

Recap on Levi Epsilons :

Consider the following notation $\varepsilon_{abc}\varepsilon_{def}$, we can write the result using cross product:

$$\begin{aligned}\varepsilon_{abc}\varepsilon_{def} &= \begin{vmatrix} \delta_{ad} & \delta_{ae} & \delta_{af} \\ \delta_{bd} & \delta_{be} & \delta_{bf} \\ \delta_{cd} & \delta_{ce} & \delta_{cf} \end{vmatrix} \\ &= \delta_{ad}(\delta_{be}\delta_{cf} - \delta_{ce}\delta_{bf}) - \delta_{ae}(\delta_{bd}\delta_{cf} - \delta_{bf}\delta_{cd}) + \delta_{af}(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd})\end{aligned}\quad (1)$$

$\varepsilon - \delta$ warmup:

Consider the following expressions:

1. $\nabla \times (\varphi \nabla \varphi) = 0$
2. $\nabla \cdot (V \times W) = W \cdot (\nabla \times V) - V \cdot (\nabla \times W)$
3. $\nabla \times (V \times W) = V(\nabla \cdot W) - W(\nabla \cdot V) + (W \cdot \nabla)V - (V \cdot \nabla)W$

General strategy:

- Assign component labels (from outside in):

$$[\nabla \times (\varphi \nabla \varphi)] = \varepsilon_{abc} \partial_b (\varphi \partial_c \varphi) = \varepsilon_{abc} (\partial_b \varphi \partial_c \varphi + \varphi \partial_b \partial_c \varphi) \quad (2)$$

Using the argument is symmetry, we note that ε_{abc} is symmetric, where $\partial_b \varphi \partial_c \varphi$ is symmetric. Therefore, the expression of equation (2) simply becomes **0**. For this specific problem, we are done.

- Use ε_{abc} for cross products

To explain this step, we now shift to considering cross products, which is the expression

$\nabla \times (\nabla \times \mathbf{a})$:

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{a}) &= \varepsilon_{abc} \delta_b (\nabla \times \mathbf{a})_c \\ &= \varepsilon_{abc} \partial_b \varepsilon_{cde} \partial_d a_e \\ &= \varepsilon_{cab} \varepsilon_{cde} \partial_b \partial_d a_e\end{aligned}\quad (3)$$

- Now, we eliminate two ε

$$= (\delta_{ad} \delta_{be} - \delta_{bd} \delta_{ae}) \partial_b \partial_d a_e \quad (4)$$

- Now, we use δ_{ij} to eliminate indices

$$= \partial_a \partial_b a_b - \partial_b \partial_b a_a \quad (5)$$

- Lastly, we identify the dot products:

$$= \nabla(\nabla \cdot \mathbf{a}) - \nabla \cdot \nabla \mathbf{a} \quad (6)$$

Vector Calculus Review

Div ($\nabla \cdot \mathbf{A}$), Gradient ($\nabla \mathbf{A}$), Curl ($\nabla \times \mathbf{A}$) and all that ...

Gradient

$\nabla := \hat{x} \partial_x + \hat{y} \partial_y + \hat{z} \partial_z$ in Cartesian coordinates.

For a scalar field $\Phi(\mathbf{x}) = \Phi(x, y, z)$, we have

$$\nabla \Phi = \hat{x} \frac{\partial \Phi}{\partial x} + \hat{y} \frac{\partial \Phi}{\partial y} + \hat{z} \frac{\partial \Phi}{\partial z} \quad (7)$$

To the first degree Taylor expansion, we have $\Delta\Phi = \nabla\Phi \cdot \Delta\mathbf{r}$. This indicates that the gradient tells us the vector pointing to the direction with most infinitesimal change in Φ .

Divergence

Consider $\mathbf{V} = v_1\hat{x} + v_2\hat{y} + v_3\hat{z}$, then the divergence:

$$\nabla \cdot \mathbf{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \partial_i v_i \quad (8)$$

Divergence measures the spreading out of vector fields about a point.

If $\nabla \cdot \mathbf{V} = 0$, we call \mathbf{V} to be “solenoidal”.

Curl

For curl, we have

$$\nabla \times \mathbf{V} = \hat{x}(\partial_y V_z - \partial_z V_y) + \hat{y}(\partial_z V_x - \partial_x V_z) + \hat{z}(\partial_x V_y - \partial_y V_x) \quad (9)$$

The curl, physically, is associated with circulation (integral of a vector around a closed curve).

If $\nabla \times \mathbf{V} = 0$, then the field is named “irrotational”

Using Jackson Cover's Equations:

- For Cartesian coordinates, this is trivial.
- For curvilinear coordinates (cylindrical and spherical), we have:

$$\nabla\Phi = \sum_i \hat{q}_i \frac{1}{h_i} \frac{\partial\Phi}{\partial q_i} \quad (10)$$

where h_i are called scaling factors, and

$$\nabla \cdot \hat{\mathbf{A}} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (A_1 h_2 h_3) + \frac{\partial}{\partial q_2} (A_2 h_1 h_3) + \frac{\partial}{\partial q_3} (A_3 h_1 h_2) \right] \quad (11)$$

Note in Jackson, the q_i are called e_i . We can then use the table of q_i and their corresponding h_i to calculate curls, div, and grad in different coordinates.

Example:

We figured that $\nabla \cdot \hat{x} = 3$ in cartesian coordinates, but let's try this in cylindrical coordinates:

$$\mathbf{x} = \hat{\rho}\rho + \hat{z}z \Rightarrow A_1 = \rho, A_2 = 0, A_3 = z. \quad (12)$$

Using formula in Jackson, you will see the divergence in this coordinate is also 3.

Vector Calculus Theorems:

Divergence Theorem

$$\int_V \nabla \cdot \mathbf{A} d^3x = \int_S \mathbf{A} \cdot \hat{n} da \quad (13)$$

Where $\mathbf{A} \cdot \hat{n}$ is the flux of \mathbf{A} through surface S

$$\int_V \nabla \psi d^3x = \int_S \psi \hat{n} da \quad (14)$$

$$\int (\nabla \times \mathbf{A}) d^3x = \int_S \hat{\mathbf{n}} \times \mathbf{A} \quad (15)$$

Stoke's Theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}} da = \oint_C \mathbf{A} \cdot d\mathbf{r} \quad (16)$$

$$\int_S (\hat{\mathbf{n}} \times \nabla \psi) da = \oint_C \psi dl \quad (17)$$

need to know:

1. intuitive ideas
2. how to interpret
3. how to apply

Common features:

- relate sums of local quantities to global quantities
- sum of derivatives in the interior is related to the value on the boundary.