QM March 29

Recap:

$|\psi(0)\rangle = |i\rangle$
$H = H_0 + H_1(t) \implies |\psi(t)\rangle = \sum_f a_f(t)e^{-i\omega_f t}|f\rangle$
where we have computed $a_f(t)$ using first order time-dependent perturb theory, and for $i \neq f$

$$P_{i\to f}(t) = |\langle f|\psi(t)\rangle|^2 = \frac{|V_{fi}|^2}{\hbar^2}\left[\left(\frac{\sin(\omega_{fi}-\omega)t/2}{(\omega_{fi}-\omega)/2}\right)\right]^2 \tag{1}$$

for $H_1(t) = 2\hat{V}\cos\omega t, \hbar\omega_{fi} = E_f - E_i, \text{and } V_{fi} = \langle f|\hat{V}|i\rangle.$

Fermi's Golden Rule:

Let us look at large t behavior of $P_{i\to f}(t)$, and claim that equation (1) at large t, we have

$$\left(\frac{\sin(\omega_{fi}-\omega)t/2}{(\omega_{fi}-\omega)/2}\right)^2 \to_{t \text{ large}} 2\pi t\delta(\omega-\omega_{fi}) \tag{2}$$

We check that

1. Equation $(2) \geq 0$
2. Width of $(2) \sim \frac{1}{t}$
3. Height of $(2) \sim t^2$
4. $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi$
   Using these and equation(2), we conclude that

$$\lim_{t\to\infty} \frac{P_{i\to f}(t)}{t} = \frac{2\pi}{\hbar^2}|\langle f|V|i\rangle|^2\delta(\omega-\omega_{fi}) \tag{3}$$

The physical interpretation of (3) is the rate of making transition from $|i\rangle \to |f\rangle$, denote as

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\delta(E - E_f + E_i) \tag{4}$$

where (4) is called the **Fermi's golden rule.**

Note that the delta function equation (4) represents a concentration of energy. This result is very satisfactory, but in the case of a single state of $|f\rangle$, it is strange that $\Gamma_{i\to f}$ is either zero or infinity.

To reconcile with this strange behavior, we now consider $|i\rangle \to \text{continium of final states } |f\rangle$ where we apply (4) to every single final state $|f\rangle$.

Therefore, we need to sum these different final states with a density of states $\rho(E_f)$. Under these condition, we should have

$$\Gamma = \sum_f \Gamma_{i\to f} \equiv \int \rho(E_f)\frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\delta(\hbar\omega - E_f + E_i)\, dE_f \tag{5}$$

Or we can write this as

$$\Gamma = \frac{2\pi}{\hbar}|\langle f|V|i\rangle|^2\rho(E_f)|_{E_f=E_i+\hbar\omega} \tag{6}$$

Note that equation (6) is a more physical form of Fermi's golden rule.

We will skip the validity of when to applying Fermi's golden rule.

Example: Fermi's golden rule $\to$ Born's approximation

$|i\rangle = \text{plane wave}$
Physically, what we will do it to "turn on" the scattering potential at $t = 0$ and then applying Fermi's golden rule. This means that:
$V(t)$ which causes scattering into various $|f\rangle$.
$\langle\vec{r}|i\rangle = \frac{1}{L^{3/2}}e^{i\vec{k}_i\cdot\vec{r}}$ which is a $3-d$ plane wave.

$\langle\vec{r}|f\rangle = \frac{1}{L^{3/2}e^{i\vec{k}_f\cdot\vec{r}}}$
Therefore, we have:

$$\begin{aligned}
\langle f|\hat{V}|i\rangle &= \int\int \langle f|\vec{r}'\rangle\langle\vec{r}'|\hat{V}|\vec{r}\rangle\,\langle\vec{r}|i\rangle\,d^3\vec{r}'\,d^3\vec{r} \\
&= \frac{1}{L^3}\int e^{-i(\vec{k}_f-\vec{k}_i)\cdot\vec{r}}\,d^3\vec{r} \\
&= \frac{1}{L^3}V(\vec{q})|_{\text{F.T.}}
\end{aligned} \tag{7}$$

We note that in 3-d, each state occupy a space $\hbar^3\left(\frac{2\pi}{L}\right)^3$ in momentum space. Therefore, we must have the number of states in a box of volume L^3 that lie in the region $d^3\vec{p}$ about $\vec{p}$ is

$$\begin{aligned}
N &= \frac{d^3\vec{p}}{(2\pi\hbar)^3}L^3 \\
&= \frac{L^3p^2dp\,d\Omega_{\hat{p}}}{(2\pi\hbar)^3} \\
&\equiv \rho(E)dE\,d\Omega_{\hat{p}}
\end{aligned} \tag{8}$$

where $\rho(E)dE$ is the number of states (for a plane wave) in a box of volume L^3 that lie in the energy range $(E, E+dE)$.

from equation(8), we find:

$$\rho(E) = \frac{L^3p^2}{(2\pi\hbar)^3}\frac{dp}{dE} \tag{9}$$

For free particle solution, we use $E = \frac{p^2}{2m}$ for equation (8) and found that

$$\rho(E) \propto \sqrt{E}$$

Note that if we combine equation(8) and equation(9) to equation (6), we can calculate Γ

Recall that the cross section $\sigma(\theta) = \frac{\Gamma}{J_{inc}}, \Gamma \sim \frac{2\pi}{\hbar}|V(\vec{q})|^2\rho(E_f), \text{and } J_{inc} = \frac{1}{L^3}\hbar\frac{k_i}{m}$
.

Recall that $|\vec{k}_f| = |\vec{k}|_i = k$, and $q = 2k\sin(\frac{\theta}{2})$ where $q = |\vec{k}_f - \vec{k}_i|$