Recap Spontaneous Emission:

An electron decay from $|i\rangle = |2lm\rangle$ to $|f\rangle = |000\rangle$ while emitting a photon γ

We wrote down the initial and final state

$$\begin{aligned} &|i\rangle \quad E_i \\ &|f\rangle \quad E_f = E_i + \hbar\omega_2 \\ &H_0 = H_{\text{hydrogen}} + H_{EM} \\ &H_1 = \frac{e}{mc}\vec{A}\cdot(-i\hbar\vec{\nabla}) \end{aligned}$$

From here, we computed $\langle f|H_1|i\rangle$ and use the golden rule.

Last time, we used $\vec{A}(\vec{r},t) = \sum_{\vec{k},\alpha} = \frac{1}{L^{3/2}}(\ldots)^{1/2}[a^\dagger_{\vec{k},\alpha}e^{i\omega_k t}e^{-i\vec{k}\cdot\vec{r}} + a_{\vec{k},\alpha}e^{i\omega_k t}e^{i\vec{k}\vec{r}}]\hat{e}$, and note that our operator is dependent on time. Therefore, this representation of $\hat{A}$ is in the Heisenberg representation. However, we want to write $\vec{A}$ in Schrodinger equation. We can write this equivalently as

$$a(t) = e^{i\hat{H}t/\hbar}\,a\,e^{-i\hat{H}t/\hbar} \quad \text{whree} \quad \hat{H} = \hbar\omega a^\dagger a$$

We can explicitly compute this and show

$$a(t) = ae^{-i\omega t}$$

Or, equivalently, we can write

$$\begin{aligned} i\hbar\frac{da}{dt} &= [a, H] \\ &= aa^\dagger a - a^\dagger aa \\ &= (1 + a^\dagger a)a - a^\dagger aa \\ &= a \end{aligned}$$

up to a factor of $\hbar\omega$. With this, we get

$$\begin{aligned} i\hbar\frac{da}{dt} &= \hbar\omega a \\ &\implies a(t) = ae^{-i\omega t} \\ &\implies a^\dagger(t) = a^\dagger e^{i\omega t} \end{aligned}$$

Therefore, in the Schrodinger picture, we can write

$$\vec{A}(\vec{r}) = \sum_{\vec{k},\alpha}\frac{1}{L^{3/2}}\left(\frac{2\pi\hbar c^2}{\omega_k}\right)[a^\dagger_{\vec{k},\alpha}e^{-i\vec{k}\cdot\vec{r}} + a_{\vec{k},\alpha}e^{i\vec{k}\cdot\vec{r}}]$$

Now we use this $\vec{A}$ in $H_1 = \frac{e}{mc}\vec{A}\cdot\vec{p}$

We can use this now to compute

$$\begin{aligned} \langle f|H_1|i\rangle &= \langle\vec{k},\alpha|A_\mu|0\rangle\langle 100|p_\mu|2lm\rangle \\ &= \frac{e}{mc}\frac{1}{L^{3/2}}\left(\frac{2\pi\hbar c^2}{\omega_k}\right)^{1/2}\vec{e}_\alpha\cdot\langle 100|\vec{p}|2lm\rangle \end{aligned}$$

where we focus on the expectation value of $\hat{p}$:

$$\vec{e}\cdot\langle 100|\hat{p}|2lm\rangle = im\omega\vec{e}\cdot\int\psi_{100}(r)\vec{r}\psi_{2lm}(r)\,d^3r$$

where we can use our selection rule

$$\begin{aligned} l &= 1 \\ m &= 0, \pm 1 \end{aligned}$$

We will not do this integral in depth, for more information, you can see Shankar Chapter 18.

We must note that for our final state, we have different modes $\vec{k}, \alpha$ where out photons can land into. Therefore, we will need to construct the density of state $\rho(E_f)$ for the photons.

We should also average over our initial states, and sum over the final states using the density of states:

$$\frac{1}{3}\sum_m |\langle 100|\vec{e}\cdot\vec{p}|2lm\rangle| = \frac{2^{15}}{3^{11}}a_0^2$$

Now we do the density of states:

$$\rho(\omega) = 2L^3\frac{4\pi k^2}{(2\pi)^3}\frac{dk}{d(\hbar\omega)}$$

Where the factor of 2 comes from the two photon polarization. We use $\omega = ck \implies \frac{d\omega}{dk} = c$, this gives:

$$\rho(\omega) = \frac{L^3k^2}{\pi^2\hbar c}$$

We then have, using golden rule:

$$\Gamma_{i\to f} = \frac{2\pi}{k}|\langle f|H_1|i\rangle|^2\delta(E_{100}+\hbar\omega - E_{2lm})$$

and

$$\begin{aligned}\Gamma &= \frac{1}{2}\sum_m \frac{2\pi}{\hbar}|\langle f|H_1|i\rangle|^2\rho(\hbar\omega)\\ &= \frac{2\pi}{\hbar}\left(\frac{e}{mc}\right)^2\frac{2\pi\hbar c^2}{\omega L^3}m^2\omega^2\left(\frac{2^{15}}{3^{11}}\right)\frac{L^3k^2}{\pi^2\hbar c}\\ &= \frac{1}{\hbar}\frac{2^{17}}{3^{11}}(a_0k)^3\frac{e^2}{a_0} \quad \text{where} \quad k = \frac{\omega}{c}\end{aligned}$$