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Minimal average cost of searching for a counterfeit coin: Restricted model☆

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Abstract

The following restricted model of coin-weighing problem is considered: there is a heavier coin in a set of n coins, $n-1$ of which are good coins having the same weight. The test device is a two-arms balance scale and each test-set is of the form $A : B$ with $|A| = |B| \leqslant \ell$, where $\ell \geqslant 1$ is a given integer. We present an optimal sequential algorithm requiring the minimal average cost of weighings when the probability distribution on the coin set is uniform distribution.

Keywords: Coin-weighing problem; Sequential algorithm; Average cost; Huffman tree

1. Introduction

One of the well-known search problems is formulated by the following model: there is a heavier coin in a set of n coins, $n-1$ of which are good coins having the same weight. The test device is a two-arms balance scale. The aim of this problem is to find an optimal algorithm which identifies the heavier coin using as few weighings as possible. Two measures are commonly utilized to estimate the efficiency of an algorithm: the *worst-case* number of weighings and the *average* number of weighings. Two classes of algorithms are usually considered: *sequential algorithm* and *predetermined algorithm*. Many results on this topic have been obtained (see [1,5,6] and references therein).

When the test device is a two-arms balance scale, each test-set is of the form $A : B$ with $|A|=|B|$ (no information can be obtained by weighing two unequal-sized sets, see [1]). A new *restricted model* has been investigated: each test-set is of the form $A : B$ with $|A|=|B| \leqslant \ell$, where $\ell \geqslant 1$ is a given integer. For more details, we refer the reader to [1,8,19]. Aigner [1] presents an optimal sequential algorithm on the restricted model requiring minimal worst-case number of weighings, and states as an open problem that how to construct an optimal algorithm on the restricted model requiring the minimal average number of weighings. In this paper, we present an optimal sequential algorithm on the restricted model requiring the minimal average number of weighings when the probability distribution on the coin set is uniform distribution. Thus the open problem is completely solved.

Let $S = \{1, 2, \ldots, n\}$ be the set of n coins. A sequential algorithm of the restricted model can be represented by a 3-ary tree T. Let $h(T)$ be the *external path length* of T, $h_{\leqslant \ell}(n) = \min h(T)$ be the minimal external path length of all

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ℓ-admissible trees with n leaves. Then the minimal average number of weighings $\bar{L}_{\leqslant \ell}(n) = h_{\leqslant \ell}(n)/n$. Essentially, the aim of the restricted model is to determine the exact value of $h_{\leqslant \ell}(n)$ for all integers $\ell \geqslant 1$ and $n \geqslant 2$. The following three theorems summarize the main results of this paper (some undefined terminologies and notations will be defined in other sections).

Theorem 1. *Given integers ℓ with $3^L \leqslant \ell < 3^{L+1}$ and n with $2 \leqslant n \leqslant 3\ell$. We have*

$$h_{\leqslant \ell}(n) = \begin{cases} H(n)+1, & (n=6 \text{ and } \ell \geqslant 1) \text{ or } (\ell = 3^{L+1}-1 \text{ and } n = 3\ell), \\ H(n), & \text{otherwise.} \end{cases}$$

Theorem 2. *Given integers ℓ with $3^L \leqslant \ell < 3^{L+1}$ and n with $3\ell < n \leqslant 5\ell$. Let*

$$\Omega_1 = \{(\ell, n) | \ell \text{ even and } n - 2\ell \leqslant 3^{L+1} - 2\},$$

$$\Omega_2 = \{(\ell, n) | (\ell \text{ even and } n - 2\ell > 3^{L+1} - 2) \text{ or } \ell \text{ odd}\}.$$

We have

$$h_{\leqslant \ell}(n) = \begin{cases} \phi_1 = n + 2H(\ell - 1) + H(n - 2\ell + 2), & (\ell, n) \in \Omega_1, \\ \phi_2 = n + 2H(\ell) + H(n - 2\ell), & (\ell, n) \in \Omega_2. \end{cases}$$

Theorem 3. *Given integers ℓ with $3^L \leqslant \ell < 3^{L+1}$ and n with $n > 5\ell$. Let $t = \lceil (n - 3\ell)/2\ell \rceil$ and*

$$\Omega_1 = \{(\ell, n) | \ell \text{ even and } n - 2t\ell \leqslant 3^{L+1} - 2\},$$

$$\Omega_2 = \{(\ell, n) | \ell \text{ even and } n - 2t\ell = 3^{L+1} - 1\},$$

$$\Omega_3 = \{(\ell, n) | (\ell \text{ even and } n - 2t\ell > 3^{L+1} - 1) \text{ or } \ell \text{ odd}\}.$$

We have

$$h_{\leqslant \ell}(n) = \begin{cases} \varphi_1 = nt - t^2\ell + t\ell + (2t-2)H(\ell) + 2H(\ell-1) + H(n - 2t\ell + 2), & (\ell, n) \in \Omega_1, \\ \varphi_2 = nt - t^2\ell + t\ell + (2t-1)H(\ell) + H(\ell-1) + H(n - 2t\ell + 1), & (\ell, n) \in \Omega_2, \\ \varphi_3 = nt - t^2\ell + t\ell + 2tH(\ell) + H(n - 2t\ell), & (\ell, n) \in \Omega_3. \end{cases}$$

This paper is organized in the following way: Section 2 gives some terminologies and notations. Theorems 1–3 are proved in Section 4. Then Section 3 contains a series of lemmas which are needed in proving Theorems 1–3.

2. Terminologies and notations

Let $S = \{1, 2, \ldots, n\}$ be the *initial set of n suspectable coins*. $A : B$ is called a *test-set* if $A, B \subset S$, $A \cap B = \emptyset$ and $|A| = |B|$ (no information can be obtained by weighing two unequal-sized sets, see [1]). A weighing $A : B$ means that we perform the weighing of A against B and A, B is placed on the left, the right pan of the two-arms balance, respectively. The outcome of one weighing must be one of the three possible *feedbacks*: "left-heavy", "right-heavy" or "equal" (denoted by $f = -1$, $f = 1$, $f = 0$, respectively):

$f = -1$ means that subset A is heavier than subset B, i.e., the heavy coin is contained in A;

$f = 1$ means that subset B is heavier than subset A, i.e., the heavy coin is contained in B;

$f = 0$ means that subsets A and B have equal weight, i.e., the heavy coin is contained in $S - A - B$.

When weighing $A : B$ is performed and we receive a feedback f, a search domain being consistent with the feedback f can be determined uniquely, denoted by S^f. Generally, for any integer $i \geqslant 1$, $S^{f_1 f_2 \cdots f_i}$ denotes the search domain determined by the feedback sequence $f_1 f_2 \cdots f_i$ of these i weighings. A search domain $S^{f_1 f_2 \cdots f_i}$ is called to be *final* if $|S^{f_1 f_2 \cdots f_i}| = 1$. A sequential algorithm of the restricted model is called *ℓ-admissible* if the weighing $A : B$ of any search domain $S^{f_1 f_2 \cdots f_i}$ satisfies $|A| = |B| \leqslant \ell$.

We call a tree 3-ary if each node has at most three sons, called 0-son, 1-son and (-1)-son, respectively. A sequential algorithm of the restricted model can be represented by a 3-ary tree T whose root corresponds to the initial search domain S and whose *leaves* correspond to the *final* search domains; each *internal node* corresponds to a search domain $S^{f_1 f_2 \cdots f_i}$.

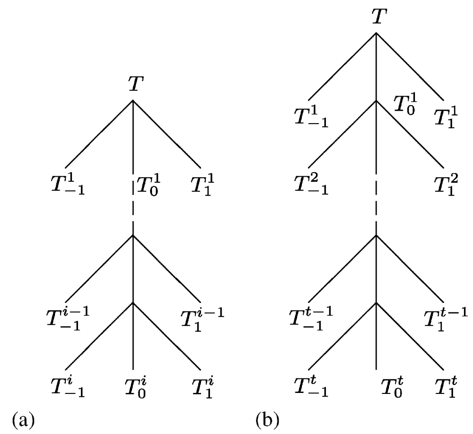

Fig. 1.

If $S^{f_1 f_2 \cdots f_i} \neq \emptyset$ then the tree T contains a node labelled by $S^{f_1 f_2 \cdots f_i}$ whose *f-son search domain* exists (labelled by $S^{f_1 f_2 \cdots f_i f}$) if $S^{f_1 f_2 \cdots f_i f} \neq \emptyset$, it does not exist otherwise. A 3-ary tree T is called *ℓ-admissible* if the corresponding algorithm is ℓ-admissible. It is obvious that an ℓ_1-admissible tree must be an ℓ_2-admissible tree if $\ell_1 \leqslant \ell_2$.

Given a 3-ary tree T, we indicate by T^1_f $(f = 0, -1, 1)$ the subtrees of T. Generally, we indicate by T^i_f the subtree of T rooted at the f-son of the node $S^{f_1 \cdots f_{i-1}}$, where $f_1 = f_2 = \cdots = f_{i-1} = 0$, i.e., T^2_f is the f-son of T^1_0, T^3_f is the f-son of $T^2_0, \ldots, T^i_f$ is the f-son of T^{i-1}_0, see Fig. 1(a). By $|T|$ we denote the number of leaves of T.

3. Huffman tree

Given a $(q+1)$-ary tree T with n leaves, let $h(i, T)$ be the length of the leaf i in T, i.e., the distance of i from the root of T to i. The *external path length* of T is defined by

$$h(T) = \sum_{i=1}^{n} h(i, T). \tag{1}$$

Let $H(n) = \min h(T)$, where the minimum is taken over all $(q+1)$-ary tree T with n leaves. We call tree T^* *Huffman tree* if $h(T^*) = H(n)$. Determining the quantity $H(n)$ and obtaining the structure of Huffman tree T^* is called the *Huffman problem*. The following Lemma 4 is the solution to Huffman problem. By $\lfloor x \rfloor$ and $\lceil x \rceil$ we denote the maximal integer $\leqslant x$ and the minimal integer $\geqslant x$, respectively.

Lemma 4. *Given an integer n with $(q+1)^L \leqslant n < (q+1)^{L+1}$. Let $n = (q+1)^L + qk + j$, where $0 \leqslant k < (q+1)^L$, $0 \leqslant j \leqslant q-1$. T^* is a Huffman tree if and only if T^* has $n - \lceil (qk+j)(q+1)/q \rceil$ leaves at level L and $\lceil (qk+j)(q+1)/q \rceil$ leaves at level $L+1$. Moreover,*

$$H(n) = n \lfloor \log_{q+1} n \rfloor + \lceil (qk+j)(q+1)/q \rceil. \tag{2}$$

Let T_L be the tree with $(q+1)^L$ leaves on level L. A Huffman tree with n leaves can be obtained from T_L by changing k leaves into internal nodes each having $q+1$ sons if $j=0$, and one more leaf into internal node having $j+1$ sons if $j>0$.

Proof. See [1,4]. □

By $\mathscr{T}_{\leqslant \ell}(n)$ we denote the class of all 3-ary ℓ-admissible trees with n leaves, and let

$$h_{\leqslant \ell}(n) = \min\{h(T) \mid T \in \mathscr{T}_{\leqslant \ell}(n)\}. \tag{3}$$

The aim of the restricted model is to determine the quantity $\bar{L}_{\leqslant \ell}(n) \triangleq h_{\leqslant \ell}(n)/n$ for all integers $n \geqslant 2$ and $\ell \geqslant 1$. A 3-ary tree T^* with n leaves is called *ℓ-optimal* if T^* is ℓ-admissible and $h(T^*) = h_{\leqslant \ell}(n)$. Essentially, determining the quantity $h_{\leqslant \ell}(n)$ and obtaining the structure of ℓ-optimal tree T^* is a special case of the Huffman problem because we are required to obtain a restricted 3-ary tree (*ℓ-admissible* tree) T^* with $h(T^*) = h_{\leqslant \ell}(n)$, where the restrictions are required by the test device (two-arms balance) and the condition that we are only allowed to test at most ℓ coins in each pan at every weighing.

The restricted model is relative to the Huffman problem. For easy citation, we now give an equivalent formula of $H(n)$. Given an integer n with $3^L \leqslant n < 3^{L+1}$, we shall represent n as

$$n = 3^L + 2k + j \quad \text{for some } 0 \leqslant k < 3^L, \quad 0 \leqslant j \leqslant 1.$$

By letting $q = 2$ in Eq. (2), we have

$$H(n) = n\lfloor \log_3 n \rfloor + \left\lceil \frac{3(2k+j)}{2} \right\rceil = n\lfloor \log_3 n \rfloor + 3k + 2j. \tag{4}$$

It follows from Lemma 4 that $H(n)$ is a lower bound of $h_{\leqslant \ell}(n)$, i.e., $h_{\leqslant \ell}(n) \geqslant H(n)$ for all integers $n \geqslant 2$ and $\ell \geqslant 1$. We also define $H(0) = H(1) = 0$.

Lemma 5. *For any two integers $3^L \leqslant n < 3^{L+1}$ and $d \geqslant 0$, we have*

$$H(n+1) - H(n) = \begin{cases} \lfloor \log_3 n \rfloor + 1 & \text{if } n \text{ even}, \\ \lfloor \log_3 n \rfloor + 2 & \text{if } n \text{ odd}, \end{cases} \tag{5}$$

$$\lfloor \log_3 n \rfloor + 1 \leqslant H(n+1) - H(n) \leqslant \lfloor \log_3 n \rfloor + 2, \tag{6}$$

$$d(\lfloor \log_3 n \rfloor + 1) \leqslant H(n+d) - H(n) \leqslant d(\lfloor \log_3 (n+d) \rfloor + 2), \tag{7}$$

$$H(n+2) - H(n) = \begin{cases} 2\lfloor \log_3 n \rfloor + 4 & \text{if } n = 3^{L+1} - 1, \\ 2\lfloor \log_3 n \rfloor + 3 & \text{otherwise}, \end{cases} \tag{8}$$

$$H(n+2d) - H(n) \leqslant d(2\lfloor \log_3 (n+2d) \rfloor + 3) \quad \text{if } n \text{ odd}, \tag{9}$$

$$H(n+2d) - H(n) \geqslant d(2\lfloor \log_3 n \rfloor + 3). \tag{10}$$

Proof. We represent n as $n = 3^L + 2k + j$ for some $0 \leqslant k < 3^L$, $0 \leqslant j \leqslant 1$. Thus $L = \lfloor \log_3 n \rfloor$.

(i) If n is odd, then $j = 0$ and thus $\lfloor \log_3 n \rfloor = \lfloor \log_3 (n+1) \rfloor = L$. So,

$$\begin{aligned} H(n+1) - H(n) &= (n+1) \cdot L + 3k + 2 - (n \cdot L + 3k) \\ &= L + 2 \\ &= \lfloor \log_3 n \rfloor + 2. \end{aligned}$$

If n is even, then $j = 1$. For $0 \leqslant k < 3^L - 1$, we have $\lfloor \log_3 n \rfloor = \lfloor \log_3 (n+1) \rfloor = L$. Therefore,

$$\begin{aligned} H(n+1) - H(n) &= H(3^L + 2(k+1)) - H(3^L + 2k + 1) \\ &= (3^L + 2(k+1)) \cdot L + 3(k+1) - [(3^L + 2k + 1) \cdot L + 3k + 2] \\ &= L + 1 \\ &= \lfloor \log_3 n \rfloor + 1. \end{aligned}$$

For $k = 3^L - 1$, i.e., $n = 3^L + 2(3^L - 1) + 1 = 3^{L+1} - 1$ and $n + 1 = 3^{L+1}$, we have $\lfloor \log_3 n \rfloor = L$ and $\lfloor \log_3 (n+1) \rfloor = L + 1$. Therefore,

$$\begin{aligned} H(n+1) - H(n) &= (n+1) \cdot (L+1) - [n \cdot L + 3 \cdot (3^L - 1) + 2] \\ &= L + 1 \\ &= \lfloor \log_3 n \rfloor + 1. \end{aligned}$$

(ii) Eq. (6) is obvious in view of Eq. (5).

(iii) Eq. (7) can be obtained by the equality

$$H(n+d)-H(n)=\sum_{i=1}^{d}(H(n+i)-H(n+i-1))$$

and the following two inequalities:

$$H(n+i)-H(n+i-1)\geqslant\lfloor\log_3(n+i-1)\rfloor+1\geqslant\lfloor\log_3 n\rfloor+1,$$
$$H(n+i)-H(n+i-1)\leqslant\lfloor\log_3(n+i-1)\rfloor+2\leqslant\lfloor\log_3(n+d)\rfloor+2.$$

(iv) If $3^L\leqslant n\leqslant 3^{L+1}-3$, then $0\leqslant k\leqslant 3^L-2$. It follows Eq. (4) that

$$\begin{aligned}H(n+2)-H(n)&=H(3^L+2(k+1)+j)-H(3^L+2k+j)\\&=(n+2)\cdot L+3(k+1)+2j-[n\cdot L+3k+2j]\\&=2\lfloor\log_3 n\rfloor+3.\end{aligned}$$

If $n=3^{L+1}-2=3^L+2(3^L-1)$, then $\lfloor\log_3 n\rfloor=L$. It follows Eq. (4) that

$$\begin{aligned}H(n+2)-H(n)&=H(3^{L+1})-H(3^{L+1}-2)\\&=3^{L+1}\cdot(L+1)-[(3^{L+1}-2)\cdot L+3(3^L-1)]\\&=2\lfloor\log_3 n\rfloor+3.\end{aligned}$$

If $n=3^{L+1}-1=3^L+2\cdot(3^L-1)+1$, Eq. (4) gives

$$\begin{aligned}H(n+2)-H(n)&=H(3^{L+1}+1)-H(3^L+2\cdot(3^L-1)+1)\\&=(n+2)\cdot(L+1)+2-[n\cdot L+3(3^L-1)+2]\\&=2L+4\\&=2\lfloor\log_3 n\rfloor+4.\end{aligned}$$

(v) Eq. (9) can be obtained by the equality

$$H(n+2d)-H(n)=\sum_{i=1}^{d}(H(n+2i)-H(n+2i-2))=\sum_{i=1}^{d}(2\lfloor\log_3(n+2i-2)\rfloor+3)$$

and the inequality $\lfloor\log_3(n+2i-2)\rfloor\leqslant\lfloor\log_3(n+2d)\rfloor$ for $1\leqslant i\leqslant d$.

(vi) Eq. (10) can be obtained by the inequality

$$H(n+2d)-H(n)=\sum_{i=1}^{d}(H(n+2i)-H(n+2i-2))\geqslant\sum_{i=1}^{d}(2\lfloor\log_3(n+2i-2)\rfloor+3)$$

and the inequality $\lfloor\log_3(n+2i-2)\rfloor\geqslant\lfloor\log_3 n\rfloor$ for $1\leqslant i\leqslant d$.

Lemma 6. *Suppose that $\ell\geqslant 1$ is odd and $m\geqslant 2\ell$. $H(i)+H(m-i)\geqslant H(\ell)+H(m-\ell)$ holds for $0\leqslant i\leqslant\ell$.*

Proof. *Case* (1) *i odd*: Now $1\leqslant i\leqslant\ell$. It follows from Eqs. (9) and (10) that

$$H(i)-H(\ell)\geqslant-\frac{\ell-i}{2}(2\lfloor\log_3\ell\rfloor+3),\tag{11}$$

$$H(m-i)-H(m-\ell)\geqslant\frac{\ell-i}{2}(2\lfloor\log_3(m-\ell)\rfloor+3).\tag{12}$$

It is obvious that $\lfloor\log_3(m-\ell)\rfloor\geqslant\lfloor\log_3\ell\rfloor$ in view of $m-\ell\geqslant\ell$. Combining with Eqs. (11) and (12), we have

$$\begin{aligned}H(i)+H(m-i)&\geqslant H(\ell)+H(m-\ell)+(\ell-i)(\lfloor\log_3(m-\ell)\rfloor-\lfloor\log_3\ell\rfloor)\\&\geqslant H(\ell)+H(m-\ell).\end{aligned}$$

Case (2) *i even*: Now $0 \leqslant i+1 \leqslant \ell$. If $i=0$, since $m \geqslant 2\ell \geqslant 2$, by Eq. (5), $H(i)+H(m)=H(m)=H(1)+H(m) \geqslant H(1)+H(m-1)$, and by case (1),

$$H(0)+H(m) \geqslant H(1)+H(m-1) \geqslant H(\ell)+H(m-\ell).$$

If $i>0$, it follows Eqs. (5) and (6) that $H(i)=H(i+1)-\lfloor \log_3 i \rfloor -1$, $H(m-i) \geqslant H(m-i-1)+\lfloor \log_3(m-i-1)\rfloor+1$. We note that $m-i-1>i$ as $m \geqslant 2\ell \geqslant 2i+2$. Thus

$$\begin{aligned} H(i)+H(m-i) &\geqslant H(i+1)+H(m-i-1)+(\lfloor \log_3(m-i-1)\rfloor - \lfloor \log_3 i \rfloor) \\ &\geqslant H(i+1)+H(m-i-1) \\ &\geqslant H(\ell)+H(m-\ell). \end{aligned}$$

The last inequality is obtained by case (1) in view of $i+1 \leqslant \ell$ and $i+1$ odd. □

Lemma 7. *Suppose that* $\ell \geqslant 1$ *is odd and* $m \geqslant 3\ell$. $H(i)+H(j)+H(m-i-j) \geqslant 2H(\ell)+H(m-2\ell)$ *holds for* $0 \leqslant i \leqslant j \leqslant \ell$.

Proof. We note that $m-j \geqslant 2\ell$ and $m-\ell \geqslant 2\ell$, by virtue of $m \geqslant 3\ell$ and $0 \leqslant i \leqslant j \leqslant \ell$. It follows Lemma 6 that

$$H(i)+H(m-j-i) \geqslant H(\ell)+H(m-j-\ell),$$

$$H(j)+H(m-\ell-j) \geqslant H(\ell)+H(m-2\ell).$$

Therefore, $H(i)+H(j)+H(m-i-j) \geqslant 2H(\ell)+H(m-2\ell)$. The proof is complete. □

Lemma 8. *Given integers* $\ell \geqslant 1$, $t>1$, $m>(2t-1)\ell$ *and* $0 \leqslant a_{-1}^i, a_1^i \leqslant \ell$ *for* $i=1,2,\ldots,t-1$. *Let* $c_{t-1}=\sum_{i=1}^{t-1}(a_1^i+a_{-1}^i)$, $\Phi_t(m)=\sum_{i=1}^{t-1}\{i\cdot(a_{-1}^i+a_1^i)+H(a_{-1}^i)+H(a_1^i)\}+t\cdot(m-c_{t-1})+H(m-c_{t-1})$. *Then*

$$\Phi_t(m) \geqslant mt-t^2\ell+t\ell+2(t-1)H(\ell)+H(m-2(t-1)\ell).$$

Proof. We proceed by induction on *t*. *Induction basis* ($t=2$). Now $m>3\ell$. If $a_1^1>0$ and $a_{-1}^1>0$, it follows from Eq. (7) that

$$H(a_1^1) \geqslant H(\ell)-(\ell-a_1^1)(\lfloor \log_3 \ell \rfloor+2), \tag{13}$$

$$H(a_{-1}^1) \geqslant H(\ell)-(\ell-a_{-1}^1)(\lfloor \log_3 \ell \rfloor+2), \tag{14}$$

$$H(m-a_1^1-a_{-1}^1) \geqslant H(m-2\ell)+(2\ell-a_1^1-a_{-1}^1)(\lfloor \log_3(m-2\ell)\rfloor+1). \tag{15}$$

It is easy to prove that Eqs. (13)–(15) hold if $a_1^1=0$ or $a_{-1}^1=0$. It is obvious that $\lfloor \log_3(m-2\ell)\rfloor \geqslant \lfloor \log_3 \ell \rfloor$ in view of $m>3\ell$. Thus,

$$H(a_1^1)+H(a_{-1}^1)+H(m-a_1^1-a_{-1}^1) \geqslant 2H(\ell)+H(m-2\ell)-(2\ell-a_1^1-a_{-1}^1),$$

$$\begin{aligned} \Phi_2(m) &= a_{-1}^1+a_1^1+H(a_{-1}^1)+H(a_1^1)+2\cdot(m-a_1^1-a_{-1}^1)+H(m-a_1^1-a_{-1}^1) \\ &\geqslant 2m-2\ell+2H(\ell)+H(m-2\ell). \end{aligned}$$

Induction step ($t>2$). If $a_1^{t-1}>0$, $a_{-1}^{t-1}>0$, it follows from Eq. (7) that

$$H(a_{-1}^{t-1}) \geqslant H(\ell)-(\ell-a_{-1}^{t-1})(\lfloor \log_3 \ell \rfloor+2), \tag{16}$$

$$H(a_1^{t-1}) \geqslant H(\ell)-(\ell-a_1^{t-1})(\lfloor \log_3 \ell \rfloor+2), \tag{17}$$

$$H(m-c_{t-2}-a_1^{t-1}-a_{-1}^{t-1}) \geqslant H(m-c_{t-2}-2\ell)+(2\ell-a_1^{t-1}-a_{-1}^{t-1})(\lfloor \log_3(m-c_{t-2}-2\ell)\rfloor+1). \tag{18}$$

It is easy to prove that Eqs. (16)–(18) hold if $a_1^{t-1}=0$ or $a_{-1}^{t-1}=0$. The assumptions $m>(2t-1)\ell$ and $0\leqslant a_{-1}^i, a_1^i\leqslant \ell$ $(i=1,2,\ldots,t-1)$ give $c_{t-2}=\sum_{i=1}^{t-2}(a_1^i+a_{-1}^i)\leqslant 2(t-2)\ell$ and $m-c_{t-2}>3\ell$, i.e., $\lfloor \log_3(m-c_{t-2}-2\ell)\rfloor \geqslant \lfloor \log_3 \ell\rfloor$. It follows Eqs. (16)–(18) that

$$\begin{aligned}&H(a_{-1}^{t-1})+H(a_1^{t-1})+H(m-c_{t-2}-a_1^{t-1}-a_{-1}^{t-1})\\&\quad\geqslant 2H(\ell)+H(m-c_{t-2}-2\ell)-2\ell+a_1^{t-1}+a_{-1}^{t-1}.\end{aligned} \tag{19}$$

By the definition of $\Phi_t(m)$ and Eq. (19), we have

$$\begin{aligned}\Phi_t(m)&=\Phi_{t-1}(m)+m-c_{t-1}+H(a_{-1}^{t-1})+H(a_1^{t-1})+H(m-c_{t-1})-H(m-c_{t-2})\\&\geqslant \Phi_{t-1}(m)+m-2\ell-c_{t-2}+2H(\ell)+H(m-2\ell-c_{t-2})-H(m-c_{t-2}),\end{aligned}$$

and

$$\Phi_{t-1}(m)=\Phi_{t-1}(m-2\ell)+2\ell(t-1)+H(m-c_{t-2})-H(m-2\ell-c_{t-2}). \tag{20}$$

Thus,

$$\Phi_t(m)\geqslant \Phi_{t-1}(m-2\ell)+m+2\ell(t-2)-c_{t-2}+2H(\ell). \tag{21}$$

We note that $m-2\ell>(2(t-1)-1)\ell$. The induction hypothesis implies that

$$\Phi_{t-1}(m-2\ell)\geqslant (m-2\ell)(t-1)-(t-1)^2\ell+(t-1)\ell+2(t-2)H(\ell)+H(m-2t\ell+2\ell).$$

This inequality, together with Eq. (21), gives

$$\Phi_t(m)\geqslant mt-t^2\ell+t\ell+2(t-1)H(\ell)+H(m-2t\ell+2\ell),$$

where the inequality $c_{t-2}\leqslant 2(t-2)\ell$ is used. □

Lemma 9. (1) *For any integer n with $2\leqslant n\leqslant 3^{L+1}$ and $n\neq 6$, there exists a 3^L-admissible tree T with n leaves such that $h(T)=H(n)$*; (2) *If there exists a good coin available at the beginning, then we can construct a 3^L-admissible tree T with n leaves such that $h(T)=H(n)$ for any integer n with $2\leqslant n\leqslant 3^{L+1}$.*

Proof. We recall that an ℓ_1-admissible tree must be an ℓ_2-admissible tree if $\ell_1\leqslant\ell_2$. So we assume $3^L<n\leqslant 3^{L+1}$ and prove these two conclusions simultaneously by induction on L. For $L=0$, i.e., $n\in\{2,3\}$, the desired 1-admissible trees are given in Fig. 2(a), (b).

Induction basis $L=1$. For $n\in\{4,5,7,8,9\}$, the desired 3-admissible trees are given in Fig. 2(c)–(g). The desired 3-admissible tree T with $h(T)=H(6)=11$ is given in Fig. 2(h), where x^* denotes the given good coin.

Induction step $L>1$. For $n=3^{L+1}$, we choose $A:B$ as the first weighing, where $A,B\subset S$ and $|A|=|B|=3^L$, then $|C|=|S-A-B|=3^L$. After this weighing, the three resulting search domain are $S^{-1}=A$, $S^1=B$ and $S^0=C$. For any feedback $f\in\{-1,1,0\}$ of the given weighing, there is at least one coin known to be good out of the search domain S^f because $|A|,|B|,|C|\geqslant 1$. The induction hypothesis implies that there are 3^{L-1}-admissible trees T_{-1}^1, T_1^1, T_0^1 such that $h(T_{-1}^1)=H(|A|)$, $h(T_1^1)=H(|B|)$, $h(T_0^1)=H(|C|)$, respectively. Therefore,

$$\begin{aligned}h(T)&=n+h(T_1^1)+h(T_{-1}^1)+h(T_0^1)\\&=n+3^L\cdot L+3^L\cdot L+3^L\cdot L\\&=H(n).\end{aligned}$$

For $3^L<n<3^{L+1}$, we represent n as $n=3^L+2k+j$, $0\leqslant k<3^L$, $0\leqslant j\leqslant 1$. It is necessary to distinguish two subcases:

Subcase (1) $0\leqslant k<2\cdot 3^{L-1}$: We choose $A:B$ as the first weighing, where $A,B\subset S$, $|A|=|B|=3^{L-1}+2\lfloor k/2\rfloor<3^L$, and then $|C|=|S-A-B|=3^{L-1}+2(k-2\lfloor k/2\rfloor)+j\leqslant 3^{L-1}+2+1<3^L$. After this weighing, the three resulting

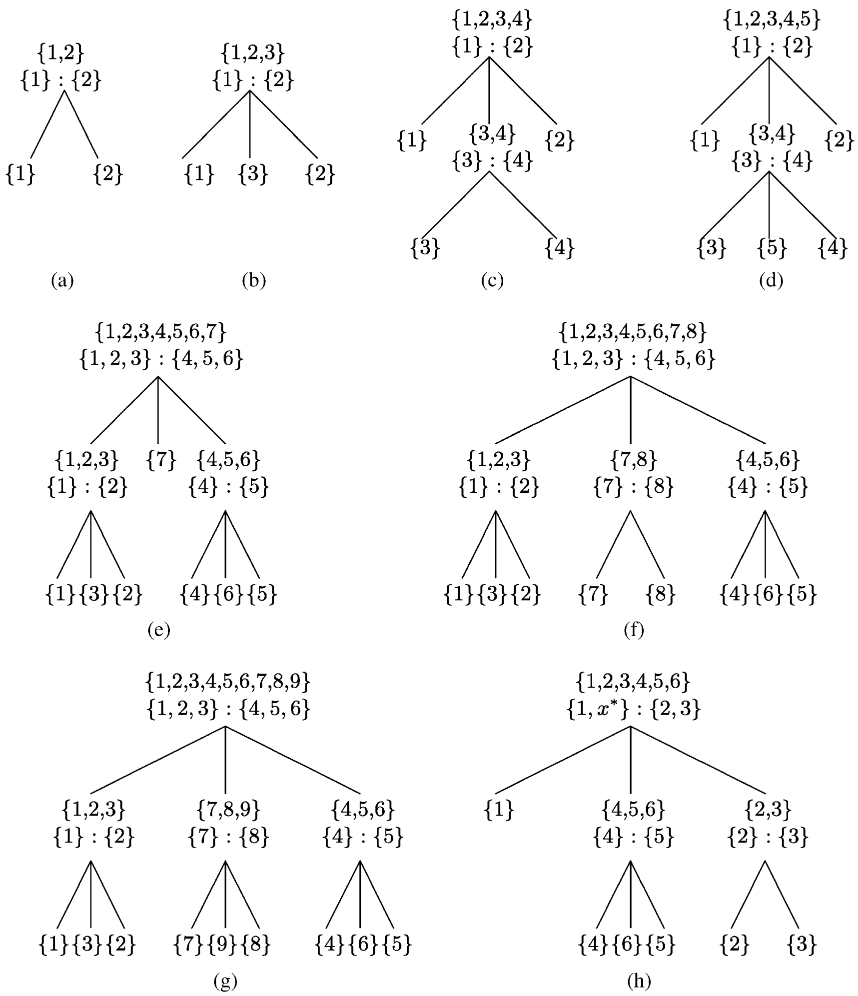

Fig. 2. Admissible trees with $h(T) = H(n)$ for $n \in \{2, 3, 4, 5, 6, 7, 8, 9\}$.

search domain are $S^{-1} = A$, $S^1 = B$ and $S^0 = C$. Similarly, there is at least one coin known to be good out of the search domain S^f, $f \in \{-1, 1, 0\}$ because $|A|, |B|, |C| \geqslant 1$. The induction hypothesis implies that there are 3^{L-1}-admissible trees T^1_{-1}, T^1_1, T^1_0 such that $h(T^1_{-1}) = H(|A|)$, $h(T^1_1) = H(|B|)$, $h(T^1_0) = H(|C|)$, respectively. Thus,

$$\begin{aligned}
h(T) &= n + h(T^1_1) + h(T^1_{-1}) + h(T^1_0) \\
&= n + 2H\left(3^{L-1} + 2\left\lfloor \frac{k}{2} \right\rfloor\right) + H\left(3^{L-1} + 2\left(k - 2\left\lfloor \frac{k}{2} \right\rfloor\right) + j\right) \\
&= n + 2\left\{\left(3^{L-1} + 2\left\lfloor \frac{k}{2} \right\rfloor\right)(L-1) + 3\left\lfloor \frac{k}{2} \right\rfloor\right\} + \left(3^{L-1} + 2\left(k - 2\left\lfloor \frac{k}{2} \right\rfloor\right) + j\right)(L-1) \\
&\quad + 3\left(k - 2\left\lfloor \frac{k}{2} \right\rfloor\right) + 2j \\
&= H(n).
\end{aligned}$$

It is obvious that T is a 3^L-admissible tree.

Subcase (2) $2 \cdot 3^{L-1} \leqslant k$: We choose $A : B$ as the first weighing, where $A, B \subset S$, $|A| = |B| = 3^L$. Then $|C| = |S - A - B| = 3^{L-1} + 2(k - 2 \cdot 3^{L-1}) + j \leqslant 3^{L-1} + 2(3^{L-1} - 1) + j = 3^L - 2 + j < 3^L$. By the same arguments, we have

$$\begin{aligned} h(T) &= n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \\ &= n + 2\cdot 3^L \cdot L + (3^{L-1} + 2(k - 2\cdot 3^{L-1}) + j)(L-1) + 3(k - 2\cdot 3^{L-1}) + 2j \\ &= H(n). \end{aligned}$$

The proof is complete. □

4. Proofs of main results

Proof of Theorem 1. (1) $n=6$ and $\ell \geqslant 1$. It is easy to check $h(T) = 12 = H(6)+1$ for any ℓ-admissible tree T, see [1, p. 82].

(2) $n < 3^{L+1}$ and $n \neq 6$. On the one hand, $h_{\leqslant \ell}(n) \geqslant H(n)$ is true for any integer ℓ; On the other hand, Lemma 9 and $3^L \leqslant \ell$ show that $h_{\leqslant \ell}(n) \leqslant h_{\leqslant 3^L}(n) \leqslant H(n)$.

(3) $3^{L+1} \leqslant n \leqslant 3\ell$. Let $\ell = 3^L + 2\bar{k} + \bar{j}$, $0 \leqslant \bar{k} < 3^L$, $0 \leqslant \bar{j} \leqslant 1$, and let $n = 3^{L+1} + 2k + j$, $0 \leqslant j \leqslant 1$. Obviously, $0 \leqslant k \leqslant 3\bar{k} + \bar{j}$.

Case (3.1) $k \equiv 0 \bmod (3)$ or $k \equiv 2 \bmod (3)$: If $k \equiv 0 \bmod (3)$, then $\lceil k/3 \rceil = k/3 \leqslant \bar{k}$; If $k \equiv 2 \bmod (3)$, then $\lceil k/3 \rceil = (k+1)/3 \leqslant \bar{k}$. So we choose $A : B$ as the first weighing, where $A, B \subset S$, $|A| = |B| = 3^L + 2\lceil k/3 \rceil \leqslant 3^L + 2\bar{k} \leqslant \ell < 3^{L+1}$. Then $|C| = |S - A - B| = 3^L + 2(k - 2\lceil k/3 \rceil) + j < 3^{L+1}$ as $k - 2\lceil k/3 \rceil \leqslant \lceil k/3 \rceil \leqslant \bar{k} < 3^L$ and $0 \leqslant j \leqslant 1$. It follows from Lemma 9(2) that there are 3^L-admissible trees T_{-1}^1, T_1^1, T_0^1 such that $h(T_{-1}^1) = H(|A|)$, $h(T_1^1) = H(|B|)$ and $h(T_0^1) = H(|C|)$. Thus,

$$\begin{aligned} h(T) &= n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \\ &= n + 2H\left(3^L + 2\left\lceil \frac{k}{3} \right\rceil\right) + H\left(3^L + 2\left(k - 2\left\lceil \frac{k}{3} \right\rceil\right) + j\right) \\ &= H(n). \end{aligned}$$

Case (3.2) $k \equiv 1 \bmod (3)$ and $k < 3\bar{k} + \bar{j}$: In this case $\lfloor k/3 \rfloor = (k-1)/3 \leqslant \bar{k} - 1$. We choose $A : B$ as the first weighing, where $A, B \subset S$, $|A| = |B| = 3^L + 2\lfloor k/3 \rfloor < 3^L + 2\bar{k} \leqslant \ell < 3^{L+1}$. Then $|C| = |S - A - B| = 3^L + 2(k - 2\lfloor k/3 \rfloor) + j = 3^L + 2(((k-1)/3) + 1) + j \leqslant 3^L + 2\bar{k} + j < 3^{L+1}$. It follows from Lemma 9(2) that there are 3^L-admissible trees T_{-1}^1, T_1^1 and T_0^1 such that $h(T_{-1}^1) = H(|A|)$, $h(T_1^1) = H(|B|)$ and $h(T_0^1) = H(|C|)$. Thus,

$$\begin{aligned} h(T) &= n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \\ &= n + 2H\left(3^L + 2\left\lfloor \frac{k}{3} \right\rfloor\right) + H\left(3^L + 2\left(k - 2\left\lfloor \frac{k}{3} \right\rfloor\right) + j\right) \\ &= H(n). \end{aligned}$$

Case (3.3) $k \equiv 1 \bmod (3)$ and $k = 3\bar{k} + \bar{j}$: In this case, $\bar{j} = 1$.

Subcase-1 $\bar{k} < 3^L - 1$: We choose $A : B$ as the first weighing, where $A, B \subset S$, $|A| = |B| = 3^L + 2\lfloor k/3 \rfloor = 3^L + 2\bar{k} < \ell < 3^{L+1}$. Then $|C| = |S - A - B| = n - 2(3^L + 2\lfloor k/3 \rfloor) = 3^L + 2(k - 2\bar{k}) + j = 3^L + 2(\bar{k}+1) + j < 3^{L+1}$. It follows from Lemma 9(2) that there are 3^L-admissible trees T_{-1}^1, T_1^1, T_0^1 such that $h(T_{-1}^1) = H(|A|)$, $h(T_1^1) = H(|B|)$ and $h(T_0^1) = H(|C|)$. Thus,

$$\begin{aligned} h(T) &= n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \\ &= n + 2H(3^L + 2\bar{k}) + H(3^L + 2(\bar{k} + \bar{j}) + j) \\ &= H(n). \end{aligned}$$

Subcase-2 $\bar{k} = 3^L - 1$ *and* $j = 0$: In this case, $\ell = 3^{L+1} - 1$ and $n = 3^{L+2} - 4 = 3\ell - 1$. We choose $A : B$ as the first weighing, where $A, B \subset S$, $|A| = |B| = 3^L + 2\lfloor k/3 \rfloor = 3^L + 2\bar{k} = 3^{L+1} - 2 < \ell < 3^{L+1}$. Then $|C| = |S - A - B| = n - 2(3^L + 2\lfloor k/3 \rfloor) = 3^L + 2(k - 2\bar{k}) = 3^L + 2(\bar{k} + \bar{j}) = 3^{L+1}$. It follows from Lemma 9(2) that there are 3^L-admissible trees T_{-1}^1, T_1^1, T_0^1 such that $h(T_{-1}^1) = H(|A|)$, $h(T_1^1) = H(|B|)$ and $h(T_0^1) = H(|C|)$.

Thus,

$$\begin{aligned} h(T) &= n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \\ &= n + 2H(3^{L+1} - 2) + H(3^{L+1}) \\ &= 3^{L+2} - 4 + 2((3^{L+1} - 2)L + 3(3^L - 1)) + 3^{L+1}(L + 1) \\ &= H(n). \end{aligned}$$

Subcase-3 $\bar{k} = 3^L - 1$ *and* $j = 1$: In this case, $\ell = 3^{L+1} - 1$ and $n = 3^{L+2} - 3 = 3\ell$. We will prove $h_{\leqslant \ell}(n) = H(n) + 1$. To prove $h_{\leqslant \ell}(n) \geqslant H(n) + 1$, it is enough to show that $h(T) \geqslant H(n) + 1$ for any ℓ-admissible tree T with n leaves. Let $A : B$ be the first weighing of T, where $|A| = |B| = i \leqslant \ell$, and T_{-1}^1, T_1^1, T_0^1 be the subtrees of T. Then $|T_1^1| = |T_{-1}^1| = i$, $|T_0^1| = n - 2i$. We have

$$h(T) = n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \geqslant n + 2H(i) + H(n - 2i). \tag{22}$$

If $i < \ell$, Lemma 7 shows that $2H(i) + H(n - 2i) \geqslant 2H(\ell - 1) + H(n - 2\ell + 2)$ as $n = 3\ell > 3(\ell - 1)$ and $\ell - 1$ odd. Therefore, $h(T) \geqslant n + 2H(\ell - 1) + H(n - 2\ell + 2) = 3^{L+2} - 3 + 2H(3^{L+1} - 2) + H(3^{L+1} + 1) = H(n) + 1$.

If $i = \ell$, Eq. (22) gives $h(T) \geqslant n + 3H(\ell) = 3^{L+2} - 3 + 3H(3^{L+1} - 1) = H(n) + 1$.

We now prove that $h_{\leqslant \ell}(n) \leqslant H(n) + 1$. It is enough to construct an ℓ-admissible tree T with $h(T) = H(n) + 1$. We choose $A : B$ as the first weighing, where $A, B \subset S$, $|A| = |B| = \ell < 3^{L+1}$. Then $|C| = |S - A - B| = \ell$. It follows from Lemma 9(2) that there are 3^L-admissible trees T_{-1}^1, T_1^1, T_0^1 such that $h(T_{-1}^1) = H(|A|)$, $h(T_1^1) = H(|B|)$ and $h(T_0^1) = H(|C|)$. Thus, $h(T) = n + h(T_{-1}^1) + h(T_{-1}^1) + h(T_0^1) = n + 3H(\ell) = n + 3H(3^{L+1} - 1) = H(n) + 1$. Obviously, T is ℓ-admissible in view of $3^L \leqslant \ell$. □

Corollary 10. *For* $3^L \leqslant \ell < 3^{L+1}$ *and* $n \leqslant 3\ell$, *if there is a good coin available at the beginning, then there exists an* ℓ-*admissible tree* T *with* n *leaves such that* $h(T) = H(n)$.

Proof. By Lemma 9(2) and Theorem 1, we need only to prove that Corollary 10 holds for $\ell = 3^{L+1} - 1$ and $n = 3\ell$. We choose $A : B \cup x^*$ as the first weighing, where x^* denotes the given good coin, and $|A| = \ell < 3^{L+1}$, $|B| = \ell - 1 < 3^{L+1}$. Then $|C| = |S - A - B| = \ell + 1 = 3^{L+1}$. It follows from Lemma 9(2) that there are 3^L-admissible trees T_{-1}^1, T_1^1, T_0^1 such that $h(T_{-1}^1) = H(|A|)$, $h(T_1^1) = H(|B|)$ and $h(T_0^1) = H(|C|)$. Thus,

$$\begin{aligned} h(T) &= n + h(T_1^1) + h(T_{-1}^1) + h(T_0^1) \\ &= n + H(3^{L+1} - 1) + H(3^{L+1} - 2) + H(3^{L+1}) \\ &= H(n). \end{aligned}$$

Obviously, T is ℓ-admissible in view of $3^L \leqslant \ell$. □

Example. We consider $\ell = 8$, $n = 3\ell = 24$. Theorem 1 shows that $h_{\leqslant \ell}(24) = H(24) + 1$. However, if there exists a good coin x^* available at the beginning, then an ℓ-admissible tree T with $h(T) = H(n)$ can be constructed as follows.

Let $S = \{1, 2, 3, \ldots, 24\}$ be the set of $n = 24$ coins. The first weighing of T is chosen as $\{1, 2, \ldots, 8\} : \{9, 10, \ldots, 15, x^*\}$, then the three resulting search domains are $S^{-1} = \{1, 2, \ldots, 8\}$, $S^1 = \{9, 10, \ldots, 15\}$, $S^0 = \{16, 17, \ldots, 24\}$. The structure of subtree T_1^1, T_{-1}^1, T_0^1 is the same with Fig. 2(e)–(g), respectively. We note that $h(T) = n + h(T_{-1}^1) + h(T_1^1) + h(T_0^1) = n + H(7) + H(8) + H(9) = 71 = H(24)$.

Proof of Theorem 2. (1) In order to get the upper bound of $h_{\leqslant \ell}(n)$, it is enough to construct an ℓ-admissible tree T with $h(T) = \phi_i$ according to $(\ell, n) \in \Omega_i$, $i \in \{1, 2\}$.

Case (1.1) $(\ell, n) \in \Omega_1$: We choose $A : B$ as the first weighing of T, where $|A| = |B| = \ell - 1$, $A, B \subset S$, $A \cap B = \emptyset$. Then $|C| = |S - A - B| = n - 2\ell + 2 \leqslant 3^{L+1} \leqslant 3\ell$. Corollary 10 shows that there exist three ℓ-admissible trees T_{-1}^1, T_1^1 and T_0^1 such that $h(T_{-1}^1) = h(T_1^1) = H(\ell - 1)$ and $h(T_0^1) = H(n - 2\ell + 2)$.

Thus,

$$\begin{aligned} h(T) &= n + h(T_{-1}^1) + h(T_1^1) + h(T_0^1) \\ &= n + 2H(\ell - 1) + H(n - 2\ell + 2) \\ &= \phi_1. \end{aligned}$$

Case (1.2) $(\ell, n) \in \Omega_2$: We choose $A : B$ as the first weighing of T, where $|A| = |B| = \ell$, $A, B \subset S$, $A \cap B = \emptyset$. Then $|C| = |S - A - B| = n - 2\ell \leqslant 3\ell$. Corollary 10 shows that there exist three ℓ-admissible trees T_{-1}^1, T_1^1 and T_0^1 such that $h(T_{-1}^1) = h(T_1^1) = H(\ell)$ and $h(T_0^1) = H(n - 2\ell)$. Thus,

$$\begin{aligned} h(T) &= n + h(T_{-1}^1) + h(T_1^1) + h(T_0^1) \\ &= n + 2H(\ell) + H(n - 2\ell) \\ &= \phi_2. \end{aligned}$$

(2) In order to prove $h_{\leqslant \ell}(n) \geqslant \phi_i$, it is enough to show that $h(T) \geqslant \phi_i$ for any ℓ-admissible tree T with n leaves, and for $(\ell, n) \in \Omega_i$, $i \in \{1, 2\}$. Let $A : B$ be the first weighing of T and $|A| = |B| = i$. Then $|T_{-1}^1| = |T_1^1| = i \leqslant \ell$ and $|T_0^1| = n - 2i$. We have

$$h(T) = n + h(T_{-1}^1) + h(T_1^1) + h(T_0^1) \geqslant n + 2H(i) + H(n - 2i). \tag{23}$$

Case (2.1) ℓ *odd*: We note that $n > 3\ell$ and $0 \leqslant i \leqslant \ell$. Lemma 7 shows that $2H(i) + H(n - 2i) \geqslant 2H(\ell) + H(n - 2\ell)$. Therefore, $h(T) \geqslant n + 2H(\ell) + H(n - 2\ell) = \phi_2$.

Case (2.2) ℓ *even*: We note that $n > 3\ell$. If $i = \ell$, then

$$2H(i) + H(n - 2i) = 2H(\ell) + H(n - 2\ell) \triangleq \lambda_1.$$

If $i \leqslant \ell - 1$, then $n > 3\ell > 3(\ell - 1)$ and $\ell - 1$ odd, so Lemma 7 shows that

$$2H(i) + H(n - 2i) \geqslant 2H(\ell - 1) + H(n - 2\ell + 2) \triangleq \lambda_2.$$

We now determine the quantity $\lambda = \min\{\lambda_1, \lambda_2\}$. Obviously,

$$\lambda_2 - \lambda_1 = 2(H(\ell - 1) - H(\ell)) + (H(n - 2\ell + 2) - H(n - 2\ell)). \tag{24}$$

Since $3^L \leqslant \ell < 3^{L+1}$ and ℓ is even, $3^L \leqslant \ell - 1 < 3^{L+1}$, i.e., $\lfloor \log_3(\ell - 1) \rfloor = L$. By Eq. (5),

$$H(\ell) - H(\ell - 1) = \lfloor \log_3(\ell - 1) \rfloor + 2 = L + 2. \tag{25}$$

Subcase (a) $n - 2\ell \leqslant 3^{L+1} - 2$ *and* ℓ *even*: In this case, $3^L < \ell < n - 2\ell < 3^{L+1} - 1$, so $\lfloor \log_3(n - 2\ell) \rfloor = L$. By Eq. (8),

$$H(n - 2\ell + 2) - H(n - 2\ell) = 2\lfloor \log_3(n - 2\ell) \rfloor + 3 = 2L + 3. \tag{26}$$

Eqs. (25) and (26) imply that $\lambda_2 - \lambda_1 < 0$. So $\lambda = \min\{\lambda_1, \lambda_2\} = \lambda_2$, i.e., $h(T) \geqslant n + 2H(\ell - 1) + H(n - 2\ell + 2) = \phi_1$.

Subcase (b) $n - 2\ell \geqslant 3^{L+1} - 1$ *and* ℓ *even*: If $n - 2\ell \geqslant 3^{L+1}$, then $3^{L+1} \leqslant n - 2\ell < n - 2\ell + 2 \leqslant 3\ell + 2 \leqslant 3^{L+2} - 1$, so $\lfloor \log_3(n - 2\ell) \rfloor = L + 1$. It follows Eq. (10) that

$$H(n - 2\ell + 2) - H(n - 2\ell) \geqslant 2\lfloor \log_3(n - 2\ell) \rfloor + 3 = 2L + 5. \tag{27}$$

Eqs. (25) and (27) imply that $\lambda_2 - \lambda_1 > 0$. If $n - 2\ell = 3^{L+1} - 1$, it follows Eq. (8) that

$$H(n - 2\ell + 2) - H(n - 2\ell) = 2\lfloor \log_3(n - 2\ell) \rfloor + 4 = 2L + 4. \tag{28}$$

Eqs. (25) and (28) imply that $\lambda_2 - \lambda_1 = 0$. So $\lambda = \min\{\lambda_1, \lambda_2\} = \lambda_1$, i.e., $h(T) \geqslant n + 2H(\ell) + H(n - 2\ell) = \phi_2$. □

Proof of Theorem 3. (1) In order to get the upper bound of $h_{\leqslant \ell}(n)$, it is enough to construct an ℓ-admissible tree T with $h(T) = \varphi_i$ according to $(\ell, n) \in \Omega_i$, $i \in \{1, 2, 3\}$. The desired tree T is shown by Fig. 1(b). We choose $A_1 : B_1$

as the first weighing of T, where $|A_1| = |B_1| = \ell$. Then $|T_0^1| = |C_1| = |S - A_1 - B_1| = n - 2\ell$ ($C_1 \triangleq S - A_1 - B_1$). If the feedback of T is 1 or -1, Corollary 10 shows that there exist two ℓ-admissible trees T_{-1}^1 and T_1^1 such that $h(T_{-1}^1) = h(T_1^1) = H(\ell)$. If the feedback of T is 0, we choose $A_2 : B_2$ as the weighing of T_0^1, where $|A_2| = |B_2| = \ell$. Then $|T_0^2| = |C_2| = |C_1 - A_2 - B_2| = n - 4\ell$ ($C_2 \triangleq C_1 - A_2 - B_2$). If the feedback of T_0^1 is -1 or 1, Corollary 10 shows that there exist two ℓ-admissible trees T_{-1}^2 and T_1^2 such that $h(T_{-1}^2) = h(T_1^2) = H(\ell)$. If the feedback of T_0^1 is 0, we choose $A_3 : B_3$ as the weighing of T_0^2, where $|A_3| = |B_3| = \ell$. And so on. If the feedback of T_0^{t-2} is -1 or 1, Corollary 10 shows that there exist two ℓ-admissible trees T_{-1}^{t-1} and T_1^{t-1} such that $h(T_{-1}^{t-1}) = h(T_1^{t-1}) = H(\ell)$. If the feedback of T_0^{t-2} is 0, we choose the weighing of T_0^{t-1} according to the values of ℓ and n:

Case (1.1) $(\ell, n) \in \Omega_1$: We choose $A_t : B_t$ as the weighing of T_0^{t-1}, where $|A_t| = |B_t| = \ell - 1$, $A_t, B_t \subset C_{t-1} = S - \sum_{i=1}^{t-1}(A_i \cup B_i)$. Then $|T_0^t| = |C_t| = n - 2t\ell + 2 \leqslant 3^{L+1} \leqslant 3\ell$ ($C_t = S - \sum_{i=1}^{t}(A_i \cup B_i)$). Corollary 10 shows that there exist three ℓ-admissible trees T_{-1}^t, T_1^t and T_0^t such that $h(T_{-1}^t) = h(T_1^t) = H(\ell - 1)$, $h(T_0^t) = H(n - 2t\ell + 2)$. Thus,

$$\begin{aligned} h(T) &= \sum_{i=1}^{t}\{i \cdot (|A_i| + |B_i|) + h(T_{-1}^i) + h(T_1^i)\} + t \cdot |C_t| + h(T_0^t) \\ &= 2\ell(1 + 2 + \cdots + (t-1)) + 2t(\ell - 1) + (2t-2)H(\ell) + 2H(\ell - 1) \\ &\quad + t(n - 2t\ell + 2) + H(n - 2t\ell + 2) \\ &= nt - t^2\ell + t\ell + (2t-2)H(\ell) + 2H(\ell - 1) + H(n - 2t\ell + 2) \\ &= \varphi_1. \end{aligned}$$

Case (1.2) $(\ell, n) \in \Omega_2$: We choose $A_t : B_t \cup x^*$ as the weighing of T_0^{t-1}, where $|A_t| = \ell$, $|B_t| = \ell - 1$, $A_t, B_t \subset C_{t-1} = S - \sum_{i=1}^{t-1}(A_i \cup B_i)$, x^* is a coin known to be good after the first weighing. Then $|T_0^t| = |C_t| = n - 2t\ell + 1 = 3^{L+1} \leqslant 3\ell$ ($C_t = S - \sum_{i=1}^{t}(A_i \cup B_i)$). Corollary 10 shows that there exist three ℓ-admissible trees T_{-1}^t, T_1^t and T_0^t such that $h(T_{-1}^t) = H(\ell)$, $h(T_1^t) = H(\ell - 1)$ and $h(T_0^t) = H(n - 2t\ell + 1)$, respectively. Thus,

$$\begin{aligned} h(T) &= \sum_{i=1}^{t}\{i \cdot (|A_i| + |B_i|) + h(T_{-1}^i) + h(T_1^i)\} + t \cdot |C_t| + h(T_0^t) \\ &= 2\ell(1 + 2 + \cdots + (t-1)) + t(2\ell - 1) + (2t-1)H(\ell) + H(\ell - 1) \\ &\quad + t(n - 2t\ell + 1) + H(n - 2t\ell + 1) \\ &= nt - t^2\ell + t\ell + (2t-1)H(\ell) + H(\ell - 1) + H(n - 2t\ell + 1) \\ &= \varphi_2. \end{aligned}$$

Case (1.3) $(\ell, n) \in \Omega_3$: We choose $A_t : B_t$ as the weighing of T_0^{t-1}, where $|A_t| = |B_t| = \ell$, $A_t, B_t \subset C_{t-1} = S - \sum_{i=1}^{t-1}(A_i \cup B_i)$. Then $|T_0^t| = |C_t| = n - 2t\ell \leqslant 3\ell$ (note that $C_t = S - \sum_{i=1}^{t}(A_i \cup B_i)$ and $t \geqslant (n - 3\ell)/2\ell$). Corollary 10 shows that there exist three ℓ-admissible trees T_{-1}^t, T_1^t and T_0^t such that $h(T_{-1}^t) = h(T_1^t) = H(\ell)$, $h(T_0^t) = H(n - 2t\ell)$. Thus,

$$\begin{aligned} h(T) &= \sum_{i=1}^{t}\{i \cdot (|A_i| + |B_i|) + h(T_{-1}^i) + h(T_1^i)\} + t \cdot |C_t| + h(T_0^t) \\ &= 2\ell(1 + 2 + \cdots + t) + 2tH(\ell) + t(n - 2t\ell) + H(n - 2t\ell) \\ &= nt - t^2\ell + t\ell + 2tH(\ell) + H(n - 2t\ell) \\ &= \varphi_3. \end{aligned}$$

(2) In order to prove $h_{\leqslant \ell}(n) \geqslant \varphi_i$, it is enough to show that $h(T) \geqslant \varphi_i$ for any ℓ-admissible tree T with n leaves, and for $(\ell, n) \in \Omega_i$, $i \in \{1, 2, 3\}$. Let T be any ℓ-admissible tree T with n leaves described by Fig. 1(b), and let $|T_j^i| = a_j^i \leqslant \ell$,

$i=1,2,\ldots,t$, $j=-1,1$. Then $|T_0^t|=n-\sum_{i=1}^{t}(a_1^i+a_{-1}^i)=n-a_1^t-a_{-1}^t-c_{t-1}$, and

$$\begin{aligned} h(T) &= \sum_{i=1}^{t}\{i\cdot(|T_{-1}^i|+|T_1^i|)+h(T_{-1}^i)+h(T_1^i)\}+t\cdot|T_0^t|+h(T_0^t) \\ &\geqslant \sum_{i=1}^{t}\{i\cdot(a_{-1}^i+a_1^i)+H(a_{-1}^i)+H(a_1^i)\}+t\cdot|T_0^t|+H(|T_0^t|) \\ &= \Phi_{t-1}+t(a_1^t+a_{-1}^t)+H(a_1^t)+H(a_{-1}^t)+t(n-a_1^t-a_{-1}^t-c_{t-1})+H(n-a_1^t-a_{-1}^t-c_{t-1}), \end{aligned}$$

where $\Phi_{t-1}\triangleq\sum_{i=1}^{t-1}\{i\cdot(a_{-1}^i+a_1^i)+H(a_{-1}^i)+H(a_1^i)\}$, $c_{t-1}\triangleq\sum_{i=1}^{t-1}(a_1^i+a_{-1}^i)$. The inequality $\lceil x\rceil<x+1$ and $t=\lceil(n-3\ell)/2\ell\rceil$ show us that $n>(2t+1)\ell$, so $m\triangleq n-a_1^t-a_{-1}^t>(2t-1)\ell$ in view of $a_1^t, a_{-1}^t\leqslant\ell$.

Lemma 8 shows that $\Phi_t(m)=\Phi_{t-1}+t(n-a_1^t-a_{-1}^t-c_{t-1})+H(n-a_1^t-a_{-1}^t-c_{t-1})\geqslant(n-a_1^t-a_{-1}^t)t-t^2\ell+t\ell+(2t-2)H(\ell)+H(n-a_1^t-a_{-1}^t-2t\ell+2\ell)$.Thus,

$$\begin{aligned} h(T) &\geqslant (n-a_1^t-a_{-1}^t)t-t^2\ell+t\ell+(2t-2)H(\ell)+H(n-a_1^t-a_{-1}^t-2t\ell+2\ell) \\ &\quad + t(a_1^t+a_{-1}^t)+H(a_1^t)+H(a_{-1}^t) \\ &= nt-t^2\ell+t\ell+(2t-2)H(\ell)+H(a_1^t)+H(a_{-1}^t)+H(n-a_1^t-a_{-1}^t-2t\ell+2\ell) \\ &= nt-t^2\ell+t\ell+(2t-2)H(\ell)+f_{n,\ell}(a_1^t,a_{-1}^t), \end{aligned} \tag{29}$$

where $f_{n,\ell}(a_1^t,a_{-1}^t)=H(a_1^t)+H(a_{-1}^t)+H(n-a_1^t-a_{-1}^t-2t\ell+2\ell)$. We are concerned with the lower bound of $h(T)$, so we calculate the minimal value of $f_{n,\ell}(a_1^t,a_{-1}^t)$ according to the values of ℓ and n.

Case (2.1) *ℓ odd*: We note that $n-2t\ell+2\ell>3\ell$ and $0\leqslant a_1^t, a_{-1}^t\leqslant\ell$. Lemma 7 shows that $f_{n,\ell}(a_1^t,a_{-1}^t)=H(a_1^t)+H(a_{-1}^t)+H(n-a_1^t-a_{-1}^t-2t\ell+2\ell)\geqslant 2H(\ell)+H(n-2t\ell)$. Therefore,

$$h(T)\geqslant nt-t^2\ell+t\ell+2tH(\ell)+H(n-2t\ell)=\varphi_3. \tag{30}$$

Case (2.2) *ℓ even*: We note that $n-2t\ell+2\ell>3\ell$. If $a_1^t=a_{-1}^t=\ell$, then

$$f_{n,\ell}(a_1^t,a_{-1}^t)=2H(\ell)+H(n-2t\ell)\triangleq\delta_3.$$

If ($a_1^t\leqslant\ell-1$ and $a_{-1}^t=\ell$) or ($a_{-1}^t\leqslant\ell-1$ and $a_1^t=\ell$), Lemma 6 shows that

$$f_{n,\ell}(a_1^t,a_{-1}^t)\geqslant H(\ell)+H(\ell-1)+H(n-2t\ell+1)\triangleq\delta_2.$$

If $a_1^t, a_{-1}^t\leqslant\ell-1$, Lemma 7 shows that

$$f_{n,\ell}(a_1^t,a_{-1}^t)\geqslant 2H(\ell-1)+H(n-2t\ell+2)\triangleq\delta_1.$$

It is obvious that $f_{n,\ell}(a_1^t,a_{-1}^t)\geqslant\delta\triangleq\min\{\delta_1,\delta_2,\delta_3\}$ for $0\leqslant a_1^t, a_{-1}^t\leqslant\ell$. We now determine the quantity δ. Since $3^L\leqslant\ell<3^{L+1}$ and ℓ is even, $3^L\leqslant\ell-1<\ell<3^{L+1}$. So $\lfloor\log_3(\ell-1)\rfloor=\lfloor\log_3\ell\rfloor=L$. By Eq. (5),

$$H(\ell)-H(\ell-1)=\lfloor\log_3(\ell-1)\rfloor+2=L+2. \tag{31}$$

Subcase (a) $(\ell,n)\in\Omega_1$: In this case, $3^L<\ell<n-2t\ell<n-2t\ell+1\leqslant 3^{L+1}-1$, so $\lfloor\log_3(n-2t\ell)\rfloor=\lfloor\log_3(n-2t\ell+1)\rfloor=L$. By Eq. (6),

$$H(n-2t\ell+1)-H(n-2t\ell)\leqslant\lfloor\log_3(n-2t\ell)\rfloor+2=L+2, \tag{32}$$

$$H(n-2t\ell+2)-H(n-2t\ell+1)\leqslant\lfloor\log_3(n-2t\ell+1)\rfloor+2=L+2. \tag{33}$$

Eqs. (31) and (32) imply that $\delta_2-\delta_3\leqslant 0$; Eqs. (31) and (33) imply that $\delta_1-\delta_2\leqslant 0$. So $\delta=\min\{\delta_1,\delta_2,\delta_3\}=\delta_1$. Eq. (29) gives $h(T)\geqslant nt-t^2\ell+t\ell+(2t-2)H(\ell)+\delta_1=\varphi_1$.

Subcase (b) $(\ell, n) \in \Omega_2$: In this case, $3^L < \ell < n - 2t\ell < n - 2t\ell + 1 = 3^{L+1}$, so $\lfloor \log_3(n - 2t\ell) \rfloor = \lfloor \log_3(n - 2t\ell + 1) \rfloor - 1 = L$. It follows Eq. (5) that

$$H(n - 2t\ell + 1) - H(n - 2t\ell) = \lfloor \log_3(n - 2t\ell) \rfloor + 1 = L + 1, \tag{34}$$

$$H(n - 2t\ell + 2) - H(n - 2t\ell + 1) = \lfloor \log_3(n - 2t\ell + 1) \rfloor + 2 = L + 3. \tag{35}$$

Eqs. (31) and (34) imply that $\delta_2 - \delta_3 < 0$; Eqs. (31) and (35) imply that $\delta_2 - \delta_1 < 0$. So $\delta = \min\{\delta_1, \delta_2, \delta_3\} = \delta_2$. Eq. (29) gives $h(T) \geqslant nt - t^2\ell + t\ell + (2t - 2)H(\ell) + \delta_2 = \varphi_2$.

Subcase (c) $n - 2t\ell > 3^{L+1} - 1$ *and* ℓ *even*: In this case, $3^{L+1} \leqslant n - 2t\ell < n - 2t\ell + 1 \leqslant 3\ell + 1 \leqslant 3^{L+2} - 2$, so $\lfloor \log_3(n - 2t\ell) \rfloor = \lfloor \log_3(n - 2t\ell + 1) \rfloor = L + 1$. It follows Eq. (6) that

$$H(n - 2t\ell + 1) - H(n - 2t\ell) \geqslant \lfloor \log_3(n - 2t\ell) \rfloor + 1 = L + 2, \tag{36}$$

$$H(n - 2t\ell + 2) - H(n - 2t\ell + 1) \geqslant \lfloor \log_3(n - 2t\ell + 1) \rfloor + 1 = L + 2. \tag{37}$$

Eqs. (31) and (36) imply that $\delta_2 - \delta_3 \geqslant 0$; Eqs. (31) and (35) imply that $\delta_1 - \delta_2 \geqslant 0$. So $\delta = \min\{\delta_1, \delta_2, \delta_3\} = \delta_3$. Eq. (29) gives $h(T) \geqslant nt - t^2\ell + t\ell + (2t - 2)H(\ell) + \delta_3 = \varphi_3$. $\square$

Further reading

It would be interesting to generalize our result to other problems: more than one counterfeit coin [2,9–11,13,14,16–18], multi-arms balance instead of the two-arms balance [3,4], b-balance instead of the two-arms balance [7], predetermined algorithm instead of the sequential algorithm [3,8,10], searching with lies [12,15,19], etc.

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