QM Berry PhaseMarch 6

The adiabatic approximation

The berry phase

When we are studying adiabatic processes, we are talking about processes that are "slow" where our extrinsic time scale T_e is much greater than the intrinsic time scale of the system T_i

For instance, suppose we have a physical pendulum with length L . In this case, the intrinsic time would be $T_i = 2\pi\sqrt{\frac{L}{g}}$, and as long as the time it takes for us to, let's say change the length of the pendulum $L(t) \to \frac{1}{2}L(t)$ in a large time T_e . We then require $T_i \ll T_e$.

The trajectory of the physical pendulum in phase space is a ellipse as we have

$$\begin{aligned}
H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = E \\
\implies \frac{p^2}{2mE} + \frac{x^2}{\frac{2E}{m\omega^2}} = 1 \\
\implies a = \sqrt{\frac{2E}{m\omega^2}}, b = \sqrt{2mE} \\
\implies I = \frac{1}{2\pi} \cdot \text{Area} = \frac{ab}{2} \\
= \frac{1}{2}\sqrt{\frac{2E}{m\omega^2}2mE} = \frac{E}{\omega}
\end{aligned}$$

Note that when we modify the length of the system, we change ω, so I is modified slowly. This is classical example of adiabatic process, but in quantum, we have operators such as

$$H = \frac{\hat{p^2}}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

And suppose the system is in a state of infinite square well with width a. This gives the wave function $\psi_0(x) = \sqrt{\frac{2}{a}}\sin(\frac{\pi}{a}x)$, and if we slowly increase the width from $a \to 2a$, we eventually will get the ground state wave function $\psi_1(x) = \sqrt{\frac{1}{a}}\sin\left(\frac{\pi}{2a}x\right)$, which is just the ground state wave function when we change $a \to 2a$
However, if we change the width of the well rapidly, the wave function will not have enough time to re-adjust, so our wave function will be at a state where it is not an eigenstate of our system. With this intuition, we write down `The adiabatic theorem:`

The adiabatic theorem:

If H changes "slowly" from $H(t_i)$ to $H(t_f)$, then a particle that start in an eigenstate $\psi_n^i(t_i)$ of $H(t_i)$ remains in the nth eigenstate $\psi_n^*(t_f)$ of $H(t_f)$

We note that this is only true for discrete energy spectrum where

$$\begin{gathered} H(t)\psi_n(t) = E_n(t)\psi_n(t) \\ H\psi_n = E_n\psi_n \end{gathered} \tag{2}$$

where the intrinsic time scale $\frac{1}{\omega} \approx \frac{\hbar}{E_{n+t}(t) - E_n(t)} \ll t_f - t_i$.
We want to note that the instantaneous $\psi_n(t)$ are not solutions to the time- dependent Schrodinger equation, but they still satisfy $\langle\psi_n(t)|\psi_m(t)\rangle = \delta_{mn}$.

Proof to the adiabatic theorem:

note that our ψ_n must also satisfy the time-dependent Schrodinger equation

$$i\hbar = \frac{\partial\psi(t)}{\partial t} = H\psi(t)$$

This means that our $\psi_n(0) \to \psi_n(t) = \psi_n(0)e^{-iE_nt/\hbar}$, which introduces a dynamic phase to our wave function. We will try to find the solution to

$$i\hbar\frac{\partial\psi(t)}{\partial t} = H(t)\psi(t) \tag{3}$$

However, the solution for (3) can be extremely complicated. We then seek to expand $\psi_n(t)$ to the instantaneous equation

$$\psi(t) = \sum_n c_n(t)\psi_n(t)e^{i\theta_n(t)} \tag{4}$$

where $\psi_n(t)$ is just the instantaneous eigenstates that satisfy (2), and we also have

$$\theta_n(t) = -\frac{1}{\hbar}\int_0^t E_n(t')\,dt' \tag{5}$$

We then combine $(3), (4)$ and get LHS:

$$\begin{aligned} i\hbar\frac{\partial\psi(t)}{\partial t} &= i\hbar\sum_n[\dot{c}_n(t)\psi_n(t)e^{i\theta_n(t)} + c_n\dot{\psi}_n e^{i\theta_n(t)} + c_n(t)\psi_n(t)i\dot{\theta}_n(t)e^{i\theta_n(t)}] \\ &= i\hbar\sum_n[\dot{c}_n\psi_n + c_n\dot{\psi}_n + ic_n\psi_n\dot{\theta}_n]e^{i\theta_n} \end{aligned}$$

And RHS

$$\begin{aligned} H(t)\psi(t) &= \sum_n c_n(t)[H(t)\psi_n(t)]e^{i\theta_n(t)} \\ &= \sum_n c_n(t)E_n(t)\psi_n(t)e^{i\theta_n(t)} \end{aligned} \tag{7}$$

as we used the simplification that

$$\sum_n (\dot{c}_n \psi_n + c_n \dot{\psi}_n) e^{i\theta_n} = 0$$

We then use the orthogonality

$$\langle \psi_n(t) | \psi_m(t) \rangle = \delta_{mn}$$

to get from the LHS equation

$$\begin{aligned}
\langle \psi_m(t) | \sum_n \dot{c}_n | \psi_n(t) \rangle e^{i\theta_n} &= -\sum_n c_n | \dot{\psi}_n(t) \rangle e^{i\theta_n} \\
\sum_n \dot{c}_n \delta_{mn} e^{i\theta_n} &= -\sum c_n \langle \psi_m(t) | \dot{\psi}_n(t) \rangle e^{i\theta_n} \\
\dot{c}_m &= -\sum_n c_n \langle \psi_m(t) | \dot{\psi}_n(t) \rangle e^{i\theta_n(t) - \theta_m(t)}
\end{aligned}$$

We recognize our instantaneous wave function ψ_n mus satisfy:

$$H(t) | \psi_n(t) \rangle = E_n(t) \psi_n(t)$$

We take the time derivative

$$\dot{H}(t) | \psi_n(t) \rangle + H(t) | \dot{\psi}_n(t) \rangle = \dot{E}(t) | \psi(t \rangle + E_n | \dot{\psi}_n(t) \rangle$$

We sandwich this by $\langle \psi_m |$

$$\langle \psi_m | \dot{H} | \psi_n \rangle + \langle \psi_m | H | \dot{\psi}_n \rangle = \dot{E}_n(t) \delta_{mn} + E_n(t) \langle \psi_m | \dot{\psi}_n \rangle = \dot{E}_n \delta_{mn} + E_n \langle \psi_m | \dot{\psi}_n \rangle$$

for case $m - n$, we straight up get Feynman- Hellman theorem

$$\langle \psi_m | \dot{H} | \psi_n \rangle = \dot{E}_m$$

and for $m \neq n$, we have

$$\langle \psi_m | \dot{H} | \psi_n \rangle = (E_n - E_m) \langle \psi_m | \dot{\psi}_n \rangle$$

where

$$c_m(t) = -c_m \langle \psi_m | \dot{\psi}_m \rangle - \sum_{n \neq m} c_n \frac{\langle \psi_m | \dot{H} | \psi_n \rangle}{E_m - E_n} \tag{10}$$

However, we assumed that the way we modify our Hamiltonian is significantly slower than the intrinsic energy difference in (10) . Therefore, we can ignore the second term in (10) and get

$$c_m(t) = -c_m \langle \psi_m | \dot{\psi}_m \rangle = -c_m \delta_m$$

where $\delta_m \equiv \int \psi_m \frac{\partial}{\partial t} \psi_m \, dx$. After solving the differential equation for c_mThis is called the Berry Phase.