QM Scattering Feb 15

We have an incident flux at $\hat{z}$ direction (plane wave) with a spherically symmetric potential.

Now, the plane wave is going to be scattered. We can put a detector at distance r away with$d\Omega = d\phi d\theta \sin\theta$ (solid angle)

And the scattering cross-section

$$\sigma(\theta, \phi) = \frac{r^2 J_{\text{scatt}}}{J_{\text{inc}}}$$

Given $V(r) \implies$ compute $\sigma(\theta, \phi)$ as a function of energy.

1-D Scenario

We have a wave traveling to the right Ae^{ikx}
After scattering, there will be a reflected wave Re^{-ikx}
and a translated wave Te^{ikx}
We then use Schrodinger Equation to solve for $\frac{T}{R}$
The Asymptotic behavior is given.

3-D Scenario

Physics intuition TM`

the wave traveling in every direction, so we first find the asymptotic behavior, then solve the Schrodinger equation.
The Asymptotic at $r \to \infty$ form of the wave function is

$$\psi \to A\left(e^{ikz} + f(\theta, \phi)\frac{e^{ikr}}{r}\right)$$

Where the first term described the incoming wave front toward $\hat{z}$ direction, and the second term describes a radially outward wave. The $\frac{1}{r}$ is due to probability conservation.

Math Derivation

Incident

$\psi_{\text{inc}}(r) = Ae^{\vec{k}\cdot\vec{r}} = Ae^{ikz}, E = \frac{\hbar^2 k^2}{2m}$

Scattering

$$\begin{aligned} k^2 &= \frac{2mE}{\hbar^2}, k > 0 \\ U(r) &= \frac{2m}{\hbar} V(r) \end{aligned}$$

By Schrodinger Equation

$$\Delta\psi + (k^2 - U(r))\psi = 0$$

Where Δ is the spherical polar coordinate
By separation of variable

$$\psi(\vec{r}) = \sum_{l,m} c_{l,m} Y_l^m(\theta, \phi) \frac{u_l(r)}{r}$$

Therefore,

$$\frac{d^2u}{dr^2} + \left[k^2 - U(r) - \frac{l(l+1)}{r^2}\right] u_l(r) = 0$$

Asymptotic behavior $r \to \infty \implies \frac{l(l+1)}{r} \to 0, U(r) \to 0 \forall U(r)$ decaying faster than $\frac{1}{r}$
Thus, we have

$$u_l'' + k^2 u_l = 0$$

with the solution

$$\begin{aligned} \frac{u_l}{r} &\sim \frac{e^{ikr}}{r} \\ &\sim \frac{e^{-ikr}}{r} \end{aligned}$$

However, we ignore the second solution as the wave function can only be "scattered out".
Therefore,

$$\psi_{\text{scatt}}(r) \sim \sum_{l,m} c_{l,m} Y_l^m(\theta, \phi) \frac{e^{ikr}}{r} \sim f(\theta, \phi) \frac{e^{ikr}}{r}$$

We will prove $\sigma(\theta,\phi) = \mid f(\theta,\phi) \mid^2$. The name of $f(\theta,\phi)$ is "scattering amplitude".

Probability Current

In QM. Fluxes are probability current densities.

$$\vec{J} = \frac{\hbar}{m}\mathrm{Im}(\psi^* \vec{\nabla}\psi)$$

$\psi_{\text{inc}} = Ae^{ikz} \implies \vec{J}_{inc} = |A|^2 \hbar \frac{k}{m}\hat{z}$
$\psi_{\text{scat}} \sim f(\theta,\phi)\frac{e^{ikr}}{r} = \vec{J}_{scatt}$

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\phi}$$

Check that $\hat{\theta}, \hat{\phi}$ component of $\vec{J}_{\text{scatt}}$ are negligible when $r \to \infty$
Compute $\vec{J}_{\text{scatt}}$ and show that

$$\vec{J}_{\text{scatt}} \approx |A|^2 \frac{\hbar k}{m} \frac{|f(\theta,\phi)|^2}{r^2}$$

Therefore,

$$\sigma(\theta,\phi) = r^2 \frac{|\vec{J}_{\text{scatt}}|^2}{|\vec{J}_{\text{scatt}}|^2}$$

and check that

$$\sigma(\theta,\phi) = |f(\theta,\phi)|^2$$

Using Green's function so solve SE

Need to solve

$$(\vec{\nabla}^2 + k^2)\psi_k(\vec{r}) = U(r)\psi_k(\vec{r}) \tag{*}$$

with suitable B.C.

Claim: It is easier to solve this problem by re-writing it as an integral equation.

$$\psi_k(\vec{r}) = \psi_k^0(\vec{r}) + \int G_k(\vec{r}-\vec{r}')U(r')\psi_k(\vec{r}')\,d^3\vec{r}' \tag{**}$$

where ψ_k^0 is the solution for homogeneous PDE, and G is a Green's function satisfying

$$(\nabla^2 + k^2)G_k(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}')$$