A Game of International Climate Policy Solved by a Homogeneous Oracle-Based Method for Variational Inequalities

Laurent Drouet, Alain Haurie, Jean-Philippe Vial, and Marc Vielle

Abstract This paper presents a game-theoretic model for the international negotiations that should take place to renew or extend the Kyoto protocol beyond 2012. These negotiations should lead to a self-enforcing agreement on a burden sharing scheme to realize the necessary global emissions abatement that would preserve the world against irreversible ecological impacts. The model assumes a noncooperative behavior of the parties except for the fact that they will be collectively committed to reach a target on cumulative emissions by the year 2050. The concept of normalized equilibrium, introduced by J.B. Rosen for concave games with coupled constraints, is used to characterize a family of dynamic equilibrium solutions in an m-player game where the agents are (groups of) countries and the payoffs are the welfare gains obtained from a Computable General Equilibrium (CGE) model. The model is solved using an homogeneous version of the oracle-based optimization engine (OBOE) permitting an implicit definition of the payoffs to the different players, obtained through simulations performed with the global CGE model GEMINI-E3.

L. Drouet (✉)
Resource Centre for Environmental Technologies, Public Research Centre Henri Tudor, Technoport Schlassgoart, 66 rue de Luxembourg, L-4002 Esch-sur-Alzette, Luxembourg
e-mail: laurent.drouet@tudor.lu

A. Haurie
ORDECSYS, Chêne-Bougeries, Switzerland
e-mail: ahaurie@ordecsys.com

J.-P. Vial
University of Geneva and ORDECSYS, Switzerland
e-mail: jpvial@ordecsys.com

M. Vielle
École Polytechnique Fédérale de Lausanne and C-ORDEE, Switzerland and Toulouse School of Economics (LERNA), TSE - Manufacture de Tabacs, 21 allées de Brienne - 31000 Toulouse, France
e-mail: marc.vielle@epfl.ch

M. Breton and K. Szajowski (eds.), *Advances in Dynamic Games*, Annals of the International Society of Dynamic Games 11, DOI 10.1007/978-0-8176-8089-3_23,

1 Introduction

This paper is a continuation of the work already reported in [8]. In this research, we study the strategic interactions of groups of countries when they negotiate the sharing of burden to stabilize the long-term concentration of greenhouse gases (GHG). We propose a dynamic game model where the strategies of each player (state or group of countries) refer to the timing of supply of emission rights (also called quotas) on an international emissions trading scheme with full banking and borrowing. The payoffs are obtained in terms of welfare gains (or losses) compared with the business as usual (BAU) situation. In our approach, these welfare gains are obtained through simulations performed with a Computable General Equilibrium (CGE) model. In this game, a coupled constraint is imposed on all players together to limit the cumulative emissions over the whole planning horizon. Games with coupled constraints have been first studied by Rosen [23] who showed that a whole family (manifold) of equilibrium outcomes should be expected, indexed over a set of weights attributed to the players. The appropriateness of the normalized equilibrium concept to deal with environmental games has been first recognized by Haurie [11] and further explored by Haurie and Zaccour [14], Haurie and Krawczyk [12], and Krawczyk [15], [16].

In this paper, we explore the manifold of normalized equilibria in a deterministic framework. In [8] a stochastic version of this game is also proposed, using the concept of S-adapted equilibria (see [13]). The added contribution of this paper compared with [8] is twofold: (i) we show that the Rosen normalized equilibria can be represented as Nash equilibria associated with different sharing schemes of the cumulative emission budget; (ii) we resort to optimization theory for general convex programming to solve the challenging equilibrium problem formulated as a system of variational inequalities. The algorithm we use is a homogeneous version of ACCPM[1] devised by Nesterov and Vial and studied in-depth in [20, 22]. The method is *oracle-based*, in that it uses a black-box mechanism, named *oracle*, to collect first-order information (function values and subgradients) of the functions entering the problem definition. This approach provides a complexity bound on the required number of iterations to reach a given level of precision. We shall also report on the most recent simulations performed with this model.

This paper is organized as follows: we present the structure of the coupled game model in Sect. 2. Then, in Sect. 3, we briefly present the multisector and multicountry CGE model of the world economy that is used for getting the payoffs; We implement the model within an oracle-based optimization framework with the homogeneous version of the Analytic Center Cutting Plane Method (ACCPM); in Sect. 5, we give the numerical results obtained for a case study where countries have to decide on their own abatement level under a global target on cumulative

[1] Acronym for Analytic Center Cutting Plane Method [19]

GHG emissions by 2050 which is consistent with a commitment to limit global temperature rise to 2 degrees Celsius above pre-industrial levels.

2 The Model

In this section, we present a deterministic game model for GHG emissions abatement, in a simple formulation on a time span comprising four periods between 2010 and 2050. We summarize the negotiation as a two-stage process: in stage 1, the negotiators determine a cumulative emission budget over the period 2010–2050 and define a sharing of this emission budget, called respective total allowances, over the set of players (each player is a group of countries); in stage 2, the countries use their total allowance to supply dynamically the international emissions trading market. In this second stage, the players behave as oligopolists supplying a sequence of markets that change over time, due to economic growth and technological progress. We do not represent stage 1 as a noncooperative game but we assume that the equilibrium outcome of stage 2 will be taken into consideration for designing a fair sharing of the cumulative emission budget. We assume that, in stage 1, the cumulative emission budget is dictated by a precautionary principle to keep the radiative forcing within a tolerable limit.

2.1 *Players, Moves, and Payoffs in the Stage 2 Game*

The game is played over 4 periods $t = 1, \ldots, 4$. M is a set of m groups of countries hereafter called players which must decide on the emission quotas they supply in each period on an international GHG emissions trading market.

We denote $\bar{e}_j(t)$ the supply of quotas decided by player j for period t. A global limit $\bar{E}$ is imposed on the cumulative emissions over the four periods $t = 1, \ldots, 4$. Therefore, the following coupled constraints are binding all players together

$$\sum_{j \in M} \sum_{t=1}^{4} \bar{e}_j(t) \leq \bar{E}. \tag{1}$$

Let $\bar{\mathbf{e}}(t) = \{\bar{e}_j(t)\}_{j \in M}$ denote the vector of emissions quotas for all players in period t. Given these quotas a general economic equilibrium is computed for the m-countries which determines a welfare gain for each player, hereafter called its payoff at t and denoted $W_j(t; \bar{e}(t))$.. Given a choice of emission quotas $\bar{\mathbf{e}} = \{\bar{\mathbf{e}}(t) \mid t = 1, \ldots, 4\}$ the payoff to player j is given by

$$J_j(\bar{\mathbf{e}}) = \sum_{t=1}^{4} \beta^{t-1} W_j(t; \bar{\mathbf{e}}(t)) \quad j \in M,$$

where β is a common discount factor.

Remark 1. The game is not defined on a dynamic system, e.g., a differential game.[2] The dynamic effect will be essentially associated with the supply over time of the respective total allowances. Therefore, there is no end-of-period effect. The limitation to four periods corresponds to the current situation where an international GHG-emission agreement is envisioned for a period extending roughly from 2010 to 2050; whence the consideration of four 10-year periods.

2.2 Normalized Equilibrium Solutions

We assume that the players behave in a noncooperative way but are bound to satisfy the global cumulative emissions constraints (1). The solution concept that we propose to use is related to the concept of normalized equilibrium introduced by Rosen [23] to deal with games where the players are bound by a coupled constraint.

Let us call $\mathscr{E}$ the set of emissions $\bar{\mathbf{e}}$ that satisfy the constraints (1). Denote also $[\bar{\mathbf{e}}^{*j}, \bar{e}_j]$ the emission program obtained from $\bar{\mathbf{e}}^*$ by replacing only the emission program $\bar{\mathbf{e}}_j^*$ of player j by $\bar{e}_j$.

Definition 1. The emission program $\bar{\mathbf{e}}_j^*$ is an equilibrium under the coupled constraints defined in (1) if the following holds for each player $j \in M$.

$$\bar{\mathbf{e}}^* \in \mathscr{E}$$
$$\forall \bar{e}_j \text{s.t.} [\bar{\mathbf{e}}^{*j}, \bar{e}_j] \in \mathscr{E} \quad J_j(\bar{\mathbf{e}}^*) \geq J_j([\bar{\mathbf{e}}^{*j}, \bar{e}_j])$$

Therefore, in this equilibrium, each player replies optimally to the emission program chosen by the other players, under the constraint that the global cumulative emission limits must be respected.

It is possible to characterize a class of such equilibria through a fixed point condition for a best reply mapping defined as follows. Let $r = (r_j)_{j \in M}$ with $r_j > 0$ and $\sum_{j \in M} r_j = 1$ be a given weighting of the different players. Then introduce the combined response function

$$\theta(\bar{\mathbf{e}}^*, \bar{\mathbf{e}}; r) = \sum_{j \in M} r_j J_j([\bar{\mathbf{e}}^{*j}, \bar{e}_j]). \tag{2}$$

It is easy to verify that, if $\bar{\mathbf{e}}^*$ satisfies the fixed point condition

$$\theta(\bar{\mathbf{e}}^*, \bar{\mathbf{e}}^*; r) = \max_{\bar{\mathbf{e}} \in \mathscr{E}} \theta(\bar{\mathbf{e}}^*, \bar{\mathbf{e}}; r), \tag{3}$$

then it is an equilibrium under the coupled constraint.

[2] The economic model used to define the payoffs of the game, GEMINI-E3, is a time-stepped model, which computes an equilibrium at each period and defines investments through an exogenously defined saving function. Therefore, there is no hereditary effect of the allocation of emission quotas and, as a consequence there is no need for a scrap value function.

Definition 2. The emission program $\bar{\mathbf{e}}^*$ is a normalized equilibrium if it satisfies (3) for a weighting r and a combined response function defined as in (2).

The RHS of (3) defines an optimization problem under constraint. Assuming the required regularity we can introduce a Kuhn–Tucker multiplier λ^0 for the constraint $\sum_{t=0}^{1} \bar{e}_j(t) \leq \bar{E}$ and form the Lagrangian

$$L = \theta(\bar{\mathbf{e}}^*, \bar{\mathbf{e}}; r) + \lambda^0 \left(\bar{E} - \sum_{j \in M} \sum_{t=1}^{4} \bar{e}_j(t) \right).$$

Therefore, by applying the standard K-T optimality conditions we can see that the normalized equilibrium is also the Nash equilibrium solution for an auxiliary game with a payoff function defined for each player j by

$$J_j(\bar{\mathbf{e}}) + \lambda^j \left(\bar{E} - \sum_{j \in M} \sum_{t=1}^{4} \bar{e}_j(t) \right),$$

where

$$\lambda^j = \frac{1}{r_j} \lambda^0.$$

This characterization has an interesting interpretation in terms of negotiation for a climate change policy. A common "tax" λ^0 is defined and applied to each player with an intensity $\frac{1}{r_j}$ that depends on the weight given to this player in the global response function.

2.3 An Interpretation as a Distribution of a Cumulative Budget

Consider an m-player concave game à la Rosen with payoff functions

$$\psi_j(x_1, x_2, \ldots, x_m), \quad x_j \in X_j \quad j = 1, \ldots, m,$$

and a coupled constraint. When the coupled constraint is scalar and separable among players, i.e. when it takes the form

$$\sum_{j=1}^{m} \varphi_j(x_j) = e,$$

the coupled equilibrium can be interpreted in an interesting way. Consider e as being a global allowance and call $\varpi_j \geq 0$ the fraction of this allowance given to player j, with $\sum \varpi_j = 1$. Then define the game with payoffs and decoupled constraints

$$\psi_j(x_1, x_2, \ldots, x_m), \quad x_j \in X_j \quad \varphi_j(x_j) \leq \varpi_j e \quad j = 1, \ldots, m.$$

A Nash equilibrium for this game is characterized, under the usual regularity conditions, by the following conditions

$$\begin{aligned} & \max_j \ \psi_j(x_1,\ldots,x_j,\ldots,x_m) - \lambda_j \varphi_j(x_j) \\ \lambda_j \ & \geq \ 0 \\ 0 \ & = \ \lambda_j(\varphi_j(x_j) - \varpi_j e). \end{aligned}$$

Now assume that at the equilibrium solution all the constraints are active and hence one may expect (in the absence of degeneracy) that all the λ_j are >0. Since the multipliers are scalars they can be written in the form

$$\lambda_j = \frac{\lambda_0}{r_j},$$

by taking

$$\lambda_0 = \sum_{j=1}^{m} \lambda_j$$

and defining

$$r_j = \frac{\lambda_0}{\lambda_j}, \quad j = 1,\ldots,m.$$

The assumption of active constraints at equilibrium leads to

$$\lambda_0 > 0 \quad \text{and} \quad \sum_{j=1}^{m} \varphi_j(x_j) - e = 0.$$

Therefore, the conditions for a normalized coupled equilibrium are met.

Notice that the assumption that all constraints are active at the Nash equilibrium is crucial. However, in a climate game where the constraints are on emissions quotas this assumptions is very likely to hold. So, in the special situation described here a normalized equilibrium is obtained by defining first a sharing of the common global allowance and then by playing the noncooperative game with the distributed constraints.

2.4 Subgame Perfectness

The game is played in a dynamic setting and the question of subgame perfectness arises naturally. The equilibrium solutions we consider are open-loop and therefore do not possess the subgame perfectness property. In fact, even though the model involves four periods, the equilibrium concept is "static" in spirit. It is used as a way to value the outcome of some sharing of a global emission budget, negotiated in stage 1.

2.5 Stage 1 Negotiations

We view the negotiations in stage 1, not as a noncooperative game, but rather as a search for equity in a "pie sharing process". More precisely, a successful international climate negotiation could involve the following agreements among different groups of countries, each group sharing similar macroeconomic interest:

- the total level of cumulative GHG emissions allowed over a part of the 21st century (typically 2010–2050), for instance based on a precautionary principle involving the overall temperature increase not to be exceeded;
- the distribution of this cumulative emission budget among the different groups, for instance using some concepts of equity;
- the fairness of the sharing will be evaluated on the basis of the equilibrium solution obtained in stage 2.

In the rest of this paper, we shall concentrate on the way to evaluate the equilibrium outcomes of the stage 2 noncooperative game.

3 Getting the Payoffs Via GEMINI-E3

This section, largely reproduced from [8], shows how the payoffs of the game are obtained from economic simulations performed with a computable general equilibrium (CGE) model.

3.1 General Overview

The payoffs of the game are computed using the GEMINI-E3 model. We use the fifth version of GEMINI-E3 describing the world economy in 28 regions with 18 sectors, and which incorporates a highly detailed representation of indirect taxation [5]. This version of GEMINI-E3 is formulated as a Mixed Complementarity Problem (MCP) using GAMS with the PATH solver [9, 10]. GEMINI-E3 is built on a comprehensive energy-economy data set, the GTAP-6 database [7], that expresses a consistent representation of energy markets in physical units as well as a detailed Social Accounting Matrix (SAM) for a large set of countries or regions and bilateral trade flows. It is the fifth GEMINI-E3 version that has been especially designed to calculate the social marginal abatement costs (MAC), i.e. the welfare loss of a unit increase in pollution abatement [4]. The different versions of the model have been used to analyze the implementation of economic instruments for GHG emissions in a second-best setting [3], to assess the strategic allocation of GHG emission allowances in the EU-wide market [6] and to analyze the behavior of Russia in the Kyoto Protocol [2, 4].

For each sector, the model computes the demand on the basis of household consumption, government consumption, exports, investment, and intermediate uses. Total demand is then divided between domestic production and imports, using the Armington assumption [1]. Under this convention, a domestically produced good is treated as a different commodity from an imported good produced in the same industry. Production technologies are described using nested CES functions.

3.2 Welfare Cost

Household's behavior consists of three interdependent decisions: 1) labor supply; 2) savings; and 3) consumption of the different goods and services. In GEMINI-E3, we suppose that labor supply and the rate of saving are exogenously fixed. The utility function corresponds to a Stone–Geary utility function [25] which is written as:

$$u_r = \sum_i \beta_{ir} \ln(HC_{ir} - \phi_{ir}),$$

where HC_{ir} is the household consumption of product i in region[3] r, ϕ_{ir} represents the minimum necessary purchases of good i, and β_{ir} corresponds to the marginal budget share of good i. Maximization under budgetary constraint:

$$HCT_r = \sum_i PC_{ir} HC_{ir}$$

yields

$$HCI_{ir} = \phi_{ir} + \frac{\beta_{ir}}{PC_{ir}} \left[HCT_r - \sum_k (PC_{kr}\phi_{kr}) \right],$$

where PC_{ir} is the price of household consumption for product i in region r.

The welfare cost of climate policies is measured comprehensively by changes in households' welfare since final demand of other institutional sectors is supposed unchanged in scenarios. Measurement of this welfare change is represented by the sum of the change in income and the "Equivalent Variation of Income" (EVI) of the change in prices, according to the classical formula. In the case of a Stone–Geary utility function, the EVI for a change from an initial situation defined by the price system ($\overline{PC_{ir}}$) to a final situation (PC_{ir}) is such as

$$\frac{\overline{HCT}_r - \sum_i \overline{PC}_{ir}\phi_{ir}}{\prod_i (\overline{PC}_{ir})^{\beta_{ir}}} = \frac{\overline{HCT}_r + EVI_r - \sum_i PC_{ir}\phi_{ir}}{\prod_i (PC_{ir})^{\beta_{ir}}}.$$

[3] The attentive reader should not be confused with the use of r as the index of regions, whereas r_j has been used before as a weight in the definition of a normalized equilibrium. The notations in this section, devoted to a brief presentation of GEMINI-E3, are not exactly the same as in the rest of the paper which describes the game and the solution method used to solve it. The same remark applies to the use of the symbols ϕ and i in this section vs. the rest of the paper.

The households' surplus is then given by

$$S_r = \left(HCT_r - \sum_i PC_{ir}\phi_{ir}\right) - \prod_i \left(\frac{PC_{ir}}{\overline{PC}_{ir}}\right)^{\beta_{ir}} \left(\overline{HCT}_r - \sum_i \overline{PC}_{ir}\phi_{ir}\right).$$

In summary, the CGE model associates a welfare gain (cost) for each country and each period with a given emissions program $\bar{e}$, which defines quotas for all countries at each period. It is important to notice that these welfare gains are obtained under the assumption that an international emissions trading scheme is put in place.

3.3 Getting the Payoffs of the Emission Quota Game

To summarize this game model, the players (i.e. the groups of regions) strategies correspond to allocations of their respective total allowances among the time periods. This determines quotas for each period. A general economic equilibrium problem is solved by GEMINI-E3 at each period. On the basis of the quotas chosen by the players, at each period, supplies and demands of emission permits are balanced and a permit price is obtained on an international carbon market. Countries which are net suppliers of permits receive revenue and in contrary countries buying permits must pay for it.[4] These financial transfers are collected by governments and are redistributed to households. These transfers influence the whole balance in the economy and determine, at the end, the terms of trade and the surpluses of each countries. The payoff $W_j(t, \bar{\mathbf{e}}(t))$ of Player (group of region) j is obtained as the discounted value of the sum of households surpluses S_j of player j on the period 2005–2050, using a 5% discount rate.

4 Oracle-Based Optimization Framework

There are different numerical methods to compute a Nash or Rosen-normalized equilibrium. Indeed one may rely to the methods solving variational inequalities [21]. Another approach is proposed in [27] and [26]. In our problem, the payoffs to players are obtained from a comprehensive economic simulation and, therefore, we do not have an explicit and analytical description of the payoff functions. This precludes the use of the various methods proposed for solving variational inequalities or concave games which exploit the analytical form of the payoff functions and their derivatives. Oracle-based methods can be used in our context, since they require only to obtain at each step a numerical information of the oracle and of a sub-(pseudo) gradient.

[4] Indeed, the buyer prefers to pay the permit price rather than support its own marginal abatement cost, while the seller can abate at a marginal cost which is lower than the market price.

In this section, we describe the implementation of an oracle-based optimization method to compute a solution to the variational inequality which characterizes the equilibrium solution. Since this implementation of an oracle-based optimization to compute Rosen normalized equilibrium solutions seems to be one of the first applications of the method, we shall develop this description in enough details to give the reader a precise idea of the mathematics involved.

4.1 Normalized Equilibrium and Variational Inequality

For concave games with differentiable payoff functions $J_j(\cdot)$, $\bar{\mathbf{e}}^*$ is a normalized equilibrium if and only if it is a solution of the following variational inequality problem

$$\langle F(\bar{\mathbf{e}}^*), \bar{\mathbf{e}}^* - \bar{\mathbf{e}} \rangle \geq 0 \quad \forall \bar{\mathbf{e}} \in \mathscr{E}, \tag{4}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product and the pseudogradient $F(\cdot)$ is defined by

$$F(\bar{\mathbf{e}}) = \begin{pmatrix} r_1 \nabla_{\bar{e}_1} J_1(\bar{\mathbf{e}}) \\ \vdots \\ r_j \nabla_{\bar{e}_j} J_j(\bar{\mathbf{e}}) \\ \vdots \\ r_m \nabla_{\bar{e}_m} J_m(\bar{\mathbf{e}}) \end{pmatrix}.$$

It has been proved in [23] that a normalized equilibrium exists if the payoff functions $J_j(\cdot)$ are continuous in $\bar{\mathbf{e}}$ and concave in $\bar{e}_j$ and if $\mathscr{E}$ is compact. In the same reference, it is proved also that the normalized equilibrium is unique if the function $-F(\cdot)$ is strictly monotone, *i.e.* if the following holds

$$\langle F(\bar{\mathbf{e}}^2) - F(\bar{\mathbf{e}}^1), \bar{\mathbf{e}}^1 - \bar{\mathbf{e}}^2 \rangle > 0 \quad \forall \bar{\mathbf{e}}^1 \in \mathscr{E}, \forall \bar{\mathbf{e}}^2 \in \mathscr{E}.$$

Remark 2. In the case where one computes a Nash equilibrium with decoupled constraints, the formulation remains the same, except that the same solution would be obtained for different weights $r_j > 0; j = 1, \ldots, m$. In that case one takes $r_j \equiv 1; j = 1, \ldots, m$.

Solving (4) is a most challenging problem. If $-F$ is monotone, the above variational inequality implies the weaker one

$$\langle F(\bar{\mathbf{e}}), \bar{\mathbf{e}}^* - \bar{\mathbf{e}} \rangle \geq 0 \quad \forall \bar{\mathbf{e}} \in \mathscr{E}. \tag{5}$$

The converse is not true in general, but it is known to hold under the assumption that the monotone operator $-F$ is continuous or maximal monotone.

The weak variational inequality (5) can be formulated as a convex optimization problem. To this end, one defines the so-called *dual-gap* function

$$\phi_D(\bar{\mathbf{e}}) = \min_{\bar{\mathbf{e}}' \in \mathscr{E}} \langle F(\bar{\mathbf{e}}'), \bar{\mathbf{e}} - \bar{\mathbf{e}}' \rangle.$$

This function is concave and nonpositive. Unfortunately, computing the value of the function at some $\bar{\mathbf{e}}$ amounts to solving a nonlinear, nonconvex problem. This disallows the use of standard convex optimization techniques. However, the definition of the dual gap function provides an easy way to compute a piece-wise linear outer approximation. Indeed, let $\bar{\mathbf{e}}^k$ be some point, we have for all optimal $\bar{\mathbf{e}}^* \in \mathscr{E}$

$$\langle F(\bar{\mathbf{e}}^k), \bar{\mathbf{e}}^* - \bar{\mathbf{e}}^k \rangle \geq 0. \tag{6}$$

This property can be used in a cutting plane scheme that will be described in the next section. Figure 1 illustrates this property.

Prior to presenting the solution method, we point out that we cannot expect to find an exact solution. Therefore, we must be satisfied with an approximate solution. We say that $\bar{\mathbf{e}}$ is an ε-approximate weak solution, in short an ε-solution, if

$$\phi_D(\bar{\mathbf{e}}) \geq -\varepsilon. \tag{7}$$

As it will be stated in the next section, it is possible to give a bound on the number of iterations of the cutting plane method to reach a weak ε-solution. Of course, the algorithm may reach an ε-solution earlier, but it cannot be checked directly since the dual gap is not computable. In practice, we use another function, the so-called primal gap function

$$\phi_P(\bar{\mathbf{e}}) = \min_{\bar{\mathbf{e}}' \in \mathscr{E}} \langle F(\bar{\mathbf{e}}), \bar{\mathbf{e}} - \bar{\mathbf{e}}' \rangle.$$

Note that the pointwise computation of ϕ_P amounts to solving a linear programming problem when $\mathscr{E}$ is linear, an easy problem. Since $-F$ is monotone, we have

$$\phi_P(\bar{\mathbf{e}}) \leq \phi_D(\bar{\mathbf{e}}).$$

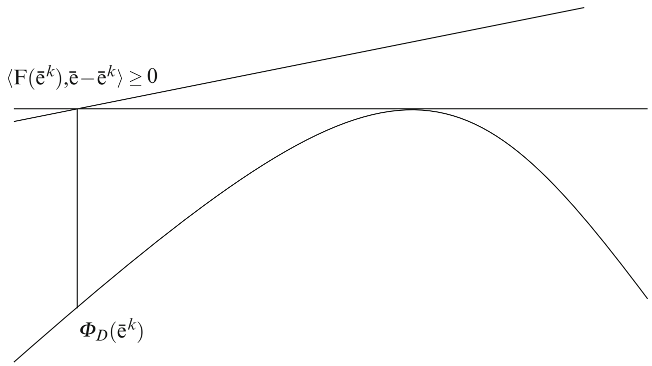

Fig. 1 Outer approximation scheme of ϕ_D

Therefore, one can substitute to the inoperative criterion (7), the more practical one

$$\phi_P(\bar{\mathbf{e}}) \geq -\varepsilon.$$

The reader must be warned that the convex optimization problem of maximizing the concave dual gap function is very peculiar. The oracle that would compute the dual gap function value would consist in solving a nonlinear, nonconvex minimization problem, a computationally intractable problem. On the other hand, the dual gap is bounded above by 0 and this bound is the optimal value. The challenge is to find a point that approximates this optimal value. To this end, one could use (6) to build a linear functional that dominates the dual gap. This piece of information constitutes the output of an oracle on which the algorithm must be built to solve the variational inequality.

In the next section, we describe the algorithm HACCPM that guarantees convergence in a number of iterations of the order of 1 over epsilon squared, where epsilon is precision achieved on the variational inequality. (That is the dual gap function at the epsilon-solution is somewhere between 0, the known optimal value, and minus epsilon.) This result is obtained under the mild assumption of monotonicity of F and a compact convex feasible set. The epsilon solution solves the so-called epsilon-weak solution of the variational inequality. As a by-product of the theoretical convergence result, one obtains that monotone variational inequalities always admit a weak solution. This property is not true for strong solutions which do not always exist under the simple monotonicity assumption. However, weak solutions are also strong solutions if F is continuous of if F is strongly convex.

Variational inequalities are reputedly hard to solve. They are significantly harder than general convex optimization problem, even though they can be formulated as the convex optimization problem of minimizing the dual gap function. The key point is that the dual gap function value is not known (computable), except at the optimum, which is 0. The remarkable theoretical fact is that one can build algorithms with a complexity bound of the same order as the best one for general convex programming.

4.2 The Homogeneous Analytic Center Cutting Plane Method

We now describe the method Homogeneous Analytic Center Cutting Plane Method (HACCPM) that solves

$$\text{Find } \bar{\mathbf{e}} \in \mathscr{E}^*_\varepsilon = \{\bar{\mathbf{e}} \in Q \mid \phi_D(\bar{\mathbf{e}}) \geq -\varepsilon\}.$$

Since ϕ_D is concave, $\mathscr{E}^*_\varepsilon$ is convex. If $\varepsilon = 0$, the set $\mathscr{E}^* = \mathscr{E}^*_0$ is the set of solutions to the weak variational inequality.

HACCPM [19] is a cutting plane method in a conic space with a polynomial bound on the number of iterations. To apply it to our problem of interest, we need to embed the original problem in an extended space. Let us describe the embedding first.

4.2.1 Embedding in an Extended Space

Let us introduce a projective variable[5] $t > 0$ and denote

$$K = \left\{ x = (y,t) \ \mid \ t > 0,\ \bar{\mathbf{e}} = \frac{y}{t} \in \mathscr{E} \right\}.$$

In this problem, $\mathscr{E}$ is linear and takes the general form

$$\mathscr{E} = \{ \bar{\mathbf{e}} \ \mid \ \langle a_i, \bar{\mathbf{e}} \rangle \leq b_i.\ i = 1 \ldots m \}.$$

Its conic version is

$$K_{\mathscr{E}} = \{ x = (y,t) \ \mid \ \langle a_i, y \rangle \leq t b_i.\ i = 1 \ldots m,\ t \geq 0 \}.$$

We associate with it the logarithmic barrier function

$$B(x) = -\sum_{i=1}^{m} \ln(t b_i - \langle a_i, y \rangle) - \ln t.$$

B is a so-called ν-logarithmically homogeneous self-concordant function, with $\nu = m + 1$. (See definition 2.3.2 in the book by Nesterov and Nemirovski [18].) In this paper $K_{\mathscr{E}}$ is a simplex; thus $m = n + 1$, where n is the dimension of $\bar{\mathbf{e}}$.

The embedding of the valid inequality

$$\langle F(\bar{\mathbf{e}}), \bar{\mathbf{e}}^* - \bar{\mathbf{e}} \rangle \geq 0,\ \ \forall \bar{\mathbf{e}}^* \in \mathscr{E}^*$$

is done similarly. For any $x = (y,t) \in \operatorname{int} K_{\mathscr{E}}$, define $\bar{\mathbf{e}}(x) = y/t \in \mathscr{E}$ and

$$\hat{G}(x) = (F(\bar{\mathbf{e}}(x)), -\langle F(\bar{\mathbf{e}}(x)), \bar{\mathbf{e}}(x) \rangle).$$

It is easy to check that for $x^* \in X^*$ with

$$X^* = \{ x = (y,t) \mid t > 0, \bar{\mathbf{e}}(x) \in \mathscr{E}^* \},$$

the following inequality holds

$$\langle \hat{G}(x), x - x^* \rangle \geq 0.$$

Finally, we define $G(x) = \hat{G}(x)/||\hat{G}(x)||$. We also associate to $\langle G(x), x - x^* \rangle > 0$ the logarithmic barrier $-\log\langle G(x), x - x^* \rangle$.

[5] The reader should not be confused by the use of t as a projective variable, whereas it was used as a time index in the game definition.

4.2.2 The Algorithm and Convergence Properties

The homogeneous cutting plane scheme, in its abstract form, can be briefly described as follows.

0) Set $B_0(x) = \frac{1}{2}||x||^2 + B(x)$.
1) kth iteration ($k \geq 0$).
   a) Compute $x_k = \arg\min_x B_k(x)$,
   b) Set $B_{k+1}(x) = B_k(x) - \ln\langle G(x_k), x_k - x\rangle$.

The algorithm is an abstract one as it assumes that the minimization of $B_k(x)$ in step (1-a) is carried out with full precision. This restriction has been removed in [20].

It is shown in [19] that $\bar{\mathbf{e}}(x^k)$ does not necessarily converge to $\mathscr{E}^*$. The correct candidate solution $\bar{\mathbf{e}}_k$ is as follows. Assume $\{x_i\}_{i=0}^{\infty}$ is a sequence generated by the algorithm. Define

$$\pi_{ik} = \frac{1}{||\hat{G}(x_i)||} \frac{1}{\langle G(x_i), x_i - x_k\rangle}, \quad P_k = \sum_{i=0}^{k-1} \pi_{ik},$$

and

$$\bar{\mathbf{e}}_k = \frac{1}{P_k} \sum_{i=0}^{k-1} \pi_{ik} \bar{\mathbf{e}}(x_i).$$

Assumption 1

1. *$\mathscr{E}$ is bounded and R is a constant such that for all $\bar{\mathbf{e}} \in \mathscr{E}$, $||\bar{\mathbf{e}}|| \leq R$.*
2. *The mapping $-F$ is uniformly bounded on $\mathscr{E}$ and is monotone, i.e., $||F(\bar{\mathbf{e}})|| \leq L$, for all $\bar{\mathbf{e}} \in \mathscr{E}$.*

The main convergence result is given by the following theorem.

Theorem 1. *HACCPM yields an ε-approximate solution after k iterations, with k satisfying*

$$\frac{k}{\sqrt{k+\nu}} \leq \frac{L(1+R^2)}{\varepsilon\theta_3} e^{\theta_2\sqrt{\nu}}. \tag{8}$$

The parameters in this formula are defined by

$$\begin{aligned}
\nu &= n + 2\\
\theta_1 &= (\sqrt{5}-1)/2 - \log(\sqrt{5}+1)/2\\
\theta_2 &= (\sqrt{5}+1)/2\\
\theta_3 &= \tfrac{1}{\theta_2}\exp(\theta_1 - \tfrac{1}{2}).
\end{aligned}$$

Note that $\theta_1, \theta_2, \theta_3$ are absolute constant, and ν is the dimension of the conic space plus 1.

To make this algorithm operational, one needs to work with an approximate minimizer of $B_k(x)$. More precisely, one can define a neighborhood of the exact minimizer with the following properties: *i*) checking whether the current iterate in the process of minimizing $B_k(x)$ belongs to this neighborhood is a direct byproduct of the computation; *ii*) the number of iterations to reach an approximate minimizer of $B_{k+1}(x)$ is bounded by a (small) absolute constant when the minimization starts from an approximate minimizer of $B_k(x)$. Finally, it is shown in [20] that the number of main iterations in the HACCPM algorithm with approximate centers, i.e., the number of generated cutting planes, is bounded by an expression similar to (8), but with slight different (absolute) values for the parameters.

In practice, the bound on the number of iterations is not used as a stopping criterion. Rather, we use the $-\varepsilon$ threshold for the primal gap ϕ_P.

5 Implementation

In the implementation, we use the method to compute the Nash equilibrium solution resulting from an allocation of a cumulative emission budget, as discussed in Sect. 2. We have used different rules of equity, based on population, Gross Domestic Product (GDP), grandfathering (i.e. historical emissions) to propose different splits of the cumulative emission budget among different groups of nations which then play a noncooperative game in the allocation of these emission allowances into quotas for each period.

During the optimization process, we compute an approximate gradient by calling the model $x + 1$ times to obtain the sensitivity information (x corresponds to the number of the variables). Let $G(\bar{\mathbf{e}})$, a response of a GEMINI-E3 model to emission quotas $\bar{\mathbf{e}}$; an approximate evaluation of a pseudosubgradient at $\bar{\mathbf{e}}$ is given by $\frac{G(\bar{\mathbf{e}})-G(\bar{\mathbf{e}}+\Delta)}{\Delta}$, where Δ is an arbitrary perturbation on the emission quotas. This introduces another source of imprecision in the procedure. However, in practice, the approach shows convergence for most of the tested instances.

5.1 Case Study

We describe here the case study developed in the EU FP7 project TOCSIN.[6] The players are 4 regions of the World:

- NAM: North American countries;
- OEC: Other OECD countries;
- DCS: Developing countries (in particular Brazil, China, India);
- EEC: Oil and gas exporting countries (in particular Russia and Middle East).

[6] Technology-Oriented Cooperation and Strategies in India and China: Reinforcing the EU dialogue with Developing Countries on Climate Change Mitigation.

Table 1 Regional GHG emission in 2005

Region	GHG emission in 2005 (MtC-eq*)
NAM	2'491
OEC	2'187
DCS	4'061
EEC	1'067

*MtC-eq: millions tons of carbon equivalent

The economy is described by the GEMINI-E3 model during the years 2004–2050. The decision variables are the emissions of the 4 regions in each of the 4 periods, denoted $\bar{\mathbf{e}}_j(t)$ where $t = 2020, 2030, 2040, 2050$ and $j \in M = \{\text{NAM,OEC,DCS, EEC}\}$. Emissions in 2005 are known and exogenous (see Table 1). The yearly emissions are interpolated linearly within each period.

The coupled constraint on global emission is then

$$8 \times \sum_{j \in M} e_j(2005) + \sum_{j \in M} [13, 10, 10, 5] \begin{bmatrix} \bar{\mathbf{e}}_j(2020) \\ \bar{\mathbf{e}}_j(2030) \\ \bar{\mathbf{e}}_j(2040) \\ \bar{\mathbf{e}}_j(2050) \end{bmatrix} \leq 519\,\text{GtC} - \text{eq}$$

Notice that here the notations have changed a little, since $\bar{\mathbf{e}}_j(t)$ refers to the yearly emission quotas of country j at the beginning of period t. The coefficients in the row matrix [13, 10, 10, 5] serve to define the total emissions per period, using linear interpolation. The 519 GtC-eq amount is the cumulative emission budget for an emission path that is compatible with a global warming of 2°C in 2100 for an average climate sensitivity of 3.5. This value (519 GtC-eq) has been obtained from simulation performed with a bottom-up model (TIAM [17]) which includes a climate module. This threshold has been established as critical by the European Union and the consequence of an overtaking will be irreversible for our ecosystem [24].

5.2 *Results*

We have tested six different splits of the emission budget called scenario 1 to scenario 6 that are presented in Table 2 . Due to the lengthy economic simulations performed by GEMINI-E3 at each call, the resolution time of the problem is important [7] and the computation of subgradients has been parallelized to reduce the oracle call time.

[7] We used a Dual 2.6 GHz Intel Xeon computer for the simulations, thus we had four available CPUs.

Table 2 Percent split of cumulative emission budget per scenario

Scenario	NAM	OEC	DCS	EEC
1	27	19	40	14
2	22	17	48	13
3	19	14	54	13
4	18	13	56	13
5	14	11	60	15
6	15	14	60	11

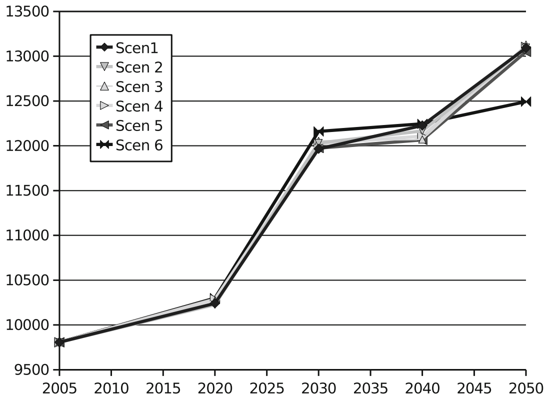

Fig. 2 World GHG emission in MtC-eq

One first interesting result concerns the World GHG emission, as it is shown in Fig. 2 the trajectory does not vary much with the rule adopted to allocate the cumulative emission budget. In Fig. 3, we display the equilibrium quotas for the different splits of a cumulative emission budget amounting to 519 GtC-eq. Of course, the equilibrium quotas at each period depend on the total allowances given to the players; however, we can note that industrialized countries (NAM and OEC) and energy exporting countries tend to allocate more quotas to the first periods of the Game. In contrary, developing countries tend to allocate their quotas to the last periods of the Game. There is a clear dichotomy between DCS and the Rest of the World which could be due to the weight given to this region in the split of the cumulative emissions budget. In all the scenarios tested, DCS represents at least 40% of the World quota and is therefore a central player.

The resulting payoffs, expressed in welfare variation in comparison with the BAU situation, are shown in Fig. 4. The welfare of the region depends on the emission budget initially given (see Table 2). Scenarios 3 and 4 are the most acceptable rules, because in all regions the welfare loss is limited to 0.5% of the household consumption. In contrary, in scenarios 1 and 6 the nonindustrialized countries bear an important welfare loss, respectively, DCS and EEC. Scenario 5 is the most likely acceptable scenario in the context of the post-Kyoto negotiation, because it leads to

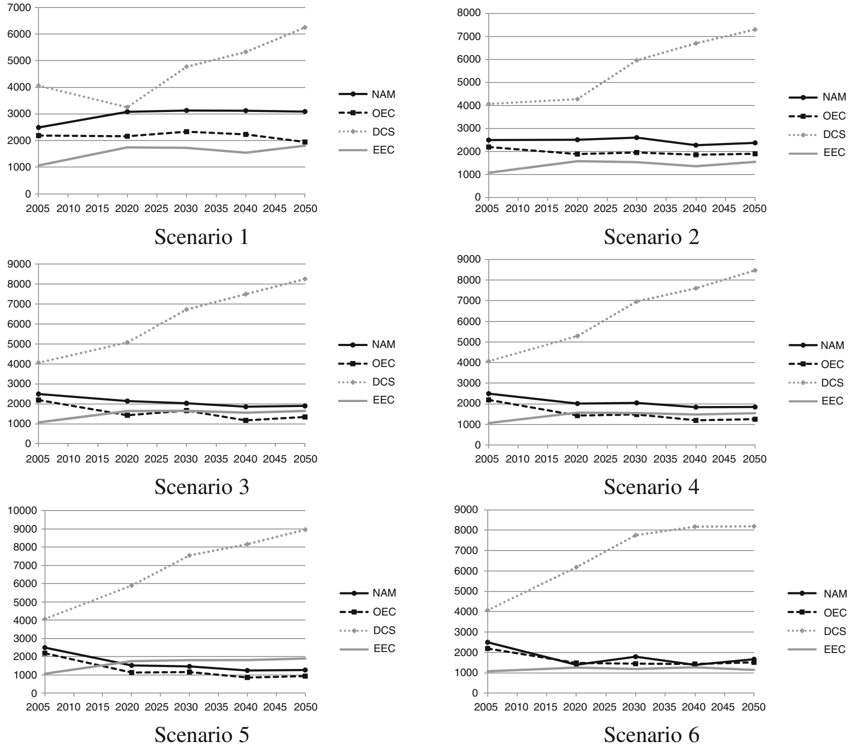

Fig. 3 Equilibrium quotas for each scenario (MtC-eq)

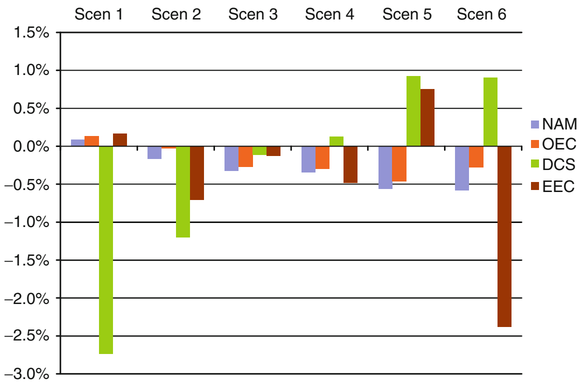

Fig. 4 Equilibrium payoffs for each scenario – sum of actualized surplus in % of households consumption

a welfare gain coming from the selling of permits by DCS and EEC. In the Kyoto Protocol, these two regions were effectively always reluctant to join the coalition of countries which are committed in constrained target of GHG.

6 Conclusion

In this paper, we have shown how an oracle-based optimization method could be implemented to compute Nash equilibrium solutions or Rosen normalized equilibrium solutions in a game where the payoffs are obtained from a large-scale macro-economic simulation model. This approach permits the use of game theoretic concepts in the analysis of economic and climate policies through the use of detailed models of economic general equilibrium. We have shown that the (always difficult) interpretation of the weights in the Rosen normalized equilibrium concept could be simplified in the case of a scalar and separable coupled constraint. In our case, the splitting of the cumulative emission budget over the planning period would be equivalent to a particular weighting in a normalized equilibrium.. We have applied this approach to a realistic description of the World economy and obtained a set of simulations showing the possible “fair” burden sharing that could emerge from negotiation for the post-2012 climate policies.

Acknowledgements Research supported by the EU-FP6 TOCSIN project and the French GICC research program. The authors thank two anonymous referees whose remarks and comments have permitted them to improve substantially this paper.

References

1. Armington, P.S.: A theory of demand for products distinguished by place of production. IMF Staff Papers **16**, 159–178 (1969)
2. Bernard, A., Haurie, A., Vielle, M., Viguier, L.: A two-level dynamic game of carbon emission trading between Russia, China, and Annex B countries. Journal of Economic Dynamics and Control **32**(6), 1830–1856 (2008)
3. Bernard, A., Vielle, M.: Comment allouer un coût global d’environnement entre pays : permis négociables versus taxes ou permis négociables et taxes? Économie Internationale **82**, 103–135 (2000)
4. Bernard, A., Vielle, M.: Measuring the welfare cost of climate change policies: A comparative assessment based on the computable general equilibrium model GEMINI-E3. Environmental Modeling & Assessment **8**(3), 199–217 (2003)
5. Bernard, A., Vielle, M.: GEMINI-E3, a General Equilibrium Model of International National Interactions between Economy, Energy and the Environment. Computational Management Science **5**(3), 173–206 (2008)
6. Bernard, A., Vielle, M., Viguier, L.: Carbon tax and international emissions trading: A swiss perspective. In Haurie, A., Viguier, L. (eds.) Coupling Climate and Economic Dynamics. Springer (2005)
7. Dimaranan, B.V.: Global Trade, Assistance, and Production: The GTAP 6 Data Base. Center for Global Trade Analysis Purdue University, Center for Global Trade Analysis, Purdue University (2006)
8. Drouet, L., Haurie, A., Moresino, F., Vial, J.-P., Vielle, M., Viguier, L..: An oracle based method to compute a coupled equilibrium in a model of international climate policy. Computational Management Science **5**(1), 119–140 (2008)
9. Ferris, M.C., Munson, T.S.: Complementarity problems in GAMS and the PATH solver. Journal of Economic Dynamics and Control **24**, 165–188 (2000)

10. Ferris, M.C., Pang, J.S.: Complementarity and Variational Problems: State of the Art. IAM Publications, Philadelphia (1997)
11. Haurie, A.: Environmental coordination in dynamic oligopolistic markets. Group Decision and Negotiation **4**, 46–67 (1995)
12. Haurie, A., Krawczyk, J.: Optimal charges on river effluent from lumped and distributed sources. Environmental Modeling and Assessment **2**, 177–199 (1997)
13. Haurie, A., Smeers, Y., Zaccour, G.: Stochastic equilibrium programming for dynamic oligopolistic markets. Journal of Optimmization Theory and Applications **66**(2) (1990)
14. Haurie, A., Zaccour, G.: Game-theoretic models of the environment. In: Annals of the International Society of Dynamic Games, volume 3, chapter Differential game models of global environmental management, pages 3–23. Birkhäuser, Boston (1995)
15. Krawczyk, J.B.: Coupled constraint Nash equilibria in environmental games. Resource and Energy Economics **27**(2), 157–181 (2005)
16. Krawczyk, J.B.: Numerical solutions to coupled-constraint (or generalised Nash) equilibrium problems. Computational Management Science **2**(1), 183–204 (2007)
17. Loulou, R.: ETSAP-TIAM: the TIMES integrated assessment model. part ii: mathematical formulation. Computational Management Science **5**(3), 41–66 (2008)
18. Nesterov, Y., Nemirovsky, A.: Interior Point Polynomial Methods in Convex Programming. SIAM, Philadelphia (1994)
19. Nesterov, Y., Vial, J.-P.: Homogeneous Analytic Center Cutting Plane Methods for convex problems and variational inequalities. SIAM J. Optim. **9**(3), 707–728 (1999)
20. Nesterov, Y., Péton, O., Vial, J.-P.: Homogeneous Analytic Center Cutting Plane Methods with approximate centers. Optimization Methods and Software **11&12**, 243–273 (1999)
21. Pang, J.S., Fukushima, M.: Quasi-variational inequalities, generalized Nash equilibria and multi-leader-follower games. Computational Management Science **2**(1), 21–56 (2005)
22. Péton, O. The homogeneous analytic center cutting plane method. Ph.D. Thesis, Université de Genève (2002)
23. Rosen, J.B.: Existence and uniqueness of equilibrium points for concave n-person games. Econometrica **33**(3), 520–534 (1965)
24. Stern, N.: Stern Review: The Economics of Climate Change. HM Treasury UK, London (2006)
25. Stone, J.R.N.: Linear expenditure systems and demand analysis: An application to the pattern of british demand. Economic Journal (1983)
26. von Heusinger, A., Kanzow, C.: SC optimization reformulations of the generalized Nash equilibrium problem. Optimization Methods and Software **23**, 953–973 (2008)
27. von Heusinger, A., Kanzow, C.: Optimization reformulations of the generalized Nash equilibrium problem using Nikaido-Isoda-type functions. Computational Optimization and Applications **43**, 353–377 (2009)