Physics 7502 HW#1

Problem 1. Qubits and Spin-1/2

Answer.

a) For any rank 2 complex matrix $\mathcal{H} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ where $\{a_{11}, a_{12}, a_{21}, a_{22}\} \in \mathbb{C}$

if $\mathcal{H}$ is Hermetian, we must have that $\mathcal{H}^\dagger = \mathcal{H}$. This constraint tells us that

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11}^* & a_{21}^* \\ a_{12}^* & a_{22}^* \end{pmatrix}$$

Therefore, we must have $a_{11} = a_{11}^*, a_{22} = a_{22}^*$. This implies that $\{a_{11}, a_{22}\} \in \mathbb{R}$. If we let $E_0 = \frac{a_{11} + a_{22}}{2}$, $\epsilon = \frac{a_{11} - a_{22}}{2}$, it is trivial that $\{E_0, \epsilon\} \in \mathbb{R}$, and $E_0 + \epsilon = a_{11}$, $E_0 - \epsilon = a_{22}$.

We also have $a_{12} = a_{21}^*$. Therefore, if we let $a_{12} = \Delta$, we must have $a_{21} = a_{12}^* = \Delta^*$. Now we can always write $\Delta = \alpha + \beta i$ whre $\alpha, \beta \in \mathbb{R}$. We simply write $\Delta = |\Delta| \exp(-i\phi)$ where $\Delta = \sqrt{\alpha^2 + \beta^2}$ and $\phi = -\arctan\left(\frac{\beta}{\alpha}\right)$

b) Condiser the expression

$$d_0\, \mathbb{I} + \vec{d} \cdot \vec{\sigma} = d_9\, \mathbb{I} = d_x \sigma_x + d_y \sigma_y + d_z \sigma_z = \begin{pmatrix} d_0 + d_z & d_x - i\, d_y \\ d_x + i\, d_y & d_0 - d_z \end{pmatrix}$$

Observe that if we let $d_0 = E_0, d_z = \epsilon, d_x = \alpha, d_y = \beta, \Delta = \alpha - \beta i$, we have the expression

$$\begin{pmatrix} E_0 + \epsilon & \alpha + \beta i \\ \alpha - \beta i & E_0 - \epsilon \end{pmatrix} = \begin{pmatrix} E_0 + \epsilon & \Delta \\ \Delta^* & E_0 - \epsilon \end{pmatrix}$$

which is nothing more but a re-write of our origonal matrix $\mathcal{H}$.

For a spin $1/2$ particle in a magnetic field, we can write its Hailtonian as

$$H = -\mu \cdot B = -g \frac{q}{2m} \vec{\sigma} \cdot$$

We realize that $\vec{d} = -\frac{gq\vec{B}}{2m}$ make our Hamiltonian H in the form of $\vec{d} \cdot \vec{\sigma}$

c) Figure 1 is a picture demonstation of the vector and what ϕ and θ means.

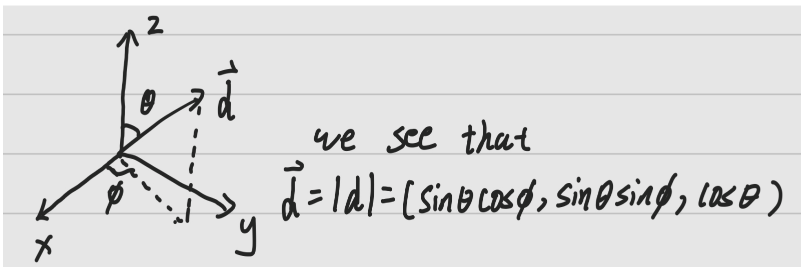

Figure 1.

Now, we use the fact that $|\mathcal{H}-\lambda\mathbb{I}|=0$ to evaluate the eigenvalue λ. This gives

$$\begin{vmatrix} d_0+d_z-\lambda & d_x-i\,d_y \\ d_x+i\,d_y & d_0-d_z-\lambda \end{vmatrix}=0$$

$$\Rightarrow \lambda^2-2d_0\lambda-(d_x^2+d_y^2+d_z^2)+d_0^2=0$$

$$\Rightarrow \lambda^2-2d_0\lambda-|d|^2+d_0^2=0$$

$$\Leftrightarrow \lambda=d_0\pm\sqrt{d_0^2-(d_0^2+|d|^2)}=d_0\pm|d|$$

Now suppose $\lambda_1=d_0+|d|$, λ_2=$d_0-|d|$, $\lambda_1$corresponds to eigenvector $v_1=\left(\begin{smallmatrix} a_1 \\ b_1 \end{smallmatrix}\right)$, λ_2 corresponds to eigenvector $v_2=\left(\begin{smallmatrix} a_2 \\ b_2 \end{smallmatrix}\right)$.

For v_1, we plug in $\lambda=\lambda_1$ and solve for

$$\begin{pmatrix} |d|\cos(\theta)-|d| & |d|\sin(\theta)\mathrm{e}^{-i\phi} \\ |d|\sin(\theta)e^{i\phi} & -|d|\cos(\theta)-|d| \end{pmatrix}\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow a_1(|d|\cos(\theta)-|d|)+b_1(|d|\sin(\theta)\mathrm{e}^{-i\phi})=0$$

let $x=\theta/2$, use the relation $\cos(2x)=\cos^2(x)-\sin^2(x), \sin(2x)=2\sin(x)\cos(x), 1=\sin^2(x)+\cos^2(x)$

$$\Rightarrow a_1\sin^2(x)=b_1\sin(x)\cos(x)\mathrm{e}^{-i\phi}\Rightarrow a_1\sin(x)\mathrm{e}^{i\phi}=b_1\cos(x)$$

Because x,ϕ are arbiratary constant, for equality to be ture, LHS of the equality must have the same expression as the RHS.

$$\Leftrightarrow a_1=\cos(x)=\cos(\theta/2), b_1=\sin(x)\mathrm{e}^{i\phi}=\sin(\theta/2)\mathrm{e}^{i\phi}$$

$$\Rightarrow v_1=\begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2)\mathrm{e}^{i\phi} \end{pmatrix}$$

Similairly for v_2, $\lambda=\lambda_2$, we solve for

$$a_2(|d|\cos(\theta)+|d|)+b_2(|d|\sin(\theta)\mathrm{e}^{-i\phi})=0$$

$$\mathrm{e}^{i\phi}a_2(\cos(2x)+1)=-b_2\sin(2x)$$

$$e^{i\phi}a_2\cos^2(x)=-b_2\sin(x)\cos(x)$$

$$a_2e^{i\phi}\cos(x)=-b_2\sin(x)$$

Using the same equality argument, this gives

$$a_2=-\sin(x)=-\sin(\theta/2), b_2=\cos(x)e^{i\phi}=\cos(\theta/2)e^{i\phi}$$

$$\Rightarrow v_2=\begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)\mathrm{e}^{i\phi} \end{pmatrix}$$

Now let's verify the orthogonality of the two eigenvectors by evaluating their inner product.

$$< v_1 | v_2 > = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \end{pmatrix} \begin{pmatrix} -\sin(\theta/2) \\ \cos(\theta/2)e^{i\phi} \end{pmatrix}$$

$$= -\sin(\theta/2)\cos(\theta/2) + \sin(\theta/2)\cos(\theta/2) = 0$$

Because the inner product is always zero, the two eigenvectors are orthogonal.

d)We have $E_{\pm} = d_0 \pm |d|$. Previously in part a, we have said that

$$d_0 = E_0, d_z = \epsilon, d_x = \alpha, d_y = \beta, \Delta = d_x - i d_y$$

Therefore, changing the variable to E_0, ϵ, Δ, we have

$$E_{\pm} = E_0 \pm \sqrt{\epsilon^2 + \Delta\Delta^*}$$

With the graph

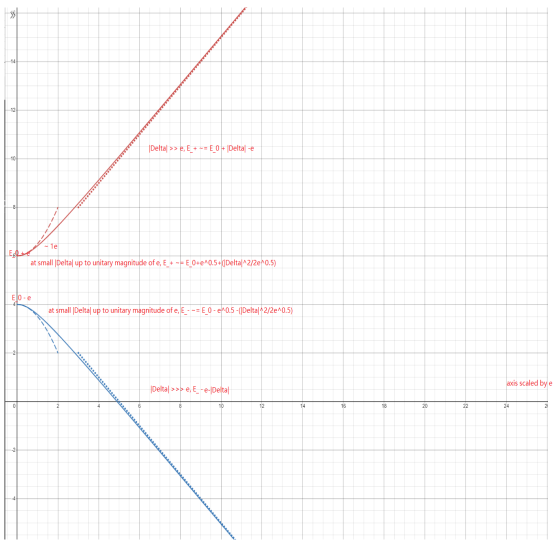

Figure 2.

e)

$$S_n \equiv \hat{n} \cdot \vec{\sigma} = \sin\theta\cos\phi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\theta\sin\phi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{in } \sigma_z \text{ basis}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} \tag{1}$$

Note that $S_n^\dagger = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix} = S_n$. Thus the operator is Hermitian.

We notice that from previous matrix,

$$\begin{pmatrix} d_0 + |d|\cos(\theta) & |d|\sin(\theta)e^{-i\phi} \\ |d|\sin(\theta)e^{i\phi} & d_0 - |d|\cos(\theta) \end{pmatrix} \tag{2}$$

I claim our new matrix is nothing but a special case of (2)where $d_0 = 0, |d| = 1$.

The euquality for the diagonal term is trivial, now lets consider $\sin(\theta)\mathrm{e}^{-i\phi}$, but using Euler's Formula immediately gives

$$\sin(\theta)\mathrm{e}^{-i\phi} = \sin(\theta)(\cos\phi - i\sin\phi) = \sin\theta\cos\phi - i\sin\theta\sin\phi$$

and

$$\sin(\theta)\mathrm{e}^{i\phi} = \sin(\theta)(\cos\phi + i\sin\phi) = \sin\theta\cos\phi + i\sin\theta\sin\phi$$

Thus we proved our claim. Therefore, for (1)

$$\lambda_1 = d_0 + |d| = 1, \lambda_2 = d_0 - |d| = -1$$

And the eigenvector stays the same as v_1, v_2 as they are independent of $|d|, d_0$

Now test what happenes if we set $\hat{n}$ to $\{\hat{x},\hat{y},\hat{z}\}$

for $\hat{x}, \theta = \pi/2, \phi = 0$

$\rightarrow S_{\hat{x}} \equiv \hat{x}\cdot\vec{\sigma} = \sin\frac{\pi}{2}\cos 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\frac{\pi}{2}\sin 0\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\frac{\pi}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \checkmark$

for $\hat{y}, \theta = \pi/2, \phi = \frac{\pi}{2}$

$\rightarrow S_{\hat{y}} \equiv \hat{y}\cdot\vec{\sigma} = \sin\frac{\pi}{2}\cos\frac{\pi}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin\frac{\pi}{2}\sin\frac{\pi}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos\frac{\pi}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y \checkmark$

for $\hat{z}, \theta = 0$

$\rightarrow S_{\hat{z}} \equiv \hat{z}\cdot\vec{\sigma} = \sin 0\cos\phi\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \sin 0\sin\phi\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \cos 0\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \checkmark$

$\square$

Problem 2. Quantum Dynamics and Measurement

Answer.

a) We want to derive the *time evolution* opretor $U(t)$ such that

$$|\psi(t)\rangle = U(t)\,|\psi(0)\rangle \tag{3}$$

We construct our infinitesimal time-evolution operator

$$U(t_0 + \mathrm{dt}, t_0) = 1 - \frac{iH\,\mathrm{dt}}{\hbar} \tag{4}$$

Because out time operator must satisfy $U(t_2,t_1)U(t_1,t_0)|\alpha\rangle = U(t_2,t_0)|\alpha\rangle$, we can exploit this and write

$$U(t+\mathrm{dt},t_0) = U(t+\mathrm{dt},t)U(t,t_0) = \left(1 - \frac{\mathrm{iH\,dt}}{\hbar}\right)U(t,t_0) \tag{5}$$

This leads to

$$U(t+\mathrm{dt},t_0) - U(t,t_0) = -i\frac{H}{\hbar}\mathrm{dt}\,U(t,t_0)$$

Rewriting this gives

$$i\hbar\frac{\partial}{\partial t}U(t,t_0) = HU(t,t_0) \tag{6}$$

Which is a diffeerential equation for the *time evolutiopn operator* for an arbitrary H

To solve for this differential equation, we simply re-write (6)

$$\frac{\partial U(t,t_0)}{\partial t} = -i\frac{H}{\hbar}U(t,t_0) \tag{7}$$

and note that

$$U(t,t_0) = \mathrm{e}^{-i\frac{H(t-t_0)}{\hbar}} \tag{8}$$

solves the differential equation for (6) if H is indepedent of time, because we explicitly make $t_0 = 0$ as our initial condition, the result is

$$U(t,0) = U(t) = \mathrm{e}^{-i\frac{Ht}{\hbar}}$$

To show that it is unitary, we simply check it in calculation

$$U^\dagger(t)U(t) = \mathrm{e}^{i\frac{Ht}{\hbar}}\mathrm{e}^{-i\frac{Ht}{\hbar}} = \mathbb{I} \tag{9}$$

b) In problem 1, we have

$$\mathcal{H} = \begin{pmatrix} E_0+\epsilon & \Delta \\ \Delta^* & E_0-\epsilon \end{pmatrix} \tag{10}$$

We express $|z+\rangle$in the eigenkets of $\mathcal{H}$. Note that

$$|z+\rangle = \cos\left(\frac{\theta}{2}\right)|\hat{n}+\rangle + \sin\left(\frac{\theta}{2}\right)|\hat{n}-\rangle \tag{11}$$

We then directly apply the time operator $U(t)$

$$\begin{aligned}
|\psi(t)\rangle &= U(t)|\psi(0)\rangle \\
&= \mathrm{e}^{-i\frac{Ht}{\hbar}}(\cos\left(\frac{\theta}{2}\right)|\hat{n}+\rangle + \sin\left(\frac{\theta}{2}\right)|\hat{n}-\rangle) \\
&= \mathrm{e}^{-i\frac{E_+t}{\hbar}}\cos\left(\frac{\theta}{2}\right)|\hat{n}+\rangle + \mathrm{e}^{-i\frac{E_-t}{\hbar}}\sin\left(\frac{\theta}{2}\right)|\hat{n}-\rangle
\end{aligned}$$

Where from problem 1, we calcualted that $E_\pm = E_0 \pm \sqrt{\epsilon^2 + \Delta\Delta^*}$

c)

$$\begin{aligned}
P(z-) &= |<z-|(\mathrm{e}^{-i\frac{E_+t}{\hbar}}\cos(\theta/2)|\hat{n}+\rangle + \mathrm{e}^{-i\frac{E_-t}{\hbar}}\sin(\theta/2)|\hat{n}-\rangle)|^2 \\
&\doteq \left|\mathrm{e}^{-i\frac{E_+t}{\hbar}}\begin{pmatrix} 0 & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\mathrm{e}^{i\phi/2} \end{pmatrix} + \mathrm{e}^{-i\frac{E_-t}{\hbar}}\begin{pmatrix} 0 & \sin\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} \\ -\cos\frac{\theta}{2}\mathrm{e}^{i\phi/2} \end{pmatrix}\right|^2 \\
&= |\mathrm{e}^{-i\left(\frac{E_+t}{\hbar}-\frac{\phi}{2}\right)}\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \mathrm{e}^{-i\left(\frac{E_-t}{\hbar}-\frac{\phi}{2}\right)}\sin\frac{\theta}{2}\cos\frac{\theta}{2}|^2 \\
&= \sin^2(\theta)\sin^2\left(\frac{(E_- - E_+)}{2\hbar}t\right) \\
&= \sin^2(\theta)\sin^2\left(\frac{\sqrt{\epsilon^2+\Delta\Delta^*}}{\hbar}t\right)
\end{aligned}$$

Where ϵ, Δ should be some known valuees.

Note that our value does not depend on E_0, our gauge energy at all. The oscillation depends only on the energy difference between the two eigenstates, which can be expressed as $2\sqrt{\epsilon^2+\Delta\Delta^*}$
By simply observing the equation, we can tell that

$$P_{\max}(t) = \sin^2(\theta)$$
$$T = \frac{\pi}{\frac{\sqrt{\epsilon^2+\Delta\Delta^*}}{\hbar}}$$
$$\omega = 2\frac{\sqrt{\epsilon^2+\Delta\Delta^*}}{\hbar}$$

Below I will graph when $\theta = \frac{\pi}{3}, \frac{\sqrt{\epsilon^2+\Delta\Delta^*}}{\hbar} = 1$. Obviously then $P_{\max} = \sin^2(\frac{\pi}{3}) = 0.75, T = \pi$

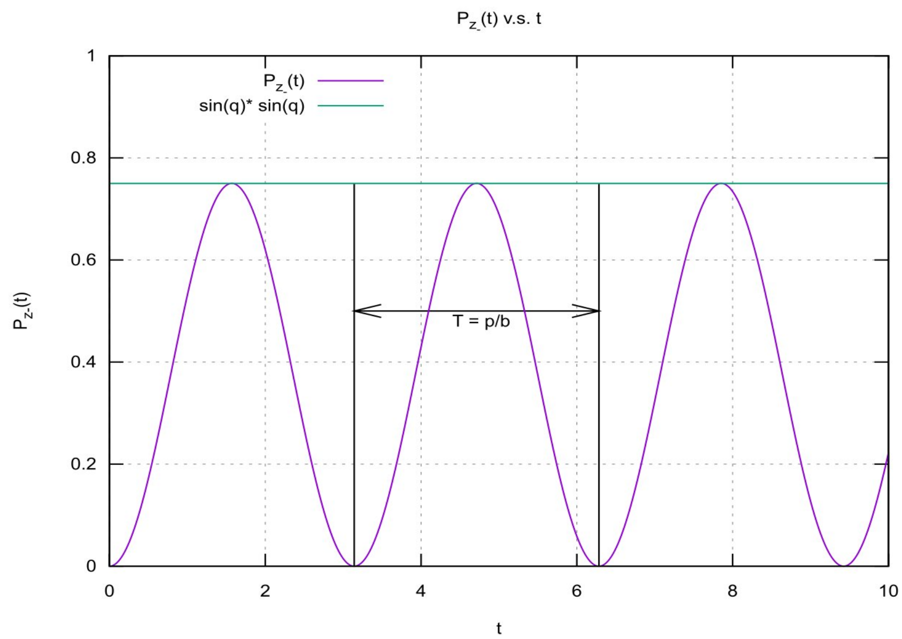

Figure 3.

□

Problem 3. Spin-half Particle in a Magnetic Field

Answer.

a) Suppose we can determine all three components of the vector **M** at the same time. This implies for any given state, if we do three measurements:

1. measure the **z** component of **M**

2. measure the **x** component of **M**

3. measure the **z** component of **M**

Measurement 3 and 1 should yield the same result.

1. Let $|\psi> = |z+> \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $M_z = \frac{\hbar}{2}\gamma S_z|\psi> \doteq \frac{\gamma\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\gamma\hbar}{2}|z+>$. Here we mearue the eigenvalue to be $\frac{\gamma\hbar}{2}$

2. Because the State is still $|z+>$, $M_x = \frac{\hbar}{2}\gamma S_x|z+> \doteq \frac{\gamma\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\gamma\hbar}{2}|z->$

3. Now we measure the z component again, $M_z = \frac{\hbar}{2}\gamma S_z|z-> = \frac{\hbar\gamma}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = -\frac{\hbar\gamma}{2}|z->$

Now, obviously our thrid measure differ from our first measurement. This means at least for state $|z+>$, the measurement for M_x and M_z is not compatible.

b) We have $\mathcal{H} = -\boldsymbol{M}\cdot\boldsymbol{B} = \frac{\gamma\hbar}{2}\begin{pmatrix} \sigma_x & \sigma_y & \sigma_z \end{pmatrix}\cdot\begin{pmatrix} 0 & 0 & B \end{pmatrix} = \frac{\gamma\hbar}{2}\sigma_z B \doteq \frac{-B\gamma\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

and $|\psi(0)> = |n+> = \cos\left(\frac{\theta}{2}\right)|z+> + \sin\left(\frac{\theta}{2}\right)e^{i\phi/2}|z->$, note that $|z+>, |z->$ are just the eigenvectors of our $\mathcal{H}$. We then have

$$
\begin{aligned}
|\psi(t)> &= U(t,0)|\psi(0)> \\
&= \cos\left(\frac{\theta}{2}\right)e^{-i\frac{\mathcal{H}t}{\hbar}}|z+> + \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{\mathcal{H}t}{\hbar}-\phi\right)}|z-> \\
&= \cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}}|z+> + \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)}|z->
\end{aligned}
$$

c) Because we measure S_z, we are going to measure the two eigenvalues of $S_z = \pm\frac{\hbar}{2}$ that corresponds to eigenstates $|z\pm>$

$$
P_{+\frac{\hbar}{2}}(t) < z+|\left(\cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}}|z+> + \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)}|z->\right)
$$

$$
\begin{aligned}
P_{+\frac{\hbar}{2}}(t) &= |<z+|\left(\cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}}|z+> + \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)}|z->\right)|^2 \\
&= \left|\cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}}\right|^2 \\
&= \cos^2\left(\frac{\theta}{2}\right) \\
P_{-\frac{\hbar}{2}}(t) &= |<z-|\left(\cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}}|z+> + \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)}|z->\right)|^2 \\
&= \left|\sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)}\right|^2 \\
&= \sin^2\left(\frac{\theta}{2}\right)
\end{aligned}
$$

d)

$$
\begin{aligned}
<\psi(t)|S_x|\psi(t)> &= \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\frac{B\gamma t}{2}} & \sin\left(\frac{\theta}{2}\right)e^{i\left(\frac{B\gamma t}{2}-\phi\right)} \end{pmatrix}\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}} \\ \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)} \end{pmatrix} \\
&= \frac{\hbar}{2}\begin{pmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\frac{B\gamma t}{2}} & \sin\left(\frac{\theta}{2}\right)e^{i\left(\frac{B\gamma t}{2}-\phi\right)} \end{pmatrix}\begin{pmatrix} \sin\left(\frac{\theta}{2}\right)e^{-i\left(\frac{B\gamma t}{2}-\phi\right)} \\ \cos\left(\frac{\theta}{2}\right)e^{i\frac{B\gamma t}{2}} \end{pmatrix} \\
&= \frac{\hbar}{2}\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i(B\gamma t+\phi)} + \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{i(B\gamma t-\phi)}\right) \\
&= \frac{\hbar}{2}\left(\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\right)(2\cos(B\gamma t-\phi)) \\
&= \frac{\hbar}{2}\sin(\theta)\cos(B\gamma t-\phi)
\end{aligned}
$$

$$
\begin{aligned}
<\psi(t)|S_z|\psi(t)> &= \left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{B\gamma t}{2}} \quad \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\left(\frac{B\gamma t}{2}-\phi\right)} \right)\frac{\hbar}{2}\left(\begin{array}{cc} 1 & 0\\ 0 & -1 \end{array}\right)\left(\begin{array}{c} \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{B\gamma t}{2}}\\ \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\left(\frac{B\gamma t}{2}-\phi\right)} \end{array}\right)\\
&= \frac{\hbar}{2}\left(\cos\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\frac{B\gamma t}{2}} \quad \sin\left(\frac{\theta}{2}\right)\mathrm{e}^{i\left(\frac{B\gamma t}{2}-\phi\right)} \right)\left(\begin{array}{c} \cos\left(\frac{\theta}{2}\right)\mathrm{e}^{i\frac{B\gamma t}{2}}\\ -\sin\left(\frac{\theta}{2}\right)\mathrm{e}^{-i\left(\frac{B\gamma t}{2}-\phi\right)} \end{array}\right)\\
&= \frac{\hbar}{2}\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)\\
&= \frac{\hbar}{2}\cos(\theta)
\end{aligned}
$$

e) We have $\frac{dL}{dt} = T = M \times B = \gamma L \times B$

$$
\begin{aligned}
\frac{dL}{dt} &= \gamma L \times B\\
\frac{dL}{dt} &= \gamma\left(L_y B_z \quad -L_x B_z \quad 0 \right)\\
\left(\frac{dL_x}{dt} \quad \frac{dL_y}{dt} \quad \frac{dL_z}{dt} \right) &= \gamma\left(L_y B_z \quad -L_x B_z \quad 0 \right)
\end{aligned}
$$

This gives the couple first order ODE for L_x, L_y and a trivial ODE for L_z

$$
\begin{aligned}
\frac{dL_z}{dt} &= 0\\
\Rightarrow L_z &= C
\end{aligned}
$$

And the coupled equation

$$
\begin{aligned}
\frac{dL_x}{dt} &= \gamma B_z L_y\\
\frac{dL_y}{dt} &= -\gamma B_z L_x\\
\Rightarrow \frac{d^2 L_x}{\mathrm{dt}^2} &= \gamma B_z \frac{dL_y}{dt}\\
&= -\gamma^2 B_z^2 L_x\\
\Rightarrow L_x &= A\cos(\gamma B_z t + \phi)\\
\frac{dL_y}{dt} &= -\gamma B_z L_x\\
&= -\gamma B_z \mathrm{A}\cos(\gamma B_z t + \phi)\\
\Rightarrow L_y &= -A\sin(\gamma B_z t + \phi)
\end{aligned}
$$

Now let's consider $\hat{L} = \left(A\cos(\gamma B_z t + \phi) \quad -A\sin(\gamma B_z t + \phi) \quad C \right)$

Note that if we let $A = \sin(\theta), C = \cos(\theta)$ fulfills that $|\hat{L}| = 1$, we then have

$\hat{L} = \left(\sin(\theta)\cos(\gamma B_z t + \phi) \quad -\sin(\theta)\sin(\gamma B_z t + \phi) \quad \cos(\theta) \right)$ Compared to $<S>$ in Quantum

$<\hat{\boldsymbol{S}}> = \frac{\hbar}{2}\left(\sin(\theta)\cos(B\gamma t - \phi) \quad \sin(\theta)\sin(B\gamma t - \phi) \quad \cos(\theta) \right)$

Note that not only the frequency $B\gamma$ is the same, the functional form of $<S>$ and L is the same with different constants.

□