QM Partial Wave Examples March 3

Recap:

We have the incident wave

$$\psi_k(\vec{r}) \sim_{r\to\infty} e^{ikz} + f(\theta)\frac{e^{ikr}}{r}$$

and scattering amplitude

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)\left[\frac{e^{2i\delta_l}-1}{2i}\right]P_l(\cos\theta)$$

Where all the information are stored in the scattering phase shift. We will need to calculate δ_l given $V(r)$, where k is E dependence. We can then write :

$\sigma(\theta) = |f(\theta)|^2$

Note that we can re-write the δ_l component in (2) as

$$\left[\frac{e^{i2\delta_l}-1}{2i}\right] = e^{i\delta_l}\sin\delta_l = \frac{1}{\cot\delta_l - i}$$

Now, we realize that there are an infinite number of δ_l to calculate. However, we will see that only "small l" (often only $l=0$) dominates "low energy scattering".

Example: Hard Sphere Potential

Hard Sphere potential is defined as:

$$V(r) = 0 \text{ for } r > 0$$
$$V(r) = \infty \text{ for } r \leq 0$$

and we wrote out last time that, at $r > r_0$

$$R_l(r) = A_l j_l(kr) + B_l n_l(kr) \tag{1}$$

with boundary condition $R_l(r_0) = 0$. This give:

$$\frac{B_l}{A_l} = -\frac{j_l(kr)}{n_l(kr)} \tag{2}$$

When $r \to \infty$,we have

$$R_l(kr) \sim_{r\to\infty} \frac{1}{kr}\left[A_l\sin\left(kr - l\frac{\pi}{2}\right) - B_l\cos\left(kr - l\frac{\pi}{2}\right)\right] = \frac{\sqrt{A_l^2+B_l^2}}{kr}\sin\left(kr - \frac{l\pi}{2} + \delta_l\right) \tag{3}$$

where

$$\delta_l(k) = \arctan\left(-\frac{B_l}{A_l}\right) \tag{4}$$

Combine (4) with (2), we have

$$\delta_l(k) = \arctan\left(\frac{j_l(kr_0)}{n_l(kr_0)}\right) \tag{5}$$

Hard sphere scattering solved!
Note: our scale for low energy is when

$$kr_0 \ll 1 \implies E \ll \frac{\hbar^2}{mr_0^2} \tag{6}$$

We recall that for $kr_0 \ll 1$:

$$j_l(kr_0) \sim \frac{x^l}{(2l+1)!!}$$
$$n_l(kr_0) \sim -\frac{(2l+1)!!}{x^{l+1}} \tag{7}$$

Combine (7) and (4) gives:

$$\tan\delta_l \approx \delta_l \approx -(\alpha)(kr_0)^{2l+1} \tag{8}$$

Note that when $k \to 0$, higher l results in phase shift approaches to 0 more rapidly. Therefore, $l=0$ dominates. We have

$$\delta_0(k) = -kr_0 \tag{9}$$

Which is a negative number. This make sense because $\delta_0 < 0$ for a repulsive potential, as our wave is pushed outward.

We then seek to compute $f_{l=0}(\theta)$. Using equation from the Recap section, we get

$$f_{l=0}(\theta) = \frac{1}{k} e^{i\delta_0} \sin(\delta_0) = \frac{1}{k} e^{-ikr_0} \sin(kr_0) \sim r_0 \tag{10}$$

Therefore, we have

$$\sigma(\theta) \approx_{kr_0 \ll 1} |f_0(\theta)|^2 = r_0^2$$

So $\sigma_{tot} = \frac{4\pi}{k^2} \sin^2\delta_0 \to 4\pi r_0^2$! What a satisfying result!

Example: Scattering Resonances & S- Wave Scattering Length a_s

We showed that for $l=0$, we can write

$$f_0 = \frac{1}{k} \frac{1}{\cot\delta_0(k) - 1} \tag{11}$$

We can show that in general,

$$\lim_{k\to 0} k \cot\delta_0(k) = -\frac{1}{a_s} + O(k^2 R_0) \tag{12}$$

where a_s defines the scattering length for the range of potential R_0 such that $kR_0 \ll 1$ (check for hard sphere case where $a_s = r_0$).

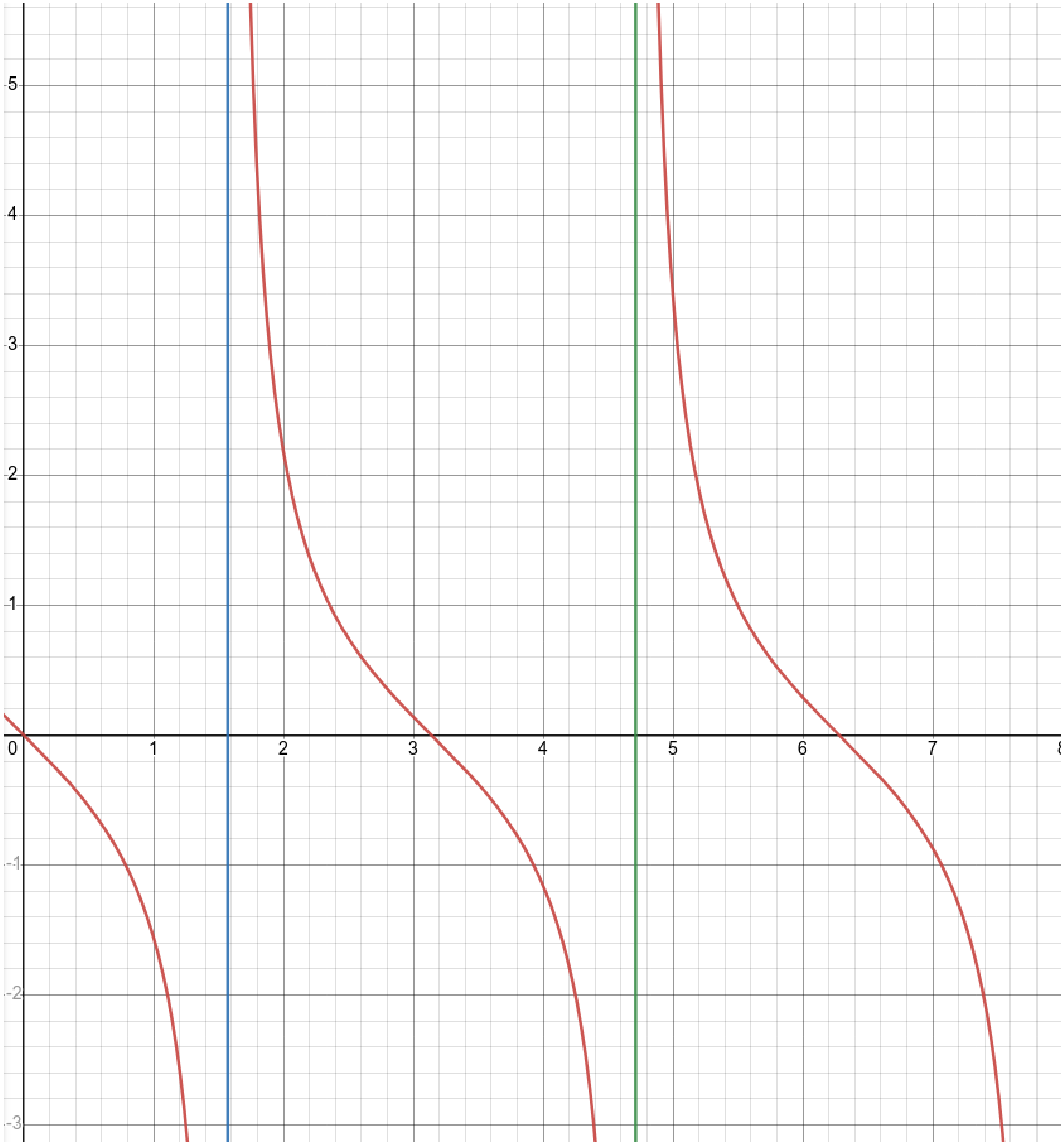
With (12) combined with (11), we have, when $k \to 0$

$$\begin{aligned} f_0 &\approx \frac{1}{-\frac{1}{a_s} + ik} \\ \implies \sigma_{tot} &= 4\pi |f_0|^2 = \frac{4\pi a_s^2}{1 + k^2 a_s^2} \end{aligned} \tag{13}$$

Note that if a_s is finite, $\sigma_{tot} \approx 4\pi a_s^2$. But if a_s diverges, we have$\sigma_{tot} \approx \frac{4\pi}{k^2} =$ unitary bound. (We must have conservation of probability) .

Question: How to make $a_s \to 0$?

Simplest example: we have a finite square well with depth $-V_0$ from 0 to r_0 satisfying $kr_0 \ll 1$. As we we keep decreasing the well, bound we have more bound states appearing. We can show that the graph of $\frac{a_s}{r_0}$ v.s. $K_0 r_0$ where $K_0 = \sqrt{\frac{2m|V_0|}{\hbar^2}}$ has the behavior



(note: this is not $\cot$)

And the corresponding σ_{tot} v.s. $K_0 r_0$ looking like