Recap:

We have derived **Fermi's golden rule** which says the transition rate from an initial state $|i\rangle$ to a final state $|f\rangle$ is:

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle f|\hat{V}|i\rangle|^2\delta(h\omega - E_f + E_i) \tag{1}$$

or the rate for all finate state $|f\rangle$ is:

$$\Gamma = \sum_f \Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle f|\hat{V}|i\rangle|^2\rho(E_f) \tag{2}$$

where $\rho(E)$ is the density of state at energy E.

We also tried to **Retrieve the Born Approximation**. We have that

$$\begin{aligned}\langle\vec{r}|i\rangle &= \frac{1}{L^{3/2}}e^{i\vec{k}_i\cdot\vec{r}}\\ \langle\vec{r}|f\rangle &= \frac{1}{L^{3/2}}e^{i\vec{k}_f\cdot\vec{r}}\end{aligned}$$

define $\vec{q} = \vec{k}_f - \vec{k}_i = 2k\sin\frac{\theta}{2}$.

We also have:

$$\langle f\rangle\hat{V}|i\rangle = \frac{1}{L^3}\int e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}V(r)\,d^3\vec{r} = V(\vec{q})|_{\text{F.T.}}$$

For the density of state, we have

$$\rho(E) = \frac{dN}{dE\,d\Omega_{\hat{p}}}$$

where

$$dN = L^3\frac{p^2dp\,d\Omega}{(2\pi\hbar)^3}$$

Therefore:

$$\rho(E) = \frac{L^3p^2}{(2\pi\hbar)^3}\frac{dp}{dE}$$

but we know that $E = \frac{p^2}{2m} \implies \frac{dE}{dp} = \frac{p}{m}$
This finally gives:

$$\rho(E) = L^3\frac{mp}{(2\pi\hbar)^3}$$

Using equation (2), we have

$$\Gamma = \left(\frac{2\pi}{\hbar}\right)^3\frac{1}{L^6}|V(\hat{q})|^2\frac{L^3m\hbar k_f}{(2\pi\hbar)^3}$$

and we have $\sigma(\theta) = \frac{\Gamma}{|\vec{J}_{inc}|}$ where $\vec{J}_{inc} = \frac{1}{L^3}\frac{\hbar\vec{k}_i}{m}$ (check this using probability current).

Combining everything (check this)

$$\sigma(\theta) = \frac{\Gamma}{|\vec{J}_{inc}|} = \frac{m^2}{(2\pi)^2\hbar^4}\left|\int e^{i\vec{q}\cdot\vec{r}}V(r)\,d^3\vec{r}\right|^2$$

in which we get back to **Born's approximation.**

Interaction Between Radiation and Matter

We look at a restricted domain where matter is treated using **QM**. To begin our analysis, we will start by treating the radiation **classically.**

Therefore, we will specify our radiation as:

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$$
$$\vec{B} = \vec{\nabla}\vec{\times} A$$

Then, for a charged particle with charge $q = -e$ (e.g. electron), we have the following Hamiltonian:

$$H = \frac{1}{2m}\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 - e\Phi + V(r)$$

Note that the above Hamiltonian did not include the spin of the electron. This means we ignore the Zeeman effect $\propto -\vec{\sigma}\cdot\vec{B}$. We will justify later why this is OK.

Note that because we use $\vec{A}$ and Φ to represent the radiation, then we have to choose a gauge for the fields. We generally choose the following gauge:

$$\vec{\nabla}\cdot\vec{A} = 0$$
$$\Phi = 0$$

Note that we set $\Phi = 0$, so we do a small perturbation in terms of $\vec{A}$ to our Hamiltonian. Therefore, we expand the term $\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2$ in terms of $\vec{A}$.

$$\left(\vec{p} + \frac{e}{c}\vec{A}\right)^2 = \left(-i\hbar\vec{\nabla} + \frac{e}{c}\vec{A}\right)^2$$

We have the following expansion:
0th order $\rightarrow \frac{\vec{p}^2}{2m}\psi = -\frac{\hbar^2}{2m}\nabla^2\psi$
1st order $\rightarrow \frac{e}{2mc}(-i\hbar)(\vec{A}\cdot\vec{\nabla} + \vec{\nabla}\cdot\vec{A})\psi = 2\vec{A}\cdot\vec{\nabla}\psi$ for our choice of gauge that $\vec{\nabla}\cdot A = 0$.

Therefore, we have

$$H = H_0 + H_1$$

where

$$H_0 = \frac{p^2}{2m} + V(r)$$
$$H_1 = \frac{e}{mc}\vec{A}\cdot p$$

Photoelectric Effect

With this setup, we can talk about the **photoelectric effect** of hydrogen.

Note that the hydrogen atom has discrete bound state for $E < 0$ and continuous unbounded state for $E > 0$.
We then consider a state starting from $|i\rangle$ a bound state to $|f\rangle$ a continuous state.

Note that the means for $|i\rangle \rightarrow |f\rangle$ is by absorbing a "photon." However, since we are considering the radiation to be **classically**, the means of the transition is actually a time-dependent perturbation:

$$E_f = E_i + \hbar\omega$$

where ω coming from the $e^{-i\omega t}$ dependence of the $\vec{A}$ field.

To simplify matters, we assume that

$$\hbar\omega = E_f - E_i \gg 13.6\text{eV}$$

We make this assumptions because we want to make our final state $|f\rangle$ to be only slightly (or not at all in limiting case) of the proton. Therefore, we need much more energy to put our final state $|f\rangle$ into a high energy state.

Because of this, our final state will just be a plane wave due to it not perturbed by the proton. Therefore, we have:

$$\langle\vec{r}|f\rangle = \frac{1}{L^{3/2}} e^{i\vec{p}_f\cdot\vec{r}/\hbar}$$

and the initial state $|i\rangle$ is just the hydrogen wave function ground state:

$$\langle\vec{r}|i\rangle = \frac{1}{\sqrt{\pi a_0^3}} e^{-r/a_0}$$

Next, we will then want to calculate:

1. $\langle f|H_1|i\rangle$
2. $\rho(E_f)$
3. apply golden rule.