A CORE-MANTLE INTERACTION IN THE ROTATION OF THE EARTH

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Abstract. Effects of an interaction between the mantle and the core of the Earth on its rotational motion are investigated. Assuming that the Earth consists of a rigid mantle and a rigid core with a frictional coupling and a kind of inertial coupling between them, the equations of motion are derived, and they are solved in a close approximation. The solution gives the expressions for the precession, the nutation, the secular changes in the obliquity and the rotational speed, the polar motion and so on as functions of the magnitudes of these forces. A numerical estimation shows that the effect of the friction on the amplitude and phase of the nutation is small for a reasonable intensity of the friction while inertial coupling force has a decisive influence on the amplitude, and an appropriately chosen value of the latter force gives a nutation which closely agrees with observations. It is also indicated that this torque remarkably lessens the rates of the secular changes in the obliquity and the rotational speed. The possibility of a periodical change in the amplitude of the polar motion is suggested as a result of the interaction between the two constituents.

1. Introduction

The theory of the rotation of the rigid Earth was studied by Woolard (1953) in full detail and his results for the value of nutation have been adopted in the Astronomical Ephemeris since 1960.

Recently Kinoshita (1977) has treated the same problem from a slightly different point of view and with some new techniques. He uses in his study Andoyer variables which have many advantages compared with the traditional Euler angles and chooses the ecliptic of date as the reference system instead of a fixed ecliptic. His study is theoretically more rigorous and elegant and even it gives some corrections to the formulae especially for the secular change of the obliquity and the Oppolzer terms, as well as small revisions to the numerical values of Woolard. Thus it may be said that by the extensive studies by Woodland and Kinoshita the theory of the rigid Earth has been learned with an almost complete accuracy.

On the other hand, observations show many evidences that the actual Earth is far from a rigid body. The change in the rotational speed, the deviation of the period of the free nutation from the theoretical value and the change in its amplitude are some examples. It should be noticed also that in any theory of the rigid Earth an inconsistency is not avoidable between the parameters for precession and for nutation so that it agrees with observational facts, i.e., the adopted nutational constant 9″.2100, which comes from observations of nutation itself, can not be deduced from the constant of precession.

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Many efforts have been made since the last century in order to take non-rigidity of the Earth into consideration. Newcomb (1892) successfully explained the prolongation of the free nutation period from its theoretical value, by attributing it to the elasticity of the Earth. Poincaré (1910) and later Jeffreys (1948) studied the nutation of an Earth composed of a rigid mantle and a liquid core and tried to explain the observational value for nutation constant. They showed the possibility of the deviation in the amplitude of the nutation from the rigid Earth, but the model gave a result contradictory with the observational fact about the period of the free nutation. In 1957, Jeffreys and Vicente proposed an improved Earth model which is composed of an elastic mantle and a liquid core and got a satisfactory value both for the nutation and for the period of the free nutation. While Molodensky (1961) independently developed a theory with a similar model and got successful results.

Some studies, however, have been devoted to the boundary interaction between the mantle and the core. Rochester (1970) categorizes the interactions into inertial coupling, viscous boundary layer friction and electromagnetic coupling, of which the first one contains the torques which occur in the models by Poincaré, etc., above. He also gives the magnitude of the electromagnetic coupling, as well as estimates the intensities of the other couplings.

Aoki (1969) studied the friction and using a simple model explained the secular change in the obliquity of the ecliptic, and later Aoki and Kakuta (1971) recomputed it with a more refined model after Molodensky. Although it is still in dispute whether a secular change in the obliquity exists or not (Duncombe and Van Flandern, 1976), the friction would be the only possible cause for this phenomenon if it exists. Kubo (1971) showed that an interaction between the mantle and the core also can cause the periodic change in the amplitude of the free nutation.

In the present paper we consider an Earth model composed of a rigid mantle and a rigid core and with two kinds of torques between them, and investigate the consequences which appear in the rotational motion of the Earth. One of the torques is that which comes from the friction, of which the existence is almost certain and of which the magnitude is known to a fairly good extent. The other torque, which we call a tensional torque, is one which will occur when the mantle and the core deviate from an equilibrium configuration so that they recover the most stable state. We shall show later (Appendix) that this torque is similar to an intertial force and has an effect on the nutation, at least, similar to that in the Poincaré's model.

It may be a too rough assumption that we consider both the mantle and the core to be rigid bodies, but it will have a significance as a first approximation now that we know very little, say, about the effects of the friction on the nutation or on the polar motion.

2. Description of the Problem

The Earth is assumed to be composed of a rigid mantle and a rigid core and two kinds of forces are considered between them:

(i) Frictional force, which includes electromagnetic coupling and is proportional to the difference of the angular velocities of the mantle and the core. The coefficient of friction can be different for the parallel and perpendicular directions to the rotational axis. Then the torque is given by

$$\mathbf{f}_m = \Lambda(\boldsymbol{\omega}_c - \boldsymbol{\omega}_m), \tag{1.1}$$

$$\mathbf{f}_c = \Lambda(\boldsymbol{\omega}_m - \boldsymbol{\omega}_c), \tag{1.2}$$

with

$$\Lambda = \begin{bmatrix} \lambda_\perp & 0 & 0 \\ 0 & \lambda_\perp & 0 \\ 0 & 0 & \lambda_\parallel \end{bmatrix}, \tag{2}$$

z-axis being in the direction of the rotation vectors.

The suffixes m and c mean the mantle and the core respectively throughout this paper.

(ii) Tensional force, which is proportional to the displacement from an equilibrium configuration between the two constituents. We assume that the equilibrium configuration is the state in which the C-axes or the axes with the largest principal moments of inertia of the mantle and the core coincide with each other, and the torque is proportional to the angle between these axes. Then the torque is given by

$$\mathbf{f}_m = \mathrm{grad}_m\, U, \tag{3.1}$$

$$\mathbf{f}_c = \mathrm{grad}_c\, U, \tag{3.2}$$

with

$$U = \tfrac{1}{2}\sigma\delta^2, \tag{4}$$

where σ is a constant coefficient and δ is the angle between the two C-axes. A slightly different force was assumed by Kubo (1971). The result, however, is similar to each other.

Both the mantle and the core are supposed to be axially symmetrical. This approximation is valid with a sufficient accuracy.

The external forces exerted by the Moon and the Sun are considered. Since they have a coupling effect with the above assumed forces, they can not be treated separately.

3. Variables

Andoyer variables (Andoyer, 1923) are used to describe the motions of the mantle and the core. These variables have an advantage that in terms of them the motion of a rigid body can be separated easily into the motions of the angular momentum vector and the axis of figure, as well as a merit that they are canonical. The meanings of Andoyer variables L, G, H, l, g, and h are as following.

Let X, Y, Z be the fixed coordinates, A, B, C, the axes of principal moments of inertia of a rigid body, and ψ, θ, ϕ, the Euler angles. Then,

L: C-axis component of the angular momentum vector $\mathbf{G}$,
G: total angular momentum $|\mathbf{G}|$,
H: Z component of $\mathbf{G}$,

l, g, and h are conjugate to L, G, and H respectively and the meanings are shown in Figure 1.

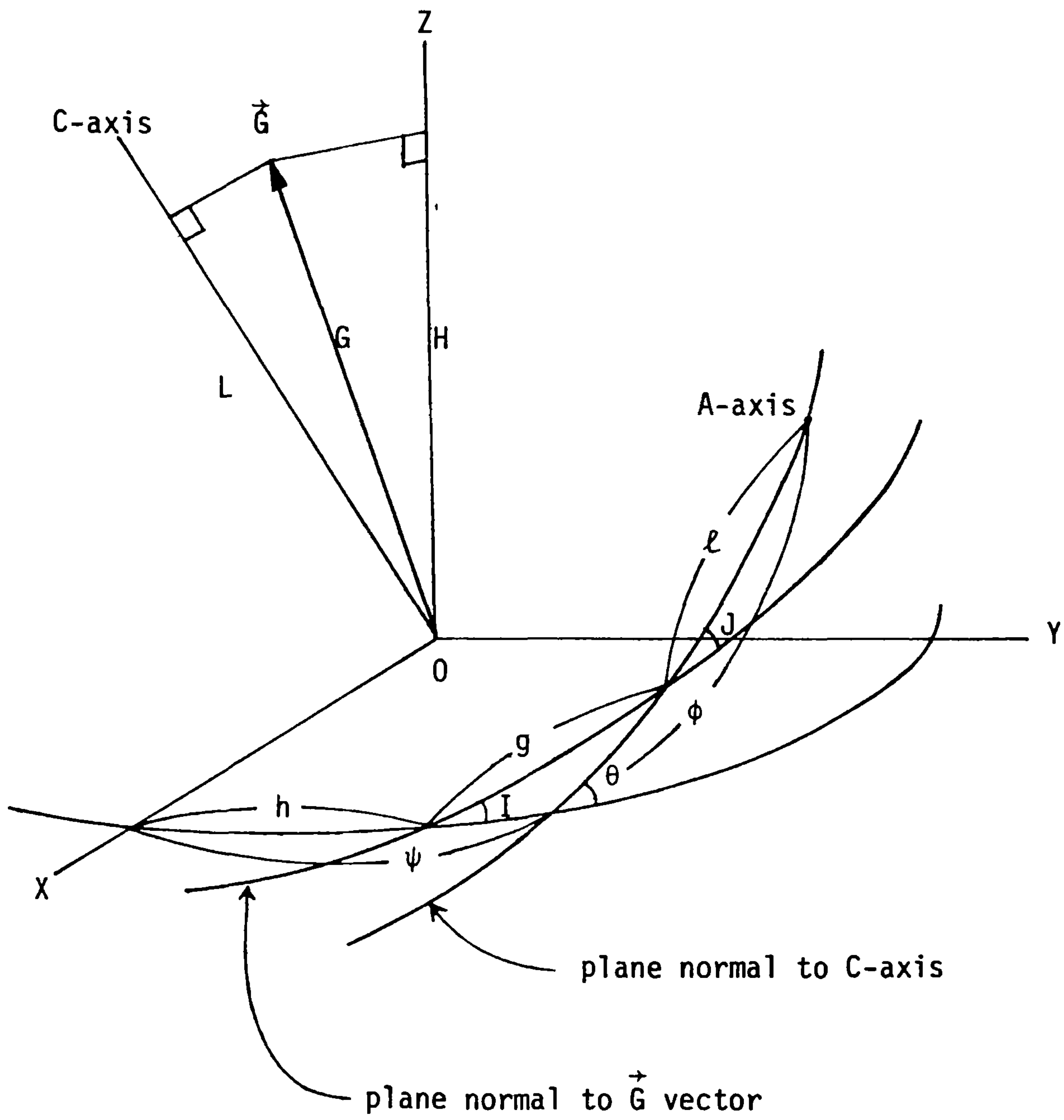

Fig. 1. Andoyer variables and Euler angles.

In terms of Andoyer variables, the Hamiltonian for the torque-free motion of a rigid body has the following simple form

$$F = \frac{1}{2}\left(\frac{\sin^2 l}{A} + \frac{\cos^2 l}{B}\right)(G^2 - L^2) + \frac{1}{2C}L^2, \tag{5}$$

where A, B, and C are the principal moments of inertia of the body. Especially in our case of $A = B$, it reduces to

$$F = \frac{1}{2}\left(\frac{1}{C} - \frac{1}{A}\right)L^2 + \frac{1}{2A}G^2. \tag{6}$$

The longitude of the node and the inclination of the plane perpendicular to the angular momentum vector are respectively given by

$$h \quad \text{and} \quad I = \cos^{-1}\frac{H}{G}\cdot \tag{7}$$

Now we consider the actual perturbed rotation of the Earth. Then the luni–solar precession and the Poisson terms of the nutation are given by

$$p = \text{secular terms of } h, \tag{8.1}$$

$$\Delta\psi = -(\text{periodic terms of } h), \tag{8.2}$$

$$\Delta\varepsilon = \text{periodic terms of } I. \tag{8.3}$$

The longitude of the node and the inclination of the figure plane (the equatorial plane) are

$$h_f = h + \frac{J}{\sin I}\sin g + O(J^2), \tag{9.1}$$

and

$$I_f = I + J\cos g + O(J^2), \tag{9.2}$$

where

$$J = \cos^{-1}\frac{L}{G}. \tag{10}$$

Those of the plane perpendicular to the rotational axis are

$$h_r = h + \frac{A-C}{A}\frac{J}{\sin I}\sin g + O(J^2), \tag{11.1}$$

and

$$I_r = I + \frac{A-C}{A}J\cos g + O(J^2). \tag{11.2}$$

In the perturbed rotational motion, the second terms on the right-hand sides of Equations (9) and (11) will contain parts which are free from the factor J, therefore of the order of J^0. These parts are called Oppolzer terms. The definition of the nutation adopted by IAU at present is equivalent to the periodic parts of h_f and I_f free from the factor J in Equations (9) (IAU Transactions, 1977).

Finally the polar motion is given by

$$x_p = \frac{G}{A\omega}J\sin l, \tag{12.1}$$

$$y_p = -\frac{G}{A\omega}J\cos l, \tag{12.2}$$

where ω is the angular speed of the rotation.

4. Equations of Motion

At first sight it seems that the problem can be easily solved applying a perturbation theory, especially by the Hori's method (Hori, 1966) which Kinoshita has used in his rigid Earth case. The fact that the present problem contains a frictional force does not make the situation essentially difficult because we have some canonical transformation theories for non-conservation system such as Hori (1971), Choi and Tapley (1973), etc.

However, we could only remove the diurnal terms by this method. As we see later, the magnitudes of the frictional force and the tensional force have the same order as the mean motions of the longer periodic terms. So we will not use a perturbation theory in this study but directly solve equations of motion which are linearized under the supposition that the deviation between the mantle and the core is small.

Using the Hamiltonian (6) we can write the equations of motion referred to a fixed ecliptic as following:

$$\dot{l}_m = \left(\frac{1}{C_m} - \frac{1}{A_m}\right)L_m + f_{lm} + \frac{\partial U}{\partial L_m} + \frac{\partial R_m}{\partial L_m}, \tag{13.1}$$

$$\dot{g}_m = \frac{G_m}{A_m} + f_{gm} + \frac{\partial U}{\partial G_m} + \frac{\partial R_m}{\partial G_m}, \tag{13.2}$$

$$\dot{h}_m = f_{hm} + \frac{\partial U}{\partial H_m} + \frac{\partial R_m}{\partial H_m}, \tag{13.3}$$

$$\dot{L}_m = f_{Lm} - \frac{\partial U}{\partial l_m} - \frac{\partial R_m}{\partial l_m}, \tag{13.4}$$

$$\dot{G}_m = f_{Gm} - \frac{\partial U}{\partial g_m} - \frac{\partial R_m}{\partial g_m}, \tag{13.5}$$

$$\dot{H}_m = f_{Hm} - \frac{\partial U}{\partial h_m} - \frac{\partial R_m}{\partial h_m}, \tag{13.6}$$

for the mantle and similar equations for the core with suffix c in place of m. f's mean the components of the torque by the friction and the derivatives of U mean those of the tensional torque. Explicit expressions for them will be derived in the following sections.

R_m and R_c are the potentials of the mantle and the core owing to the luni–solar forces, and they are with a sufficient accuracy given by

$$R_m = \kappa^2(C_m - A_m)\left\{\frac{M_☾}{r_☾^3} P_2(\sin \delta_{☾m}) + \frac{M_\odot}{r_\odot^3} P_2(\sin \delta_{\odot m})\right\} \tag{14}$$

and a similar expression for R_c.

5. Torque of Frictional Force

5.1. A TORQUE AND ITS COMPONENTS

We first consider a general problem of how the components of a torque are expressed referred to Andoyer variables. The equations of motion of a rigid body in terms of Euler angles under an external torque N are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\psi}}\right) - \frac{\partial T}{\partial \psi} = N_\psi, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\phi}}\right) = N_\phi, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = N_\theta, \tag{15}$$

with

$$T = \tfrac{1}{2}\{A(\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi)^2 + B(\dot{\psi}\sin\theta\cos\phi - \dot{\theta}\sin\phi)^2 + C(\dot{\psi}\cos\theta + \dot{\phi})^2\}, \tag{16}$$

where

N_ψ: Z-component of **N**,
N_ϕ: C-axis component of **N**,
N_θ: component of **N** along the line of nodes.

(See Figure 2.)

Then the components of the torque referred to Andoyer variables are (cf. Brouwer and Clemence, 1961, for example)

$$\begin{aligned} f_{li} &= -N_\psi \frac{\partial \psi}{\partial L_i} - N_\phi \frac{\partial \phi}{\partial L_i} - N_\theta \frac{\partial \theta}{\partial L_i}, \\ f_{Li} &= +N_\psi \frac{\partial \psi}{\partial l_i} + N_\phi \frac{\partial \phi}{\partial l_i} + N_\theta \frac{\partial \theta}{\partial l_i}, \\ &(l_i, = l, g, h;\ L_i = L, G, H) \end{aligned} \tag{17}$$

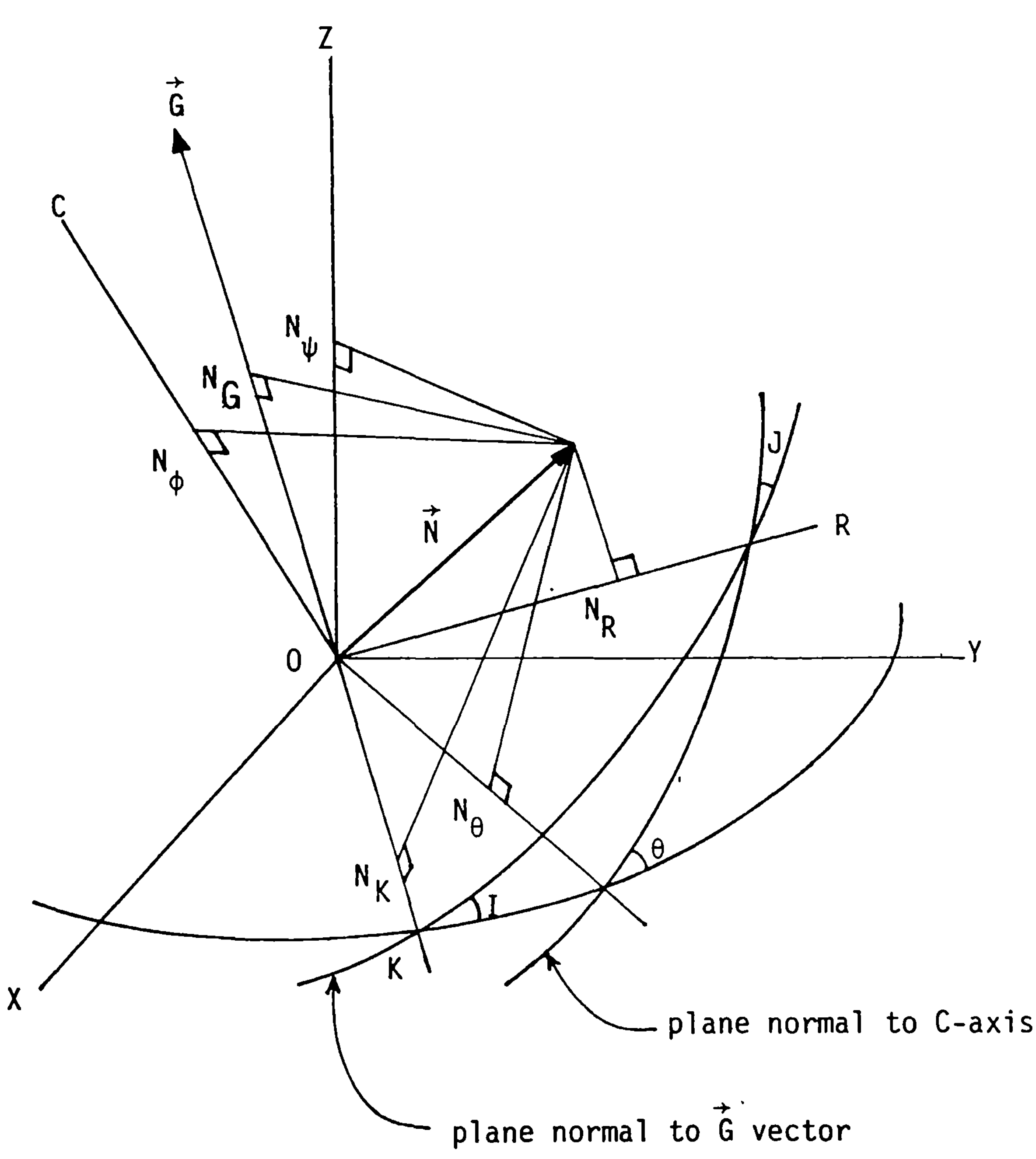

Fig. 2. A torque and components.

where

$$\frac{\partial\psi}{\partial l}=0,\qquad \frac{\partial\phi}{\partial l}=1,\qquad \frac{\partial\theta}{\partial l}=0,$$

$$\frac{\partial\psi}{\partial g}=\frac{\sin J\cos(\phi-l)}{\sin\theta},\qquad \frac{\partial\phi}{\partial g}=\frac{\sin I\cos(\psi-h)}{\sin\theta},\qquad \frac{\partial\theta}{\partial g}=\frac{-\sin I\sin J}{\sin\theta}\sin g,$$

$$\frac{\partial\psi}{\partial h}=1,\qquad \frac{\partial\phi}{\partial h}=0,\qquad \frac{\partial\theta}{\partial h}=0,$$

$$\frac{\partial\psi}{\partial L}=-\frac{1}{G}\frac{\sin(\phi-l)}{\sin\theta\sin J},\qquad \frac{\partial\phi}{\partial L}=\frac{1}{G}\frac{\cos\theta\sin(\phi-l)}{\sin\theta\sin J},\qquad \frac{\partial\theta}{\partial L}=-\frac{1}{G}\frac{\cos(\phi-l)}{\sin J},$$

$$\frac{\partial\psi}{\partial H}=\frac{1}{G}\frac{\cos\theta\sin(\psi-h)}{\sin\theta\sin I},\ \frac{\partial\phi}{\partial H}=-\frac{1}{G}\frac{\sin(\psi-h)}{\sin\theta\sin I},\qquad \frac{\partial\theta}{\partial H}=-\frac{1}{G}\frac{\cos(\psi-h)}{\sin I},$$

$$\frac{\partial\psi}{\partial G}=-\frac{\partial\psi}{\partial L}\cos J-\frac{\partial\psi}{\partial H}\cos I,\quad \frac{\partial\phi}{\partial G}=-\frac{\partial\phi}{\partial L}\cos J-\frac{\partial\phi}{\partial H}\cos I,$$

$$\frac{\partial\theta}{\partial G}=-\frac{\partial\theta}{\partial L}\cos J-\frac{\partial\theta}{\partial H}\cos I. \tag{18}$$

After some manipulations, we have

$$f_l=\frac{N_R}{G\sin J},\quad f_g=-\frac{N_R}{G}\cot J-\frac{N_K}{G}\cot I,\quad f_h=\frac{N_K}{G\sin I},$$
$$f_L=N_\phi,\quad f_G=N_G,\quad f_H=N_\psi, \tag{19}$$

where the meanings of N_R, N_K and N_G are shown in Figure 2.

5.2. Torque by frictional force

Next we consider the torque by the friction between the mantle and the core.

From Equations (1),

$$\mathbf{N}_m=\begin{bmatrix}\lambda_\perp(\omega_c-\omega_m)_{\xi m}\\ \lambda_\perp(\omega_c-\omega_m)_{\eta m}\\ \lambda_\parallel(\omega_c-\omega_m)_{\zeta m}\end{bmatrix}\quad\text{and}\quad \mathbf{N}_c=\begin{bmatrix}\lambda_\perp(\omega_m-\omega_c)_{\xi c}\\ \lambda_\perp(\omega_m-\omega_c)_{\eta c}\\ \lambda_\parallel(\omega_m-\omega_c)_{\zeta c}\end{bmatrix}, \tag{20}$$

where we have expressed the torque vector in such coordinate systems in the mantle and the core respectively that ζ-axis is coincident with $\mathbf{G}$ vector, η-axis with OR in Figure 2 and ξ-axis is taken so that ξ, η, ζ axes make a right-hand system. Then we have

$$\begin{bmatrix}\omega_{m\xi m}\\ \omega_{m\eta m}\\ \omega_{m\zeta m}\end{bmatrix}=\omega_m\begin{bmatrix}-\sin J_m'\\ 0\\ \cos J_m'\end{bmatrix} \tag{21}$$

and

$$\omega_{c\xi m} = \omega_c\{(-\cos I_c \sin J_c' \cos g_c + \sin I_c \cos J_c')(\cos I_m \cos \Delta h \cos g_m - \sin \Delta h \sin g_m) - \sin J_c' \sin g_c(\cos I_m \sin \Delta h \cos g_m + \cos \Delta h \sin g_m) - (\sin I_c \sin J_c' \cos g_c + \cos I_c \cos J_c') \sin I_m \cos g_m\}, \tag{22.1}$$

$$\omega_{c\eta m} = \omega_c\{(-\cos I_c \sin J_c' \cos g_c + \sin I_c \cos J_c')(-\cos I_m \cos \Delta h \sin g_m - \sin \Delta h \cos g_m) + \sin J_c' \sin g_c(\cos I_m \sin \Delta h \sin g_m - \cos \Delta h \cos g_m) + (\sin I_c \sin J_c' \cos g_c + \cos I_c \cos J_c') \sin I_m \sin g_m\}, \tag{22.2}$$

$$\omega_{c\zeta m} = \omega_c\{(-\cos I_c \sin J_c' \cos g_c + \sin I_c \cos J_c') \sin I_m \cos \Delta h - \sin I_m \sin J_c' \sin \Delta h \sin g_c + (\sin I_c \sin J_c' \cos g_c + \cos I_c \cos J_c') \cos I_m\}, \tag{22.3}$$

and similar expressions for $\omega_{m\xi c}$, $\omega_{c\xi c}$, etc., where

$$J_m' = \frac{C_m - A_m}{A_m} J_m + O(J_m^3), \quad J_c' = \frac{C_c - A_c}{A_c} J_c + O(J_c^3), \tag{23}$$

and

$$\Delta h = h_m - h_c. \tag{24}$$

All the components f_l, f_g, f_h, f_L, f_G and f_H are obtained from Equations (19), (20), (21) and (22).

For example,

$$f_{lm} = \frac{\lambda_\perp \omega_{c\eta m}}{G_m \sin J_m} = \frac{\lambda_\perp \omega_c}{G_m \sin J_m}\{(-\cos I_c \sin J_c' \cos g_c + \sin I_c) \times (-\cos I_m \cos \Delta h \sin g_m - \sin \Delta h \cos g_m) + \sin J_c' \sin g_c (\cos I_m \sin \Delta h \sin g_m - \cos \Delta h \cos g_m) + (\sin I_c \sin J_c' \cos g_c + \cos I_c) \sin I_m \sin g_m\}. \tag{25}$$

In this calculation the terms of $O(J)$ have been neglected.

6. Torque of Tensional Force

The tensional force is given by the potential (4), or

$$U = \tfrac{1}{2}\sigma\delta^2 = \sigma(1 - \cos\delta) + O(\sigma\delta^4), \tag{26}$$

where δ is the angle between C_m- and C_c-axes and

$$\cos\delta = (C_{mx}C_{cx} + C_{my}C_{cy} + C_{mz}C_{cz})/C_mC_c. \tag{27}$$

Neglecting the $O(\sigma\delta^4)$ terms, U is expressed in terms of Andoyer variables as following:

$$\begin{aligned} U = \sigma[1 &- \cos \Delta h\{\sin J_m \sin J_c \sin g_m \sin g_c + (\cos I_m \sin J_m \cos g_m + \\ &+ \sin I_m \cos J_m)(\cos I_c \sin J_c \cos g_c + \sin I_c \cos J_c)\} + \\ &+ \sin \Delta h\{\sin J_m \sin g_m(\cos I_c \sin J_c \cos g_c + \sin I_c \cos J_c) - \\ &- \sin J_c \sin g_c(\cos I_m \sin J_m \cos g_m + \sin I_m \cos J_m)\} - \\ &- (\sin I_m \sin J_m \cos g_m - \cos I_m \cos J_m)(\sin I_c \sin J_c \cos g_c - \\ &- \cos I_c \cos J_c)]. \end{aligned} \tag{28}$$

By differentiating U with respect to L_m, G_m, . . . , L_c, G_c, . . . , we get the components of the torque.

For example,

$$\begin{aligned} \frac{\partial U}{\partial L_m} = \frac{\sigma}{G_m}\{&\cot J_m \sin J_c \cos \Delta g - \cos \Delta I \cos J_c - \\ &- \cot J_m \cos J_c \sin \Delta I \cos g_m - \sin \Delta I \sin J_c \cos g_c + \\ &+ \sin \Delta h(-\cot J_m \sin J_c \cos I \sin \Delta g - \\ &- \cot J_m \cos J_c \sin I \sin g_m - \sin I \sin J_c \sin g_c)\}, \end{aligned} \tag{29}$$

where

$$\Delta I = I_m - I_c, \tag{30.1}$$

$$\Delta g = g_m - g_c, \tag{30.2}$$

and I is used in place of I_m or I_c when the distinction is not necessary.

7. Equations of Motion—Explicit Expression

From Equations (13), (25), (29), etc., we obtain the explicit form of the equations of motion except that the external forces still remain implicit.

We neglect the terms of $O(J)$ except in $\dot{L} - \dot{G}$, where the terms of $O(J^2)$ must be retained. Further we put $I_m = I_c = I$ and $\omega_m = \omega_c = \omega$, wherever the differences are not essential.

Thus the equations of motion for the mantle are

$$\begin{aligned} \dot{l}_m = \left(\frac{1}{C_m} - \frac{1}{A_m}\right)L_m &+ \frac{\partial R_m}{\partial L_m} + \\ &+ \frac{\sigma}{G_m}\{-\Delta I + \frac{1}{J_m}(J_c \cos \Delta g - \sin I \sin \Delta h \sin g_m - \\ &+ \frac{\sigma}{G_m}\{-\Delta I + \frac{1}{J_m}(J_c \cos \Delta g - \sin I \sin \Delta h \sin g_m - \\ &- \sin \Delta I \cos g_m)\}, \end{aligned} \tag{31.1}$$

$$\dot{g}_m = \frac{G_m}{A_m} + \frac{\partial R_m}{\partial G_m} + \frac{\lambda_\perp \omega}{G_m}\left\{\cos I \sin \Delta h - \frac{1}{J_m}(J_c' \sin \Delta g - \right.$$
$$\left. - \sin I \sin \Delta h \cos g_m)\right\} + \frac{\sigma}{G_m}\left\{\cos \Delta I + \cot I \sin \Delta I - \right.$$
$$\left. - \frac{1}{J_m}(J_c \cos \Delta g - \sin I \sin \Delta h \sin g_m - \sin \Delta I \cos g_m)\right\}, \tag{31.2}$$

$$\dot{h}_m = \frac{\partial R_m}{\partial H_m} - \frac{\lambda_\perp \omega}{G_m}\frac{\sin I_c}{\sin I_m} \sin \Delta h + \frac{\sigma}{G_m}(\cot I_m \sin I_c \cos \Delta h - \cos I_c), \tag{31.3}$$

$$\dot{L}_m = \lambda_\parallel\{-\omega_m + \omega_c(\sin I_c \sin I_m \cos \Delta h + \cos I_c \cos I_m)\}, \tag{31.4}$$

$$\dot{G}_m = \dot{L}_m + O(J), \tag{31.5}$$

$$\dot{H}_m = -\frac{\partial R_m}{\partial h_m} + \lambda_\parallel(\cos I_c \omega_c - \cos I_m \omega_m) +$$
$$+ (\lambda_\perp - \lambda_\parallel)\omega_c \sin I_m(-\sin I_c \cos I_m \cos \Delta h + \cos I_c \sin I_m) -$$
$$- \sigma \sin I_m \sin I_c \sin \Delta h, \tag{31.6}$$

and finally we need

$$\dot{G}_m - \dot{L}_m = -\frac{\partial R_m}{\partial g_m}$$
$$+ \lambda_\perp \omega(-J_m' J_m + J_m J_c' \cos \Delta g + J_m \sin I \sin \Delta h \sin g_m) -$$
$$- \sigma(J_m J_c \sin \Delta g - J_m \sin \Delta I \sin g_m +$$
$$+ J_m \sin I \sin \Delta h \cos g_m). \tag{31.7}$$

8. Precession, Nutation and Secular Change in the Obliquity

8.1. Solution for h and I

First we solve the equations of motion for h and I. Neglecting $(\Delta h)^3$ and $(\Delta I)^3$, we have from (31.3)

$$\dot{h}_m = -\frac{\lambda_\perp}{C_m}\frac{\sin I_c}{\sin I_m}\Delta h - \frac{\sigma}{G_m \sin I_m}\Delta I - \frac{\sigma}{2G_m}\cos I (\Delta h)^2 + p_m + u_m(t). \tag{32.1}$$

Using $\dot{I} = (\dot{G}\cos I - \dot{H})/(G \sin I)$, we have from Equations (31.5) and (31.6)

$$\dot{I}_m = -\frac{\lambda_\perp}{C_m}\Delta I + \frac{\sigma \sin I_c}{G_m}\Delta h - \frac{\lambda_\perp}{2C_m}\sin I \cos I (\Delta h)^2 + v_m(t). \tag{32.2}$$

In these equations

$$p_m = \left(\frac{\partial R_m}{\partial H_m}\right)_{\text{sec.}}, \tag{33.1}$$

$$u_m(t) = \left(\frac{\partial R_m}{\partial H_m}\right)_{\text{per.}} \equiv \sum_k a_{mk} \cos n_k t, \tag{33.2}$$

$$v_m(t) = \frac{1}{G_m \sin I}\frac{\partial R_m}{\partial h_m} - \frac{\cot I}{G_m}\frac{\partial R_m}{\partial g_m} \doteqdot \frac{1}{G_m \sin I}\frac{\partial R_m}{\partial h_m}$$

$$\equiv -\sin I \sum_k b_{mk} \sin n_k t. \tag{33.3}$$

The terms which contain g_m have been omitted since they have J as a factor. We have corresponding equations for the core. In Equations (33), $n_k t$ is the abbreviation for $i_1 l + i_2 l' + i_3 F + i_4 D + i_5 \Omega$, which is a linear function of time, the meanings of l, l', F, D, and Ω being the same as Woolard's. Although it contains h through $\Omega = \Omega_0 - h$, h may be considered constant in this case.

It should be noticed that the Poisson terms of the nutation of the mantle become

$$\Delta\psi = -\int u_m(t)\,\mathrm{d}t = -\sum_k \frac{a_{mk}}{n_k} \sin n_k t, \tag{34.1}$$

$$\Delta\varepsilon = \int v_m(t)\,\mathrm{d}t = \sin I \sum_k \frac{b_{mk}}{n_k} \cos n_k t, \tag{34.2}$$

if there is no interaction between the mantle and the core (i.e., the nutation for the independent motion of the mantle), while those for the rigid Earth are given by

$$\Delta\psi = -\int \left(\frac{\partial R}{\partial H}\right)_{\text{per.}} \mathrm{d}t = -\int \left(\frac{C_m}{C}\frac{\partial R_m}{\partial H_m} + \frac{C_c}{C}\frac{\partial R_c}{\partial H_c}\right)_{\text{per.}} \mathrm{d}t$$

$$= -\sum_k \frac{1}{Cn_k}(C_m a_{mk} + C_c a_{ck}) \sin n_k t, \tag{35.1}$$

$$\Delta\varepsilon = \int \frac{1}{G \sin I}\frac{\partial R}{\partial h}\,\mathrm{d}t = \frac{1}{G \sin I}\int \left(\frac{\partial R_m}{\partial h_m} + \frac{\partial R_c}{\partial h_c}\right) dt$$

$$= \sin I \sum_k \frac{1}{Cn_k}(C_m b_{mk} + C_c b_{ck}) \cos n_k t, \tag{35.2}$$

where

$$R = \kappa^2 (C - A)\left\{\frac{M_{\text{☾}}}{r_{\text{☾}}^3} P_2(\sin \delta_{\text{☾}}) + \frac{M_\odot}{r_\odot^3} P_2(\sin \delta_\odot)\right\}, \tag{36}$$

with $C = C_m + C_c$ and $A = A_m + A_c$.

Also use is made of the relation

$$\frac{\partial R_m}{\partial H_m} = \frac{C}{C_m} \cdot \frac{C_m - A_m}{C - A}\frac{\partial R}{\partial H}, \quad \frac{\partial R_m}{\partial h_m} = \frac{C_m - A_m}{C - A}\frac{\partial R}{\partial h}, \quad \text{etc.} \tag{37}$$

From Equations (32) and the corresponding equations for the core,

$$\Delta\dot{h} = -\gamma\lambda_\perp \Delta h - \frac{\gamma\sigma}{\omega \sin I_0}\Delta I + \Delta p + \gamma'\lambda_\perp \frac{\cos I_0}{\sin I_0}(\Delta I)(\Delta h) +$$

$$+ \frac{\gamma'_\sigma}{\omega}\frac{\cos I_0}{\sin^2 I_0}(\Delta I)^2$$

$$- \frac{\gamma'\sigma}{2\omega}\cos I_0 (\Delta h)^2 - p' \tan I_0\, \Delta I + \sum_k \Delta a_k \cos n_k t, \tag{38.1}$$

$$\Delta \dot{I} = \sin I_0 \left[\frac{\gamma\sigma}{\omega}\Delta h - \frac{\gamma\lambda_\perp}{\sin I_0}\Delta I - \frac{\gamma'\lambda_\perp}{2}\cos I_0(\Delta h)^2 - \sum_k \Delta b_k \sin n_k t\right],$$

where (38.2)

$$\gamma = \frac{1}{C_m} + \frac{1}{C_c}, \quad \gamma' = \frac{1}{C_m} - \frac{1}{C_c}, \quad I_0 = \frac{C_m I_m + C_c I_c}{C}, \tag{39.1}$$

p : precession of a rigid Earth with the ratio $(C - A)/C$ and the obliquity I_0,

$$\Delta p = \left\{\frac{C(C_m - A_m)}{C_m(C - A)} - \frac{C(C_c - A_c)}{C_c(C - A)}\right\}p, \quad p' = \frac{C_c p_m + C_m p_c}{C}, \tag{39.2}$$

$$\Delta a = a_m - a_c \text{ and } \Delta b = b_m - b_c. \tag{39.3}$$

Notice that

$$p_m = \frac{C(C_m - A_m)}{C_m(C - A)}\frac{\cos I_m}{\cos I_0}p \quad \text{and} \quad p_c = \frac{C(C_c - A_c)}{C_c(C - A)}\frac{\cos I_c}{\cos I_0}p. \tag{40}$$

Equations (38) are linear non-homogeneous differential equations if we neglect the second order terms. Therefore we first solve the equations neglecting these terms. Then we obtain the average values of Δh and ΔI, which can be considered constant if $\lambda_\perp$ and σ are small and as a result ΔI and Δh are so large that the periodic terms in them are negligible (roughly ΔI and $\Delta h > 1''$). Finally, substituting these values into the second order terms, we again solve the linear non-homogeneous differential equations (38). The result is

$$\begin{aligned}\Delta h = {} & \frac{\lambda_\perp \Delta p}{\gamma(\lambda_\perp^2 + \sigma'^2)} + \frac{P\sin I_0\,\lambda_\perp - Q\sigma'}{\gamma(\lambda_\perp^2 + \sigma'^2)\sin I_0} \\ & + \sum_k \frac{\Delta a_k + \Delta b_k}{2}\frac{\sin(n_k t + \varepsilon_1)}{\{\gamma^2\lambda_\perp^2 + (\gamma\sigma' + n_k)^2\}^{1/2}} + \\ & + \sum_k \frac{\Delta a_k - \Delta b_k}{2}\frac{\sin(n_k t + \varepsilon_2)}{\{\gamma^2\lambda_\perp^2 + (\gamma\sigma' - n_k)^2\}^{1/2}} + \\ & + D_1\,e^{-\gamma(\lambda_\perp + i\sigma')t} + D_2\,e^{-\gamma(\lambda_\perp - i\sigma')t}\end{aligned} \tag{41.1}$$

$$\begin{aligned}\Delta I = {} & \sin I_0\left[\frac{\sigma'\Delta p}{\gamma(\lambda_\perp^2 + \sigma'^2)} + \frac{P\sin I_0\,\sigma' + Q\lambda_\perp}{\gamma(\lambda_\perp^2 + \sigma'^2)\sin I_0} + \right. \\ & + \sum_k \frac{\Delta a_k + \Delta b_k}{2}\frac{\cos(n_k t + \varepsilon_1)}{\{\gamma^2\lambda_\perp^2 + (\gamma\sigma' + n_k)^2\}^{1/2}} - \\ & \left. - \sum_k \frac{\Delta a_k - \Delta b_k}{2}\frac{\cos(n_k t + \varepsilon_2)}{\{\gamma^2\lambda_\perp^2 + (\gamma\sigma' - n_k)^2\}^{1/2}}\right] + \\ & + i\sin I_0\{D_1\,e^{-\gamma(\lambda_\perp + i\sigma')t} - D_2\,e^{-\gamma(\lambda_\perp - i\sigma')t}\},\end{aligned} \tag{41.2}$$

where

$$\sigma' = \sigma/\omega, \quad i = \sqrt{-1}, \tag{42}$$

$$\tan \varepsilon_1 = \frac{\gamma\lambda_\perp}{n_k + \gamma\sigma'}, \quad \tan \varepsilon_2 = \frac{\gamma\lambda_\perp}{n_k - \gamma\sigma'}, \tag{43}$$

P and Q are the constant terms of the second order in Δh and ΔI respectively, and D_1 and D_2 are constants of integration. In solving these equations we have used the condition $\gamma\sigma'$, $\gamma\lambda_\perp \gg |\Delta p|$, and considered I_0 to be constant.

8.2. Precession and Secular Change in the Obliquity

Since the exponential terms in Equations (41) damp rapidly, we omit them. Substituting the other terms into (32), we have

$$\dot{h}_m = p + \frac{\sigma'(\Delta p)^2}{\gamma^2 C(\lambda_\perp^2 + \sigma'^2)} \frac{\cos^2 I_0 - \sin^2 I_0}{\cos I_0} + \text{(Periodic Terms)}, \tag{44.1}$$

$$\dot{I}_m = -\frac{\lambda_\perp(\Delta p)^2}{\gamma^2 C(\lambda_\perp^2 + \sigma'^2)} \sin I_0 \cos I_0 + \text{(Periodic Terms)}, \tag{44.2}$$

and similar equations for the core.

The secular term in $\dot{h}_m$ means the precession. The corresponding term for the core is the same. It follows that under the condition $\gamma\sigma'$, $\gamma\lambda_\perp \gg |\Delta p|$ the precessions of the mantle and the core coincide with each other but slightly different from p. This comes from the fact that in the present model the total angular momentum vector does not coincide with the combined C-axis even in the average. The secular term in $\dot{I}_m$ means the secular change in the obliquity. The corresponding term for the core is the same. If we put $\lambda_\perp = \lambda_\parallel = \lambda$ and $\sigma' = 0$, it agrees with Aoki's expression (Aoki, 1969).

In Figure 3 the numerical values are shown. The unit of $\lambda_\perp$ and σ' throughout Figures 3–9 is $G = C\omega = 5.25 \times 10^{40}$ gr cm^2 s^{-1}.

The following values have been taken from Malkus (1963);

$$\frac{C - A}{C} = 3.27 \times 10^{-3}, \quad \frac{C_m}{C} = 0.889, \quad \frac{C_c - A_c}{C_c} = 2.00 \times 10^{-3}, \tag{45}$$

which give

$$\frac{C_m - A_m}{C_m} = 3.43 \times 10^{-3}. \tag{46}$$

Also $p = -5037''$/century has been used.

It is seen in Figure 3 that σ' lessens the rate of the secular change in the obliquity to a large extent.

Rochester's value for $\lambda_\perp$ (Rochester, 1970) corresponds to $10^{-6.18}$ G, and a probable value for σ' is $10^{-3.65}$ G (see Appendix). For these values, the assumption that ΔI and Δh are large enough to be considered constant is not valid. But more refined calculations show that Equations (44) are still sufficiently exact, and give

$$\dot{h}_m = p + 0''.0033/\text{century and } \dot{I}_m = -0''.0000048/\text{century}. \tag{47}$$

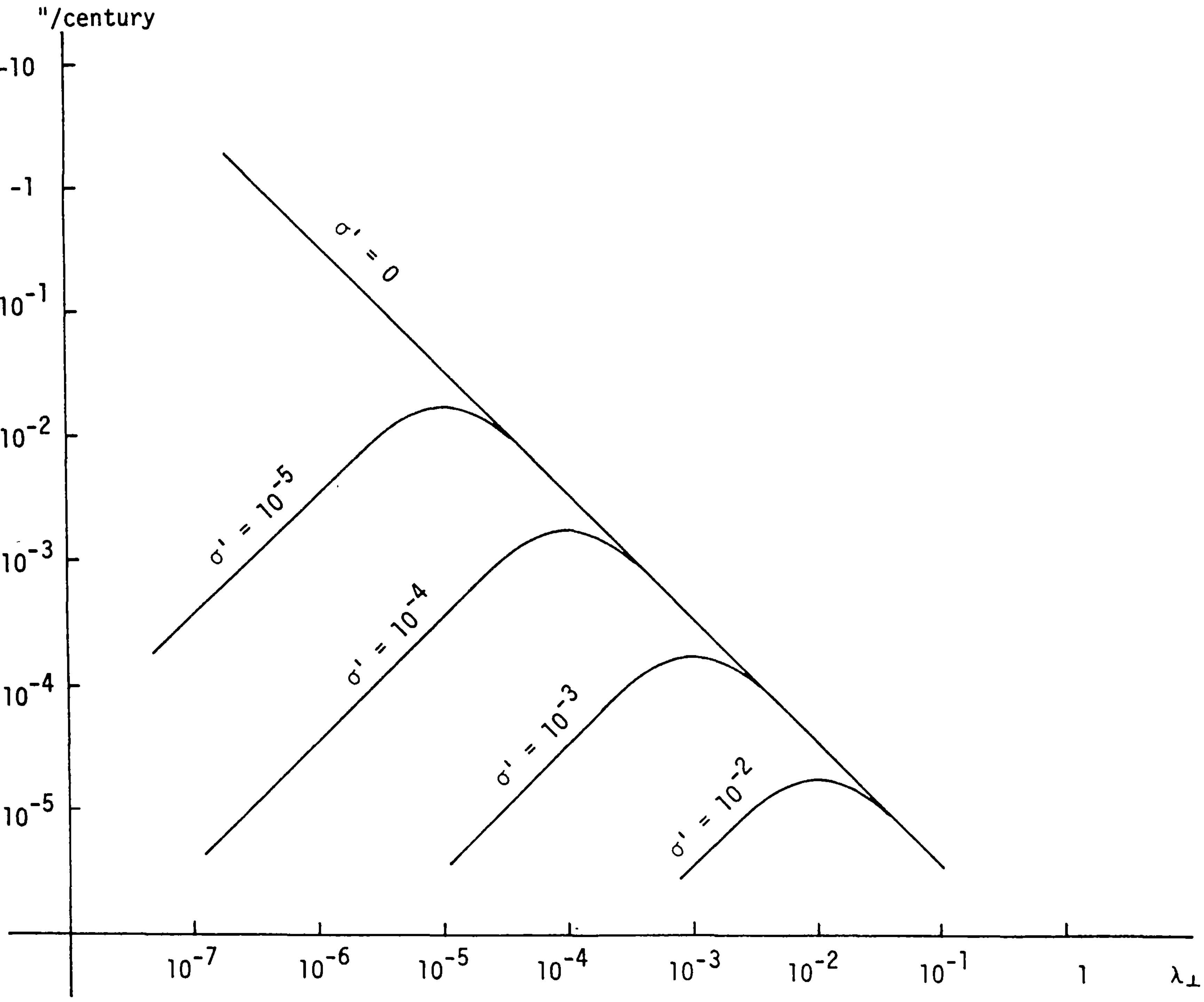

Fig. 3. Secular change in the obliquity.

The omitted exponential terms in Equations (41) could produce oscillations in the angular momentum vector of the mantle, with the periods of $\pm$ 400 days and the damping time of about 400 years. They could also cause nearly diurnal oscillations in the polar motion whose periods are (1 ± 0.00227) sidereal days.

8.3. Nutation (Poisson terms)

By integrating the periodic terms of (44), we obtain the nutation. If it is denoted by $\Delta\psi$ and $\Delta\varepsilon$,

$$\Delta\psi = -\sum_k \frac{1}{n_k}\left\{a_{mk} \sin n_k t - \frac{\Delta a_k + \Delta b_k}{2} A^{(+)} \sin (n_k t + \varepsilon_1') + \right.$$

$$\left. + \frac{\Delta a_k - \Delta b_k}{2} A^{(-)} \sin (n_k t + \varepsilon_2')\right\}, \tag{48.1}$$

$$\Delta\varepsilon = \sin I \sum_k \frac{n_k}{1}\left\{b_{mk} \cos n_k t - \frac{\Delta a_k + \Delta b_k}{2} A^{(+)} \cos (n_k t + \varepsilon_1') \right. -$$

$$- \frac{\Delta a_k - \Delta b_k}{2} A^{(-)} \cos (n_k t + \varepsilon_2') \Big\} , \tag{48.2}$$

where

$$A^{(\pm)} = \frac{(\lambda_\perp^2 + \sigma'^2)^{1/2}}{C_m\{\gamma^2\lambda_\perp^2 + (\gamma\sigma' \pm n_k)^2\}^{1/2}} , \tag{49}$$

$$\tan \varepsilon_1' = \tan (\varepsilon_1 - \varepsilon) = \frac{-\lambda_\perp n_k}{\gamma(\lambda_\perp^2 + \sigma'^2) + \sigma' n_k} ,$$

$$\tan \varepsilon_2' = \tan (\varepsilon_2 + \varepsilon) = \frac{\lambda_\perp n_k}{-\gamma(\lambda_\perp^2 + \sigma'^2) + \sigma' n_k} , \tag{50}$$

in which $\tan \varepsilon = \lambda_\perp/\sigma'$.

It is easily seen that

(a) if $\lambda_\perp \to \infty$ or $\sigma' \to \infty$, Equations (48) reduce to (35), which is the nutation of the rigid Earth, and

(b) if $\lambda_\perp \to 0$ and $\sigma' \to 0$, Equations (48) reduce to (34), which is the nutation in the independent motion of the mantle. In general, the nutation in the present model is expressed by

$$\Delta\psi = -\sum_k \frac{A_{mk}}{n_k} \sin (n_k t + E_{mk}), \tag{51.1}$$

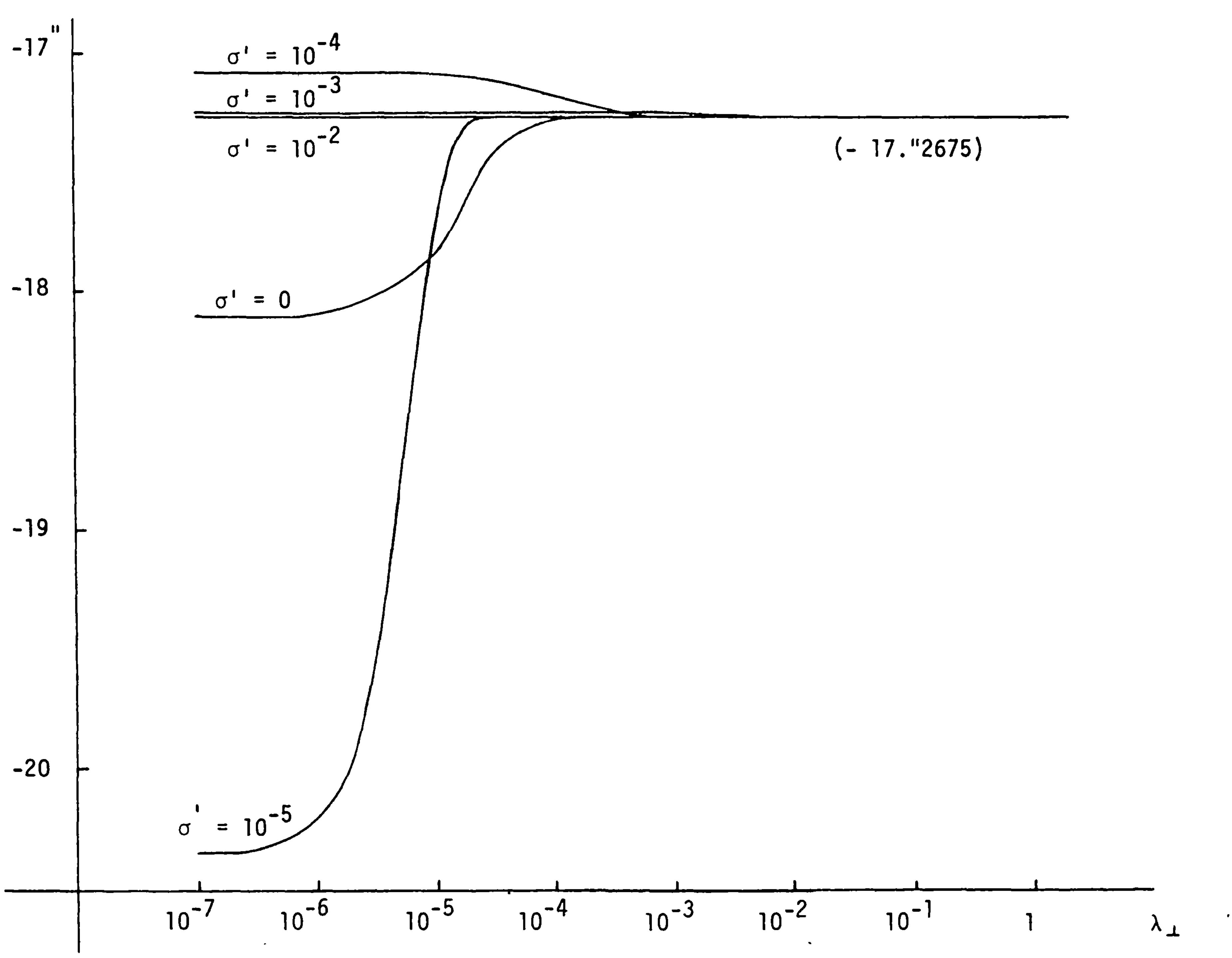

Fig. 4. Amplitude of the term with the argument Ω in $\Delta\psi$.

$$\Delta\varepsilon = \sin I \sum_k \frac{B_{mk}}{n_k} \cos (n_k t + E'_{mk}), \tag{51.2}$$

i.e., different amplitudes and phase angles from those of the rigid Earth.

Figure 4 through Figure 7 show a numerical result. We have chosen a single term with the argument Ω (the period 18.6 years) as an example. Amplitudes 9".2277 for $\Delta\varepsilon$ and $-17".2675$ for $\Delta\psi$ are the theoretical values for the rigid Earth obtained by Kinoshita (1977). It should be noticed that the effect of friction is almost negligible near the value of $10^{-6.18}$ G while the value of σ' has an important role in determining the amplitude. The corresponding value of σ' to the nutation constant 9".2100 is $10^{-3.54}$ G.

We compare the nutation resulting from the present model with those obtained from observations and other models in Table I. Notwithstanding the simplicity of the present model our values agree with observations or the other models within the observation errors.

9. Secular Changes in the Speed of Rotation

Next we solve Equation (31.5) and the corresponding equation for the core. We put

$$\sin I_c \sin I_m \cos \Delta h + \cos I_c \cos I_m = \cos \beta, \tag{52}$$

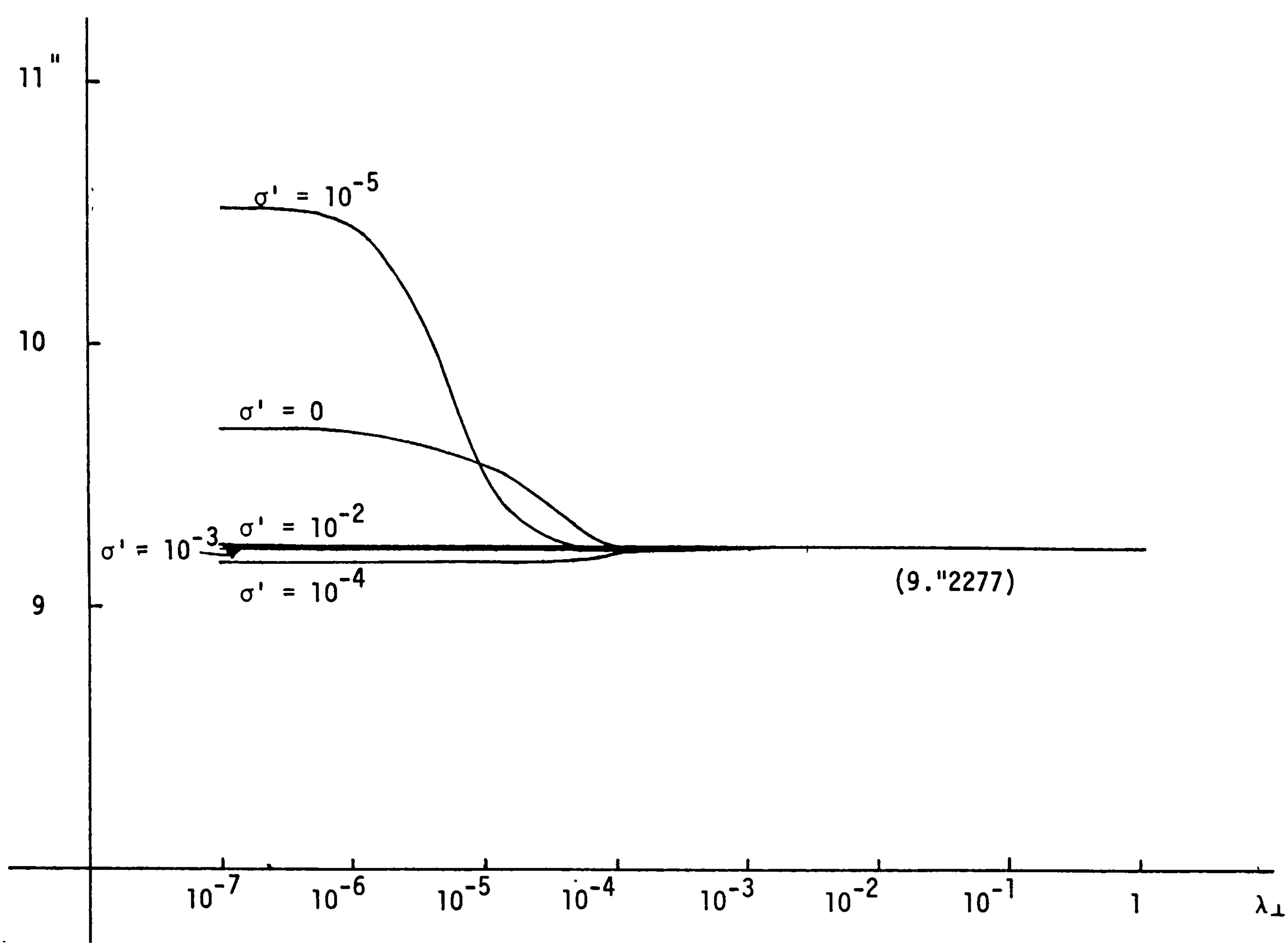

Fig. 5. Amplitude of the term with the argument Ω in $\Delta\varepsilon$.

TABLE I

Nutations given by some models and observations for 1900.0, without Oppolzer terms

						Kubo[b] $\lambda_{\perp} = 10^{-6.18}$ G					
						$\sigma' = 10^{-3.54}$ G		$\sigma' = 10^{-3.60}$ G		$\sigma' = 10^{-3.65}$ G	
Arg.	Per.	Rigid (Kinoshita)	IAU	Astron. Obs.[a]	Molodensky-II[b]	Amp.	Phase	Amp.	Phase	Amp.	Phase
					$\Delta\psi$						
Ω	$-6798^{\mathrm{d}}.4$	$-17''.2675$	$-17''.2327$	$-17''.192$	$-17''.1915$	$-17''.2021$	$-0^{\circ}.000$	$-17''.1928$	$-0^{\circ}.000$	$-17''.1843$	$-0^{\circ}.001$
$2L$	182.6	−1.2698	−1.2729	(0.533)[c]	−1.3072	−1.3053	*+0.001*	−1.3081	*+0.001*	−1.3101	*+0.001*
2Ω	−3399.2	+0.2079	+0.2088		+0.2063	+0.2065	*−0.001*	+0.2063	*−0.001*	+0.2063	*−0.001*
$L-\Gamma$	365.3	+0.1258	+0.1261	(0.031)[c]	+0.1415	+0.0842	*+1.031*	+0.1724	*+0.544*	+0.1455	*+0.103*
$3L-\Gamma$	121.7	−0.0496	−0.0497		−0.0512	−0.0513	*+0.001*	−0.0513	*+0.001*	−0.0515	*+0.001*
$L+\Gamma$	365.2	+0.0213	+0.0214		+0.0217	+0.0224	*−0.054*	+0.0211	*−0.063*	+0.0216	*−0.008*
$2☾$	13.7	−0.2041	−0.2037	(0.087)[c]	−0.2082	−0.2136	*+0.000*	−0.2136	*+0.000*	−0.2136	*+0.000*
$☾-\Gamma'$	27.5	+0.0677	+0.0675		+0.0709	+0.0709	*+0.000*	+0.0709	*+0.000*	+0.0709	*+0.000*
					$\Delta\varepsilon$						
Ω	$-6798^{\mathrm{d}}.4$	$+9''.2277$	$+9''.2100$	$+9''.209$	$+9''.2043$	$+9''.2098$	$-0^{\circ}.000$	$+9''.2069$	$-0^{\circ}.000$	$+9''.2042$	$-0^{\circ}.001$
$2L$	182.6	+0.5509	+0.5522	(0.553)[c]	+0.5694	+0.5700	*+0.002*	+0.5704	*+0.002*	+0.5709	*+0.002*
2Ω	−3399.2	−0.0902	−0.0904		−0.0896	−0.0897	*−0.001*	−0.0897	*−0.001*	−0.0896	*−0.001*
$L-\Gamma$	365.3	0	0	(0.031)[c]	+0.0050	−0.0178	*−1.948*	+0.0173	*+2.177*	+0.0065	*+0.917*
$3L-\Gamma$	121.7	+0.0215	+0.0216		+0.0223	+0.0223	*+0.001*	+0.0223	*+0.001*	+0.0223	*+0.001*
$L+\Gamma$	365.2	−0.0092	−0.0093		−0.0095	−0.0092	*+0.056*	−0.0097	*+0.057*	−0.0095	*+0.011*
$2☾$	13.7	+0.0885	+0.0884	(0.087)[c]	+0.0906	+0.0926	*+0.000*	+0.0927	*+0.000*	+0.0927	*+0.000*
$☾-\Gamma'$	27.5	0	0		+0.0004	+0.0001	*+0.021*	+0.0001	*+0.021*	+0.0001	*+0.021*

[a] Yokoyama, Manabe *et al.* Includes Oppolzer terms.

[b] Based on the rigid Earth by Kinoshita.

[c] $\dfrac{\Delta\psi \cos\varepsilon + \Delta\varepsilon}{2}$.

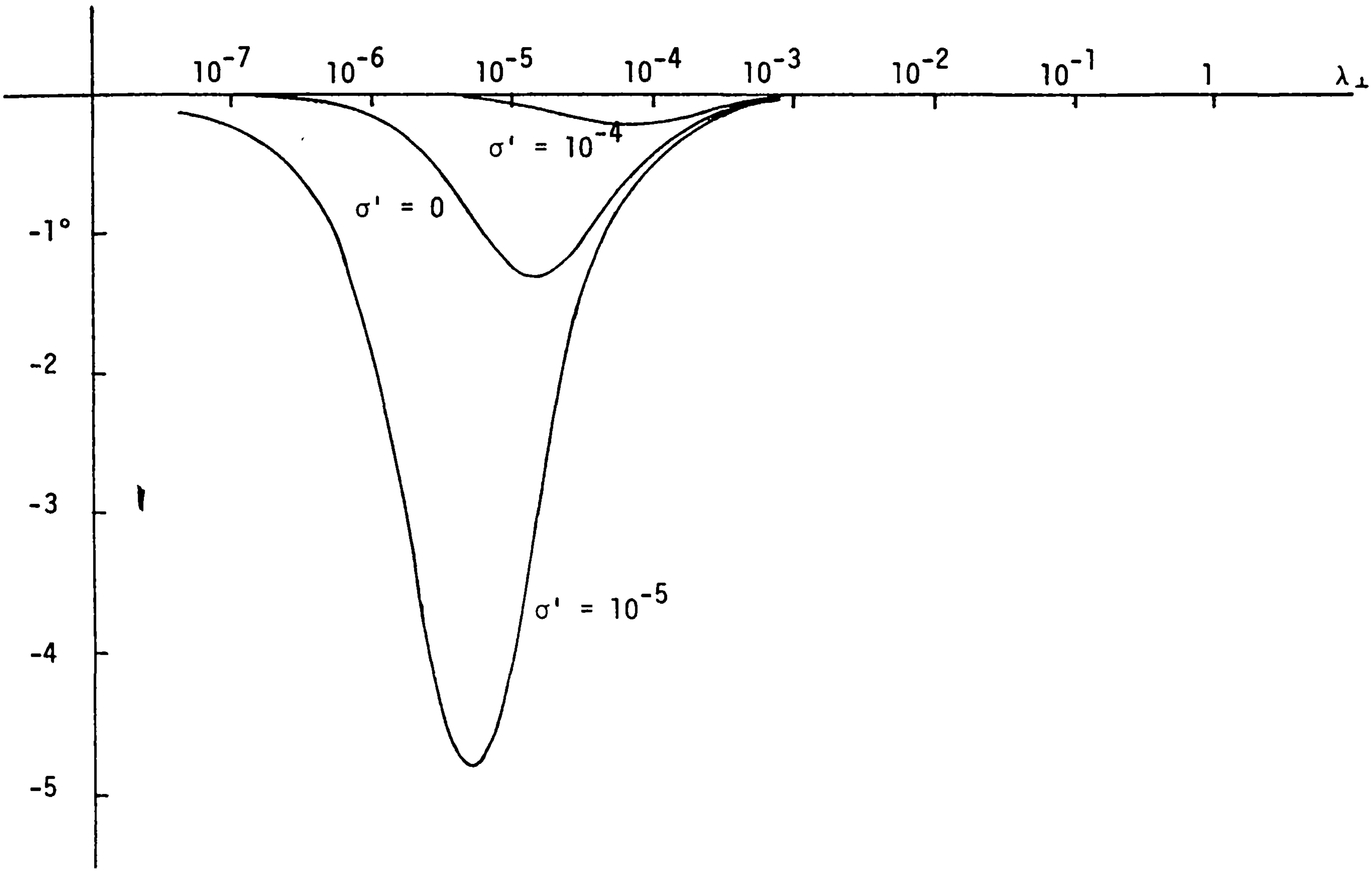

Fig. 6. Deviation of phase angle of the term with the argument Ω in $\Delta\psi$.

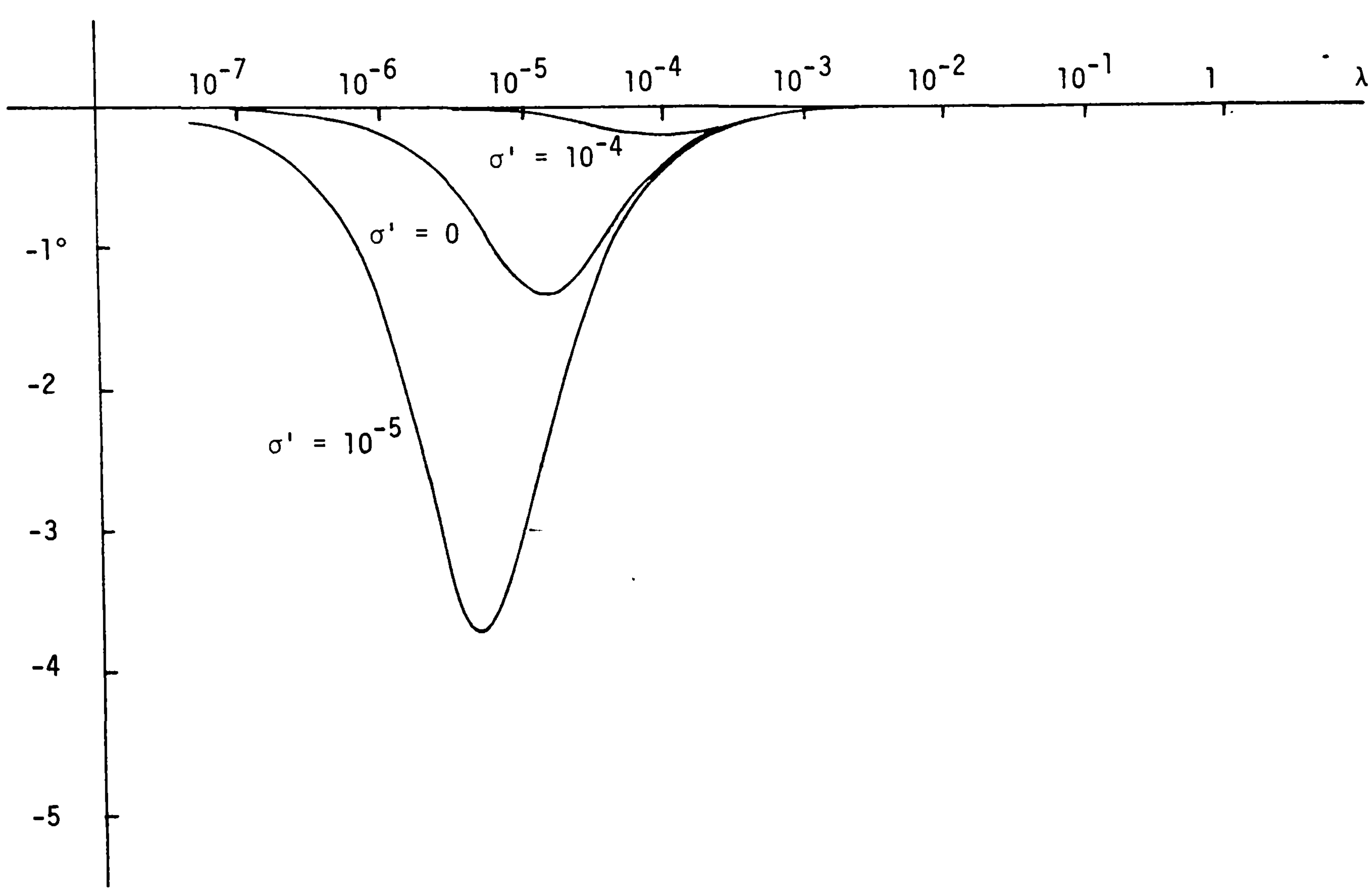

Fig. 7. Deviation of phase angle of the term with the argument Ω in $\Delta\varepsilon$.

β being the angle between the angular momentum vectors of the mantle and the core. β can be considered roughly constant having the value of 1''.7 for the above values of $\lambda_\perp$ and σ'. Equation (31.5) becomes

$$\dot{G}_m = \lambda_\parallel(-\omega_m + \omega_c \cos\beta) = \lambda_\parallel\left(-\frac{1}{C_m}G_m + \frac{\cos\beta}{C_c}G_c\right). \tag{53.1}$$

The corresponding equation for the core is

$$\dot{G}_c = \lambda_\parallel(-\omega_c + \omega_m \cos\beta) = \lambda_\parallel\left(\frac{\cos\beta}{C_m}G_m - \frac{1}{C_c}G_c\right). \tag{53.2}$$

Solving these linear homogeneous differential equations (53),

$$G_m = G_1\, \mathrm{e}^{-\lambda_\parallel \gamma''(\sin^2\beta)t} + G_2\, \mathrm{e}^{\lambda_\parallel(-\gamma + \gamma''(\sin^2\beta)t}, \tag{54.1}$$

$$G_c = \frac{\gamma''(C_c + C_m\cos^2\beta)}{\cos\beta}\left\{\frac{C_c}{C_m}G_1\, \mathrm{e}^{-\lambda_\parallel\gamma''(\sin^2\beta)t} + G_2\, \mathrm{e}^{\lambda_\parallel(-\gamma + \gamma''(\sin^2\beta)t}\right\}, \tag{54.2}$$

where G_1 and G_2 are constants of integration and $\gamma'' = C^{-1}$.

Since the second terms on the right-hand sides damp rapidly, we neglect them. Then the secular change in the speed of rotation is given by

$$\frac{\dot{\omega}_m}{\omega_m} \doteqdot \frac{\dot{G}_m}{G_m} = -\frac{\lambda_\parallel}{C}\sin^2\beta. \tag{55}$$

From Equations (41),

$$\sin^2\beta \doteqdot (\Delta I)^2 + \sin^2 I\,(\Delta h)^2 = \frac{(\Delta p)^2}{\gamma^2(\lambda_\perp^2 + \sigma'^2)}\sin^2 I.$$

Hence

$$\frac{\dot{\omega}_m}{\omega_m} = -\frac{1}{\gamma^2 C}\frac{\lambda_\parallel(\Delta p)^2}{(\lambda_\perp^2 + \sigma'^2)}\sin^2 I. \tag{56}$$

Comparing this expression with (44.2), we have a relation

$$\begin{pmatrix}\text{secular change in the}\\ \text{speed of rotation}\end{pmatrix} = \frac{\lambda_\parallel}{\lambda_\perp}\tan I \times \begin{pmatrix}\text{secular change in}\\ \text{the obliquity}\end{pmatrix}. \tag{57}$$

This relation is valid again only for sufficiently large ΔI and Δh, but exact enough for the present case. Supposing $\lambda_\parallel \sim \lambda_\perp$,

$$\dot{\omega}_m \sim -10^{-6}\ \mathrm{s\ day^{-1}\ century^{-1}}. \tag{58}$$

This is far smaller than the observed value. Therefore the friction between the mantle and the core cannot be a major cause for the secular change of the rotational speed.

10. Motion of the Axis of Figure and Polar Motion

10.1. Solution for g and J

Finally we solve the equations of motion for g and J. From Equation (31.2)

$$\begin{aligned}\dot{g}_m = {} & \frac{G_m}{A_m} + \frac{\lambda_\perp \omega}{G_m} \cos I \sin \Delta h + \frac{\sigma}{G_m} \cos \Delta I + \frac{\partial R_m}{\partial G_m} + \\ & + \frac{\lambda_\perp \omega}{G_m J_m} \{-J_c' \sin (g_m - g_c) + \sin I \sin \Delta h \cos g_m\} - \\ & - \frac{\sigma}{G_m J_m} \{J_c \cos (g_m - g_c) - \sin \Delta I \cos g_m - \sin I \sin \Delta h \sin g_m\}.\end{aligned} \tag{59.1}$$

Using $\dot{J} \doteqdot (\dot{\mathrm{G}} - \dot{\mathrm{L}})/G \cdot J$, we have from (31.7),

$$\begin{aligned}\dot{J}_m = {} & -\frac{1}{G_m J_m} \frac{\partial R_m}{\partial g_m} + \\ & + \frac{\lambda_\perp \omega}{G_m} \{-J_m' + J_c' \cos (g_m - g_c) + \sin I \sin \Delta h \sin g_m\} - \\ & - \frac{\sigma}{G_m} \{J_c \sin (g_m - g_c) - \sin \Delta I \sin g_m + \sin I \sin \Delta h \cos g_m\}.\end{aligned} \tag{59.2}$$

In Equation (59.1), the second and the third terms can be included into the first term in the first order theory, and we neglect them. The terms of the derivatives of R_m in Equations (59) would produce Oppolzer terms, which would be different from those for the rigid Earth in the similar way to the Poisson terms. However, since the Oppolzer terms are small themselves, we do not consider their change in his study and we omit the terms of the derivatives of R_m in the following discussion.

Let $x_m = J_m \cos g_m$, $y_m = J_m \sin g_m$, etc. Then Equations (59.1) and (59.2) become respectively

$$\begin{aligned}x_m \dot{x}_m + y_m \dot{y}_m = {} & \frac{\lambda_\perp \omega}{G_m} \left\{ -\frac{C_m - A_m}{A_m} (x_m^2 + y_m^2) + \frac{C_c - A_c}{A_c} (x_m x_c + y_m y_c) + \right. \\ & \left. + (\sin I \sin \Delta h) y_m \right\} - \frac{\sigma}{G_m} \{(y_m x_c - x_m y_c) - (\sin \Delta I) y_m + \\ & + (\sin I \sin \Delta h) x_m\},\end{aligned} \tag{60.1}$$

$$\begin{aligned}x_m \dot{y}_m - \dot{x}_m y_m = {} & \frac{G_m}{A_m} (x_m^2 + y_m^2) + \frac{\lambda_\perp \omega}{G_m} \left\{ -\frac{C_c - A_c}{A_c} (y_m x_c - x_m y_c) + \right. \\ & \left. + (\sin I \sin \Delta h) x_m \right\} - \frac{\sigma}{G_m} \{(x_m x_c + y_m y_c) - (\sin \Delta I) x_m \\ & - (\sin I \sin \Delta h) y_m\}.\end{aligned} \tag{60.2}$$

Let $z_m = x_m + i y_m$ and $z_c = x_c + i y_c$. Then, combining Equations (60) and dividing the both sides by $\bar{z}_m(= x_m - iy_m)$, we have

$$\dot{z}_m = i\frac{G_m}{A_m}z_m + \frac{\lambda_\perp\omega}{G_m}\left\{-\frac{C_m - A_m}{A_m}z_m + \frac{C_c - A_c}{A_c}z_c + i\sin I\sin \Delta h\right\} - \\ -\frac{\sigma}{G_m}\{iz_c - i\sin \Delta I + \sin I\sin \Delta h\}, \tag{61}$$

and similar equation for $\dot{z}_c$. Write them as

$$\dot{z}_m = q_m z_m + r_m z_c + s_m, \tag{62.1}$$

$$\dot{z}_c = q_c z_c + r_c z_m + s_c, \tag{62.2}$$

where

$$q_m = -\frac{C_m - A_m}{A_m}\frac{\lambda_\perp\omega}{G_m} + i\frac{G_m}{A_m}, \tag{63.1}$$

$$r_m = \frac{C_c - A_c}{A_c}\frac{\lambda_\perp\omega}{G_m} - i\frac{\sigma}{G_m}, \tag{63.2}$$

$$s_m = -\frac{\sigma}{G_m}\sin I\sin \Delta h + i\left(\frac{\lambda_\perp\omega}{G_m}\sin I\sin \Delta h + \frac{\sigma}{G_m}\sin \Delta I\right), \tag{63.3}$$

and the similar expressions for the core.
Considering Δh and ΔI to be constant in the same sense as in Section 8, Equations (62) are linear non-homogeneous differential equations and we have the following general solution.

$$z_m = \alpha e^{\lambda_1 t} + \frac{\lambda_2 - q_c}{r_c}\beta\, e^{\lambda_2 t} - \frac{q_c s_m - r_m s_c}{q_m q_c - r_m r_c}, \tag{64.1}$$

$$z_c = \frac{\lambda_1 - q_m}{r_m}\alpha\, e^{\lambda_1 t} + \beta\, e^{\lambda_2 t} - \frac{q_m s_c - r_c s_m}{q_m q_c - r_m r_c}, \tag{64.2}$$

where α and β are constants of integration and λ_1 and λ_2 are the roots of the characteristic equation

$$(\lambda - q_m)(\lambda - q_c) - r_m r_c = 0. \tag{65}$$

10.2. Motion of the Axis of Figure

Denote the position of the axis of figure relative to the angular momentum vector by Δh_f and ΔI_f. Then from Equations (9),

$$\Delta h_{fm} = \frac{J_m \sin g_m}{\sin I} = \frac{(z_m)_{\text{imag.}}}{\sin I} \\ = \frac{1}{\sin I}\left\{a_1\, e^{k_1 t}\sin(n_1 t + \mu_1) + a_2\, e^{k_2 t}\sin(n_2 t + \mu_2) - \right. \\ \left. - \left(\frac{q_c s_m - r_m s_c}{q_m q_c - r_m r_c}\right)_{\text{imag.}}\right\}, \tag{66.1}$$

$$\begin{aligned} \Delta I_{fm} &= J_m \cos g_m = (z_m)_{\text{real}} \\ &= a_1\, e^{k_1 t} \cos(n_1 t + \mu_1) + a_2\, e^{k_2 t} \cos(n_2 t + \mu_2) - \\ &\quad - \left.\left(\frac{q_c s_m - r_m s_c}{q_m q_c - r_m r_c}\right)_{\text{real}} \right\}, \end{aligned} \tag{66.2}$$

where k_1 and k_2 are the real parts of λ_1 and λ_2 respectively, n_1 and n_2 are the imaginary parts, and a_1, a_2, μ_1 and μ_2 are arbitrary constants.

The first and the second terms on the right-hand sides of Equations (66) are the free nutation in the present model. n_1 and n_2 reduce to the periods of free nutations of the mantle and the core respectively when $\lambda_\perp$ and σ' vanish. The third terms are similar to the Oppolzer terms and should be added to the nutation. These terms are very small, but reaches $+0\overset{''}{.}0004$ in ΔI_{fm}.

Figure 8 shows the values of k_1 and k_2. They are little affected by σ' and give a damping time of about 10^5 years near $\lambda_\perp = 10^{-6.18}$G.

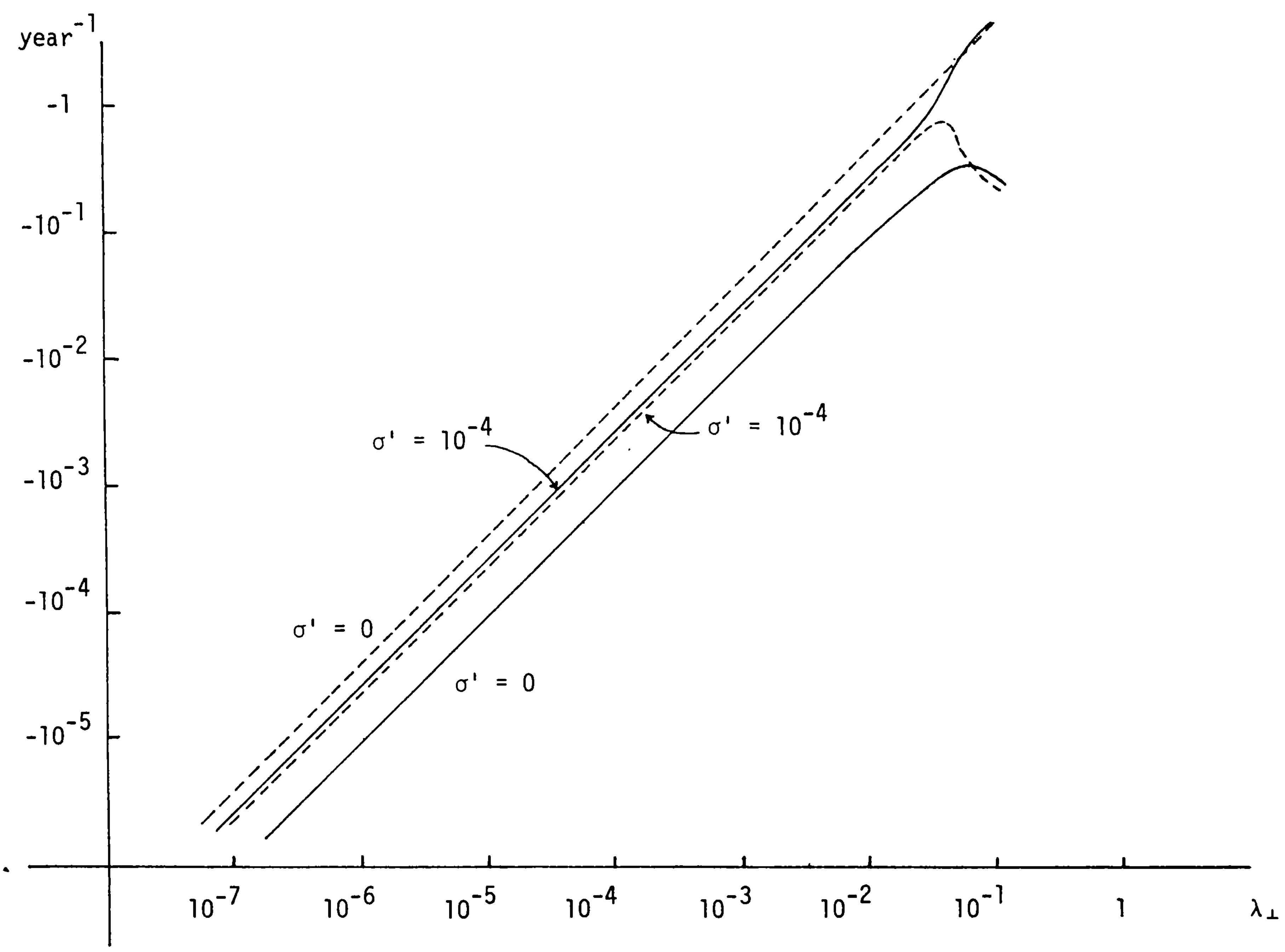

Fig. 8. k_1 (solid lines) and k_2 (dotted lines).

10.3. Polar motion

If the nutation is defined by the rotational axis of the mantle, except for the Oppolzer terms, the polar motion should be defined as the motion of this axis on the surface of the mantle. In this case we have from Equations (12),

$$\frac{x_p}{y_p} = \pm \frac{C_m}{A_m} J_m \frac{\sin}{\cos} l_m. \tag{67}$$

Put $\tau = l_m + g_m$. Then $\tau \cong \omega t + \tau_0$, and Equations (67) become

$$\begin{aligned}\frac{x_p}{y_p} &= \pm\frac{C_m}{A_m} J_m \frac{\sin}{\cos} (\tau - g_m) \\ &= \pm\frac{C_m}{A_m}\Bigg[a_1\, e^{k_1 t} \frac{\sin}{\cos}\{(\omega - n_1)t + \mu_1'\} + a_2\, e^{k_2 t} \frac{\sin}{\cos}\{(\omega - n)t + \mu_2''\} \\ &\quad - \left(\frac{q_c s_m - r_m s_c}{q_m q_c - r_m r_c}\right)_{\text{real.}} \frac{\sin}{\cos}\tau \pm \left(\frac{q_c s_m - r_m s_c}{q_m q_c - r_m r_c}\right)_{\text{image}} \frac{\cos}{\sin}\tau\Bigg].\end{aligned} \tag{68}$$

The first and the second terms on the right-hand sides are the free nutation in the present model. $\omega - n_1$ and $\omega - n_2$ reduce to the periods of free nutations of the mantle and the core respectively when $\lambda_\perp$ and σ' vanish.

Since the periods of these two terms are near each other, they may have a beat effect and bring about a gradual and periodical change in the amplitude of the polar wobble which is constant except for small oscillations in the rigid Earth. This will

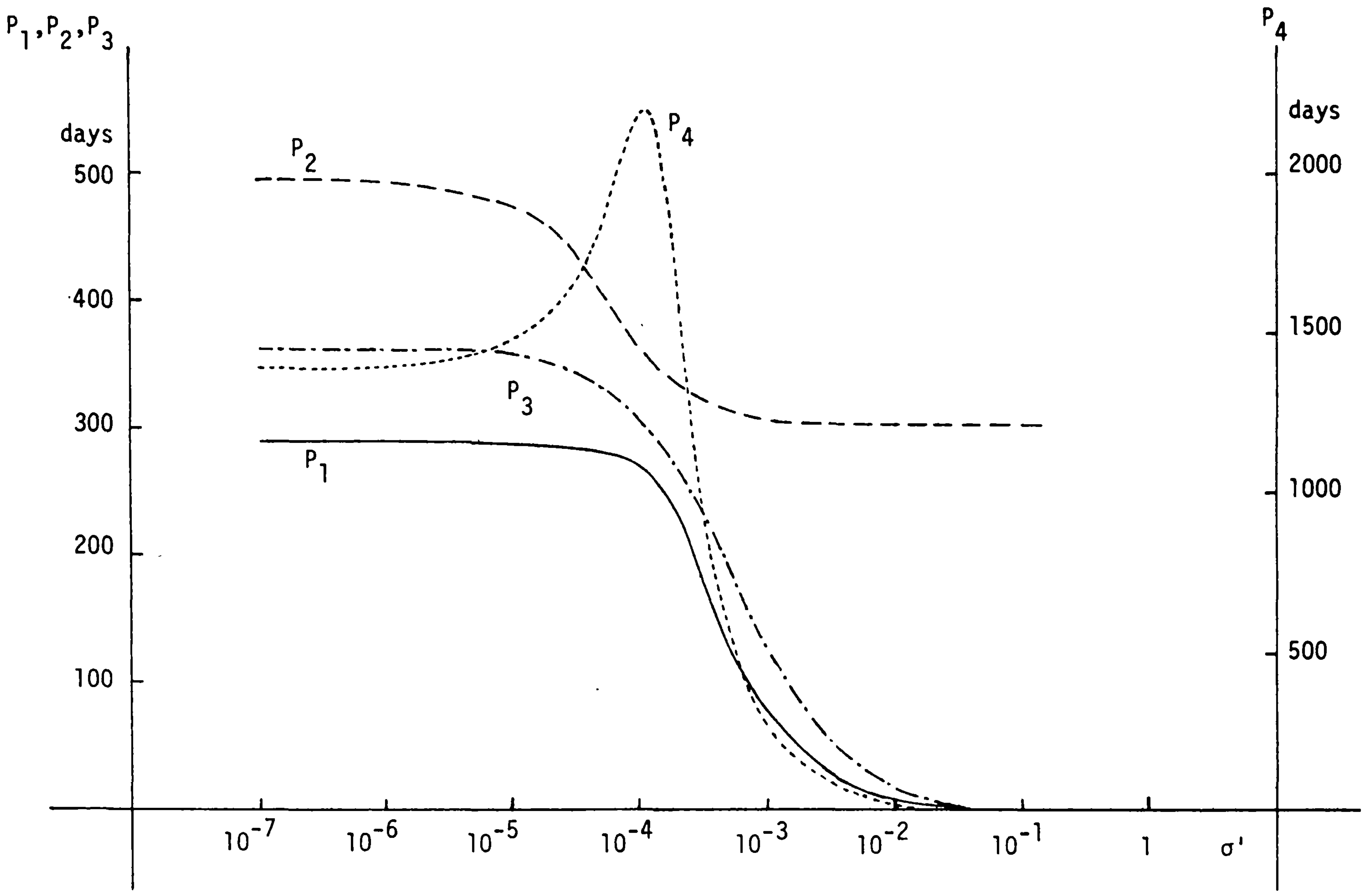

Fig. 9. $P_1 = 2\pi/(\omega - n_1)$, $P_2 = 2\pi/(\omega - n_2)$, $P_3 = 4\pi/(2\omega - n_1 - n_2)$ and $P_4 = 4\pi/(n_1 - n_2)$.

occur if $a_1 \simeq a_2$ in Equations (68). The period of the wobble in this case becomes $4\pi/(2\omega - n_1 - n_2)$ while the period of the change in its amplitude is given by $4\pi/(n_1 - n_2)$. Figure 9 shows these periods together with $P_1 = 2\pi/(\omega - n_1)$ and $P_2 = 2\pi/(\omega - n_2)$ as functions of σ'. They are almost independent of $\lambda_\perp$ as far as $\lambda_\perp$ is not very large. The present model represents the observational facts about the wobble fairly well but not sufficiently in quantity. The last two terms of Equations (68) are diurnal terms, of which the period is exactly one sidereal day.

11. Concluding Remarks

We have investigated the rotation of an Earth which consists of a rigid mantle and a rigid core and has an interaction between them through a frictional force and a tensional force. With the use of Andoyer variables the motion can be separated into two parts; the motion of the angular momentum vector and the motion of the axis of figure relative to the angular momentum vector. The former contains the precession, the Poisson terms, or main parts, of the nutation and the secular changes in the obliquity and the rotational speed, and the latter includes the Oppolzer terms and the polar motion.

The results obtained by this investigation are summarized as following. Some of them may depend largely on the nature of the model used and have to be revised if models are different, but most of them would remain correct as a first approximation for more improved models.

(1) The effect of the friction on the nutation is very small in the vicinity of the intensity of the friction which is considered probable, while the influence of tensional force is essential in determining the amplitudes of the nutation. Hence it is possible to make the theoretical values agree with those obtained by observations.

(2) These forces also influence the phase angles of the nutation, although the change is small.

(3) Secular changes in the obliquity and the speed of rotation are very sensitive to the magnitudes of the frictional and the tensional forces. The magnitudes favorable to explain the amplitude of the nutation produce very small secular changes.

(4) The motion of the axis of figure relative to the angular momentum vector is affected by the interaction between the mantle and the core. The period of the free nutation, or the polar motion circle, is different from that of the rigid earth and its amplitude may undergo a gradual periodical change.

(5) The damping time of the free nutation is about 10^5 years for the probable intensity of friction.

(6) The Oppolzer terms have not been discussed fully in this paper, although it is likely that they are different from those in the rigid Earth.

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Appendix

Nature of the tensional force

We shall compare our tensional force with the force which the rigid mantle is exerted by the liquid core in the Poincaré's model (Poincaré, 1910). The latter force is categorized as an inertial coupling by Rochester (1970).

Consider a single term of luni-solar potential which gives such u_m and v_m in Equations (33) as expressed by $a_m \cos nt$ and $-\sin I a_m \sin nt$ respectively. Let the amplitudes of the corresponding term of the nutation for the Earth with a liquid core and for the rigid Earth be denoted by $\tilde{\omega}$ and $\tilde{\omega}_0$ respectively. Then Poincaré shows

$$\frac{\tilde{\omega}}{\tilde{\omega}_0} = \frac{1 + n/\varepsilon\omega}{1 + (n/\varepsilon\omega)(C_m/C)}, \tag{A.1}$$

where $\varepsilon = (C_c - A_c)/C_c$.

While, with the tensional force we have from Equations (48) for the same term of perturbation,

$$\begin{aligned}\tilde{\omega} &= \frac{1}{n}\left\{a_m - (a_m - a_c)\frac{\sigma}{C_m(\gamma\sigma' + n)}\right\} \\ &= \frac{1}{Cn}\,\frac{(C_m a_m + C_c a_c) + (nC_c/\sigma')C_m a_m}{1 + (nC_c/\sigma')(C_m/C)},\end{aligned}$$

since $\Delta a - \Delta b = 0$ in this case and ε_1' is zero. When σ' increases, we have $\tilde{\omega}_0$ for the rigid Earth,

$$\tilde{\omega}_0 = \frac{1}{Cn}(C_m a_m + C_c a_c).$$

Hence

$$\frac{\tilde{\omega}}{\tilde{\omega}_0} = \frac{1 + C_m a_m/(C_m a_m + C_c a_c)\cdot(nC_c/\sigma')}{1 + (nC_c/\sigma')(C_m/C)}.$$

Put $\sigma' = \varepsilon C_c \omega$. Then, giving numerical values to C_c/C_m and a_c/a_m, it becomes

$$\frac{\varpi}{\varpi_0} = \frac{1 + 0.93(n/\varepsilon\omega)}{1 + (n/\varepsilon\omega)(C_m/C)}. \tag{A.2}$$

Comparing Equations (A.1) and (A.2) we can see a similarity of the tensional force to an intertial force, at least concerning the effect on the nutation. The value of σ' corresponding to $\varepsilon C_c \omega$ is $10^{-3.65}$ G.

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