QM T-D Pert. March 27

Time-dependent perturbation theory

$$H = H_0 + \lambda H_1(t)$$
$$|\psi(0)\rangle = |i\rangle$$

and we want to solve for $|\psi(t)\rangle$ for first degree in λH_1

$$P_{i\to f} = |\langle f|\psi(t)\rangle|^2$$
$$P_{i\to f}(t) = \frac{|V_{fi}|^2}{\hbar^2}\left|\frac{1-e^{i(\omega_{fi}+\omega)t}}{\omega_{fi}+\omega} + \frac{1-e^{i(\omega_{fi}-\omega)t}}{\omega_{fi}-\omega}\right|^2 \tag{1}$$

With the following Initial condition:

$$H_1(t) = 2V\cos(\omega t)\Theta(t)$$
$$h\omega_{fi} = E_f - E_i$$
$$\langle f|H|i\rangle \equiv V_{fi}$$

Sinusoidal Potential

Let us understand (1) in the case of two discrete levels with sinusoidal perturbation $\omega > 0$.

We fix observation time t and vary perturbation frequency ω. Note that it seems if $\omega = \pm\omega_{fi}$ we would observe resonance.

Now we have two cases:
1. if $E_f > E_i$ resonance happens when $\omega \to \omega_{fi} > 0$. This is called **resonant absorption**.
2. if $E_f < E_i$ resonance happens when $\omega \to -\omega_{fi} > 0$. This is called **resonant emission.**

Without loss of generality, let us only consider the case of resonant absorption.

Let us define

$$\left|\frac{1-e^{i(\omega_{fi}+\omega)t}}{\omega_{fi}+\omega}\right| \equiv A_+$$
$$\left|\frac{1-e^{i(\omega_{fi}+\omega)t}}{\omega_{fi}-\omega}\right| \equiv A_-$$

Note that we can the following math equation is true:

$$\frac{1-e^{i\theta t}}{\theta} = -ie^{i\theta t/2}\frac{\sin\left(\frac{\theta t}{2}\right)}{\frac{\theta}{2}}$$

This gives

$$A_\pm = -ie^{i(\omega_{fi}\pm\omega)t/2}\frac{\sin\left[\frac{(\omega_{fi}\pm\omega)t}{2}\right]}{\frac{\omega_{fi}\pm\omega}{2}}$$

note that We only consider the A_- term for the absorption case. Then, for $|\omega-\omega_{fi}| \ll \omega_f$, we get

$$P_{i\to f}(t:\omega) \approx \frac{|V_{fi}|^2}{\hbar^2}\left|\frac{\sin\left[\frac{(\omega_{fi}-\omega)t}{2}\right]}{\frac{\omega_{fi}-\omega}{2}}\right|^2$$

Note that the central maxima width of the diagram if $\frac{4\pi}{t}$, the second peak is less than 5% of the central peak.
And finally, when $\omega \to \omega_{fi}, P \to \frac{|V_{fi}|^2t^2}{\hbar^2}$.

Validity of our result

1. Near ω of absorption resonance, $|A_+(\omega)| \ll |A_-(\omega)|$
    1. $2|\omega_f| \gg \frac{4\pi}{t}$. This means the two peaks are well-separated
    2. This also means $t \gg \frac{1}{|\omega_{ft}|} \approx \frac{1}{\omega}$
2. $P_{i\to f}(t;\omega=\omega_{fi}) = \frac{|V_{fi}|^2}{\hbar^2}t^2$, but our first degree perturbation requires $P_{i\to f} \ll 1$, so $t \ll \frac{\hbar}{|V_{fi}|}$

Putting 1. and 2. together, we have

$$\frac{1}{\omega} \approx \frac{1}{|\omega_{fi}|} \ll t \ll \frac{\hbar}{|V_{fi}|}$$

Therefore, we also have $\hbar|\omega_{fi}| \gg |V_{fi}|$. This means the separation of energy levels must be much larger than our perturbation potential energy, which makes sense!