QM Scattering Feb 17

Recap: Scattering
$\psi_k(\vec{r}) \sim_{r\to\infty} A\left[e^{ikz} + \frac{f(\theta,\phi)e^{ikr}}{r}\right]$
Where the first term is the incident wave, and the second term is the outgoing spherical wave
$\sigma(\theta,\phi) = |f(\theta,\phi)|^2$ where $f(\theta,\phi)$ is the scattering amplitude
This lecture: find $f(\theta,\phi)$ give V

Integral Equation

We have the integral equation formulation of Schrodinger's Equation

$$\psi_k(\vec{r}) = \psi_k^0(\vec{r}) + \int G(\vec{r}-\vec{r}')U(r')\psi_k(r')\,d^3\vec{r}' \tag{1}$$

where ψ_k^0 is the free particle solution

$$(\nabla^2 + k^2)\psi_k^0 = 0 \tag{2}$$

and G satisfy

$$(\nabla^2 + k^2)G(\vec{r}-\vec{r}') = \delta(\vec{r}-\vec{r}') \tag{3}$$

Task #1: $\text{integral equation} \equiv \text{Schrodinger's equation}$
We prove $(\nabla^2 + k^2)$ acting on (1)

$$(\nabla^2 + k^2)\psi_k = (\nabla^2 + k^2)\psi_k^0 + \int (\nabla^2 + k^2)G_k(\vec{r}-\vec{r}')U(r')\psi(r')\,d^3\vec{r'}$$

Using (2) and (3), we have

$$\begin{aligned}(\nabla^2 + k^2)\psi_k &= 0 + \int \delta(\vec{r}-\vec{r}')U(r')\psi_k(r')\,d^3\vec{r'} \\ (\nabla^2 + k^2)\psi_k &= U(\vec{r})\psi_k(\vec{r})\end{aligned}$$

which is just the Schrodinger's equation

Task #2: Find $G_k(\vec{r}-\vec{r}')$

$$(\nabla^2 + k^2)G_k(\vec{r}) = \delta(\vec{r}) \tag{let r'=0}$$

We will check that

$$G_k^{\pm} = -\frac{1}{4\pi}\frac{e^{\pm ikr}}{r} \tag{HW \#5}$$

Note that we used $\nabla^2 G = \nabla \cdot \nabla G$, and work this out in spherical polar coordinates.
Another tip: $\nabla^2(\frac{1}{r}) = -4\pi\delta^{(3)}(r)$
and
$G^+ \to \text{outgoing sph. wave}\checkmark$
$G^- \to \text{imploding sph. wave}\times$
Keep only G^+ in equation (1)

$$\psi_k(\vec{r}) = A\left[e^{ikz} - \frac{1}{4\pi}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}U(r')\psi_k(\vec{r}')\,d^3\vec{r}'\right] \tag{4}$$

Note that our Green's function has B.C. build in. Therefore, this is more convenient than Schrodinger's equation

Note also (4) is valid for all value of r .

Task 3: Let's use (4) to see if we can retrieve large r behavior and find $f(\theta,\phi)$

$$\int d^3\vec{r}' \text{ in equation } (4) \text{ is restricted to } |\vec{r}' \leq R_0|$$

Look at $|\vec{r}'| \leq R_0 \ll |\vec{r}|$
Under these conditions,

$$\begin{aligned}
|\vec{r}-\vec{r}'| &= (r^2 + r'^2 - 2rr'\cos\alpha)^{1/2} \\
&= r\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\alpha\right]^{1/2} \\
&\approx r\left[1 - \frac{r'}{r}\cos\alpha + O\left(\frac{r'}{r}\right)^2\right] \\
&\approx r - r'\cos\alpha + \ldots \\
&\approx r - r'\cdot\hat{r}
\end{aligned}$$

Plug into (4)

$$\psi_k(\vec{r}) \approx_{r\to\infty} A\left[e^{ikz} - \frac{1}{4\pi}\frac{e^{ik\vec{r}}}{r}\int e^{-ik\vec{r}'\cdot\hat{r}}U(r')\psi_k(\vec{r}')\,d^3\vec{r}'\right] \tag{5}$$

Note that we didn't make the approximation that $r - r' \approx r$ on the phase term because the maximum phase is 2π . Therefore, even though $r' \ll r$, we can not ignore it in the phase term
Note that the integral term in (5) is simply $f(\theta,\phi)$

$$\implies f(\theta,\phi) = -\frac{1}{4\pi}\int e^{-ik\vec{r}'\cdot\hat{r}}U(r')\psi_k(\vec{r}')\,d^3\vec{r}' \tag{6}$$

Note that $\psi_k(\vec{r}')$ is still in (6) . Therefore, we are not done.

Solving the Integral Equation

$$\psi = \psi^0 + GU\psi$$

Suppose $U\psi \ll K\psi^0$ where K is the kinetic energy of the incident wave. And use the recursive approximation.

$$\begin{aligned} \psi &= \psi + GU(\psi^0 + GU\psi) \\ &= \psi^0 + GU\psi^0 + (GUGU\psi^0 + GUGUGU\psi^0 \ldots) \end{aligned} \tag{7}$$

Where if we use the first two terms on (7) , the method is called "Born Approximation"

Born Approximation

$$f(\theta,\phi) = -\frac{1}{4\pi}\int e^{ik\vec{r}'\cdot r}U(r')e^{ik\hat{z}\cdot\vec{r}'}\,d^3\vec{r}' \tag{8}$$

where the third term in (8) is $\psi_k^0(\vec{r}')$ with $\vec{k} = k\hat{z}$
Now we finally can compute $\sigma_{\text{Born}}(\theta,\phi)$.