QM Time Depedent Deg March 24

We want to study the type of the problem where

$$H = H_0 + \lambda H_1(t) \tag{1}$$

where H_0 is exactly solved (with ϵ_n and $|n\rangle$) and we want to study λH_1

We start with TDSE

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = (H_0 + \lambda H_1(t))|\psi\rangle \tag{2}$$

We will express

$$\begin{aligned} |\psi(t)\rangle &= \sum_n c_n(t)|n\rangle \\ &= \sum_n a_n(t) e^{-i\epsilon_n t/\hbar}|n\rangle \end{aligned} \tag{3}$$

where we separated the dynamic phase out.
Combine (3) and (1) yields:

$$\sum_n \left[i\hbar \frac{da_n}{dt} + \epsilon_n a_n(t)\right] e^{-i\epsilon_n t/\hbar}|n\rangle = \sum_n [\epsilon_n a_n(t) + \lambda H_1(t) a_n(t)] e^{-i\epsilon_n t/\hbar}|n\rangle$$

where we can see that $\epsilon_n a_n(t)$ terms cancel.
We then tank an inner product with $\langle m|$ and we know $\langle m|n\rangle = 1$. This get us:

$$i\hbar \frac{da_m}{dt} e^{-i\epsilon_m t/\hbar} = \lambda \sum_n \langle m|H_1(t)|n\rangle e^{-i\epsilon_n t/\hbar} a_n(t)$$

Define $\hbar\omega_{mn} \equiv (\epsilon_m - \epsilon_n) \implies$

$$i\hbar \dot{a}_m = \lambda \sum_n \langle m|H_1|n\rangle e^{i\omega_{mn}t} a_n(t)$$

which is an exact result. Next, we approximate a_n as

$$a_n(t) = a_n^0 + \lambda a_n^1 + \lambda^2 a_n^2 + \dots$$

Please check that

$$i\hbar \dot{a}_m^{r+1} = \sum_n \langle m|H_1|n\rangle e^{i\omega_{mn}t} a_n^r(t) \tag{4}$$

where a_n^0 are known and is defined by $\psi(0) = \sum_n a_n^0 |n\rangle$. We will only do this for first order P.T. Therefore, $r = 0$ for $\dot{a}_n^1$.

Let $H_1(t) = 0$ for $t < 0$, and initial condition $\psi(0) = |i\rangle \leftrightarrow a_n^0 = \delta_{n,i}$
simplify notation: $a_n^1(t) = a_n(t)$.
Apply $(4) \implies$

$$i\hbar \frac{da_f}{dt} = \langle f|H_1(t)|i\rangle e^{i\omega_{fi}t}$$

Solving this get

$$a_f(t) = \delta_{fi} - \frac{i}{\hbar} \int_0^t \langle f|H_1(t')|i\rangle e^{i\omega_{fi}t'}\, dt' \tag{5}$$

Note that when t=0, the integral term in (5) is trivially zero. However, then $t > 0$, we can see that some probabilities leaks from $|i\rangle$ into other states $|f\rangle$ where $\langle f|\psi(t)\rangle = a_f(t)e^{i\epsilon_f t/\hbar}$.

Note that this approximation is valid when the integral part of (5) is small in comparison to 1 , or $|a_f(t)| \ll \forall f \neq i$.

Two perturbations that are very important:

1. Constant perturbation $H_1(t) = \Theta(t)$
2. sinusoidal $H_1(t) = 2V_0 \cos(\omega t)\Theta(t)$
   Let us focus on 2. as 1. is nothing but the limit of $\omega \to 0$.

For $f \neq i$

$$a_f(t) = -i\hbar\langle f|\hat{V}_0|i\rangle \int_0^t e^{i(\omega_{fi}+\omega)t'} + e^{i(\omega_{fi}-\omega)t'}\, dt'$$

Where the integral becomes

$$\frac{e^{i(\omega_{fi}+\omega)t}-1}{i(\omega_{fi}+\omega)} + \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}$$

Therefore,

$$P_{i\to f} = \frac{|V_{fi}|^2}{\hbar^2}|\frac{e^{i(\omega_{fi}+\omega)t}-1}{i(\omega_{fi}+\omega)} + \frac{e^{i(\omega_{fi}-\omega)t}-1}{i(\omega_{fi}-\omega)}|^2$$

Note that for constant case, we let $\omega \to 0$

We will look in detail at two physically distinct consequences:

1. Two discrete states $|i\rangle$ and $|f\rangle$ in a two level system with a t-dependent Hamiltonian.
2. A discrete level $|i\rangle$ together with a continuum of final states $\implies$ Fermi's golden rule.