrm{d}^3\vec{\boldsymbol{p}} e^{i\vec{\boldsymbol{p}}\cdot(x-y)} e^{-i\sqrt{\vec{\boldsymbol{p}}^2+m^2}t} \\ &= \frac{1}{(2\pi)^3} \frac{1}{|x-y|} \int_0^\infty \mathrm{d}pp \sin(p(x-y)) e^{-i\sqrt{p^2+m^2}t} \end{aligned} \tag{2}$$

To compute this integral, we have to countour integrate this with a countour closes at infinity. We will dive into this later, but note that the final integrand in euqation (2) is a finite number N. Consequently, the propergater becomes:

$$\left\langle x \middle| e^{-i\widehat{H}t} \middle| y \right\rangle = \frac{N}{(2\pi)^3} \frac{1}{|x-y|} \tag{3}$$

Therefore, no matter how seperated $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ are, the evaluation of this propergater is never zero. This is simply not possible whene $|x - y| > ct$

In QFT, there must be such a propergater with the following properties:

1. Goes from x to y if $t_x < t_y$
2. Goes from y to x if $t_y < t_x$

Review on Lorentz Symmetry

Lorentz Invariance in Classical Theory

To transfrom from rest frame into a moving frame with some velocity $\vec{v}$ (boost). This happenes through a 4-vector:

$$\begin{pmatrix} x_{0'} \\ x_{1'} \\ x_{2'} \\ x_{3'} \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4}$$

We introduce a metric space time $g_{\mu\nu}$. In Flat space, we call this $\eta_{\mu,\nu}$, which is the Minkowski metric:

$$\eta_{\mu,\nu} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{5}$$

We have A^μ which are countravariant, and A_μ which is co variant. We can use $g_{\mu,\nu}$ to lower or rise indicies. For instance:

$$g_{\mu,\nu} X^\nu = X_\mu \text{ and } X^\mu = g_{\mu,\nu} X_\nu \tag{6}$$

Additionally, we can use $g_{\mu,\nu}$ to represent dot products:

$$A_\mu B_\mu = A^\mu g_{\mu\nu} B^\nu \tag{7}$$

This metric defines the “distances” between two quantities. Therefore, it shows the “shape” of the space, and the “disstance” $\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x^\nu g^{\mu\nu}$. In flatspace with $\eta_{\mu\nu}$, we have:

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x^\nu \eta_{\mu\nu} = \mathrm{d}x_0^2 - \sum_{i=1} \mathrm{d}x_i^2 \tag{8}$$

. We also need to take derivatives:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \text{ Gives covariant vector} \tag{9}$$

and

$$\partial^\mu = \frac{\partial}{\partial x_\mu} \text{ Gives contravariant vector} \tag{10}$$

We have transformation laws for covariant and contravariant vectors:

$$\frac{\partial x'^\mu}{\partial x^\nu} A^\nu = A'^\mu \text{ For covariant} \tag{11}$$

$$\frac{\partial x'_\nu}{\partial x'_\mu} A_\nu = A'_\mu \text{ For covariant} \tag{12}$$

We can then prove that the innerproducts btween two vectors are the same:

$$A'_\mu B'^\mu = \frac{\partial x'^\alpha}{\partial x^\beta} A_\alpha \frac{\partial x'^\alpha}{\partial x^\beta} B^\alpha = \delta_{\alpha,\beta} A_\alpha B^\beta \tag{13}$$

A more simple mathematical object than four vector is a scalar φ, where $\partial_\mu \varphi$ also transforms like a four vector:

$$\partial_\mu \varphi = \frac{\partial \varphi}{\partial x^\mu} \to \frac{\partial x_\nu}{\partial x'^\mu} \frac{\partial \varphi}{\partial x^\nu} = \frac{\partial \varphi}{\partial x'^\mu} \tag{14}$$

and

$$(\partial_\mu \varphi)(d^\mu \varphi) = \text{Lorentz scalar} \tag{15}$$

Similiarly, $\varphi^2, \varphi^3, \varphi^4$ are also Lorentz vector.

Classical Field Theory

$S \equiv$ action, which is a **functional** (function of function).

In CM, $S = \int \mathrm{d}t L(x(t), \dot{x}(t), t)$ where $x(t)$ is the trajectory. In CM, $x(t)$ extremizes the action S.

To this end, we will need to get the extreme of functional that extremize S.

For field theory, instead of trajectories, we are considering fields. Therefore, for aciton, we have:

$$S = \int \mathrm{d}^4 x L(\varphi(x), \partial_\mu \varphi(x), t) \tag{16}$$

We hen seek to find the field $\varphi(x)$ that extremizes S. We first take some variation of the field:

$$\varphi(x) \to \varphi(x) + \delta\varphi \text{ and } \partial_\mu \varphi \to \partial_\mu \varphi + \delta \partial_\mu \varphi \tag{17}$$

We then consider the variation of the action:

$$\delta S = \int \mathrm{d}^4 x L(\varphi + \delta\varphi, \partial_\mu \varphi + \delta \partial_\mu \varphi, t) - \int \mathrm{d}^4 x L(\varphi, \partial_\mu \varphi, t) = 0 \tag{18}$$

Doing a first degree taylor expansion yields:

$$\delta\varphi \frac{\partial L}{\partial \varphi} + \delta \partial_\mu \varphi \frac{\partial L}{\partial \partial_\mu \varphi} = \delta S = 0 \tag{19}$$

We can see that the second term in (19) (omitting one term using same end point values) becomes:

$$\begin{gathered}
\int \mathrm{d}^4 x \delta\varphi \frac{\partial L}{\partial \varphi} - \int \mathrm{d}^4 x \partial_\mu \left(\frac{\partial L}{\partial \partial_\mu \varphi} \delta\varphi \right) = 0 \\
\int \mathrm{d}^4 x \left[\frac{\partial L}{\partial \varphi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi} \right] \delta\varphi = 0 \\
\frac{\partial L}{\partial \varphi} - \partial_\mu \frac{\partial L}{\partial \partial_\mu \varphi} = 0 \text{ (E-L euqation)}
\end{gathered} \tag{20}$$

One "trivial" example is the scalar field φ. $L = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi$, using E-L euqation, this is trivially gives $\partial_\mu (\partial^\mu \varphi) = 0$. This gives: $\partial_t^2 - \boldsymbol{\nabla}^2 \varphi = 0$, which is nothing but the wave equation.

Symmetry and Conservation Laws

Noether's Theorem: For each conserved quantity, there is a symmetry of L. Consider $\varphi(x) \to \varphi(x) + \alpha\delta(x)$, the Lagrangian is then $L(\varphi + \alpha\delta(x), \partial_\mu \varphi + \alpha\delta\partial_\mu \varphi) \Leftrightarrow L(\varphi, \partial_\mu \varphi)$. While this is certainly true, but the transformation can also modify the surface term of the Lagrangian and it wouldn't matter in action minimization:

$$L \to L + \alpha \partial_\mu J^\mu \tag{21}$$

where J is just some surface term quantity. For the transfoamrion of $\varphi \to \varphi + \delta\alpha, \partial_\mu \varphi \to \partial_\mu \varphi + \delta\alpha\partial_\mu\varphi$, we have $\Delta L = \alpha\partial_\mu \left(\frac{\partial L}{\partial\partial_\mu\varphi}\delta\right) + \alpha\left(\frac{\partial L}{\partial\varphi} - \partial_\mu \frac{\partial L}{\partial\partial_\mu\varphi}\delta\right)$. Howver, this second term is zero due to E-L equation. Therefroe, as long as our J^μ satisfy:

$$\alpha\partial_\mu \left(\frac{\partial L}{\partial\partial_\mu\varphi}\delta\right) - \alpha\partial_\mu J^{\mu(x)} = 0 \tag{22}$$

will satisfy the conservation propety