QM Scattering Feb 19

From last time, we say the wave function after scattering could be written as

$$\Psi = \Psi^0 + GU\Psi \tag{1}$$

which is an integral equation, and we chose:
$G^+ = \frac{e^{i\vec{k}\cdot r}}{r} \equiv \Psi_{scatt}$, and $\Psi_0 = e^{i\vec{k}\cdot\vec{z}} \equiv J_{inc}$
Now, using Born's approximation, we have

$$\Psi = \Psi^0 + GU\Psi^0 + GUGU\Psi^0 + \dots \tag{2}$$

Solving for the exact equation for $f(\theta, \phi)$ gives

$$f(\theta, \phi) = \frac{m}{2\pi\hbar^2} \int e^{-i\vec{k}\cdot\vec{r}'} V(r') \Psi_k(r')\, d^3r' \tag{3}$$

Combining (2) and (3) gives

$$f_{Born}(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int e^{-i(\vec{k}_f - \vec{k}_i)\cdot\vec{r}'} V(r')\, d^3r' \tag{4}$$

Note of equation (4)

1. The Born's approximation only use the first term of (2) , which indicates $\Psi \approx \Psi^0$.
2. Note: The Born's approximation is valid when the potential term $V(r')$ is small compared to the initial kinetic energy $\frac{\hbar^2 k_i^2}{2m}$.
3. $f_{Born} \sim \mathscr{F}(V(r'))$
4. f_{born} depends on energy $E = \hbar\frac{k^2}{2m}$, which relate with $\vec{q}, |\vec{q}|$
5. f_{born} depends on θ in the form of $e^{-i\vec{q}\cdot\vec{r}'}$
6. f_{born} has do dependency on ϕ because our problem have cylindrical symmetry around $\hat{z}$

Note for wave vector $\vec{k}$
we have our initial wave vector $\vec{k}_i$ and final wave vector $\vec{k}_f$, and the angle between them θ, and we also have $|\vec{k}_i| = |\vec{k}_f|$ because the collision is elastic,
we then have

$$\vec{q} \equiv \vec{k}_f - \vec{k}_i = 2k \sin\frac{\theta}{2}$$

Then we can re-write (4) as

$$f_{Born}(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int e^{-i\vec{q}\cdot\vec{r}'} V(r')\, d^3r' \tag{5}$$

We have Spherical Symmetry,so equation (5) can be simplified

$$f_{born}(\theta) = \int e^{-i\vec{q}\cdot\vec{r}'} V(r')\, d^3r' = \int_0^\infty \int_0^\pi \int_0^{2\pi} r'^2 \sin\theta e^{iqr'\cos\theta}\, d\phi\, d\theta\, dr' \tag{6}$$

By just doing integration, we have

$$f_{born}(\theta) = \frac{2m}{\hbar^2} \int_0^\infty r' \frac{\sin(qr')}{q} V(r')\, dr'$$

Example 1: $V(r) = g\frac{e^{-\mu r}}{r}$

$$f_{born}(\theta) = -\frac{2mg}{\hbar^2 q} \int_0^\infty e^{-\mu r'} \frac{e^{iqr} - e^{-iqr}}{2i}\, dr' = -\frac{2mg}{\hbar^2(q^2+\mu^2)}$$

$$\sigma_{born}(\theta) = \frac{4m^2g^2}{\hbar^4\left[4k^2\sin^2\frac{\theta}{2} + \mu^2\right]^2}$$

Note, when energy is large $\implies$ $k =$ large, $\sigma_{born}(\theta) \to 0$

Be cautious that we can not take $\mu \to 0$, because if we do so, $V(r) \to \frac{1}{r}$ which is no longer "local", but let's do it anyway. This will yield us

$$\sigma_{born}(\theta) \text{ "="} \frac{g^2}{16E^2\sin^4\frac{\theta}{2}}$$

Now we want to know, when is this approximation valid?

1. We already discussed at high kinetic energy compared with $V(r)$
2. $|\Psi_{scatt}(r)| \ll |\Psi_{inc}(r)|$, where $\Psi_{inc}(r) = e^{ik\cdot\vec{z}}$ $|\Psi_{scatt}| = \frac{m}{\hbar^2 k}|\int_0^\infty V(r')\sin(kr')e^{ikr'}\, dr'|$
   where $\sin(kr')e^{ikr'} = \frac{e^{2ikr'}-1}{2i} \approx -\frac{1}{2i} \implies \frac{m}{\hbar^2 k}|\int_0^\infty V(r)\, dr'| \ll 1 \implies \frac{m|V_0|r_0}{\hbar k} \ll 1$. This gives $E = \hbar^2\frac{k^2}{2m} \gg \frac{|V_0|^2}{\frac{\hbar^2}{mr_0^2}}$

Not gonna lie I don't quite get the physics intuition here beyond the fact that we must have $E \gg V_0$. I suppose we just require $r_0^2|V_0|^2$ to be small now?