QM Berry Phase Examples March 10

Recap:

If we have a Hamiltonian $H(t)$, we first diagonalize it and get instantaneous eigenstates $H(t)\psi_n(t) = E_n\psi_n(t)$. Usually we use a path $\vec{R}(t)$ to represent such time dependence. Using the fact that $\dot{H}$ is small, we can write down an approximation

$$\psi_n(\vec{R}(t))e^{i\theta_n(t)}e^{i\delta_m(t)} \tag{1}$$

where $\theta_n(t)$ is the dynamic phase that come from $H(t)$, and we have the equation for the geometric phase δ_n:

$$\delta_n(t) = i\int_{\vec{R}_i}^{\vec{R}_f} \langle\psi_n(\vec{R})|\nabla_R\psi_n(\vec{R})\rangle\,\vec{R} \tag{2}$$

And if we have a closed loop where $\vec{R}_i = \vec{R}_f$, we have

$$\delta_n = \oint \vec{A}_n(\vec{R})d\vec{R} \tag{3}$$

were $\vec{A}_n = i\,\langle\psi_n|\nabla_R\psi_n\rangle$.

Example: Uniform $\vec{B}$ field with spin 1/2 particles

We have $H(t) = \frac{e}{m}\vec{S}\cdot\vec{B}(t)$
where $\vec{B}(t) = B_0(\sin\alpha\cos\omega t, \sin\alpha, \sin\omega t, \cos\alpha)$
and $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$
we have

$$H(t) = \hbar\frac{\omega_1}{2}\begin{pmatrix}\cos\alpha & e^{-i\omega t\sin\alpha} \\ e^{i\omega t}\sin\alpha & -\cos\alpha\end{pmatrix} \tag{4}$$

with intrinsic frequency $\omega_1 = \frac{eB_0}{m}$. The adiabatic condition is when $\omega_1 \gg \omega$

if we diagonalize (4) and eigen-decompose it, we will ge

$$\chi_+(t) = \begin{pmatrix} coa\frac{\alpha}{2} \\ e^{i\omega t}\sin\frac{\alpha}{2} \end{pmatrix} \quad E_+ = \frac{\hbar\omega_1}{2}$$
$$\chi_-(t) = \begin{pmatrix} e^{-i\omega t\sin\alpha/2} \\ -\cos\frac{\alpha}{2} \end{pmatrix} \quad E_- = -\frac{\hbar\omega_1}{2}$$

Just solve the equation will grant us (check)

$$\chi(t) = \left[\cos\frac{\lambda t}{2} - i\omega_1 - \frac{\omega\cos\alpha}{\lambda}\sin\frac{\lambda t}{2}\right]e^{-i\omega t/2}\chi_+(t)$$
$$+ i\left[\frac{\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2}\right]e^{i\omega t/2}\lambda_-(t)$$

where $\lambda = \sqrt{\omega^2 + \omega_1^2 - 2\omega\omega_1\cos\alpha}$.

Let us consider a transition from $\chi_+(0) \to \chi_-$, we have the probability for such transition to be

$$P(t) = |\langle\chi_-(t)|\chi(t)\rangle|^2 = \left(\frac{\omega}{\lambda}\sin\alpha\sin\frac{\lambda t}{2}\right)^2$$

If we use the condition of adiabaticity $\omega_1 \gg \omega_0$, we have

$$\lambda \approx \omega_1 - \omega\cos\alpha$$
$$P(t) \approx \left(\frac{\omega}{\omega_1}\sin\alpha\sin\frac{\lambda t}{2}\right)^2 \ll 1$$

So there is a small (but none zero) chance that our state flip from λ_+ to λ_-

Now, what happens under a full rotation?

Using the condition for adiabaticity which gives us $\lambda \approx \omega_1 - \omega\cos\alpha$
We can re-write $\chi(t)$ to be

$$\chi(t) \approx e^{-i\lambda t/2}e^{-i\omega t/2}\chi_+(t)$$
$$= e^{-i\omega_1 t/2}e^{i\omega\cos(\alpha)t/2}e^{-i\omega t/2}\chi_+(t)$$

We can then get from inspection that $\theta_+(t) = -\frac{\omega_1 t}{2}$ and $\delta_+(t) = \frac{\omega t}{2}(\cos\alpha - 1)$

We realize that $\delta_+(T)$ after one complete rotation is nothing but $-\frac{1}{2}\Omega$, the solid angle of the circle swept by $\vec{B}$ on the top of the sphere.

We discussed a situation where we have circular motion, but what if we have a general enclosed path?
We still have that ($\alpha \to \phi, \beta \to \theta$)

$$\chi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

We want to compute A_+ as defined in (3), so we will need to calculate $\nabla_B \chi$. Since we are in spherical coordinate, we have

$$\nabla_B \chi = \frac{1}{r}\begin{pmatrix} \frac{1}{2}\sin\frac{\theta}{2} \\ \frac{1}{2}e^{i\phi}\cos\frac{\theta}{2} \end{pmatrix}\hat{\theta} + \frac{1}{r\sin\theta}\begin{pmatrix} 0 \\ ie^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}\hat{\phi}$$

This gives

$$\langle \chi_+ | \nabla_B \chi_+ \rangle = \frac{i\sin^2\frac{\theta}{2}}{r\sin\theta}\hat{\phi} = \frac{i}{2}\frac{1}{r}\tan\frac{\theta}{2}\hat{\phi} \tag{5}$$

using (3) gives:

$$\delta_n = \oint \hat{A}_n \cdot d\vec{R} = \int (\nabla_R \times A_n)\, d\vec{a} = \int \vec{B}_n\, d\vec{a} \tag{6}$$

we can see that A_n actually becomes a fictitious magnetic field! This is called the `Berry field`

combine (5) and (3) gives

$$\vec{A}_= = i\langle \chi_+ | \nabla_B \chi_+ \rangle = -\frac{1}{2r}\tan\frac{\theta}{2}\hat{\phi}$$

This gives

$$\vec{B}_+ = \nabla_B \times \vec{A}_+ = -\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{2r}\tan\frac{\theta}{2}\right)\hat{r} = -\frac{1}{2r^2}\hat{r} \tag{7}$$

note that our $\vec{B}_+$ points radially! This is equivalent to a mono-pole magnetic charge at the origin!

Because $d\vec{a} = \hat{r} r^2 d\Omega$, combine (6) and (7) gives

$$\delta_+(T) = \int \vec{B} \cdot \, d\vec{a} = -\frac{1}{2}\Omega$$

So the result we got from a circular close contour is actually general to any close contour!