Geomagnetic Core-Mantle Coupling

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Abstract. The electrically conducting lower mantle is penetrated by changing magnetic fields originating in the earth's core. The Lorentz force ($\mathbf{j} \times \mathbf{B}$) interaction of these fields with the electrical currents there produces a mechanical torque on the mantle. By using the Maxwell stress representation the expression for this torque is shown to reduce to the surface integral $-1/\mu \int (\mathbf{r} \times \mathbf{B}) B_r dS$ over the core-mantle boundary. The result is important in the study of possible changes in the length of day and the geographical position of the pole, due to angular momentum transfer between the electromagnetically coupled core and mantle.

List of symbols.

$\mathbf{B}$, magnetic flux density.
B_i, cartesian component of $\mathbf{B}$.
B_r, B_θ, B_ϕ, spherical polar components of $\mathbf{B}$.
$\mathbf{e}_i$, cartesian unit vector in x_i direction.
i, j, k, s, t, cartesian tensor subscripts.
$\mathbf{j}$, electrical conduction current density.
M, magnetic part of Maxwell stress dyadic.
M_{ij}, cartesian component of M.
$\mathbf{n}$, unit vector normal to a surface.
$\mathbf{r}$, position vector relative to origin at earth's center.
r, θ, ϕ, spherical polar coordinates.
S_c, core-mantle boundary, assumed to be spherical.
S_m, boundary between conducting and insulating regions of mantle.
S_∞, surface at infinity.
T, dyadic.
T_{ij}, cartesian component of T.
V_m, colume of conducting region of mantle.
V_∞, volume of space above S_m.
x_i, cartesian coordinate.
$\mathbf{\Gamma}$, Lorentz magnetic torque on mantle.
Γ_i, cartesian component of $\mathbf{\Gamma}$.
δ_{ij}, Kronecker delta symbol.
ϵ_{ijk}, Levi-Civita tensor density or alternating tensor.
μ, permeability.
ψ, magnetic potential.
∇, gradient operator.
$\nabla \cdot$, divergence operator on a vector.
$\nabla \times$, curl operator on a vector.
div, divergence operator on a dyadic.

Introduction. The effects of changing magnetic fields in the earth's core on the angular momentum of the mantle have been examined by *Bullard et al.* [1950], *Takeuchi and Elsasser* [1954], and more recently by *Rochester* [1960], *Roden* [1961], *Smylie* [1961], and *Kakuta* [1961]. These studies require the calculation of the Lorentz torque on the mantle owing to the interaction, within the electrically conducting lower mantle, of the electrical currents there with the magnetic field of deep interior origin. In the mks system this torque is

$$\mathbf{\Gamma} = \int_{V_m} \mathbf{r} \times (\mathbf{j} \times \mathbf{B})\, dV \qquad (1)$$

$\mathbf{B}$ consists of fields generated by magnetohydrodynamic interactions within the core and generated by Faraday induction from the changing electric fields in the lower mantle. The upper mantle is treated as an insulator. Actually, its slight conductivity is sufficient to shield the lower mantle from the highly variable fields of external origin [*Lahiri and Price,* 1939]. Any coupling between $\mathbf{B}$ and the field originating outside the solid earth is therefore neglected in the studies so far carried out.

The relative motion of the mantle and the sources of field in the core (revealed by the westward drift) is slow enough so that only a negligible part of the total electromagnetic field energy resides in the electrical part of the field. The electrical fields therefore exert no appreciable mechanical forces and can be neglected insofar as dynamical effects on the mantle are concerned [*Elsasser,* 1956, p. 137]. The temporal changes in $\mathbf{B}$ at points in the core or mantle are so slow that the displacement current can be neglected, as in most geophysical contexts, and the necessary Maxwell equations are

$$\mu \mathbf{j} = \nabla \times \mathbf{B} \tag{2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3}$$

By making some approximations *Takeuchi and Elsasser* [1954, p. 43] reduced the expression for the axial (x_3) component of (1), responsible for acceleration of the rate of rotation of the mantle, to a form equivalent to

$$\Gamma_3 = -\frac{1}{\mu} \int_{S_c} B_r B_\phi r \sin \theta \, dS \tag{4}$$

Rochester [1960, pp. 541-542] was able to show, for fields of exponential time dependence, that (4) is in fact not an approximation, but exact. *Smylie* [1961] extended the proof to the case of fields varying in time in the most general manner allowed by Maxwell's equations. His discussion took into account the equatorial components of (1), and he obtained equation 15 of this paper. However both these proofs involved carrying out a multipole expansion of the poloidal and toroidal parts of **B** and taking advantage of the boundary conditions on individual multipole field components at the top of the conducting region of the mantle. Such a roundabout approach is unnecessary. The result can be obtained much more directly by using some of the standard theorems of vector and dyadic analysis, and the condition that $\nabla \times \mathbf{B}$ vanish in the upper layers of the mantle.

One of the essential steps in the proof, the reduction of the volume integral in (1) to a surface integral over the entire boundary of V_m, is analogous to the way in which the expression for the Lorentz force on a conducting volume is reduced to a surface integral via the Maxwell stress representation. After some search, however, the author has found the corresponding discussion for torque in only two places in the literature [*Page and Adams* 1940, pp. 274–275; *Fano, Chu, and Adler* 1960, pp. 424–426]. In both places the mathematics seems unnecessarily lengthy, even taking into account that the whole electromagnetic field (with displacement current and magnetization allowed for) is considered. In view of the rarity and length of these discussions, and because of the usefulness of the result, a briefer treatment of this stage in the proof is presented using cartesian tensor notation [*Jeffreys*, 1931].

Maxwell stress representation of torque. Let

$$\mathsf{M} = M_{ij} \mathbf{e}_i \mathbf{e}_j$$

be the magnetic part of the Maxwell stress dyadic

$$\mathsf{M} = \frac{\mathbf{BB}}{\mu} - \frac{B^2}{2\mu} \mathbf{e}_i \mathbf{e}_i \tag{5}$$

$$M_{ij} = \frac{B_i B_j}{\mu} - \frac{B^2}{2\mu} \delta_{ij}$$

Stratton [1941, pp. 98–99] shows that

$$\operatorname{div} \mathsf{M} = \mathbf{j} \times \mathbf{B} \tag{6}$$

and a brief proof of (6) using cartesian tensor notation is given in the appendix to the present paper.

It can be shown that for any dyadic T

$$\mathbf{r} \times \operatorname{div} \mathsf{T} - \operatorname{div} (\mathbf{r} \times \mathsf{T}) = \epsilon_{ijk} T_{ij} \mathbf{e}_k \tag{7}$$

But

$$\epsilon_{ijk} T_{ij} = 0 \tag{8}$$

when T is symmetric, as is the case for the Maxwell stress dyadic. (Equation 7 can be used to show the symmetry of the mechanical stress tensor in deformable matter in which the internal forces have no resultant moment [*Landau and Lifshitz*, 1959, p. 6].)

Making use in turn of (6), (7), (8), and the tensor divergence theorem [*Stratton*, 1941, p. 100], the expression (1) for the torque on the mantle can be written

$$\begin{aligned} \mathbf{\Gamma} &= \int_{V_m} \mathbf{r} \times \operatorname{div} \mathsf{M} \, dV \\ &= \int_{V_m} \operatorname{div} (\mathbf{r} \times \mathsf{M}) \, dV \\ &= \int_{S_c + S_m} \mathbf{r} \times \mathsf{M} \cdot \mathbf{n} \, dS \end{aligned} \tag{9}$$

Reduction of the torque to a core-mantle boundary integral. Insofar as the field **B** of deep interior origin is concerned, the region V_∞ bounded below by the surface S_m and extending outward to infinity is an insulator. In this region

$$\mathbf{j} = \nabla \times \mathbf{B} = 0 \tag{10}$$

and we can represent the field in the usual way as the gradient of a potential function

$$\mathbf{B} = \mu \nabla \psi \tag{11}$$

Because of (10)

$$\operatorname{div} \mathsf{M} = 0$$

throughout V_∞. Therefore, by the same arguments that led to (9),

$$\int_{V_\infty} \mathbf{r} \times \operatorname{div} \mathsf{M}\, dV = \int_{S_m + S_\infty} \mathbf{r} \times \mathsf{M}\cdot\mathbf{n}\, dS = 0 \tag{12}$$

At infinity $|\mathbf{B}|$ tends to zero at least as rapidly as r^{-3} so that the integral over S_∞ itself tends to zero at least as rapidly as r^{-3}. Then (12) reduces to

$$\int_{S_m} \mathbf{r} \times \mathsf{M}\cdot\mathbf{n}\, dS = 0 \tag{13}$$

This result can also be obtained directly (though the proof is arduous) by using (11) and expanding the magnetic potential ψ in spherical harmonics with time-dependent coefficients, assuming S_m spherical and making use of the orthogonality relations among the surface harmonics.

From (13) we see that (9) reduces to

$$\mathbf{\Gamma} = \int_{S_e} \mathbf{r} \times \mathsf{M}\cdot\mathbf{n}\, dS = \frac{1}{\mu}\int_{S_e} \left[(\mathbf{r} \times \mathbf{B})\mathbf{B}\cdot\mathbf{n} - \frac{B^2}{2}\mathbf{r} \times \mathbf{n}\right] dS \tag{14}$$

If the core-mantle boundary is assumed spherical then on it $\mathbf{n}$ points in the direction opposite to $\mathbf{r}$ and (14) becomes

$$\mathbf{\Gamma} = -\frac{1}{\mu}\int_{S_e} (\mathbf{r} \times \mathbf{B})\mathbf{B}_r\, dS \tag{15}$$

Since

$$(\mathbf{r} \times \mathbf{B})\cdot\mathbf{e}_3 = r \sin \theta B_\phi$$

(4) follows directly from (15). Similarly (15) leads to the following expressions for the equatorial (x_1 and x_2) components of torque:

$$\Gamma_1 = \frac{1}{\mu}\int_{S_e} B_r(B_\phi \cos \theta \cos \phi + B_\theta \sin \phi) r\, dS$$

$$\Gamma_2 = \frac{1}{\mu}\int_{S_e} B_r(B_\phi \cos \theta \sin \phi - B_\theta \cos \phi) r\, dS$$

Applications. The concept of the Maxwell stress suggests, and (14) confirms, that the dynamical response of the mantle to the presence of the changing internal magnetic field of the earth is entirely determined by the core-mantle boundary values of this field. This fact greatly reduces the amount of work that needs to be done when calculating torques in studies of the kind referred to in the introduction.

The investigations of electromagnetic core-mantle coupling so far carried out have been concerned with the possibility of attributing the irregular changes in the length of day, at the rate of a few milliseconds per decade, to the transfer of rotational momentum between the core and the mantle. If this transfer is effected by electromagnetic coupling, the rotational acceleration of the mantle is determined by the torque given by (4). The tightness of the coupling is measured by a time constant for the change in speed of the mantle following an impulsive change in field strength at the core boundary. For a uniformly conducting mantle this time constant is inversely proportional to the conductivity [*Bullard et al.*, 1950; *Rochester*, 1960]. An upper bound of about 10^3 mho/m to the mean electrical conductivity of the lower mantle is set by observations of changes in field strength made at the earth's surface [*Runcorn*, 1955]. With this value the time constant on the Bullard-Rochester model is about 25 years.

Although the boundary values of the field are affected by the distribution of electrical conductivity in the overlying mantle, the deduction of (14) from (1) is independent of the particular conductivity distribution. This was assumed by *Roden* [1961] in his study of the effects of several possible conductivity distributions in the lower mantle on the tightness of the axial component of the electromagnetic core-mantle coupling. Roden's work confirmed a conjecture of *Rochester* [1960], that the tightness of the coupling is primarily determined by the conductivity in a relatively small thickness of the mantle just above the core boundary. He found that the time constant for the coupling could be made as small as 10 years without violating the restrictions placed on the conductivity distribution by

Runcorn's conclusion and by the estimates of conductivity in the lower mantle due to *McDonald* [1957]. *Kakuta* [1961] has obtained a value for the time constant of this same order.

Mr. D. E. Smylie of the University of Toronto is using the expressions for the equatorial components of (15) in a study of the effect of electromagnetic core-mantle coupling on wobble of the mantle relative to its axis of rotation.

Appendix

Writing $\nabla = \mathbf{e}_i\, \partial/\partial x_i$ we have

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = \epsilon_{ijk}\epsilon_{ist} B_s \frac{\partial B_k}{\partial x_j} \mathbf{e}_t$$

$$= B_j \frac{\partial B_k}{\partial x_j} \mathbf{e}_k - B_k \frac{\partial B_k}{\partial x_j} \mathbf{e}_j$$

$$= (\mathbf{B}\cdot\nabla)\mathbf{B} - \nabla(B^2/2)$$

On the other hand, using (5) we have

$$\mu \operatorname{div} \mathsf{M} = \frac{\partial}{\partial x_j}\left[\left(\mathbf{BB} - \frac{B^2}{2}\mathbf{e}_i\mathbf{e}_i\right)\cdot\mathbf{e}_j\right]$$

$$= \frac{\partial}{\partial x_j}\left(\mathbf{B}B_j - \frac{B^2}{2}\mathbf{e}_j\right)$$

$$= (\mathbf{B}\cdot\nabla)\mathbf{B} + \mathbf{B}(\nabla\cdot\mathbf{B}) - \nabla(B^2/2)$$

and (6) follows immediately upon using (2) and (3).

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