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Parallel sorting algorithms

Theory and implementation

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Course Parallel Programming
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Overview

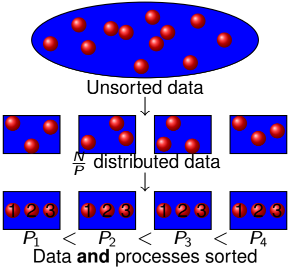
Complexity of sorting algorithms

Basic operations

- **Compare algorithm** : Comparison algorithm complexity is supposed $\mathcal{O}(1)$. But in distributed parallel context, one must consider the distribution of the initial data to account for the cost of data exchange between processes !
- **Exchange algorithm** : Exchange algorithm complexity is supposed $\mathcal{O}(1)$. But same consideration to do as **compare algorithm** ;
- Sequential “compare–and–exchange” algorithm :

```
if (a>b) { // Comparison
  // Exchange
  tmp = a;
  a   = b;
  b   = tmp;    }
```

Potential speed-up



- Best sequential sorting algorithms (for arbitrary sequences of numbers) have average time complexity $O(n \log n)$
- hence, the best speedup one can expect from using n processors is $\frac{O(n \log n)}{n} = O(\log n)$
- there are such parallel algorithms, but the hidden constant is very large (F. Thomson Leighton : Introduction to parallel algorithms and architectures (1991))
- In general, a practical useful $O(\log n)$ algorithm may be difficult to find.

Beware, it may be a bad idea to take n processes to sort n data (granularity).

Parallelization of a naive algorithm

Naive algorithm

- Count the number of numbers that are smaller than a number *a* in the list
- this gives the position of *a* in the sorted list
- this procedure has to be repeated for all elements of the list ; hence the time complexity is $n(n-1) = O(n^2)$ (not so good sequential algorithm)

Implementation

```
for ( i = 0; i < n; i++ ) {// For each value
  x = 0;
  for ( j = 0; j < n; j++ )// Computing the new pos.
    if (a[i] > a[j]) x++;
  b[x] = a[i];
}
```

Works well if there are no repetitions of the numbers in the list (in case of repetitions the code must be changed slightly).

Rank sort : Parallel code

Embarrassingly “ideal” algorithm

Parallel code, using n processes (for n values to sort)

```
x = 0;
for ( j = 0; j < n; j++ )
  if ( a[rank] > a[j] ) x++;
b[x] = a[rank];
```

Complexity

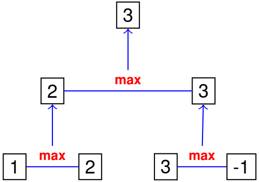
- n processors work in parallel to find the ranks of all numbers of the list ;
- Parallel time complexity is $O(n)$, better than any sequential sorting algorithm !
- Usable for GPGPU units.

More parallelization...

Parallel code using n^2 processes (for n values to sort)

Parallel algorithm

- In the case n^2 processes may be used, the comparison of each `a[0],...,a[n-1]` with `a[i]` may be done in parallel as well
- Incrementing the counter is still sequential, hence the overall computation requires $1 + n$ steps ;
- If a tree structure is used to increment the counter, then the overall computation time is $O(\log_2 n)$



(but, as one expects, processor efficiency is very low)

These are just theoretical results : it is not efficient to use n or n^2 processors to sort n numbers.

Data partitioning

Context

- Usually the number n of values is much larger than the number p of processes ;
- In such cases, each process will handle a part of the data (a sublist of the data)

Distributed sorted container

- local container is sorted ;
- if $p_i < p_j$ then $\forall a_i \in p_i, \forall a_j \in p_j, a_i \leq a_j$

Global scheme of parallel sort algorithm

For a process :

- Sort his local data ;
- Run a merge sort algorithm to concatenate its list with that received from another process ;
- Keep the bottom half (or the top half) of the sorted list.

Parallel compare and exchange operations

Asymmetric algorithm

- Process p_i sends local value A to process p_j ;
- Process p_j compares value A with some local values B_j ;
- Send the B_j which are larger (or lesser) than A. If no B_j is larger (or lesser) than A, send back A ;

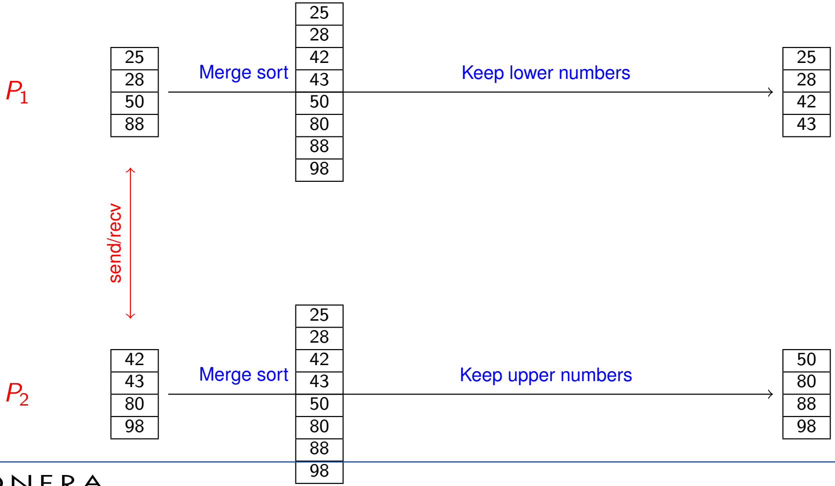
Symmetric algorithm

- Processes p_i and p_j send some value to the other ;
- Each process compares his value with the received value ;
- Each process keeps his value or the received value relative to the comparison result ;

Remarks

- Data exchanges between processes is very expensive, so find some algorithms which minimize data exchanges ;
- In general, the receive operation doesn't know the number of values to receive $\Rightarrow$ one must probe the received message to get the number of data to receive, allocate the relative buffer and receive the data !

Scheme of a general algorithm for parallel sort algorithm



Overview

Sequential bubble sort algorithm

Bubble sort algorithm

- Simplest, but not an efficient sequential sorting algorithm ;
- Compare/exchange complexity :

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = O(n^2)$$

Figure – Analogic bubble sort

Sequential code

```
for (int i=n-1; i>0; --i)
    for (int j=0; j<i; ++j) {
      k = j+1;
      if (a[j]>a[k]) std::swap(a[j],a[k]);
    }
```

Odd-Even sort algorithm

- Parallelized bubble sort
- Based on idea that the bodies of the main loop may be overlapped

“scalar” Algorithm : Iteration between even and odd phase

- Even phase

0 ⟷ 1   2 ⟷ 3

```
if (rank%2==0) {
  recv(&temp, (rank+1)%nbp);
  send(&value,(rank+1)%nbp);
  if (temp < A) A = temp;   }
```

```
if (rank%2==1) {
  send(&value,rank-1);
  recv(&temp, rank-1);
  if (temp > A) A = temp;   }
```

- Odd phase

0   1 ⟷ 2   3

```
if (rank%2==0) {
  recv(&temp, (rank+nbp-1)%nbp);
  send(&value,(rank+nbp-1)%nbp);
  if (temp > A) A = temp;   }
```

```
if (rank%2==1) {
  send(&value,(rank+1)%nbp);
  recv(&temp, (rank+1)%nbp);
  if (temp < A) A = temp;   }
```

Example of even–odd parallel bubble sort

Example : Sorting 8 numbers on 8 processes

Step	P_0		P_1		P_2		P_3		P_4		P_5		P_6		P_7
0	4	$\leftrightarrow$	2		7	$\leftrightarrow$	8		5	$\leftrightarrow$	1		3	$\leftrightarrow$	6
1	2		4	$\leftrightarrow$	7		8	$\leftrightarrow$	1		5	$\leftrightarrow$	3		6
2	2	$\leftrightarrow$	4		7	$\leftrightarrow$	1		8	$\leftrightarrow$	3		5	$\leftrightarrow$	6
3	2		4	$\leftrightarrow$	1		7	$\leftrightarrow$	3		8	$\leftrightarrow$	5		6
4	2	$\leftrightarrow$	1		4	$\leftrightarrow$	3		7	$\leftrightarrow$	5		8	$\leftrightarrow$	6
5	1		2	$\leftrightarrow$	3		4	$\leftrightarrow$	5		7	$\leftrightarrow$	6		8
6	1	$\leftrightarrow$	2		3	$\leftrightarrow$	4		5	$\leftrightarrow$	6		7	$\leftrightarrow$	8
7	1		2	$\leftrightarrow$	3		4	$\leftrightarrow$	5		6	$\leftrightarrow$	7		8

Odd-even parallel algorithm per block

Per block algorithm

- Replace a value per process with a sorted set of values per process
- Use sort-fusion algorithm to exchange values
- Data comparison complexity : $\frac{N}{nbp}\log_2\left(\frac{N}{nbp}\right) + (nbp-1).\frac{2N}{nbp}$
- Data communication complexity : $(nbp-1).\frac{2N}{nbp}$

Implementation

```
sort(values);// Quick sort of local values
for (iter=0; iter<nbp-1; ++iter) {// Odd-even algorithm
  if (iter is odd) {
    if (rank is even and rank > 0) {
      recv(buffer, rank-1); send(values, rank-1);
      values = fusionSort(buffer, values, keepMax);
    } else if (rank is odd and rank < nbp-1) {
      send(values, rank+1); recv(buffer, rank+1);
      values = fusionSort(buffer, values, keepMin);
    }
  } else if (iter is even) {
    if (rank is even and rank < nbp-1) {
      recv(buffer, rank+1); send(values, rank+1);
      values = fusionSort(buffer, values, keepMin);
    } else if (rank is odd) {
      send(values, rank-1); recv(buffer, rank-1);
      values = fusionSort(buffer, values, keepMax);
    }
  }
}
```

Shear sort algorithm

Two dimensional sorting

Basic idea

- Look at the array as a two-dimensional array (one row per process)
- The goal is to sort this 2D array in snakelike style : even rows increasing, odd rows decreasing ;
- Two phases : In **even phase**, sort per row, in **odd phase**, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

Example

4	14	8	2
10	3	13	16
7	15	1	5
12	6	11	9

Original number

Remarks

- Embarrassingly parallel algorithm for shared memory !
- But not well adapted for distributed parallel architecture as is ;
- How to change this algorithm for distributed parallel architecture ?

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Example

2	4	8	14
16	13	10	3
1	5	7	15
12	11	9	6

Phase 1 – row sort

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- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

Example

1	4	7	3
2	5	8	6
12	11	9	14
16	13	10	15

Phase 2 – col sort

Remarks

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Shear sort algorithm

Two dimensional sorting

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- Two phases : In **even phase**, sort per row, in **odd phase**, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

Example

1	3	4	7
8	6	5	2
9	11	12	14
16	15	13	10

Phase 3 : row sort

Remarks

- Embarrassingly parallel algorithm for shared memory !
- But not well adapted for distributed parallel architecture as is ;
- How to change this algorithm for distributed parallel architecture ?

Shear sort algorithm

Two dimensional sorting

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- Look at the array as a two-dimensional array (one row per process)
- The goal is to sort this 2D array in snakelike style : even rows increasing, odd rows decreasing ;
- Two phases : In **even phase**, sort per row, in **odd phase**, sort per column increasing from top to bottom ;
- After $\log_2(N) + 1$ phases, the array is snakelike-style sorted.

Example

1	3	4	2
8	6	5	7
9	11	12	10
16	15	13	14

Phase 4 : col sort

Remarks

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Shear sort algorithm

Two dimensional sorting

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Example

1	2	3	4
8	7	6	5
9	10	11	12
16	15	14	13

Final 5 : row sort

Remarks

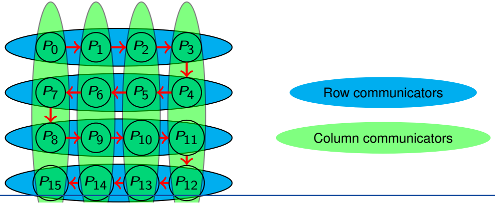
- Embarrassingly parallel algorithm for shared memory !
- But not well adapted for distributed parallel architecture as is ;
- How to change this algorithm for distributed parallel architecture ?

Shear sort algorithm for parallel distributed memory architecture

Implementation ideas

- Same principal as odd-even algorithm : replace a value with some sets of values S_i (one set per process) ;
- Define a relation order : $S_i < S_j$ iff $\max(S_i) < \min(S_j)$ (In set, values ordered as increasing order)
- Use odd-even algorithm to parallelize the phase of sorting per row or column ;
- Grouping processes in new communicators per rows and per columns ;
- Play with rank numbering to alternate between increasing order and decreasing order for rows ;

Processes repartition



- Use `MPI_Comm_split(comm,color,key,&newcomm)` to define row and columns communicators ;
- Processes calling this function with same color are inside the same new communicator ;
- key is a value to number the processes inside the new communicator.

Overview

Sequential quick-sort algorithm

Reminder

- Optimal sequential sorting algorithm ($\mathcal{O}(n\log_2(n))$) based on divide-and-conquer algorithm class ;
- Select a number r called pivot and split the list into two sublists : one with all elements at most equal to r, the other holding all elements greater than r ;
- This procedure is recursive, applied until one element lists are obtained (which are sorted)
- **Example** :

4 2 7 8 5 1 3 6

Sequential code

```
void quicksort( T* list, T* start, T* end ) {
    auto pivot = choosePivot(start,end);
    if (start < end) {
        split(list, start, end, pivot);
        quicksort(list, start, pivot-1);
        quicksort(list,pivot+1, end   );
    }
}
```

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Sequential quick-sort algorithm

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    }
}
```

Naive parallelization of quicksort algorithm

Ideas

- The class of quick-sort algorithm suggests to apply a divide-and-conquer parallelization method ;
- The main problem of this approach is that the tree distribution (induced by the lengths of the sublists) heavily depends on pivot selection ; in the worst case, the tree may consists of a single path (as a list) ;
- **Analysis** : provided an equal distribution of values within sublists is ensured, one gets :
  - Comparisons : $n + \frac{n}{2} + \frac{n}{4} + \cdots \approx 2n$
  - Communications : $t_s + \frac{n}{2}t_d + t_s + \frac{n}{4}t_d + \cdots \approx \log_2(n)t_s + nt_d$ where t_s is time to start a communication and t_d the time to transfer one element to another process.
- But only last iteration uses full parallelization !

Hyperquick sort algorithm

Binary numbering of Hypercube

0

Figure – Dimension 0

Hyperquick sort algorithm

Binary numbering of Hypercube

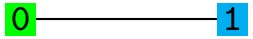


Figure – Dimension 1

Hyperquick sort algorithm

Binary numbering of Hypercube

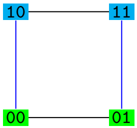


Figure – Dimension 2

Hyperquick sort algorithm

Binary numbering of Hypercube

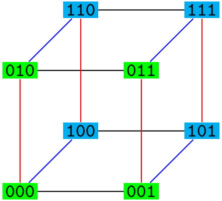


Figure – Dimension 3

Hyperquick sort algorithm

Binary numbering of Hypercube

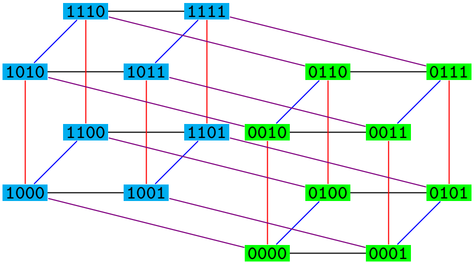


Figure – Dimension 4

Hyper quick sort algorithm (2)

Numbering vertices of hyper-cube

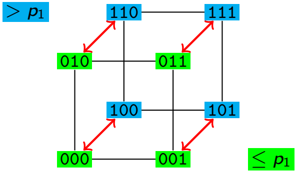
- The binary numbers of two linked nodes have a difference of one bit ;
- The distance between two nodes in a hypercube (minimal number of nodes to access to go from first node to second node) is the number of bit which differ in their binary number ;
- It's the **Gray code** numbering.

Ideas of the hyper quick sort algorithm

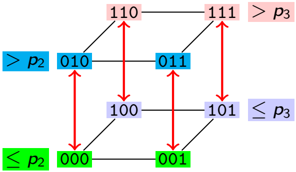
- Initially, Data are distributed across all processes ;
- Each process sorts its local data ;
- Loop on dimension of the hypercube and for dimension d, consider pair of processes $\left(p; p + 2^d\right)$
- Process p chooses his median value as pivot and sends value lesser than pivot to process $p + 2^d$;
- Process $p + 2^d$ receives pivot and data from process p and keeps value greater than pivot ;
- Each process keeps sorted value, using fusion sort algorithm to keep sorting.

Illustration of hyper quick sort

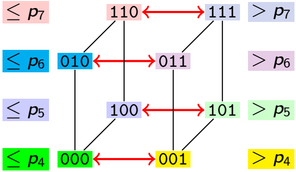
Phase 1 :



Phase 2 :



Phase 3 :



Hyperquicksort : complexity analysis

- Suppose we run algorithm on $nbp = 2^d$ processes (hypercube dimension d) ;
- Each process holds initially $NI = \frac{N}{nbp}$ values ;
- **Initial sorting :** $NI.\log_2(NI)$ comparisons ;
- **Pivot selection :** $\mathcal{O}(1)$ (one takes middle list element) for each dimension ;
- **Pivot broadcasting :**
  - Broadcast one pivot in a k hypercube : $k(t_s + t_d)$;
  - For all iterations : $(d + \cdots + 1)(t_s + t_d) = \frac{d(d-1)}{2}(t_s + t_d)$ comparisons ;
- **Split list from pivot value :** For sorted list of size x, $\log_2(x)$ comparisons ;
- **Exchange part of list :** To exchange $\frac{x}{2}$ data : $2(t_s + \frac{x}{2}t_d)$
- **Fusion merge sort :** $\frac{x}{2}$ comparisons

Total (for ideal balance)

- $N_l.\log_2(N_l) + \left(\log_2(N_l) + \frac{N_l}{2}\right).d$ comparisons
- $\left(\frac{d(d-1)}{2} + 2d\right)t_s + \left(\frac{d(d-1)}{2} + d.N_l\right).t_d$ for message ;

Overview

Bitonic sequences

Definition of a bitonic sequence

- A sequence of values $\{a_i\}_{i\in[1;N]}$ than can be split in two subsequences (of consecutive numbers), one increasing and one decreasing ; e.g :

$$\exists i \in [1; N] \text{ verifying } \begin{cases} a_1 \leq a_2 \leq \cdots \leq a_i \\ a_i \geq a_{i+1} \geq \cdots \geq a_N \end{cases}$$

  Example : 3, 5, 8, **19**, 17, 14, 12, 11

- **Or** a sequence which may be brought to this form by a circular shifting of the elements of the sequence

  Example : 12, 11, 3, 5, 8, **19**, 17, 14

Remarks

- The first subsequence can be increasing **or** decreasing.
- So a bitonic sequence (without considering circular shifting) can be increasing-decreasing or decreasing-increasing.
- A monotone bitonic sequence is a sorted sequence (increasing or decreasing) ;
- All sequences with three or less elements are bitonic.

Splitting a bitonic sequence

Bitonic split

Let $\{a_i\}_{i\in[1;N]}$ be a bitonic sequence. So the subsequences

$$\left\{\begin{array}{lcl} \{b_i\}_{i\in\left[1;\frac{N}{2}\right]} & = & \left\{\min(a_1, a_{1+\frac{N}{2}}), \min(a_2, a_{2+\frac{N}{2}}), \cdots, \min(a_{\frac{N}{2}-1}, a_N)\right\} \\ \{c_i\}_{i\in\left[1;\frac{N}{2}\right]} & = & \left\{\max(a_1, a_{1+\frac{N}{2}}), \max(a_2, a_{2+\frac{N}{2}}), \cdots, \max(a_{\frac{N}{2}-1}, a_N)\right\} \end{array}\right.$$

- are bitonic sequences ;
- $\forall i \in \left[1; \frac{N}{2}\right]$; $b_i \leq c_i$

Example

$$1\ \ 5\ \ 8\ \ 7\ \ 6\ \ 4\ \ 3\ \ 2 \overset{\text{split}}{\Longrightarrow} 1\ \ 4\ \ 3\ \ 2\ \ 6\ \ 5\ \ 8\ \ 7$$

Sorting a bitonic sequence (SBS algorithm)

Algorithm

Apply split procedure on bitonic sequence and repeat this splitting procedure on subsequences until having one element per subsequence to obtain sorted sequence.

Example

1. 1 5 8 7 6 4 3 2
2. 1 4 3 2 6 5 8 7
3. 1 2 3 4 6 5 8 7
4. 1 2 3 4 5 6 7 8

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Example

1.

1	5	8	7	6	4	3	2

2.

1	4	3	2	6	5	8	7

3.

1	2	3	4	6	5	8	7

4.

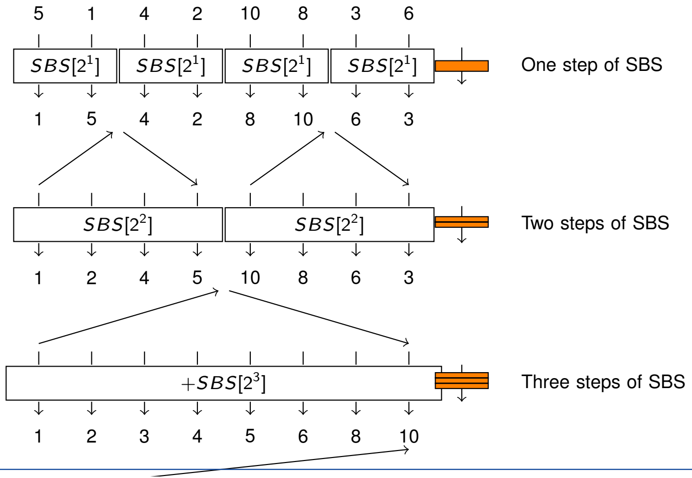
1	2	3	4	5	6	7	8

Building a bitonic sequence

Procedure

1. Split the sequence in two-elements subsequences (which are bitonics !) ;
2. Sort subsequences alternating increasing and decreasing sorting (using SBS algorithm) ;
3. Concatenate two adjacent lists to get a longer bitonic sequence ;
4. Repeat from step 2 until the full list becomes a bitonic sequence.

Example on 8 elements



Bitonic sort analysis

Complexity of bitonic sort :

- With $n = 2^k$, there are k phases, each involving $1, 2, \ldots, k$ steps, respectively ;
- The total number of steps is

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2} = O(k^2) = O(\log_2^2 n) \quad (1)$$

- Total complexity is $O(n \log_2^2 n)$

Adapt bitonic sort to distributed parallel architecture

Main ideas

- First sort local data with the fastest sort algorithm in increasing values if rank is even and decreasing values if rank is odd ;
- Pairing inside a subcommunicator processes to define a bitonic sequence ;
- And apply bitonic sort algorithm, grouping processes per four, height and so on. . .
- Sub-communicators built here are very similar to the sub-communicators build with hyperquick sort algorithms !

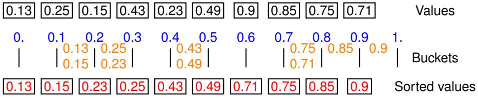
Overview

Bucket Sort algorithm

Algorithm

Distribute elements of a list into a fixed number of buckets, and sort the elements inside each buckets.

- Setup an array of initially empty "buckets"
- **Scatter** : Put each element of the list in the right bucket
- Sort the elements inside each bucket
- **Gather** : Visit each bucket in order and put all elements back in original list



- Complexity at best : $N + \frac{N}{k}\log_2(\frac{N}{k})$ comparisons (k = numbers of buckets)
- Difficulty : How to choose the intervals for the buckets ?

Parallel bucket sort

Main ideas for the parallel algorithm

- Each process is seen as one bucket ;
- Compute intervals for buckets :
  - Sort local data
  - Take $nbp + 1$ values at regular intervals
  - Gather values in bucket array
  - Extract $nbp + 1$ values to find intervals for buckets

Analysis

- Local sort : $\frac{N}{nbp}\log_2(\frac{N}{nbp})$ comparisons ;
- Gather bucket array : $nbp(t_s + (nbp + 1)t_d)$
- Sort bucket array : $\frac{nbp}{\log}{}_2(nbp)$ comparisons ;
- Distribute data in buckets : $(nbp - 1)\left(t_s + \frac{N}{nbp^2}t_d\right)$
- Sort local data : $2(nbp - 1)\frac{N}{nbp}$ comparisons (fusion sort)