Physics 7701 Introduction

Maxwell Equations

For reference in Jackson, see section i or Zangwill chapter 2.

$$\vec{\nabla} \cdot \vec{D} = \rho \to \vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$$
$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J} \to \vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$
$$\vec{\nabla} \times \vec{E} - \frac{\partial \vec{B}}{\partial t} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

These Maxwell equations all functions of $\vec{x}, t$, and are local equations, which means the Maxwell equations are evaluated at a small neighborhood around a point $\vec{x}_1$.
Additionally, we have the Lorentz force equation:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

To comfortably solve those equations, we note that the knowledge of **vector calculus** and **differential equations** are going to be needed.

Note: because differential operator are linear, superposition could be imposed for Maxwell equations.

Application 1

What is the basic physics principle underlying a ground fault circuit interrupter?

1. Charge conservation - if there is current leakage, charge conservation would be broken.
2. Two current carrying wire with opposite current would have zero effect if the current on the two wires is exact.
3. Induced current in a coil around the loop of the wires if the current conservation is broken, which triggers the GFCI.

Current Conservation

Does the principle of the current conservation (or charge conservation) is apparent in the Maxwell's equation?
Note that the concept of current conservation is the same as if the following equality is true:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Which indicates that the change in charge density over time has to be explained by a divergence of current.

Prove

From Maxwell's equations, we have:

$$\vec{\nabla} \cdot E = \frac{\rho}{\epsilon_0} \tag{1}$$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} \tag{2}$$

Taking the gradient for both sides of equation (2) gives:

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) - \frac{1}{c^2}\frac{\partial}{\partial t}\vec{\nabla} \cdot \vec{E} = \mu_0 \vec{\nabla} \cdot \vec{J} \tag{3}$$

Note that you can exchange the order of time derivative and gradient.
Combine equation (3) and equation (1), using the identity that $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = 0$:

$$-\frac{1}{c^2 \epsilon_0 \mu_0}\frac{\partial \rho}{\partial t} = \vec{\nabla} \cdot \vec{J} \to \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

Kronecker Delta and Levi Epsilon

Definitions

Below is the definition for Kronecker Delta

$$\delta_{ij} = 1 \text{ if } i = j$$
$$\delta_{ij} = 0 \text{ if } i \neq j$$

Below is the definition for Levi Epsilon

$$\epsilon_{ijk} = 1 \text{ if } i, j, k \text{ are cyclic}$$
$$\epsilon_{ijk} = -1 \text{ if } i, j, k \text{ are anti-cyclic}$$
$$\epsilon_{ijk} = 0 \text{ if } i, j, k \text{ for other cases}$$

Using Einstein's sum convention, we can write vector operators such as dot products and cross products using this convention:

1. $\vec{A} \cdot \vec{B} = A_1B_1 + A_2B_2 + A_3B_3 = \sum_i A_iB_i = \sum_i \sum_j A_i\delta_{ij}B_j$
2. $\left(\vec{A} \times \vec{B}\right)_i = \epsilon_{ijk}A_jB_k$

Additionally, we can derive some properties of these two special functions:

1. $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$
2. $\delta_{ii} = 3$

With such notation in mind, we can simplify $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \nabla^2\vec{A}$:

Prove

Let us only consider the i component of the expression $\left[\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)\right]_i$
Using δ, ϵ, and Einstein sum, we can simplify, this expression as:

$$\begin{aligned}
\left[\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right)\right]_i &= \epsilon_{ijk}\partial_j\left(\vec{\nabla} \times \vec{A}\right)_k \\
&= \epsilon_{ijk}\partial_j(\epsilon_{klm}\partial_lA_m) \\
&= \epsilon_{kij}\epsilon_{klm}\partial_j\partial_lA_m \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\partial_j\partial_lA_m \\
&= \partial_i\partial_jA_j - \partial_j\partial_jA_i
\end{aligned}$$

Doing so for other components will revleal the relationship that

$$\vec{\nabla} \times (\nabla \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot A) - \nabla^2 A$$