

FT-HMC code for 2D $U(1)$

Nobuyuki Matsumoto*

Abstract

Brief description of the code. Many features mimic the qlat [1].

*E-mail address: nobuyuki.matsumoto@riken.jp

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1. Physical system

Path integral

$U(1)$ lattice gauge theory on T^2 :

$$Z_{\beta,\theta} \equiv \int (dU) e^{-S(U) + i\theta Q(U)}, \quad (1.1)$$

where $U_{x,\mu}$ are $U(1)$ -valued link variables (see figure 1):

$$U_{x,\mu} = e^{i\phi_{x,\mu}} \quad (1.2)$$

and (dU) the Haar measure:

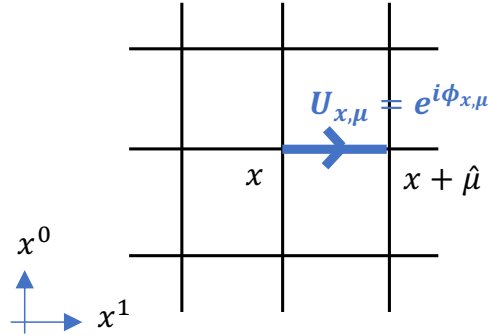


Figure 1: 2D $U(1)$ lattice.

$$(dU) = \prod_{x,\mu} \left(\frac{d\phi_{x,\mu}}{2\pi} \right). \quad (1.3)$$

We choose as $S(U)$ the Wilson action [2]:

$$S(U) \equiv -\beta \sum_x \cos \kappa_x \quad (1.4)$$

and $Q(U)$ the integer-valued topological charge [3, 4]:

$$Q(U) \equiv \frac{1}{2\pi} \sum_x \kappa_x, \quad (1.5)$$

where κ_x is the plaquette angle:

$$\begin{aligned}\kappa_x &\equiv \frac{1}{i} \ln (U_{x,0} U_{x+0,1} U_{x+1,0}^\dagger U_{x,1}^\dagger) \\ &= \phi_{x,0} + \phi_{x+0,1} - \phi_{x+1,0} - \phi_{x,1} \pmod{2\pi},\end{aligned}\tag{1.6}$$

where it is assumed that we take the principal branch of the logarithm:

$$\kappa_x \in [-\pi, \pi).\tag{1.7}$$

Classically, the gauge potential $A_{x,\mu}$ in the continuum is defined as:

$$\phi_{x,\mu} = a A_{x,\mu},\tag{1.8}$$

where a is the lattice spacing. By scaling β as:

$$\beta = \frac{1}{(ag)^2},\tag{1.9}$$

we have

$$S(U) \sim \frac{a^2}{2g^2} \sum_x F_{01}^2 \sim \frac{1}{4g^2} \int d^2x F_{\mu\nu}^2\tag{1.10}$$

and

$$Q(U) \sim \frac{1}{2\pi} a^2 \sum_x F_{01} \sim \frac{1}{2\pi} \int d^2x F_{01},\tag{1.11}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu.\tag{1.12}$$

Exact solutions

We compute the partition function:

$$Z_{\beta,\theta} = \int (dU) \prod_x e^{\beta \cos \kappa_x + \frac{i\theta}{2\pi} \kappa_x}.\tag{1.13}$$

We can use the Fourier expansion:

$$e^{\beta \cos \kappa + \frac{i\theta}{2\pi} \kappa} = \sum_{n \in \mathbb{Z}} \lambda_n(\beta, \theta) e^{in\kappa},\tag{1.14}$$

where

$$\lambda_n(\beta, \theta) \equiv \int_{-\pi}^{\pi} \frac{d\kappa}{2\pi} e^{-i\nu\kappa} e^{\beta \cos \kappa} \Big|_{\nu=n-\theta/2\pi}\tag{1.15}$$

$$\sim \frac{e^\beta}{\sqrt{2\pi\beta}} \sum_{k \geq 0} (\nu, k) \left(\frac{-1}{2\beta} \right)^k \Big|_{\nu=n-\theta/2\pi}\tag{1.16}$$

with

$$(\nu, k) \equiv \frac{\Gamma(\nu + k + 1/2)}{k! \Gamma(\nu - k + 1/2)} = \frac{\{4\nu^2 - 1^2\}\{4\nu^2 - 3^2\} \cdots \{4\nu^2 - (2k - 1)^2\}}{2^{2k} \cdot k!}. \quad (1.17)$$

For $\theta = 0$, $\lambda_n(\beta, \theta)$ is identical to the modified Bessel function $I_\nu(\beta)$, but they are different generically.¹ We now have

$$Z_{\beta, \theta} = \int (dU) \prod_x \sum_{n_x} \lambda_{n_x}(\beta, \theta) e^{in_x \kappa_x} = \sum_n \lambda_n(\beta, \theta)^{\hat{V}}, \quad (1.18)$$

where \hat{V} is the lattice volume (in the lattice units).

We compute some expectation values for the case $\theta = 0$. The average plaquette can be calculated as

$$\begin{aligned} \langle \cos \kappa_x \rangle &= \frac{1}{\hat{V}} \frac{d}{d\beta} \ln Z_{\beta, \theta=0} \\ &= \frac{\sum_n [(I_{n+1}(\beta) + I_{n-1}(\beta))/2] \cdot I_n(\beta)^{\hat{V}-1}}{\sum_n I_n(\beta)^{\hat{V}}} \\ &\rightarrow \frac{I_1(\beta)}{I_0(\beta)} \sim \left(1 - \frac{1}{2\beta}\right) \quad (\beta \rightarrow \infty), \end{aligned} \quad (1.19)$$

where we used that $(0, 0) = (1, 0) = 1$, $(0, 1) = -1/4$ and $(1, 1) = 3/4$. We define the average energy density by:

$$e \equiv \frac{1}{a^4 \hat{V}} \sum_x (1 - \cos \kappa_x) \rightarrow \frac{1}{2V} \int d^2x F_{01}^2 \quad (a \rightarrow 0), \quad (1.20)$$

where $V \equiv a^2 \hat{V}$ is the physical volume of the system. This is actually a divergent quantity as can be seen from that:

$$\langle e \rangle \sim \frac{g^2}{2a^2}. \quad (1.21)$$

It is also notable that there is no correlation length in this system, which can be seen from the above calculation by modifying β to β_x and by taking the derivatives with respect to β_x :

$$\langle \cos^2 \kappa_x \rangle = \frac{\sum_n [(I_{n+2}(\beta) + I_n(\beta)) + I_{n-2}(\beta)]/4 \cdot I_n(\beta)^{\hat{V}-1}}{\sum_n I_n(\beta)^{\hat{V}}}, \quad (1.22)$$

$$\langle \cos \kappa_x \cos \kappa_y \rangle = \frac{\sum_n [(I_{n+1}(\beta) + I_{n-1}(\beta))^2/4] \cdot I_n(\beta)^{\hat{V}-2}}{\sum_n I_n(\beta)^{\hat{V}}} \quad (x \neq y). \quad (1.23)$$

¹The integral (1.15) is an even function of ν , but $I_\nu(\beta)$ is not.

We also calculate the susceptibility:

$$\begin{aligned}
\chi_Q &\equiv \frac{1}{V} \langle Q^2 \rangle \\
&= -\frac{1}{V} \frac{d^2}{d\theta^2} \ln Z_{\beta, \theta} \Big|_{\theta=0} \\
&= -\frac{1}{V} \frac{1}{Z_{\beta, \theta}} \hat{V} \sum_n \lambda_n^{\hat{V}-1} \cdot \frac{d^2 \lambda_n}{d\theta^2} \Big|_{\theta=0} \\
&\sim -\frac{1}{a^2} \frac{1}{\lambda_0(\beta, \theta)} \frac{d^2 \lambda_0}{d\theta^2} \Big|_{\theta=0},
\end{aligned} \tag{1.24}$$

where we used the fact $(d\lambda_n/d\theta)|_{\theta=0} = 0$. To get the continuum limit, we take the derivative of eq. (1.16) with respect to ν . Using the polygamma function:

$$\psi^{(m)}(z) \equiv \frac{d^m}{dz^m} \ln \Gamma(z), \tag{1.25}$$

we have:

$$(2\pi) \frac{d}{d\theta} \lambda_n(\beta, \theta) \sim \frac{e^\beta}{\sqrt{2\pi\beta}} \sum_{k \geq 0} (\nu, k) A(\nu, k) \left(\frac{-1}{2\beta} \right)^k \Big|_{\nu=n-\theta/2\pi}, \tag{1.26}$$

$$(2\pi)^2 \frac{d^2}{d\theta^2} \lambda_n(\beta, \theta) \sim \frac{e^\beta}{\sqrt{2\pi\beta}} \sum_{k \geq 0} (\nu, k) \{A(\nu, k)^2 + B(\nu, k)\} \left(\frac{-1}{2\beta} \right)^k \Big|_{\nu=n-\theta/2\pi} \tag{1.27}$$

with

$$A(\nu, k) \equiv \psi^{(0)}(\nu + k + 1/2) - \psi^{(0)}(\nu + k - 1/2), \tag{1.28}$$

$$B(\nu, k) \equiv \psi^{(1)}(\nu + k + 1/2) - \psi^{(1)}(\nu + k - 1/2). \tag{1.29}$$

Therefore,

$$\begin{aligned}
\chi_Q &\sim -\frac{1}{a^2} \frac{1}{I_0(\beta)} \frac{d^2 \lambda_0}{d\theta^2} \Big|_{\theta=0} \\
&\sim -\frac{1}{a^2} \cdot 1 \cdot \left(\frac{-1}{4} \right) \cdot (-8) \cdot \left(\frac{-1}{2\beta} \right) = g^2,
\end{aligned} \tag{1.30}$$

where we used that $A(0, 0) = A(0, 1) = 0$, $B(0, 0) = 0$ and $B(0, 1) = -8$. The susceptibility has a finite value in the continuum limit.

2. Field-transformed HMC

The contents expressed here was first developed by [5] for $SU(3)$ on T^4 .

Transformation

$$U_{x, \mu} \rightarrow U_{x, \mu}^{(\epsilon)} \equiv \mathcal{F}_\epsilon(U)_{x, \mu} \equiv e^{\epsilon \tilde{\partial}_{x, \mu} \tilde{S}(U)} U_{x, \mu}. \tag{2.1}$$

We choose the kernel $\tilde{S}(U)$ to be the plaquettes:

$$\tilde{S}(U) = \sum_x \cos \kappa_x. \quad (2.2)$$

We recursively define for $\ell \geq 1$:

$$S_{\ell\epsilon}(U) \equiv S_{(\ell-1)\epsilon}(\mathcal{F}_\epsilon(U)) - \ln \det \mathcal{F}_{\epsilon*}(U), \quad (2.3)$$

where $S_0(U) = S(U)$ and $\mathcal{F}_{\epsilon*}(U)$ is the Jacobian of the transformation (2.1). By acting $d \equiv \sum_{x,\mu} d\phi_{x,\mu} \partial_{x,\mu}$ to eq. (2.1), we find

$$\mathcal{F}_{\epsilon*}(U)_{x,\mu|y,\nu} = \delta_{xy} \delta_{\mu\nu} + \epsilon \partial_{y,\nu} \partial_{x,\mu} \tilde{S}(U). \quad (2.4)$$

To the leading order in ϵ ,

$$S_{\ell\epsilon}(U) = S_{(\ell-1)\epsilon}(U) + \epsilon \sum_{x,\mu} \partial_{x,\mu} \tilde{S}(U) \partial_{x,\mu} S(U) - \epsilon \sum_{x,\mu} \partial_{x,\mu}^2 \tilde{S}(U). \quad (2.5)$$

Explicitly,

$$S_{\ell\epsilon}(U) = S_{(\ell-1)\epsilon}(U) + \epsilon \sum_x \left[4\beta \sin^2 \kappa_x - \beta \sum_{\mu=\pm 0, \pm 1} \sin \kappa_{x+\mu} \sin \kappa_x + 4 \cos \kappa_x \right]. \quad (2.6)$$

Note that, from eq. (2.4),

$$\partial_{y,\nu} = \partial_{x,\mu}^{(\epsilon)} \mathcal{F}_{\epsilon*}(U)_{x,\mu|y,\nu}. \quad (2.7)$$

Therefore, from eq. (2.3),

$$\partial_{y,\nu} S_{\ell\epsilon}(U) = \sum_{x,\mu} [\delta_{xy} \delta_{\mu\nu} + \epsilon \partial_{y,\nu} \partial_{x,\mu} \tilde{S}(U)] \partial_{x,\mu}^{(\epsilon)} S_{(\ell-1)\epsilon}(U^{(\epsilon)}) - \text{tr} [\mathcal{F}_{\epsilon*}(U)^{-1} \partial_{y,\nu} \mathcal{F}_{\epsilon*}(U)]. \quad (2.8)$$

In practice, we deal with finite ϵ . To ensure that the map is invertible, we perform the even/odd partitioning. In terms of the angle variables, the transformation can be rewritten as:

$$\phi_{x,\mu}^{(\epsilon)} = \phi_{x,\mu} - \epsilon (\sin \kappa_{x'} - \sin \kappa_{x''}), \quad (2.9)$$

where

$$x' = x, x'' = x - 1 \quad (\mu = 0) \quad (2.10)$$

$$x' = x - 0, x'' = x \quad (\mu = 1). \quad (2.11)$$

To invert eq. (2.9), we rewrite the equation as:

$$f(X) = Z_{x,\mu}(\phi)|_{\phi=\phi^{(\epsilon)}-\epsilon X}, \quad X = f(X), \quad (2.12)$$

where

$$Z_{x,\mu}(\phi) \equiv \partial_{x,\mu} \tilde{S}(U) = -\sin \kappa_{x'} + \sin \kappa_{x''}. \quad (2.13)$$

Then, for $\epsilon < 1/2$ the map f defines the contraction mapping because:

$$\begin{aligned} |f(X) - f(Y)| &= |-\sin(\varphi + \epsilon X) + \sin(\varphi' - \epsilon X) + \sin(\varphi + \epsilon Y) - \sin(\varphi' - \epsilon Y)| \\ &= \left| -2 \cos \left(\varphi + \frac{\epsilon}{2}(X - Y) \right) \sin \left(\frac{\epsilon}{2}(X - Y) \right) \right. \end{aligned} \quad (2.14)$$

$$\begin{aligned} &\quad \left. + 2 \cos \left(\varphi' + \frac{\epsilon}{2}(X - Y) \right) \sin \left(\frac{\epsilon}{2}(X - Y) \right) \right| \\ &\leq 2\epsilon |X - Y|. \end{aligned} \quad (2.15)$$

3. Code

The code consists of the `main.c` and the following header files:

util.h some utility functions (print functions for vectors, projection from \mathbb{R} to $U(1)$).

lattice.h defines the `Lattice` class, which describes the geometry of the lattice.

coord.h defines the `Coord` class, which describes a point x on the lattice.

scalar_field.h defines the `ScalarField` class, which describes the field $\phi_{x,\mu}$ on the lattice.

We use it for the angular variables.

action.h defines the `WilsonAction` class, which describes the action function S .

corr.h defines the `Corr` class, which gives the correlator functions $\cos \kappa_x \cdot \cos \kappa_y$.

rnd.h defines the `Rnd` class, which gives a set of mt19937 generators on each site or link.

hmc.h defines the `HMC` class, which gives the *ordinary* HMC algorithm.

kernel.h defines the `Kernel` class, which describes the kernel \tilde{S} for the field transformations.

field_trsf.h defines the `FieldTrsf` class, which describes the infinitesimal map \mathcal{F}_ϵ .

ft_hmc.h defines the `FT_HMC` class, which gives the FT-HMC algorithm.

The present code calculates and records the

- average plaquette
- two-point functions

- topological charge.

The plaquette and the topological charge are recorded for both the physical field U and the auxiliary field V .

The code is aimed to be run on laptops and uses `c++17` for `filesystem`. The remaining part only uses `c++11`. It can run with `openmp`. No MPI calls. It generates 500 configurations with two-step Wilson-flowed HMC in 30 sec for 8×8 lattice on my workstation (with OpenMP). It creates `./result/` directory, in which the observable data are stored, and a few log files. The contents written in this paragraph are mostly described in `main.c`.

One can compile and run the code with `make IS_FTHMC=0; make run`. The included `analysis.ipynb` can be used to analyze observable data. The exact solutions (or their approximated functions) are written here for the checks.

References

- [1] L. Jin, “Qlattice,” <https://github.com/jinluchang/Qlattice>
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