Recap on Levi Epsilons：

Consider the following notation $\varepsilon_{abc}\varepsilon_{def}$, we can write the result using cross prodduct:

$$\begin{aligned}\varepsilon_{abc}\varepsilon_{def} &= \begin{vmatrix}\delta_{ad} & \delta_{ae} & \delta_{af} \\ \delta_{bd} & \delta_{be} & \delta_{bf} \\ \delta_{cd} & \delta_{ce} & \delta_{cf}\end{vmatrix} \\ &= \delta_{ad}\left(\delta_{be}\delta_{cf} - \delta_{ce}\delta_{bf}\right) - \delta_{ae}\left(\delta_{bd}\delta_{cf} - \delta_{bf}\delta_{cd}\right) + \delta_{af}\left(\delta_{bd}\delta_{ce} - \delta_{be}\delta_{cd}\right)\end{aligned} \tag{1}$$

$\varepsilon - \delta$ warmup:

Consider the following expressions:

1. $\boldsymbol{\nabla} \times (\varphi \boldsymbol{\nabla} \boldsymbol{\varphi}) = 0$
2. $\boldsymbol{\nabla} \cdot (\boldsymbol{V} \times \boldsymbol{W}) = \boldsymbol{W} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V}) - \boldsymbol{V} \cdot (\boldsymbol{\nabla} \times \boldsymbol{W})$
3. $\boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{W}) = \boldsymbol{V}(\boldsymbol{\nabla} \cdot \boldsymbol{W}) - \boldsymbol{W}(\boldsymbol{\nabla} \cdot \boldsymbol{V}) + (\boldsymbol{W} \cdot \boldsymbol{\nabla})\boldsymbol{V} - (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{W}$

General strategy:

- Assign component labels (from outside in):

$$[\boldsymbol{\nabla} \times (\varphi \boldsymbol{\nabla} \varphi)] = \varepsilon_{abc}\partial_b(\varphi \partial_c \varphi) = \varepsilon_{abc}(\partial_b \varphi \partial_c \varphi + \varphi \partial_b \partial_c \varphi) \tag{2}$$

Using the argument is symmetry, we note that ε_{abc} is symmetric, where $\partial_b \varphi \partial_c \varphi$ is symmetric. Therefore, the expression of equation (2) simply becomes **0**. For this specific problem, we are done.

- Use ε_{abc} for cross products

To explain this step, we now shift to considering cross products, which is the expression $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a})$:

$$\begin{aligned}\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{a}) &= \varepsilon_{abc}\delta_b(\boldsymbol{\nabla} \times \boldsymbol{a})_c \\ &= \varepsilon_{abc}\partial_b \varepsilon_{cde}\partial_d a_e \\ &= \varepsilon_{cab}\varepsilon_{cde}\partial_b \partial_d a_e\end{aligned} \tag{3}$$

- Now, we eliminate two ε

$$= (\delta_{ad}\delta_{be} - \delta_{bd}\delta_{ae})\partial_b \partial_d a_e \tag{4}$$

- Now, we ust δ_{ij} to eliminate indices

$$= \partial_a \partial_b a_b - \partial_b \partial_b a_a \tag{5}$$

- Lastly, we identify the dot products:

$$= \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{a}) - \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \boldsymbol{a} \tag{6}$$

Vector Calculus Review

Div ($\boldsymbol{\nabla} \cdot \boldsymbol{A}$), Gradient ($\boldsymbol{\nabla} \boldsymbol{A}$), Curl ($\boldsymbol{\nabla} \times \boldsymbol{A}$) and all that ...

Gradient

$\boldsymbol{\nabla} := \hat{\boldsymbol{x}}\partial_x + \hat{\boldsymbol{y}}\partial_y + \hat{\boldsymbol{z}}\partial_z$ in Cartesian coordinates.

For a scalar field $\Phi(\boldsymbol{x}) = \Phi(x, y, z)$, we have

$$\boldsymbol{\nabla}\Phi = \hat{\boldsymbol{x}}\frac{\partial \Phi}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial \Phi}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial \Phi}{\partial z} \tag{7}$$

To the first degree Taylor expansion, we have $\Delta\Phi = \nabla\Phi \cdot \Delta\boldsymbol{r}$ This indicates that the gradient tells us the vector poiting to the direction with most infinitesimal change in Φ.

Divergence

Consider $\boldsymbol{V} = v_1\hat{\boldsymbol{x}} + v_2\hat{\boldsymbol{y}} + v_3\hat{\boldsymbol{z}}$, then the divergence:

$$\nabla \cdot \boldsymbol{V} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \partial_i v_i \tag{8}$$

Divergence measaures the spreading out of vectors fields about a point.

If $\nabla \cdot \boldsymbol{V} = 0$, we call $\boldsymbol{V}$ to be "solenoidal".

Curl

For curl, we have

$$\nabla \times \boldsymbol{V} = \hat{\boldsymbol{x}}\left(\partial_y V_z - \partial_z V_y\right) + \hat{\boldsymbol{y}}(\partial_z V_x - \partial_x V_z) + \hat{\boldsymbol{z}}\left(\partial_x V_y - \partial_y V_x\right) \tag{9}$$

The curl, physically, is associated with circulation (integral of a vector around a closed curve).

If $\nabla \times \boldsymbol{V} = 0$, then the field is named "irrotational"

Using Jackson Cover's Equations:

- For Cartesian coordinates, this is trivial.
- For curvelinear coordinates (cylindrical and spherical), we have:

$$\nabla\Phi = \sum_i \hat{\boldsymbol{q}}_i \frac{1}{h_i} \frac{\partial\Phi}{\partial q_i} \tag{10}$$

where h_i are called scaling factors, and

$$\nabla \cdot \widehat{\boldsymbol{A}} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}(A_1 h_2 h_3) + \frac{\partial}{\partial q_2}(A_2 h_1 h_3) + \frac{\partial}{\partial q_3}(A_3 h_1 h_2)\right] \tag{11}$$

Note in Jackosn, the q_i are called e_i. We can then use the table of q_i and their corresponding h_i to calculate curls, div, and grad in different coordinates.

Example:

We figured that $\nabla \cdot \hat{\boldsymbol{x}} = 3$ in cartesian coordinates, but let's try this in cylindrical coordinates:

$$\boldsymbol{x} = \hat{\boldsymbol{\rho}}\rho + \hat{\boldsymbol{z}}z \Rightarrow A_1 = \rho, A_2 = 0, A_3 = z. \tag{12}$$

Using formula in Jackson, you will see the divergence in this coordinate is also 3.

Vector Calculus Theorems:

Divergence Theorem

$$\int_V \nabla \cdot \boldsymbol{A} \mathrm{d}^3 x = \int_S \boldsymbol{A} \cdot \hat{\boldsymbol{n}} \mathrm{d}a \tag{13}$$

Where $\boldsymbol{A} \cdot \hat{\boldsymbol{n}}$ is the flux of $\boldsymbol{A}$ through surface S

$$\int_V \nabla\psi \mathrm{d}^3 x = \int_S \psi \hat{\boldsymbol{n}} \mathrm{d}a \tag{14}$$

$$\int (\boldsymbol{\nabla} \times \boldsymbol{A}) \mathrm{d}^3 x = \int_S \hat{\boldsymbol{n}} \times \boldsymbol{A} \tag{15}$$

Stoke's Theorem

$$\int_S (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \hat{\boldsymbol{n}} \mathrm{d}a = \oint_C \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{r} \tag{16}$$

$$\int_S (\hat{\boldsymbol{n}} \times \boldsymbol{\nabla} \psi) \mathrm{d}a = \oint_C \psi \mathrm{d}l \tag{17}$$

need to know:
1. intuitive ideas
2. how to interpret
3. how to apply

Common features:
- relate sums of local quantities to global quantities
- sum of derivatives in the interior is related to the value on the boundary.