Physics 7701 August 29

Linear Algebra with Einstein Notation

Consider the matrix-vector multiplication

$$\boldsymbol{A}\vec{\boldsymbol{B}} = \vec{\boldsymbol{C}}$$

Where the bold now is a matrix. $\boldsymbol{A}$. We then have:

$$A_{ij}B_j = C_i$$

Additionally, we have matrix-matrix multiplication

$$\boldsymbol{A}\boldsymbol{B} = \boldsymbol{C}$$
$$\Rightarrow A_{ij}B_{jk} = C_{ik}$$

This works with both squared and none squared matricies.

Assume $i = j = 3$

$$\delta_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

We note that the aboive matrix is symmetric

$$\varepsilon_{1jk} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

We note that the above matrix is antisymmetric if it satisfy $A_{ij}A_{ik} = \delta_{jk}$? We note that effcevtive we have $\left(A^T\right)_{ij} = A_{ji}$. therefore, we have

$$A^T_{ji}A_{ik} = \delta(jk) = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

This indicates that $\boldsymbol{A}$ is an orthogonal matrix.

Additionally, A_{aa} is $\mathrm{Tr}(A)$, and $\varepsilon_{ijk}A_{1i}A_{2j}A_{3k} = \det(\boldsymbol{A})$

Dirac Delta Function

In PS1, problem 2, we found out that, in cylindrical coordinates:

$$\boldsymbol{\nabla}^2\psi(\rho, \varphi, z) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}$$

For $\rho > 0$, we figured out that $\boldsymbol{\nabla}^2\ln(\rho) = 0$. What happenes at $\rho = 0$ when $\ln(\rho) \to -\infty$?

To uncover the δ function by regulating the $\ln(\rho)$ term. To tis end, we can add a small $\varepsilon \in \mathbb{R}$. This gives:

$$\ln(\rho) \to \ln\left(\sqrt{\rho^2+\varepsilon^2}\right) = \frac{1}{2}\ln(\rho^2+\varepsilon^2)$$
$$= \nabla^2 \ln(\rho) \to \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho^2}{\rho^2+\varepsilon^2}\right)$$
$$= 2\frac{\varepsilon^2}{\left(\rho^2+\varepsilon^2\right)^2}$$

Note that if $\rho = 0, \varepsilon \to 0$, then $\nabla^2 \ln(\rho) \to \infty$. This is out first check that the function goes to an δ function.

We then check if the integral is finite:

$$\int_0^\infty 2\frac{\varepsilon^2}{\left(\rho^2+\varepsilon^2\right)^2}\mathrm{d}\rho$$
$$\text{let } \rho' = \frac{\rho}{\varepsilon}$$
$$= 2\int_0^\infty \mathrm{d}\rho' \frac{\rho'}{\left(1+\rho'^2\right)^2} = 1$$

which is finite this, combined with that the integrad itself diverges, indicates that $\nabla^2 \ln(\rho) \propto \delta(\rho)$

Doing a similiar analysis gives:

$$\boldsymbol{E} = q(4\pi\varepsilon_0)\frac{\hat{\boldsymbol{r}}}{r^2} \Rightarrow \nabla \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\boldsymbol{r})$$

and because $\nabla \cdot \boldsymbol{E} = \frac{\rho(\boldsymbol{r})}{\varepsilon_0}$, this really make sense at the charge density of a point charge is inifinity where that charge is.

Chapter 6 by Lea titled "Generalized Functions", the δ function is in that chapter. We will need to know how to:
1. Applying properties of delta/theta Functions.
2. Identiying of delta functions in physics contexts.
3. Fourier representation of the delta function.
4. Differential equations with impulse terms.

We note that the physical context of the delta function will be **idealizations** of the physics. For instance, the function $f(\boldsymbol{x}) = q\delta^3(\boldsymbol{x} - \boldsymbol{x_0}) = q\delta(x)\delta(y)\delta(z)$

Delta Sequences:

There are sequences of functions φ_n such that:

$$\lim_{n\to\infty} \varphi_n(x) = \delta(x)$$

Two examples that we will see and prove in PS02 of such sequence is:

$$\varphi_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2}$$

and

$$\varphi_n(x) = \frac{1 - \cos(nx)}{n\pi x^2}$$

To prove those, we need to demonstrate the "sifting property":

$$\int_{-\infty}^{\infty} f(x)\delta(x)\mathrm{d}x = f(0) \to \int_{-\infty}^{\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

The test function $f(x)$ will need to satisfy $\int_{-\infty}^{\infty} \|f(x)\|^2 \mathrm{d}x < \infty$

One property of δ function is:

$$\delta(ax) = \frac{1}{|a|}\delta(x)$$

$$\begin{aligned}\int_{-\infty}^{\infty} f(u)\delta(au)\mathrm{d}u &= \int_{-\infty}^{\infty} f\left(\frac{x}{a}\right)\delta(x)\frac{1}{a}\mathrm{d}x \\ &= \frac{1}{a}f(0)\end{aligned}$$

where the absolute value comes from the change of variable step. If $a < 0$, we will have to pick up a minus sign and fip the intergral.

The Derivative of δ function:

$$\int_{-\infty}^{\infty} \delta(x)' f(x)\mathrm{d}x = \int_{-\infty}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}x}\delta(x)\right] f(x)\mathrm{d}x$$

Doing integration by parts yields:

$$\begin{aligned}&= -\int_{-\infty}^{\infty} \delta(x)\frac{\mathrm{d}}{\mathrm{d}x}f(x)\mathrm{d}x + \delta(x)f(x)_{-\infty}^{\infty} \\ &= -f'(0) \\ &\Rightarrow \int_{-\infty}^{\infty} \delta^{(n)}(x)f(x)\mathrm{d}x = (-1)^n f^{(n)}(x)\end{aligned}$$

One more important identity is:

$$\int_{-\infty}^{\infty} \delta(g(x))f(x)\mathrm{d}x = \sum_i^N \frac{f\left(x_{0_i}\right)}{\left|g'\left(x_{0_i}\right)\right|}$$

where x_{0_i} is the ith root of $g(x)$

Example:

suppose $g(x) = (x-a)(x-b)$, consider:

$$\int_{-\infty}^{\infty} \delta((x-a)(x-b))f(x)\mathrm{d}x$$

near $x = a \to \delta((x-a)(a-b)) = \frac{1}{|a-b|}\delta(x-a)$ because variation in $x - b$ is slow and using the identity above. Similiarly, near $x = b \to \delta((x-b)(b-a)) = \frac{1}{|b-a|}\delta(x-b)$

therefore,

$$\int_{-\infty}^{\infty} \delta((x-a)(x-b))f(x)\mathrm{d}x = \frac{1}{|a-b|}f(a) + \frac{1}{|b-a|}f(b)$$

More generally, near $x = x_{0_i}$, $g(x) = g\left(x_{0_i}\right) + \left(x - x_{0_i}\right)g'\left(x_{0_i}\right)$