QM March 8

Recap:

We realize that when the change in Hamiltonian H over time $\dot{H}$ is small when compared with the equivalent time energy difference between instantaneous eigenstates $E_n - E_m$, which means $\left|\sum_{n\neq m}\frac{\langle\psi_m|\dot{H}|\psi_n\rangle}{E_n-E_m}\right| \ll 1$, we can use the adiabatic approximation. We obtain our final coefficient relation that

$$\dot{c}_m(t) = -c_m(t)\langle\psi_m||\dot{\psi}_m\rangle \tag{1}$$

Geometric phase:

The solution for (1) is that

$$c_m(t) = c_m(0)e^{i\delta_m(t)} \tag{2}$$

where we obtain `geometric phase`

$$\delta_m(t) = i\int_0^t \langle\psi_m(t')|\frac{\partial}{\partial t}|\psi_m(t')\rangle \, dt' \tag{3}$$

so that we can write our wave function with the coefficient values in (2) and get

$$\Psi(t) = \sum_n c_m(0)\psi_n(t)e^{i\theta_n(t)}e^{i\delta_m(t)} \tag{4}$$

If we have the special case where

$$c_n(0) = 1$$
$$c_m(0) = 0 \forall m \neq n$$

We then get the case where the total wave function (4) gets reduced to a stationary state ψ_n with the dynamic phase and the geometric phase:

$$\Psi(t) = \psi_n(t) e^{i\theta_n(t)} e^{i\delta_n(t)} \tag{5}$$

We note that δ_m is real, because

$$\langle \psi_m(t) | \psi_m(t) \rangle = 1$$
$$\frac{\partial}{\partial t} \langle \psi_m(t) | \psi_m(t) \rangle = 0$$
$$\langle \dot{\psi}_m | \psi_m \rangle + \langle \dot{\psi}_m | \psi_m \rangle = 0$$
$$2\mathrm{Re} \langle \psi_m | \dot{\psi}_m \rangle = 0$$

and combine the i in front of (3) gives that $\delta_m \in \mathbb{R}$.

Closed Contour

If ψ has $\vec{R}(t)$ dependence, we can write that

$$\frac{\partial}{\partial t} \psi_m(\vec{R}(t)) = \frac{\partial \psi_m}{\partial R_1} \frac{\partial R_1}{\partial t} + \frac{\partial \psi_m}{\partial R_2} \frac{\partial R_2}{\partial t} + \frac{\partial \psi_m}{\partial R_3} \frac{\partial R_3}{\partial t} = \nabla_R \psi_m \cdot \frac{\vec{\partial} R}{\partial t}$$

Using this formulation with (3) will result in

$$\delta_n(t) = i \int_0^t \langle \psi_m | \dot{\psi}_m \rangle \, dt' = i \int_0^t \langle \psi_n | \nabla_R \psi_m \rangle \frac{d\vec{R}}{dt} \, dt := \int_{R_i}^{R_f} \vec{A}_n(\vec{R}) \, d\vec{R}$$

However, "phases" are not real physical observable. Therefore, we consider the gauge dependence of $\delta_n(t)$ so that:

$$\psi_n(\vec{R}(t)) \to \psi'_n(\vec{R}(t)) = \psi_n(\vec{R}(t)) e^{i\zeta_n(\vec{R}(t))} \tag{6}$$

This gauge transformation gives:

$$\vec{A}_n \rightarrow \vec{A}'_n = \vec{A}_n - \nabla_R \zeta_n(\vec{R})$$

$$\delta_n(t) \rightarrow \delta'_n(t) = \delta_n(t) - \zeta(\vec{R}_f) + \zeta(\vec{R}_i)$$

and we if have a closed contour, we have $\vec{R}_i = \vec{R}_f$, and therefore

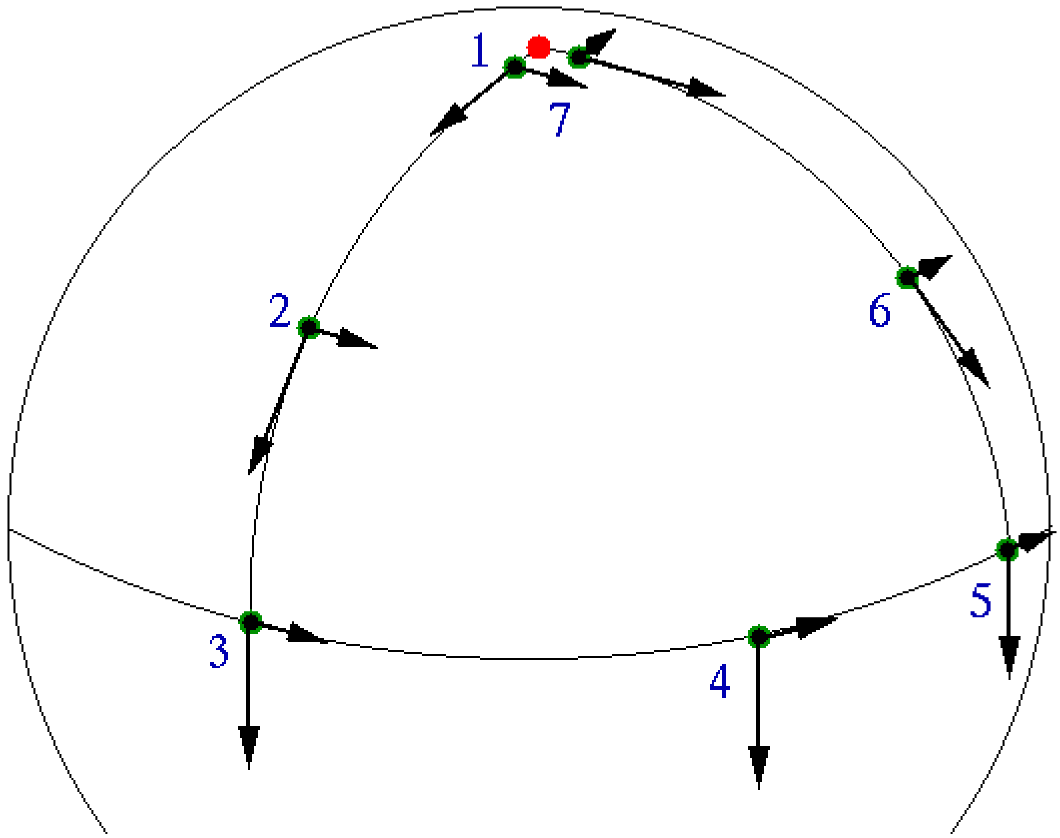
$$e^{i\zeta_n(\vec{R}_i)} = e^{i\zeta_n(\vec{R}_f)}$$

Therefore, we must have that

$$\zeta_n(\vec{R}_f) - \zeta_n(\vec{R}_i) = 2\pi n \text{ for } n \in \mathbb{Z}$$

Therefore, for a closed contour, $\delta_n(t)$ is actually a physical phase as a gauge transformation **cannot** cancel $\delta_n(t)$. Therefore, $\delta_n(t)$ is physical.

A classical example:



Note that any rotator starting from 1 , and go through a closed contour as the diagram indicated, we realize the plane of the rotation changes.

This process is called a `Non-holonomic transport`, or `paralled transformation` in differential geometry.

On the example above, I claim that the shift in the plane of rotation $\theta = \Omega$, the solid angle of the surface. This is because

$$A = \frac{2\pi R^2}{2\pi/\theta} \to \theta = \frac{A}{R^2} = \Omega$$

This happens because the surface of a sphere has intrinsic curvature (not homeomorphic to $\mathbb{R}^2$).

Measurement of Berry Phase

We know our wave function have a total phase (dynamic + Berry)

$$\Psi_n(t) = \psi_n(t) e^{i\theta_n(t)} e^{i\delta_n(t)}$$

and Berry phase is nothing but a T period angle change of $\vec{R}(t)$, so $\delta_n(T) =$ Berry phase. But note that after T, our dynamic phase becomes 1. Therefore

$$\Psi = \psi_0 + e^{i\delta_n} \to |\Psi|^2 = |\psi_0|^2(1 + \cos\delta_n)$$

Example: Spin in a rotating magnetic field.

Let's consider we have a rotating spin $\frac{1}{2}$ particle under a uniform magnetic field. If we express the strength of $\vec{B}$ over time, we have:

$$\vec{B} = B_0(\sin\alpha\cos(\omega t), \sin\alpha\sin(\omega t), \cos\alpha)$$

in polar coordinates with angle α and our spin $\frac{1}{2}$ particle is rotating with a frequency ω in the uniform magnetic field, with our Hamiltonian

$$H(t) = \frac{e}{m}\vec{S}\cdot\vec{B}$$

and $\omega_1 = \frac{eB_0}{m}$ which is called the cyclotron frequency. Note that the adiabatic condition is when $\omega_1 \gg \omega$.

Just diagonalize $H(t)$ gives our instantaneous eigenstates

$$\chi_+(t) = \begin{pmatrix} \cos\frac{\alpha}{2} \\ e^{i\omega t\sin\alpha/2} \end{pmatrix} \quad E_+ = \frac{\hbar\omega_1}{2}$$

$$\chi_-(t) = \begin{pmatrix} e^{-i\omega t}\sin\frac{\alpha}{2} \\ -\cos\frac{\alpha}{2} \end{pmatrix} \quad E_- = \frac{-\hbar\omega_1}{2}$$