Recap Zero Point Energy:

See Sakurai for a (more math intensive) derivation.

For a quantized EM field, we have the operator

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},\alpha}(\ldots)[a^\dagger_{\vec{k},\alpha}e^{i(\ldots)} + a_{\vec{k},\alpha}e^{i(\ldots)}]\hat{e}_{\vec{k},\alpha}$$

where the operators $a^\dagger, a$ has the algebra

$$\begin{aligned}[a_{\vec{k}\alpha}, a^\dagger_{\vec{k}'',\alpha'}] &= \delta_{kk'}\delta\alpha\alpha' \\ [a_{\vec{k},\alpha}, a_{\vec{k}'\alpha'}] &= [a^\dagger_{\vec{k},\alpha}, a^\dagger_{\vec{k}',\alpha'}] = 0\end{aligned}$$

where $\vec{k}_\alpha$ is the wave vector, and our gauge choice $\vec{\nabla}\cdot A = 0$ implies $\hat{e}_{\vec{k},\alpha}\cdot\vec{k}_\alpha = 0$
Note that this just means that we have a transverse wave, where $\vec{k}\perp\vec{E}\perp\vec{B}$.

We can put down the Hamiltonian purely due to this EM field as:

$$\mathcal{H} = \sum_{\vec{k},\alpha}\hbar\omega_k\left[a^\dagger_{k\alpha}a_{k\alpha} + \frac{1}{2}\right]$$

Note that even in the case of the ground state where $a^\dagger a = 0$. We define this state to be the vacuum state where

$$|\text{vacuum}\rangle = |n_{\vec{k},\alpha} = 0\rangle \quad \forall(\vec{k},\alpha)$$

and the vacuum state is an eigenstate of the Hamiltonian.

$$\mathcal{H}|\text{vacuum}\rangle = E_0|\text{vacuum}\rangle$$

and we can calculate the vacuum state as

$$\begin{aligned}E_0 &= \sum_{\vec{k},\alpha}\frac{1}{2}\hbar\omega_k \\ &= \hbar\frac{c}{2}\sum_\alpha\sum_{\vec{k}}\sqrt{k_x^2+k_y^2+k_z^2} \\ &\sim \int_0^{k_f}\vec{k}\,d^3k \sim k^4 \to \infty\end{aligned}$$

An important consequence is that the ground state energy diverges. For most purposes, this doesn't matter as we only care about the energy difference. However, in many cases (general relativity as the most famous case), the ground state energy does affect the observable quantities. This remains to be unsolved to this day.

Casimir Effect of EM Field

Suppose we have two metallic plates separated L apart with area L^2. Now, what happens if we disturb the vacuum if we add another plate distance r away from one of the plate in the L^3 box created?

We can calculate the energy difference between the two by the equation

$$\Delta E = \{E_0(r) - E_0(L-r)\} - E_0(L) \tag{*}$$

What Casimir discovered is that there is a Casimir force per unit area between the plates

$$f = -\frac{1}{L_yL_z}\frac{\partial(\Delta E_0)}{\partial r} \equiv \text{ Casimir fource}$$

In the $3D$ example, we have

$$E_0(r) = 2\sum_{n=1}^\infty\int\int\frac{1}{2}\hbar c\sqrt{\left(\frac{n\pi}{r}\right)^2 + k_y^2 + k_z^2}\,\frac{dk_y}{2\pi}\,\frac{dk_z}{2\pi}$$

The factor of 2 comes from 2 polarization.
We can do this summation using Euler- Maclaurin formula. (See Sakurai).

For Physics intuition purpose, let us only consider the $1D$ case with 1 polarization. For this case, we have;

$$\sum_{n=1}^{\infty} \frac{1}{2} \hbar c \left(\frac{n\pi}{r} \right) = \frac{\pi \hbar c}{2r} \sum_{n=1}^{\infty} n \tag{1}$$

Note that equation (1) is still divergent (without analytical continuation), and it is refereed as **ultraviolet divergent**. It is a very famous sum that will sum to $-\frac{1}{12}$.

We can regulate this UV divergence by saying $k < \frac{1}{a}$ contribute (UV cutoff) and take the $a \to 0$ limit in the end. To this end, we transform (1) as:

$$\sum_{n=1}^{\infty} n \to \sum_{n=1}^{\infty} n \exp\left(-\frac{n\pi a}{r}\right) \tag{2}$$

Note as as long as $a \neq 0$, (2) converges. For now, we don't care about the specific cut-off value for a.

$$\begin{aligned} E_0(r) &= \frac{\pi \hbar c}{2r} \sum n e^{-n\pi a/r} \\ &= \frac{\hbar c}{2} \left(-\frac{\partial}{\partial a}\right) \sum_{n=1}^{\infty} e^{-n\pi a/r} \\ &= -\frac{\hbar c}{2} \frac{\partial}{\partial a} \left[\frac{1}{1 - e^{-\pi a/r}} \right] \end{aligned}$$

Note that because eventually we will take $a \to 0$, it is fine to assume $a \ll r$ so we can expand $\frac{1}{1-e^{x}}$ for $x \ll 1$. Please check if this is true, then we have:

$$E_0(r) = \frac{\hbar c \pi}{2r} \left[\frac{1}{\pi^2} \left(\frac{r}{a}\right)^2 - \frac{1}{12} + O\left(\frac{a}{r}\right)^2 \right] \tag{3}$$

Note that as $a \to 0$, equation (3) still diverges. Note that the $-\frac{1}{12}$ term comes from $\sum_{n=1}^{\infty} n$. Even though this is true, we have to realize that $E_0(r)$ is not a measurable thing. Instead, we want to compute $(*)$ which is:

$$\Delta E_0 = E_0(r) + E_0(L-r) - E_0(L)$$

and we also want to calculate the Sasimir force in $1D$:

$$F = -\frac{\partial(\Delta E)}{\partial r} = -\frac{\partial E_0(r)}{\partial r} - \frac{\partial E(L-r)}{\partial r}$$

We then use equation (3) (and substitute $l - r$ for r) to get:

$$F = -\frac{\hbar c}{2} \left[\frac{1}{a^2} + \frac{\pi}{12 r^2} + \dots \right] - \frac{\hbar c}{2} \left[-\frac{1}{a^2} + \frac{\pi}{12(L-r)^2} + \dots \right]$$

Note that the diverging term cancels! after taking the limit $a \to 0$, we get

$$F_{1D} = -\frac{\pi \hbar c}{24 r^2}$$

The negative sign indicates the force is attractive. For $3-D$ case derived in Sakurai, we have

$$\frac{F_{3D}}{L_y L_z} = \frac{-\pi^2}{240} \frac{\hbar c}{r^4}$$

Now, we notice that the attractive force $\propto \frac{1}{r^2}$ in $1D$ case, and $\propto \frac{1}{r^4}$ in $3D$ case.

We did **not** derive the vacuum energy. Instead, we found a way to disturb the vacuum and measure that perturbation.