Recap:

Classical E&M

Solve the wave equation for vector potential in the gauge in a box of size L^3

$$\Phi = 0$$
$$\vec{\nabla} \cdot \vec{A} = 0$$

implies

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k},\alpha} = \hat{e}_{\vec{k},\alpha}[A_{k,\alpha} e^{-i(\omega_k t - i\vec{k}\cdot\vec{r})} + A^*_{k,\alpha} e^{i(\omega_k t - \vec{k}\cdot\vec{r})}] \tag{1}$$

Where a state $\vec{k}, \alpha$ is a mode and $\vec{k} = \frac{2\pi}{L}(n_x, n_y, n_z)$ and α is the transverse polarization.

Note that α is transverse, so the solution $\hat{e} \perp \alpha$ which is apparent in the gauge we chose, but this is a general statement.

Also note that $A_{k,\alpha}$ and $A^*_{k,\alpha}$ is nothing but the complex amplitude of their corresponding waves in (1), and because the velocity of EM wave is c, we then have $\omega = ck$.

The energy stored in the EM field is (algebra in Sakurai)

$$\mathbf{E} = \frac{L^3}{4\pi} \sum_{\vec{k},\alpha} \left(\frac{\omega_k}{c}\right)^2 (A^*_{k,\alpha} A_{k,\alpha} + A_{k,\alpha} A^*_{k,\alpha}) \tag{2}$$

Note that the reason we wrote $A_{k,\alpha}$ in this way is because A will become an operator in QM, and they don't necessarily commute.

This is kind of difficult, because we can have an infinite amount of modes $\vec{k}$. We try to convert a classical $\mathbf{E}$ in equation (2) to a quantum Hamiltonian $\mathcal{H}$. Be careful about commutation relation! By doing this, we elevate the constant $A_{k,\alpha} \to \hat{a}_{k,\alpha}$ and $A^*_{k,\alpha} \to \hat{a}^\dagger_{k,\alpha}$ where $\hat{a}, \hat{a}^\dagger$ are operators.

We define a algebra on the operator

$$\begin{aligned} [\hat{a}_{k,\alpha}, \hat{a}^\dagger_{k',\alpha'}] &= \delta_{k,k'}\delta_{\alpha,\alpha'} \\ [\hat{a}_{k,\alpha}, \hat{a}_{k',\alpha'}] &= 0 \\ [\hat{a}^\dagger_{k,\alpha}, \hat{a}^\dagger_{k',\alpha'}] &= 0 \end{aligned} \tag{3}$$

The intuition of defining such operator is from the raising and lowering operator of the harmonic oscillators.

Now we write down a quantum Hamiltonian that look like (2)

$$\hat{\mathcal{H}} = \sum_{\vec{k},\alpha} \frac{\hbar\omega_k}{2} (\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,\alpha} + \hat{a}_{k,\alpha}\hat{a}^\dagger_{k,\alpha}) \tag{4}$$

Note the correlation between (2) and (4).

Looking at a specific node, we have, from (3)

$$aa^\dagger - a^\dagger a = 1$$
$$aa^\dagger = 1 + a^\dagger a$$

Using this relation, we can re-write (4) as :

$$\hat{\mathcal{H}} = \sum_{k,\alpha} = \hbar\omega_k \left(\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,a} + \frac{1}{2}\right) \tag{5}$$

Please note the similarity of (5) with the Hamiltonian of a harmonic oscillator. This is saying that each mode $\vec{k}, \alpha$ look like a quantum harmonic oscillator with frequency ω_k.

Note that in order to get the pre-factor of (4), we have replaced the classical object

$$A_{k,\alpha} \to \frac{1}{L^{3/2}}\left(\frac{2\pi\hbar c^2}{\omega_k}\right)^{1/2} \hat{a}_{k,\alpha}$$

The detail of this is not physically relevant.

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we can re-write (4) by separating the mode label:

$$\mathcal{H} = \sum_{\alpha=1,2,\dots} \sum_{\vec{k}} \left(\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,\alpha} + \frac{1}{2}\right)\hbar\omega_k$$

where the two sums still resemble the sum over all modes, but note that because, in quantum harmonic oscillator, we have $\hat{N} = \hat{a}^\dagger a$, this means that here, $\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,\alpha}$ resembles the number of photons in mode $\vec{k}, \alpha$, the $1/2$ is nothing but the "zero point energy", and the $\hbar\omega_k$ is the energy of photon in mode $\vec{k}, \alpha$ with $\omega = ck$.

Vacuum State:

The ground state of $\mathcal{H}$ is called the **vacuum state** $|0\rangle$. Analogous to harmonic oscillator, $|0\rangle$ is defined by the conditions:

$$\begin{aligned} &\hat{a}_{k,\alpha}|0\rangle = 0 \\ \implies &\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,\alpha}|0\rangle = 0 \\ \implies &\text{no photonsin the vacuum} \end{aligned}$$

We can see that $\mathcal{H}|0\rangle = \sum_{\vec{k},\alpha} \frac{1}{2}\hbar\omega_k|0\rangle$, but because we have a infinite sum over a value, this indicates that the vacuum has a infinite amount of energy or some constant. We denote this ground state energy E_0.

Even though this is problematic mathematically, we don't really bother to calculate the ground state energy E_0 because, when doing the experiment, all we measure is the difference of energy from the ground state.

Now we want to know if there are $\vec{E}, \vec{B}$ in the vacuum. Note that classically, we have

$$\begin{aligned} \vec{E} &= -\frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \vec{\nabla} \times \vec{A} \end{aligned}$$

where both of these are linear combination of $\hat{a}$ and $a^\dagger$ summed over modes.

If we want

$$\langle 0\rangle\vec{E}|0\rangle \quad \text{and} \quad \langle 0|\vec{B}|0\rangle$$

we can use the linear combination of

$$\langle 0|\hat{a}_{k,\alpha}|0\rangle \quad \text{and} \quad \langle 0|\hat{a}^\dagger_{k,\alpha}|0\rangle$$

and realize both of the component will be 0. Therefore, it turns out that

$$\langle 0|\vec{E}|0\rangle = \langle 0|\vec{B}|0\rangle = 0$$

This make sense as, on average, the ground state which is the vacuum state should not have and $\vec{E}$ or $\vec{B}$ field.

But note that

$$\langle 0|\vec{E}^2|0\rangle \neq 0 \quad \text{and} \quad \langle 0|\vec{B}^2|0\rangle \neq 0$$

This is true be cause when we square them, there will be term like $\hat{a}\hat{a}^\dagger$ which has none zero expectation value. This means that there is a fluctuation of $\vec{E}$ and $\vec{B}$ field in vacuum.

Note that this consequence is the same as a harmonic oscillator

$$\begin{aligned} \langle 0|x|0\rangle &= 0 \\ \langle 0|x^2|0\rangle &\neq 0 \end{aligned}$$

The fact that we have fluctuation of $\vec{E}$ and $\vec{B}$ field in the vacuum ties to the fact that we have none zero vacuum energy E_0.

Excited states:

In parallel with the harmonic oscillators, we can show that

$$|\vec{k},\alpha\rangle := \hat{a}^\dagger_{\vec{k},\alpha}|0\rangle$$

is an eigenstate of $\mathcal{H}$. We can check that

$$\mathcal{H}|\vec{k},\alpha\rangle = \hbar\omega_k|\vec{k},\alpha\rangle$$

The reader should **check this** in detail!

We call $|\vec{k},\alpha\rangle$ the 1- photon state with a photon of energy $h\omega_k = \hbar ck$ in the mode $\vec{k},\alpha$.
Note that this is a single quantum excitation of a EM field.

We can discuss the momentum of a photon:
Intuitively, we already know that $E = \hbar\omega_k$, so the momentum should be $\hbar\vec{k}$.

Classically, the momentum $\vec{p}_{cl} = \frac{1}{4\pi c}\int \vec{E}\times\vec{B}\,d^3r$. If we were to make $\vec{p}_{cl}$ to be a operator $\hat{p}$, and we can make $\vec{E},\vec{B}$ to be expressed in $\vec{A}$, which we can quantize using $\hat{a},\hat{a}^\dagger$. After a huge amount of algebra, we reach a quantum answer that the quantum momentum:

$$\hat{p} = \sum_{\vec{k},\alpha}\hbar\vec{k}\,\hat{a}^\dagger_{k,\alpha}\hat{a}_{k,\alpha}$$

which implies

$$\hat{p}|\vec{k},\alpha\rangle = \hbar\vec{k}|k,\alpha\rangle$$

Now we can calculate the mass of photon

$$\begin{gathered} E = \hbar ck \quad \text{and} \quad E^2 = p^2c^2 + m^2c^4 \\ p = \hbar k \implies m^2c^4 = 0 \implies m_\gamma = 0 \end{gathered}$$

We can also calculate the spin (angular momentum) of a photon.
One can start with the classical result, which is not so familiar to many, so, instead, let us use a plausibility argument that the wave equation in coordinate space is

$$\langle\vec{r}|\vec{k}\alpha\rangle = \frac{e^{i\vec{k}\cdot\vec{r}}}{L^{3/2}}\hat{e}_{\vec{k},\alpha}$$

where $\hat{e}_{\vec{k},\alpha}$ is the polarization vector. This implies angular momentum $S = 1\hbar$, so one might propose that we have 3 S_z state $S_z = 1, 0, -1$.

It turns out that the photon only has $S_z = \pm 1$ states (right/left polarization). The reason why we only have two sates is because we have two transverse polarization. At $v = c$, we can never get to the rest frame of $v = c$. Therefore, we could never get to the longitudinal frame which is traveling at the speed of light.