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Lattice properties of acyclic pipe dreams

Propriétés de treillis des arrangements de tuyaux acycliques

Thèse de doctorat de l'université Paris-Saclay

École doctorale n° 580, sciences et technologies de l'information et de la communication (STIC)
Spécialité de doctorat : Informatique mathématique
Graduate School : Informatique et sciences du numérique. Référent : Faculté des sciences d'Orsay.

Thèse préparée dans les unités de recherche du **Laboratoire interdisciplinaire des sciences du numérique (Université Paris-Saclay, CNRS)** et du **Laboratoire d'Informatique de l'École Plytechnique (École Polytechnique, CNRS)**, sous la direction de **Florent HIVERT**, professeur, le co-encadrement de **Vincent PILAUD**, chargé de recherche

Thèse soutenue à Paris-Saclay, le 18 octobre 2023, par

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THESE DE DOCTORAT

NNT : 2023UPASG065

université PARIS-SACLAY

ÉCOLE DOCTORALE
Sciences et technologies de l'information et de la communication (STIC)

Titre : Propriétés de treillis des arrangements de tuyaux acycliques

Mots clés : Treillis de Tamari, Théorie des treillis, Associaèdre, Complexes simpliciaux et polytopes, Groupes de Coxeter, Complexes de sous-mots

Résumé : Cette thèse s'inscrit dans le domaine de la combinatoire algébrique. Certains algorithmes de tri peuvent être décrits par des diagrammes appelés réseaux de tri, et l'exécution de ces algorithmes sur des permutations se traduit alors par des arrangements de courbes sur ces réseaux. Ces arrangements donnent des modèles pour des structures combinatoires classiques : par exemple, le treillis de Tamari, dont les relations de couverture sont les rotations sur les arbres binaires, et qui est un quotient bien connu de l'ordre faible sur les permutations.

Les complexes de sous-mots généralisent les réseaux de tris et les arrangements de courbes aux groupes de Coxeter. Ils ont des liens profonds en algèbre et géométrie, notamment dans le calcul de Schubert, l'étude des variétés grassmanniennes et la théorie des algèbres amassées. Cette thèse s'intéresse aux structures de treillis sur certains complexes de sous-mots, généralisant les treillis de Tamari. Plus précisément, elle étudie la relation définie par les extensions linéaires des facettes d'un complexe de sous-mot.

Dans un premier lieu, nous nous intéressons aux complexes de sous-mots définis sur un mot triangulaire du groupe symétrique, que nous représentons par des arrangements de tuyaux triangulaires. Nous prouvons alors que cette relation définit un quotient de treillis d'un intervalle de l'ordre faible ; par ailleurs, nous pouvons également utiliser cette relation pour définir un morphisme de treillis de cet intervalle au graphe des flips du complexe de sous-mots restreint à certaines de ses facettes. Dans un second lieu, nous étendons notre étude aux complexes de sous-mots définis sur les mots alternants du groupe symétrique. Nous montrons que cette même relation définit également un quotient de treillis ; en revanche, le morphisme associé n'a plus pour image le graphe des flips, mais le squelette du polyhèdre de brique, un objet défini sur les complexes de sous-mots pour étudier des réalisations du multi-associaèdre. Enfin, nous discutons des possibles extensions de ces résultats aux groupes de Coxeter finis, ainsi que de leurs applications pour généraliser certains objets définis en type A comme les treillis de nu-Tamari.

Title : Lattice properties of acyclic pipe dreams

Keywords : Tamari lattice, Lattices theory, Associahedron, Simplicial complexes and polytopes, Coxeter groups, Subword complexes

Abstract : This thesis comes within the scope of algebraic combinatorics. Some sorting algorithms can be described by diagrams called sorting networks, and the execution of the algorithms on input permutations translates to arrangements of curves on the networks. These arrangements modelize some classical combinatorial structures : for example, the Tamari lattice, whose cover relations are the rotations on binary trees, and which is a well-known quotient of the weak order on permutations.

Subword complexes generalize sorting network and arrangements of curves to Coxeter groups. They have deep connections in algebra and geometry, in particular in Schubert calculus, in the study of grassmannian varieties, and in the theory of cluster algebras. This thesis focuses on lattice structures on some subword complexes, generalizing Tamari lattices. More precisely, it studies the relation defined by linear extensions of the facets of a subword complex.

At first we focus on subword complexes defined on a triangular word of the symmetric group, which we represent with triangular pipe dreams. We prove that this relation defines a lattice quotient of a weak order interval ; moreover, we can also use this relation to define a lattice morphism from this interval to the restriction of the flip graph of the subword complex to some of its facets. Secondly, we extent our study to subword complexes defined on alternating words of the symmetric group. We prove that this same relation also defines a lattice quotient ; however, the image of the associated morphism is no longer the flip graph, but the skeleton of the brick polyhedron, an object defines on subword complexes to study realizations of the multiassociahedron. Finally, we discuss possible extensions of these results to finite Coxeter groups, as well as their applications to generalize some objects defined in type A such as nu-Tamari lattices.

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Remerciements

Je souhaite d'abord remercier mes encadrants, Florent Hivert et Vincent Pilaud. Le début de ma thèse, en automne 2020, pouvait difficilement tomber plus mal, et pourtant ils ont été présents tout du long. Malgré les nombreuses responsabilités qui leur incombaient, ils m'ont permis de tirer le meilleur de ces trois ans. Ils m'ont guidé dans la recherche mais aussi dans mon développement en général, et m'ont offert un grand nombre d'excellentes opportunités de rencontrer d'autres chercheurs et de présenter mon travail.

Merci aussi à Cesar Ceballos et Nantel Bergeron, avec qui collaborer a été un plaisir, à mes rapporteurs Jean Fromentin et Christian Stump pour leurs commentaires appréciés et leur immense travail de relecture qui permet à cette thèse d'être un peu plus lisible, et à tous les membres du jury pour leur présence aujourd'hui. Merci à toute l'équipe GALaC pour le très bon environnement de travail dont j'ai pu profiter ces trois dernières années. Je pense particulièrement à Viviane, qui a été d'excellent conseil quand j'en avais grand besoin, à Benjamin, Nathalie et Nicolas en particulier pour leur soutien quand je cherchais des financements pour voyager, à François pour les discussions du midi, et à Balthazar, Daniel, Hugo, João, Léo, Loïc, Nicolas, Quention, Tianjiao et Valentin pour le soutien psychologique entre stagiaires et doctorants. Toujours dans le monde académique, merci à Chiara, Clément, Doriann, Eva et Germain pour leur présence et leur amitié ces dernières années, et à Monica en particulier pour m'avoir embarquée dans l'escalade. Toutes les difficultés de cette thèse m'auraient paru beaucoup plus décourageantes sans votre présence à tous.

Merci à Tito pour ses relectures, à Mi ma jumelle de cœur, à Basile, Lwenn et Drew, Nath et Orphée, Robin, Sarah, Valentin et Ju pour ce groupe d'amis que j'aime tant. Sans vous, ces études m'auraient parues bien longues... Merci à Séréna d'être une excellente amie depuis si longtemps. Merci à François et Catherine pour leur présence continue et toujours agréable. Merci à Éva, Pascale et Éliane de m'avoir permis de ne pas penser qu'à mes études ces dernières années.

Je remercie enfin ma famille qui a toujours été présente pour moi. Merci à mes parents dont le support sans faille n'a jamais faibli, que j'ai besoin d'une aide matérielle ou juste de leur parler. Je n'aurais pas pu être si sereine tout au long de ma thèse sans eux. Merci également à mes frères, Étienne et Hippolyte, qui ont aussi été là et se sont très bien occupé d'Onyx quand j'en ai eu besoin. Merci à mes grands-parents, Papi Jean-Claude, Mamie Mathé, Papi Michel et Mamie Sadie, pour les longues discussions au téléphone le soir et pour leur accueil quand j'avais besoin de changer d'air. Merci à Alice de m'avoir donné Onyx, à Aude et Arnaud ainsi qu'à Frédérique de m'avoir accueillie chez eux à l'occasion, et à Hélène et Nicolas pour les longues et intéressantes discussions sur l'avenir. Merci enfin à tous et à mes cousins et cousines, Liliane et Suzanne, Adam et Anna-Rose, et Anatole, Augustin et Mahaut, pour cette famille qui fait que je ne me suis jamais vraiment sentie seule.

Enfin, je pense à tatie Danielle et à grand-mamie. Elles ne sont plus là aujourd'hui mais j'aurais aimé pouvoir partager ce moment avec elles.

Finalement, un grand merci à mon chat Onyx, chasseur de moustiques et d'idées noires et peluche quand il le veut bien.

Introduction

Cette thèse donne une présentation de mon travail sur le lien entre l'ordre faible et les facettes acycliques de complexes de sous-mots dans les groupes de Coxeter.

La combinatoire est l'étude d'objets mathématiques discrets, comme les permutations, les arbres, les posets ou les graphes, et en particulier de leur énumération. La combinatoire algébrique s'intéresse plus spécifiquement aux structures algébriques définies sur ces objets: par example, l'opération de composition sur les permutations de n éléments définit un groupe. Comprendre ces structures est utile dans de nombreux autres domaines, en particulier en algorithmique, où cela aide à définir et à optimiser des algorithmes sur ces objets.

Contexte

Permutations. Une **permutation** est une bijection vers lui-même d'un ensemble $[n] = \{1, 2, \ldots, n\}$; l'ensemble des permutations de taille n est noté $\mathfrak{S}_n$. En général, nous emploierons la notation en ligne $\sigma(1)\sigma(2)\ldots\sigma(n)$ pour représenter une permutation $\sigma \in \mathfrak{S}_n$. Des problèmes mettant en jeu des permutations sont étudiés depuis au moins la Grèce antique, car elles fournissent de nombreuses questions intéressantes à tous les degrés de difficulté et apparaissent au cœur de questions concrètes très variées. Par exemple, les propriétés des permutations des racines jouent un rôle dans la description par É. Galois des équations polynomiales résolubles par radicaux, mais les permutations sont également utilisées en physique quantique pour décrire les états de particules, ou en cryptologie où elles ont joué un rôle central dans la cryptanalyse de la machine Enigma allemande. Aujourd'hui, elles sont très présentes dans l'étude des algorithmes de tri. Par exemple, le **tri à bulles** est un algorithme de **tri par transpositions simples**, dans le sens où la seule opération utilisée est la comparaison de deux éléments consécutifs, suivie de leur échange si leurs positions respectives ne sont pas celles attendues. Son exécution sur la liste $[5, 8, 3, 7, -1]$ est illustrée sur la gauche de la Fig. 1 par un **arrangement de pseudo-lignes**: les lignes horizontales, du haut vers le bas, représentent les indices de la liste qui est triée (ici de 1 à 5), et la comparaison de deux éléments

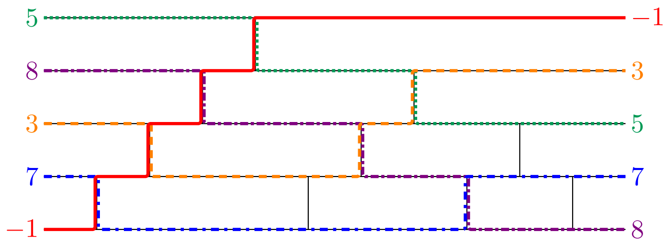


Figure 1: Deux algorithmes de tri appliqués à la liste $[5, 8, 3, 7, -1]$.

est montrée par une barre verticale entre les deux indices pertinents. Les valeurs de la liste sont représentées par des lignes brisées colorées qui vont de la gauche (leur indice initial) vers la droite, et deux de ceux lignes se croisent sur une barre verticale si leurs valeurs sont échangées après la comparaison. Le lecteur trouvera une présentation générale des permutations dans les algorithmes dans [CLRS01] (en particulier dans le chapitre 27 sur les réseaux de tri) et dans le chapitre 5 de [Knung].

L'étude de ces algorithmes et la recherche du nombre de comparaisons nécessaires pour trier n'importe quelle permutation de taille n mènent rapidement à introduire les **inversions** d'une permutation. Un inversion est une paire $i < j$ telle que $\omega^{-1}(i) > \omega^{-1}(j)$, c'est-à-dire que i apparaît après j dans ω. Le nombre maximum d'inversions dans une permutation de $\mathfrak{S}_n$ est $\frac{n(n-1)}{2}$, et ce nombre est atteint par la permutation maximale $\omega_0 = n\ (n-1)\ \ldots\ 1$; c'est le nombre minimal de comparaisons pour un algorithme de tri par transpositions simples. Le tri à bulles est donc optimal pour le nombre d'échanges d'éléments consécutifs, mais ce n'est pas le seul: par exemple, le **tri pair-impair**, illustré en partie droite de la Fig. 1, utilise le même nombre d'échanges et est parallélisable.

On appelle $\mathrm{inv}(\omega)$ l'ensemble des inversions d'une permutation, et cet ensemble la **caractérise**, car il donne la position relative de n'importe quelle paire de nombres (puisque toute paire $i < j$ qui n'est pas une inversion est une **non-inversion** telle que $\omega^{-1}(i) < \omega^{-1}(j)$). Il est donc naturel de comparer les ensembles d'inversions des permutations de $\mathfrak{S}_n$: l'**ordre faible** sur les permutations est défini par $\pi \leqslant \omega$ si $\mathrm{inv}(\pi) \subseteq \mathrm{inv}(\omega)$. Sa structure sur les permutations de taille 3 et 4 est donnée en Fig. 2, avec les petites permutations en bas et les grandes en haut, et $\pi \leqslant \omega$ si il y a un chemin qui monte de π vers ω. Remarquez que ses **couvertures** (les arêtes du graphe) sont toutes de la forme $UijV < UjiV$ pour $i < j$: c'est ainsi qu'on ajoute ou retire une et une seule inversion à une permutation. Cet ordre a été beaucoup étudié: en particulier, G. T. Guilbaud et P. Rosenstiehl ont démontré dans [GR63] qu'il s'agit d'un **treillis**, c'est-à-dire que pour toute paire de permutations ω_1, ω_2, il existe une permutation minimum plus grande qu'elles deux (leur **borne supérieure**) et une permutation maximum plus petite qu'elles deux (leur **borne inférieure**). De plus, le **diagramme de Hasse** de cet ordre sur $\mathfrak{S}_n$ (le graphe représenté en Fig. 2) est le squelette d'un polytope de dimension $(n-1)$, le **permutaèdre**, défini pour

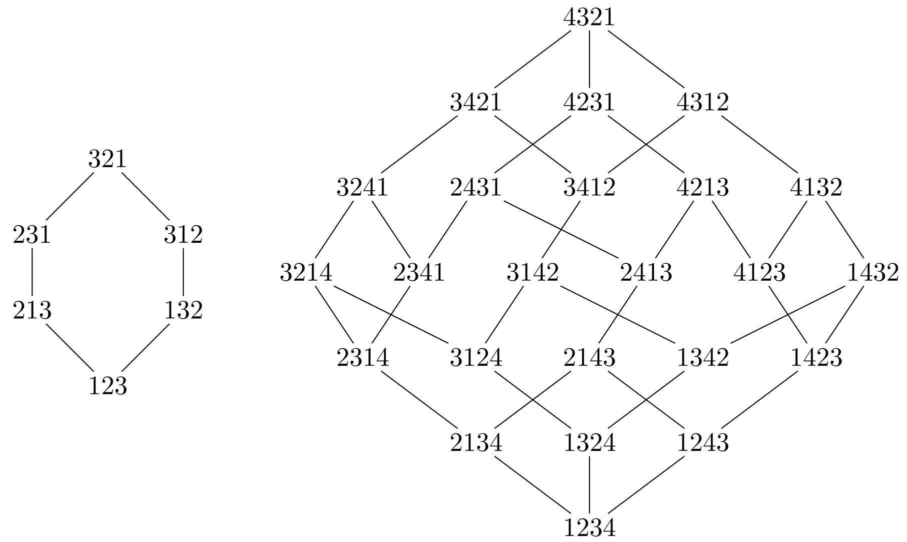


Figure 2: L'ordre faible sur $\mathfrak{S}_3$ et $\mathfrak{S}_4$.

la première fois en 1911 dans [Sho11]. Pour plus d'informations au sujet de la combinatoire des permutations, on pourra se référer à [Bon12].

Arbres binaires. Un **arbre binaire** peut être défini récursivement comme une feuille (un arbre vide) ou bien un nœud qui a deux arbres binaires comme sous-arbres gauche et droit ; un exemple est représenté en Fig. 3. L'ensemble $\mathcal{BT}_n$ des arbres binaires à n nœuds est l'une des nombreuses familles combinatoires comptées par les nombres de Catalan C_n, avec les triangulations d'un $(n+2)$-gone, les parenthésages possibles de $n+1$ objets en paires ou les chemins de Dyck à $2n$ pas. Les arbres binaires sont souvent utilisés comme une structure de données dans des algorithmes : par exemple, un **arbre binaire de recherche** est un arbre dont chaque nœud contient des données indexées par des clés (comme des nombres ou des chaînes de charactères) tel que si un nœud contient une clé k, toutes les clés de son sous-arbre gauche sont inférieures à k et toutes les clés de son sous-arbre droit sont supérieures à k. La complexité de l'accès à une clé est alors proportionnelle à la profondeur du nœud qui la contient, comme illustré en Fig. 3: à chaque nœud, on sait si la clé est présente dans ce nœud, dans son sous-arbre gauche ou dans son sous-arbre droit, et l'ajout d'une donnée indexée par une clé s'effectue similairement. Pour plus d'informations sur les utilisations des arbres binaires en algorithmique, voir [CLRS01] (en particulier la partie 3, mais aussi les chapitres 6 et 18) et [Knung] (chapitres 2 et 6).

Comme la complexité des opérations de recherche et d'ajout dans un arbre binaire sont proportionnelles à sa hauteur (la profondeur maximale de ses nœuds), il est nécessaire que ces arbres soient **équilibrés**, c'est-à-dire que les hauteurs des sous-

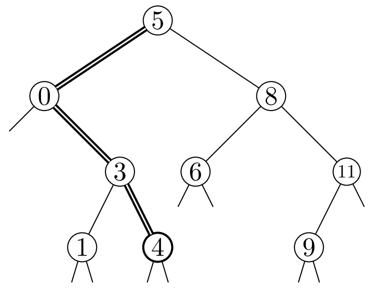
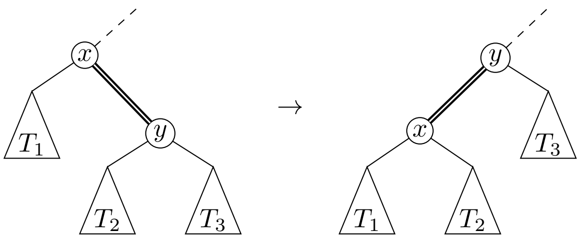


Figure 3: Recherche de la clé 4 dans un arbre binaire de recherche.

arbres gauche et droit d'un nœud soient toujours (presque) égales. Cela garantit que la hauteur globale de l'arbre soit minimale ou presque pour son nombre de nœuds. C'est dans ce but que G. Adelson-Velsky et E. Landis ont introduit les **rotations** dans [AVL62], une opération qui permet de rendre un arbre plus équilibré tout en préservant ses propriétés d'arbre binaire de recherche. Cette opération appliquée à une arête $x \to y$ de l'arbre est illustrée en haut de Fig. 4: les sous-arbres de x et y sont réarrangés de façon à diminuer la profondeur du sous-arbre de x contenant y et d'augmenter celle de l'autre sous-arbre, et le nœud y passe d'enfant de x à son parent. La rotation est une **rotation gauche** si y est le fils droit de x (comme en Fig. 4) et une **rotation droite** dans le cas contraire. Le **treillis de Tamari**, introduit par D. Tamari dans [Tam62] (à l'origine sur les parenthésages de $n+1$ objets), est l'ordre défini sur $\mathcal{BT}_n$ par $T_1 \leqslant T_2$ si on peut passer de T_1 à T_2 par une suite de rotations gauches. Son diagramme de Hasse sur $\mathcal{BT}_3$ et $\mathcal{BT}_4$ est donné en bas de la Fig. 4, et comme son nom l'indique, il a été démontré dans [HT72] qu'il s'agit d'un treillis.

Ce diagramme de Hasse sur $\mathcal{BT}_n$ est le squelette d'un polytope de dimension $n-1$, l'associaèdre: cet objet a été théorisé séparément par D. Tamari et J. D. Stasheff ([Sta63]), puis réalisé comme un polytope convexe de nombreuses manières. Une présentation de l'histoire de ces réalisations peut être trouvée dans l'introduction de [CSZ14] ou dans [PSZ23]; l'associaèdre de dimension 3 (qui représente le treillis de Tamari sur $\mathcal{BT}_4$) est dessiné en Fig. 5. La structure combinatoire de ce polytope a été source de nombreuses questions intéressantes: par exemple, son diamètre a été majoré par D. Sleator, R. Tarjan et W. Thurston dans [STT88] en 1988, mais il a fallu attendre 2014 pour que L. Pournin démontre dans [Pou14] que cette majoration est optimale à partir de la dimension 11.

De l'ordre faible au treillis de Tamari. Les couvertures de l'ordre faible et du treillis de Tamari correspondent dans les deux cas à de petits changements locaux, dans le premier cas l'échange de deux valeurs consécutives et dans le second la rotation de deux nœuds adjacents. Nous allons voir que ces deux opérations sont en réalité étroitement liées. Pour chaque arbre binaire à n nœuds, il existe exactement



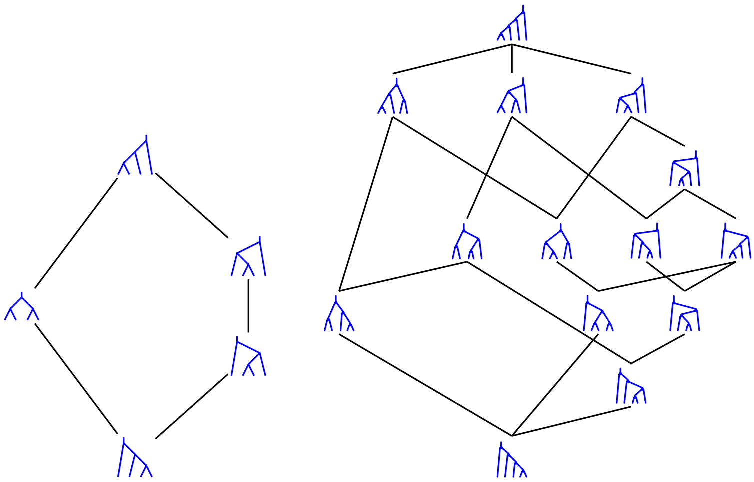


Figure 4: L'ordre des rotations gauches sur les arbres binaires à 3 et 4 nœuds.

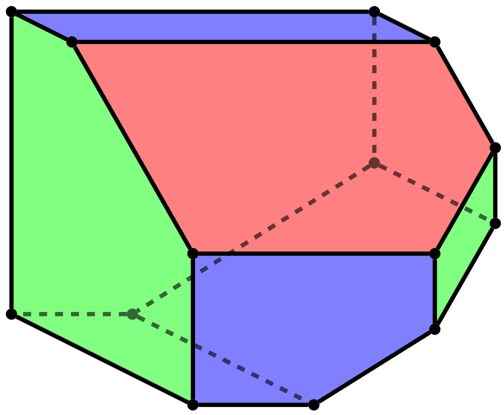


Figure 5: L'associaèdre de dimension 3.

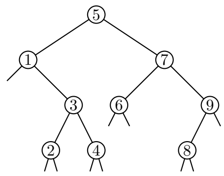


Figure 6: L'étiquetage en arbre binaire de recherche d'un arbre binaire.

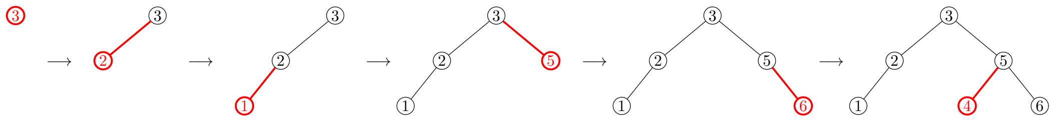


Figure 7: Insertion de 321564 dans un arbre binaire de recherche.

un étiquetage de ces nœuds avec les valeurs de 1 à n qui respecte les règles des arbres binaires de recherche. Cet étiquetage correspond à l'ordre dans lequel les nœuds sont parcourus lors d'un parcours infixe de l'arbre, défini récursivement en effectuant le parcours infixe du sous-arbre gauche, puis en considérant le nœud racine, puis le parcours infixe du sous-arbre droit. Par exemple, cet étiquetage pour l'arbre présenté en Fig. 3 est donné en Fig. 6. Une **extension linéaire** d'un arbre binaire de recherche T est alors une permutation telle que si le nœud étiqueté par j est un descendant du nœud étiqueté par i dans T, alors $\omega^{-1}(j) > \omega^{-1}(i)$ (la valeur j apparaît après la valeur i dans ω). Par exemple, les permutations 513247698 et 579861342 sont toutes deux des extensions linéaires de l'arbre en Fig. 6.

En général, une permutation ω est une extension linéaire d'exactement un arbre binaire de recherche, et cet arbre peut être obtenu en insérant chaque élément de $[n]$ dans un arbre binaire de recherche dans l'ordre donné par ω : d'abord $\omega(1)$ est placé à la racine, puis $\omega(2)$ est un des fils de la racine (gauche si $\omega(2) < \omega(1)$ et droit sinon)... Par example, l'insertion de la permutation 321564 est donné en Fig. 7. Ceci définit une **application d'insertion** surjective (mais pas injective) de $\mathfrak{S}_n$ vers $\mathcal{BT}_n$. Cette application montre un lien profond entre l'ordre faible et le treillis de Tamari: une couverture de l'ordre faible $UijV < UjiV$ est soit à l'intérieur d'une des fibres de cette application, ou bien son image est une rotation gauche. Le premier cas se produit quand j n'est pas le fils de i, c'est-à-dire si il existe $k \in U$ tel que $i < k < j$ (donc i est dans le sous-arbre gauche de k et j dans son sous-arbre droit), et le second quand j est le fils droit de i, et alors la rotation image est appliquée à l'arête $i \to j$. F. Hivert, J.-C. Novelli et J.-Y. Thibon ont démontré dans [HNT05] que l'application d'insertion est un **morphisme d'ordre** de l'ordre faible vers le treillis de Tamari.

La **congruence sylvestre** est la relation d'équivalence sur $\mathfrak{S}_n$ pour laquelle deux permutations sont équivalentes si elles sont les extensions linéaires d'un même arbre

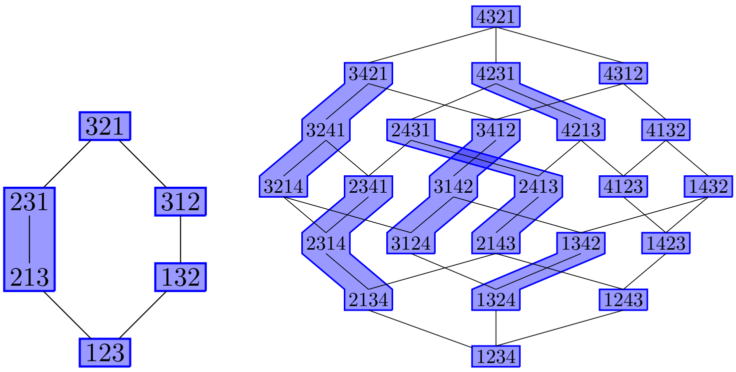


Figure 8: La congruence sylvestre sur $\mathfrak{S}_3$ et $\mathfrak{S}_4$.

binaire de recherche. Elle peut également être définie comme la clôture réflexive et transitive des relations $UijV \equiv UjiV$ si U contient au moins une lettre k telle que $i < k < j$. Cela signifie en particulier qu'une permutation est **minimale** dans sa classe de congruence si elle évite le motif 231 (il n'y a pas de triplet $i < k < j$ tel que $\omega^{-1}(k) < \omega^{-1}(j) < \omega^{-1}(i)$) et **maximale** si elle évite le motif 213; les caractéristiques des **congruences de treillis** nous apprenent donc que le treillis de Tamari est isomorphe à la restriction de l'ordre faible à ces permutations évitant le motif 231 (ou 213). La congruence sylvestre sur $\mathfrak{S}_3$ et $\mathfrak{S}_4$ est donnée en Fig. 8. Nous verrons en Section 2.2.1 que cette congruence et un équivalent de l'application d'insertion peuvent être définis en utilisant des arrangements de pseudo-lignes sur le réseau de tri du tri à bulles (voir Fig. 1).

L'existence d'un tel lien entre ordre faible et treillis de Tamari était connu avant que l'approche algorithmique que nous venons de détailler, et qui utilise l'insertion dans les arbres binaires de recherche, ne soit découverte : elle avait été démontrée par L. Billera et B. Strumfels dans [BS94] et par A. Tonks dans [Ton97] en étudiant les relations entre une réalisation bien choisie de l'associaèdre et du permutaèdre, par J.-L. Loday et M. Ronco dans [LR98] en plongeant l'algèbre de Hopf des arbres binaires dans une sous-algèbre de l'algèbre de Hopf de Malvenuto-Reutenauer des permutations, ou encore par N. Reading dans [Rea05] en utilisant des outils de théorie des treillis. Cette diversité des approches possibles est un aspect central de la combinatoire algébrique et a été utilisée pour proposer de multiples généralisations possibles du treillis de Tamari, comme les treillis cambriens (définis par N. Reading dans [Rea06]) ou les treillis de ν-Tamari (définis par L.-F. Préville-Ratelle et X. Viennot dans [PRV15]). L'objectif de cette thèse est de trouver un contexte aussi général que possible dans lequel cette multitude de points de vue donne toujours des résultats intéressants. Le cadre algébrique le plus adapté que nous ayons trouvé est celui des complexes de sous-mots dans les groupes de Coxeter finis.

Groupes de Coxeter et complexes de sous-mots. Nous avons précédemment expliqué que les couvertures de l'ordre faible sont les relations $UijV < UjiV$; une description équivalente serait de dire que ce sont les relations $\omega < \omega\tau_k$, avec τ_k la transposition simple $(k, k+1)$ et avec $\omega(k) < \omega(k+1)$. Cela signifie que l'ordre faible sur les permutations est profondément lié à la structure du groupe symétrique comme le groupe généré par les transpositions simples $(\tau_k)_{1 \leqslant k < n}$. Les **groupes de Coxeter finis** sont une famille de groupes définie par H. S. M. Coxeter dans [Cox34, Cox35] qui généralisent cette idée: ils sont générés par une famille d'involutions, les **réflexions simples**, et sont entièrement caractérisés par les relations entre ces réflexions. Ils peuvent également être décrits comme certains sous-groupes des isométries d'espaces euclidiens générés par des réflexions orthogonales. Les interactions entre ces deux définitions équivalentes ont donné naissance à un champ d'étude particulièrement riche, où des arguments algébriques démontrent des résultats géométriques et inversement. Par exemple, N. Reading utilise une réalisation géométrique de l'ordre faible pour démontrer un grand nombre de ses propriétés dans [Rea16a]. On trouvera la description complète et une étude approfondie des groupes de Coxeter en partant de leur définition algébrique dans [BB05], tandis que [Hum90] propose un travail similaire en partant d'une définition géométrique.

Plutôt que de partir des arbres binaires, la réalisation du treillis de Tamari qui peut se généraliser aisément aux groupes de Coxeter utilise les arrangements de pseudo-lignes. Leur généralisation se fait via les **complexes de sous-mots**, définis sur des mots de l'alphabet des réflexions simples d'un groupe de Coxeter, et qui représentent les **sous-mots réduits** de ces mots dont le produit est fixé. Ces objets ont été définis par A. Knutson et E. Miller dans [KM04] pour généraliser les **arrangements de tuyaux**, une famille d'objets combinatoires introduits dans [KM05] pour développer les polynômes de Schubert dans la base des monômes habituels. On rappelle que les polynômes de Schubert forment une base des polynômes à plusieurs variables bien adaptée à l'étude de leurs symétries partielles. Ils sont apparus d'abord en géométrie algébrique et sont intrinsèquement liés à la théorie des fonctions symétriques, comme par exemple pour la règle de Littlewood-Richardson [LS85, Mac91].

A. Woo a prové dans [Woo04] que certains complexes de sous-mots étaient isomorphes au treillis de Tamari ; cela a été étendu dans [Stu11] par C. Stump aux multitriangulations (car les triangulations donnent une autre réalisation du treillis de Tamari), ainsi que dans [PP12] par V. Pilaud et M. Pocchiola aux pseudotriangulations et aux multipseudotriangulations. L'ensemble des k-triangulations d'un n-gone est conjecturé comme étant isomorphe au complexe de limites d'un polytope simplicial de dimension $k(n-2k-1)-1$; pour tenter (sans y parvenir) de démontrer cette conjecture, V. Pilaud et F. Santos ont défini le **polytope de briques** d'un réseau de tri dans [PS12]. Malgré cet échec, il s'est avéré que ce polytope donne une réalisation des associaèdres généralisés et sa définition a été étendue à une famille bien plus importante de complexes de sous-mots dans n'importe quel groupe de Coxeter par V. Pilaud et C. Stump dans [PS15a], puis enfin à n'importe quel complexe de sous-mots dans ce cadre par D. Jahn et C. Stump dans [JS21]. V. Pilaud et F. Santos ont démontré dans [PS12, Pil18a] que des polytopes de brique réalisent les treillis cambriens ; en fait, ils sont même des translatés des associaèdres généralisés de C. Hohlweg et C. Lange présentés dans [HL07].

Cette thèse approfondira l'étude des liens entre les ordres faibles et les polyèdres de briques de complexes de sous-mots, et tentera en particulier de répondre à la question suivante : quand le polyèdre de brique d'un complexe de sous-mots est-il un quotient de treillis d'un intervalle de l'ordre faible ? Nous définirons une application de cet intervalle vers les sommets du polyèdre de briques, en utilisant une approche géométrique ainsi qu'une approche algorithmique, et nous étudierons ses fibres sou l'angle de la théorie des treillis.

Contenu

Dans le chapitre 1, nous introduirons les objets mathématiques que nous utiliserons dans cette thèse. Le lecteur ou la lectrice avertie pourra sauter tout ou partie de ce chapitre et s'y référer lorsque les résultats qui y sont présentés sont cités plus tard. Nous commencerons par définir les **posets** et les **treillis**, nous décrirons certaines de leurs propriétés, et nous donnerons quelques outils qui sont utilisés pour les étudier. En particulier, nous définirons et caractériserons les **congruences de treillis** et les **quotients de treillis**, deux notions qui seront centrales dans la suite de notre travail. Nous appliquerons ensuite ces outils au **poset des régions d'un arrangement d'hyperplans**, un objet géométrique à la structure combinatoire intéressante. Nous définirons ensuite les **groupes de Coxeter** en partant d'un point de vue géométrique, puis d'un point de vue algébrique, et présenterons certaines de leurs propriétés, avant de nous concentrer sur un ordre partiel défini sur leurs éléments, l'**ordre faible**. Cet ordre a plusieurs définitions équivalentes, algébriques ou géométriques, et nous verrons comment mêler ces deux points de vue sur les groupes de Coxeter permet d'obtenir une vue d'ensemble de sa structure. Enfin, nous définirons formellement les **complexes de sous-mots** et donnerons un aperçu du vocabulaire et des résultats les concernant que nous utiliserons. En particulier, nous nous concentrerons sur le **polytope de briques** et les **facettes acycliques** d'un complexe de sous-mots, car ils se révèleront indispensables aux objets que nous étudierons.

Nos premiers résultats originaux sont présentés en chapitre 2. Dans ce chapitre, nous travaillerons sur les **arrangements de tuyaux triangulaires**, une représentation graphique des complexes de sous-mots définis sur le réseau de tri du tri à bulles dessiné à gauche en figure 1. Il est connu que certains de ces arrangements de tuyaux donnent une réalisation du treillis de Tamari, qui est un quotient de l'ordre faible ; notre but sera de généraliser ce résultat autant que possible. Nous démontrerons d'abord avec le théorème 2.2.12 que pour tout complexe de sous-mots représenté par des arrangements de tuyaux triangulaires, les extensions linéaires des facettes acycliques définissent une congruence de treillis d'un intervalle de l'ordre faible, la **congruence de pipe dreams**. Nous définirons également une **application d'insertion** de cet intervalle vers les arrangements de tuyaux acycliques et caractériserons l'image de l'ordre faible en terme de **flips sur les facettes acyclicques** avec le théorème 2.2.17, et donnerons **deux algorithmes** qui calculent efficacement les valeurs de cette application ainsi qu'une définition alternative de la congruence de pipe dreams n'utilisant que la clôture réflexive et transitive d'une **relation** sur les permutations en section 2.2.5. Enfin, nous démontrerons avec la proposition 2.3.1 que l'image de l'ordre faible par l'application d'insertion est exactement le **squelette du polytope de briques** du complexe de sous-mots associé, et nous appliquerons ce résultat pour obtenir une réalisation polyédrale des treillis de ν-Tamari dans le corollaire 2.3.7.

Nous étendrons ensuite ces résultats aux **arrangements de tuyaux alternants**, une classe plus étendue d'arrangements de tuyaux sur des formes non-triangulaires. Ils sont une représentation graphique de complexes de sous-mots sur les mots alternants du groupe symétrique inspirée par une réalisation des treillis cambriens. Nous démontrerons en premier lieu avec le théorème 3.2.6 que la **congruence de pipe dreams** définie par les extensions linéaires des arrangements de tuyaux acycliques est toujours une **congruence de treillis** d'un intervalle de l'ordre faible. En revanche, l'**application d'insertion** de cet intervalle vers les facettes acycliques n'est plus toujours surjective, et l'image de l'ordre faible est plus difficile à décrire : nous donnerons dans la proposition 3.2.11 une caractérisation de cette image moins satisfaisante que celle obtenue dans le chapitre 2. Nous démontrerons également en section 3.2.5 que les **deux algorithmes** définis sur les arrangements de tuyaux triangulaires dans la section 2.2.5 fonctionnent toujours sur les arrangements de tuyaux alternants. Enfin, nous montrerons avec la proposition 3.3.1 que l'image de l'ordre faible par l'application d'insertion est une **partie du squelette du polytope de briques**, et donnerons une condition qui garantit qu'il s'agisse de l'intégralité de ce squelette dans le théorème 3.3.4. Ce dernier théorème fournit une caractérisation bien plus satisfaisante de l'image de l'ordre faible.

Nous terminons cette thèse dans le chapitre 4 avec une discussion des possibilités de généralisation des résultats précédents à n'importe quel groupe de Coxeter fini. Nous commençons par démontrer que la **congruence de sous-mots** et l'**application d'insertion** sont toujours bien définis par les extensions linéaires des facettes acycliques, et nous donnons un exemple dans lequel cette congruence n'est **pas** une congruence de treillis. Nous utilisons le travail déjà effectué sur les groupes de Coxeter de type A pour proposer **plusieurs conjectures** généralisant les résultats du chapitre 3. En particulier, nous suggérons par les conjectures 4.2.1 et 4.2.3 à 4.2.5 que la congruence de sous-mots est une **congruence de treillis** si le mot sur lequel le complexe de sous-mots est défini est **alternant**, et que dans ce cas l'image de l'ordre faible par l'application d'insertion est toujours une partie du squelette du polytope de brique associé. Ces conjectures ont été testées par de nombreuses expérimentations informatiques dans divers groupes de Coxeter. De plus, nous utilisons des idées tirées du chapitre 3 pour suggérer une preuve possible de toutes ces conjectures en partant d'un lemme conjecturé, le lemme 4.2.7. Nous concluons notre travail en évoquant deux concepts définis dans les groupes de Coxeter de type A qui semblent être liés à notre travail, les **chute moves** et les **treillis de ν-Tamari**, et en proposant une façon d'utiliser ce lien pour étendre leurs définitions à d'autres groups de Coxeter finis.

Introduction

This thesis presents an overview of my work on the links between the weak order and acyclic facets of subword complexes in Coxeter groups.

Combinatorics is the study of discrete mathematical objects, like permutations, trees, posets or graphs, and in particular of their enumeration. Algebraic combinatorics focuses more specifically on the algebraic structures defined on these objects: for example, the composition of permutations on n elements is a group. Understanding those structures is useful in many other fields, in particular in algorithmics, where it helps define and optimize algorithms on these objects.

Context

Permutations. A **permutation** is a bijection from a set $[n] = \{1, 2, \dots, n\}$ to itself; the set of permutations of size n is denoted by $\mathfrak{S}_n$. In general, we use the one-line notation $\sigma(1)\sigma(2)\dots\sigma(n)$ to represent a permutation $\sigma \in \mathfrak{S}_n$. Problems involving permutations have been studied since at least ancient Greece, as they provide many interesting questions at all levels of difficulty and appear in a variety of concrete questions. For example, properties of the permutations of the roots play a role in the description by E. Galois of solvable polynomial equations, but permutations are also used in quantum physics to describe states of particles, or in cryptology where they were central in the cryptanalysis of the German Enigma machine. Today they are proheminent in the study of sorting algorithms. For example, the **bubble sort** is a **simple transpositions sorting algorithm**, in the sense that its only operation is comparing two consecutive values of a list and exchanging them if they are not in the intended order. Its execution on the list $[5, 8, 3, 7, -1]$ is illustrated on the left of Fig. 9 as a **pseudoline arrangement**: the horizontal lines from top to bottom represent the indices of the list being sorted (here from 1 to 5), and the comparison of two elements is drawn as a vertical bar between the two relevant indices. The values of the list are represented by colored curves going from left (their initial index) to right, with two lines crossing on a vertical bar if the values are exchanged after the comparison. For a general overview of permu-

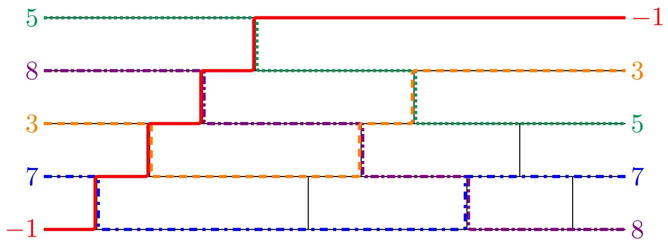


Figure 9: Two sorting algorithms on the list $[5, 8, 3, 7, -1]$.

tations in algorithms, see [CLRS01] (in particular Chapter 27 on sorting networks) and [Knung, Chapter 5].

Studying these algorithms and determining how many comparisons are necessary to sort any permutation of size n leads quickly to introducing the **inversions** of a permutation. An inversion is a pair $i < j$ such that $\omega^{-1}(i) > \omega^{-1}(j)$, i.e i appears after j in ω. The maximal number of inversions in a permutation of $\mathfrak{S}_n$ is $\frac{n(n-1)}{2}$, reached by the maximal permutation $\omega_0 = n\ (n-1)\ \ldots\ 1$, and this is the minimal number of comparisons for a simple transposition sorting algorithm. The bubble sort is thus optimal in terms of how many exchanges of consecutive elements are made, but it is not the only one: for example, the **odd-even sort**, illustrated on the right of Fig. 9, uses the same number of exchanges but is parallelizable.

We denote by $\mathrm{inv}(\omega)$ the set of inversions of a permutation, and it **characterizes** the permutation, as it gives the relative position of any two numbers in it (since any pair $i < j$ that is not an inversion is a **noninversion** such that $\omega^{-1}(i) < \omega^{-1}(j)$). It is therefore natural to compare those inversion sets along all the permutations in $\mathfrak{S}_n$: the **weak order** on permutations is defined by $\pi \leqslant \omega$ if $\mathrm{inv}(\pi) \subseteq \mathrm{inv}(\omega)$. Its structure is represented for permutations of size 3 and 4 in Fig. 10, with the smaller permutations at the bottom and the larger at the top, and $\pi \leqslant \omega$ if there is a path going up from π to ω. Note that its covers (the edges of the graph) are all of the form $UijV < UjiV$ with $i < j$: this is how one can add or remove exactly one inversion to a permutation. This order has been extensively studied: in particular, G. T. Guilbaud and P. Rosenstiehl proved in [GR63] that it is a **lattice**, i.e for any two permutations ω_1, ω_2, there exists a minimal permutation greater than both (their **join**) and a maximal permutation lower than both (their **meet**). Moreover, the **Hasse diagram** of this order on $\mathfrak{S}_n$ (the graph given in Fig. 10) is the skeleton of an $(n-1)$-dimensional polytope, the **permutohedron**, first defined in 1911 in [Sho11]. For more informations on the combinatorics of permutations, one may read [Bon12].

Binary trees. A **binary tree** can be recursively defined as a leaf (an empty tree) or a node with two binary trees as its left and right subtrees; an example is given in Fig. 11. The sets $\mathcal{BT}_n$ of binary trees with n nodes are one of the

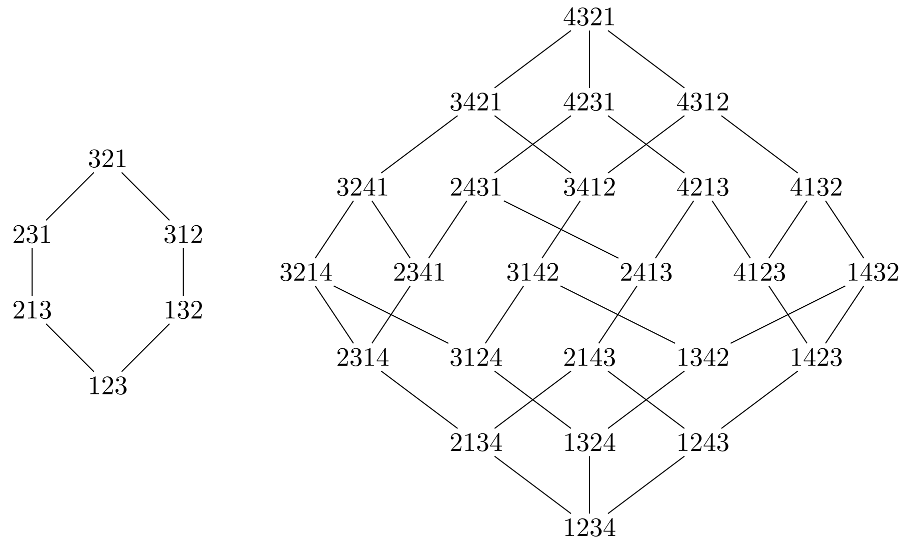


Figure 10: The weak order on $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

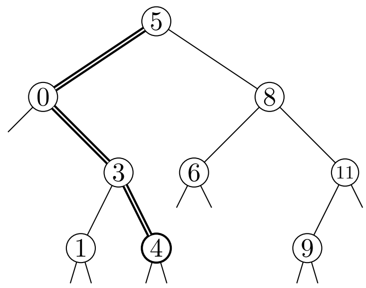


Figure 11: The search for key 4 in a binary search tree.

many combinatorial families counted by Catalan numbers C_n, like triangulations of an $(n + 2)$-gon, the possible bracketings of $n + 1$ objects into pairs or the Dyck paths with $2n$ steps. Binary trees are commonly used as a data structure in several algorithms: for example, a **binary search tree** is a tree containing data indexed by keys (like numbers or character strings) such that for each node with a key k, the keys in its left subtree are all smaller than k and the keys in the right subtree are all larger than k. The complexity of accessing a key is then proportional to the depth of its node, as illustrated in Fig. 11: at each node, we know whether the key is at the node, in its right subtree or in its left subtree, and adding a key proceeds similarly. For more information on the algorithmics uses of binary trees, see [CLRS01] (in particular Part 3, but also chapters 6 and 18) and [Knung] (Chapters 2 and 6).

Since the complexity of search and add operations in binary trees are proportional to its height (the maximal depth of its nodes), we need trees to be **balanced**, i.e with the height of the subtrees of any node being (almost) equal. This guarantees that the height of the tree is minimal regarding its number of nodes. To that aim, G. Adelson-Velsky and E. Landis introduced **rotations** in [AVL62] as an operation that can make trees more balanced while keeping their binary search tree properties. This operation, applied on an edge $x \to y$ of the tree, is illustrated at the top of Fig. 12: the subtrees of x and y are rearranged so as to decrease the height of the subtree of x containing y, and increase the height of the other subtree, and y becomes the parent of x instead of its child. The rotation is a **left rotation** if y is the right child of x (like in Fig. 12) and a **right rotation** otherwise. The **Tamari lattice**, introduced by D. Tamari in [Tam62] (initially on bracketings of $n+1$ elements), is the order defined on $\mathcal{BT}_n$ by $T_1 \leqslant T_2$ if one can go from T_1 to T_2 by a sequence of left rotations. Its Hasse diagram on $\mathcal{BT}_3$ and $\mathcal{BT}_4$ is given at the bottom of Fig. 12, and as its name indicates, it was proven to be a lattice [HT72].

This Hasse diagram on $\mathcal{BT}_n$ is the skeleton of an $(n-1)$-dimensional polytope, the **associahedron**: this object was theorized separately by D. Tamari and J. D. Stasheff ([Sta63]), then realized as a convex polytope in many different ways. An overview of the history of these realizations can be found in the introduction of [CSZ14] or in [PSZ23]; the 3-dimensional associahedron (representing the Tamari lattice on $\mathcal{BT}_4$) is given in Fig. 13. The combinatorial structure of this polytope has yielded many interesting questions: for example, its diameter was bounded by D. Sleator, R. Tarjan and W. Thurston in [STT88] in 1988, but this bound was only proved to be tight (in dimension $\geqslant 11$) in 2014 by L. Pournin in [Pou14].

From the weak order to the Tamari lattice. The covers of the weak order and the Tamari lattice both consist of small local changes, the exchange of two consecutive values in the first case and the left rotation in the second case. We will show that these two operations are actually closely linked. For each binary tree with n nodes, there is exactly one way to label all the nodes from 1 to n that gives a binary search tree. This labeling corresponds to the order in which the nodes are visited during the in-order traversal of the tree, defined recursively by doing an in-order traversal of the left subtree, then visiting the node, then doing an in-order traversal of the right subtree. For example, this labeling of the tree in Fig. 11 is given in Fig. 14. A **linear extension** of a binary search tree T is then a permutation such that if the node labeled with j is a descendant of the node labeled i in T, then $\omega^{-1}(j) > \omega^{-1}(i)$ (i.e j appears after i in ω). For example, both 513247698 and 579861342 are linear extensions of the tree in Fig. 14.

In general, a permutation ω is a linear extension of exactly one binary search tree, and this tree can be created by inserting each number in $[n]$ into a binary search tree in the order given by ω: first $\omega(1)$ is at the root, then $\omega(2)$ is the left or right child of the root (left if $\omega(2) < \omega(1)$ and right otherwise)... For example, the insertion of 321564 is given in Fig. 15. This defines a surjective (but not injective) **insertion map** from $\mathfrak{S}_n$ to $\mathcal{BT}_n$. This map shows a deep link between the weak

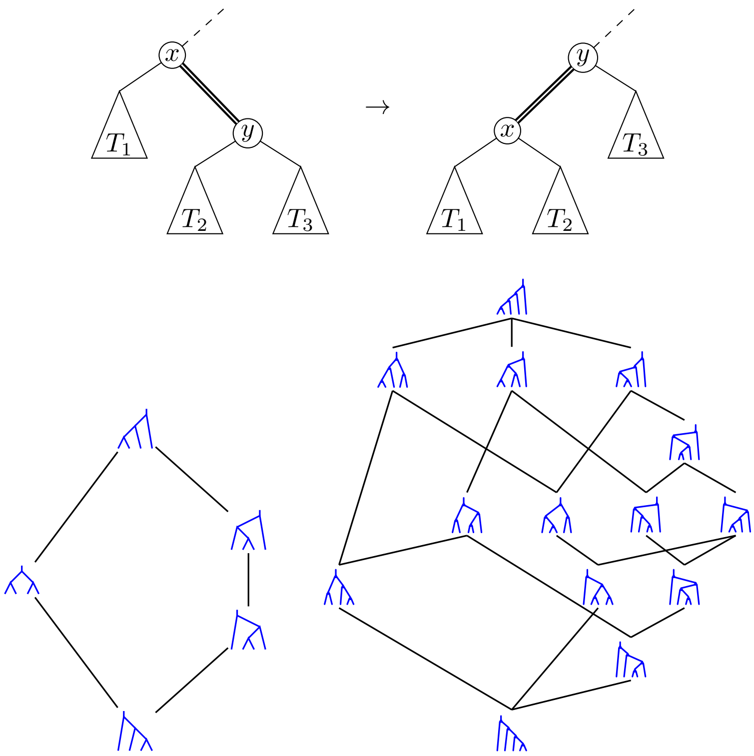


Figure 12: The left rotation order on trees with 3 and 4 nodes.

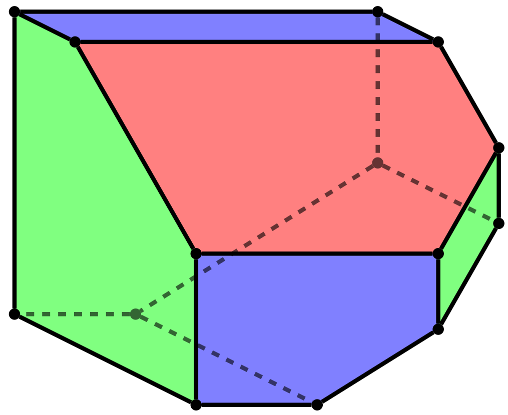


Figure 13: The 3-dimensional associahedron.

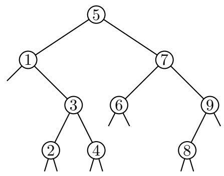


Figure 14: The binary search tree labeling of a binary tree.

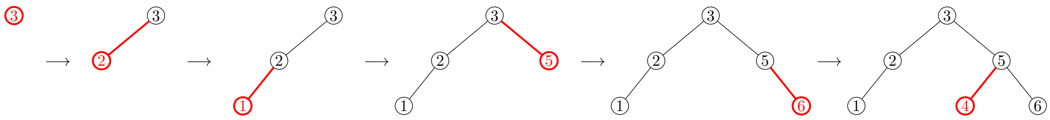


Figure 15: Inserting 321564 into a binary search tree.

order and the Tamari lattice: a cover of the weak order $UijV < UjiV$ is either inside a fiber of the insertion map or its image is a left rotation. The first case happens when j is not a child of i, i.e there exists $k \in U$ such that $i < k < j$ (so i is in the left subtree of k and j in its right subtree), and the second one when j is the right child of i, and then the rotation happens on the edge $i \to j$. F. Hivert, J.-C. Novelli and J.-Y. Thibon proved in [HNT05] that the insertion map is an **order morphism** from the weak order to the Tamari lattice.

The **sylvester congruence** is the equivalence relation where two permutations are equivalent if they are linear extensions of the same binary search tree. It can also be defined as the reflexive and transitive closure of the relations $UijV \equiv UjiV$ if U contains some letter k such that $i < k < j$. This means that a permutation is **minimal** in its congruence class if it avoids the pattern 231 (i.e there is no $i < k < j$ such that $\omega^{-1}(k) < \omega^{-1}(j) < \omega^{-1}(i)$) and **maximal** if it avoids the pattern 213; the characteristics of **lattice congruences** thus tell us that the Tamari lattice is isomorphic to the restriction of the weak order to these 231-avoiding (or 213-avoiding) permutations. The sylvester congruence on $\mathfrak{S}_3$ and $\mathfrak{S}_4$ is given in Fig. 16. As we will detail in Section 2.2.1, this congruence and an equivalent of the insertion map can be defined using pseudoline arrangements on the sorting network of the bubble sort (see Fig. 9).

The existence of such a link between weak order and Tamari lattice was known before the algorithmic approach of the insertion into binary search trees was introduced: it had been proved by L. Billera and B. Sturmfels in [BS94] and by A. Tonks in [Ton97] by studying the link between a specific realization of the associahedron and the permutohedron, by J.-L. Loday and M. Ronco in [LR98] by embedding the Hopf algebra of planar binary trees as a sub Hopf algebra of the Malvenuto-Reutenauer Hopf algebra of permutations, or by N. Reading in [Rea05] by using lattice theory tools. This diversity of possible points of view is central to algebraic

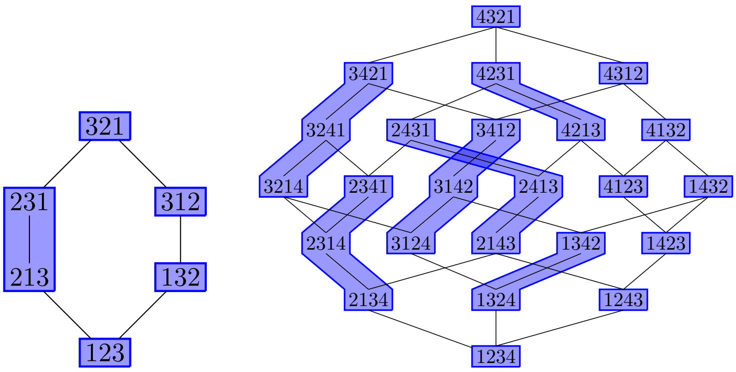


Figure 16: The sylvester congruence on $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

combinatorics and has been used to suggest several generalization of the Tamari lattice, like Cambrian lattices (defined by N. Reading in [Rea06]) or ν-Tamari lattices (defined by L.-F. Préville-Ratelle and X. Viennot in [PRV15]). The aim of this thesis will be to find a context as general as possible in which this multi-approach study still yields interesting results. The algebraic framework we found best suited for this is the subword complexes on finite Coxeter groups.

Coxeter groups and subword complexes. We said previously that the covers of the weak order are the relations $UijV < UjiV$; another way to describe them are the relations $\omega < \omega\tau_k$ with τ_k the simple tranposition $(k, k+1)$ and for $\omega(k) < \omega(k+1)$. This means that the weak order on permutations is deeply linked to the structure of the symmetric group as the group generated by the simple transpositions $(\tau_k)_{1\leqslant k<n}$. The **finite Coxeter groups** are a family of groups introduced by H. S. M. Coxeter in [Cox34, Cox35] generalizing this: they are generated by a family of involutions, the **simple reflections**, and are entirely described by the relations between these reflections. They can also be described as certain subgroups of isometries on Euclidean spaces generated by orthogonal reflections. The interplay of these two equivalent definitions gives rise to a very rich field of studies, with algebraic arguments proving geometric results and vice-versa. For example, N. Reading in [Rea16a] uses a geometric realization of the weak order to prove many of its properties. A description and study of Coxeter groups starting from the algebraic point of view can be found in [BB05], while [Hum90] does similar work starting from a geometric definition.

Rather than binary trees, the realization of the Tamari lattice that has a simply defined equivalent in Coxeter groups is the one using pseudoline arrangements. This equivalent is **subword complexes**, defined on words on the alphabet of simple reflections of a Coxeter group, and representing the **reduced subwords** of these words with a fixed product. They were first defined by A. Knutson and E. Miller in [KM04] as a generalization of **pipe dreams**, a family of combinatorial objects introduced in [KM05] to expand the Schubert polynomials in the usual monomial basis. Recall that Schubert polynomials are a basis of multivariate polynomials suited to the study of their partial symmetries. They first appeared in algebraic geometry and are deeply connected with the theory of symmetric functions, for example the Littlewood-Richardson rule [LS85, Mac91].

A. Woo proved in [Woo04] that some subword complexes were isomorphic to the Tamari lattice; this was extended by C. Stump in [Stu11] to multitriangulations (as triangulations provide a realization of the Tamari lattice) and by V. Pilaud and M. Pocchiola in [PP12] to pseudotriangulations and multipseudotriangulations. The set of k-triangulations of an n-gon has been conjectured to be isomorphic to the boundary complex of a simplicial $(k(n-2k-1)-1)$-dimensional polytope; in an unsuccessful attempt to prove this conjecture, V. Pilaud and F. Santos defined the **brick polytope** of a sorting network in [PS12]. However, this polytope was found to be a realization of the generalized associahedra and its definition was extended to a much wider family of subword complexes on all finite Coxeter groups by V. Pilaud and C. Stump in [PS15a], and finally extended to any subword complex by D. Jahn and C. Stump in [JS21]. V. Pilaud and F. Santos proved in [PS12, Pil18a] that brick polytopes realized the Cambrian lattices; in fact, they were a translation of the generalized associahedra of C. Hohlweg and C. Lange introduced in [HL07].

This thesis will further study the links between weak orders and brick polyhedra of subword complexes, and in particular try to answer the following question: when is the brick polyhedron of a subword complexe a lattice quotient of a weak order interval? We will define a map from this interval to vertices of the brick polyhedron using both a geometric and an algorithmic approach, and study its fibers from the point of view of lattice theory.

Content

Chapter 1 will introduce the mathematical objects that we will use in this thesis. An informed reader may skip all or parts of this chapter and refer to it as the results are cited later in the text. We will start by defining **posets** and **lattices**, describe some of their properties, and give some of the basic tools used to work with them. In particular, we will define and characterize **lattice congruences** and **lattice quotients**, two notions that will be central in later parts. We will then apply these tools to the **poset of regions of a hyperplane arrangement**, a geometric object with an interesting combinatorial structure. We will then define **Coxeter groups** from both a geometric and an algebraic point of view and present some of their properties, before focusing on a partial order defined on their elements, the **weak order**. This order has several equivalent definitions, some algebraic and some geometric, and we will see how both aspects of Coxeter groups come together to give a full picture of its structure. Finally, we will formally define **subword complexes** and give a survey of the vocabulary and results we will use to work on them. In particular, we will focus on the **brick polyhedron** and **acyclic facets** of a subword complex, as they will later prove to be central to the objects we will study.

Our first original results are presented in Chapter 2. In this chapter, we will work on **triangular pipe dreams**, a graphical representation of subword complexes defined on the sorting network of the bubble sort drawn on the left of Fig. 9. Some of these pipe dreams were proved to be a realization of the Tamari lattice, which is a lattice quotient of the weak order; our goal will be to extend this result as much as possible. We will first prove with Theorem 2.2.12 that for any subword complex represented by triangular pipe dreams, the linear extensions of acyclic facets define a lattice congruence of a weak order interval, the **pipe dream congruence**. We will also define an **insertion map** from this interval to acyclic pipe dreams and characterize the image of the weak order in terms of **flips on acyclic facets** with Theorem 2.2.17, as well as give **two algorithms** efficiently computing the values of this map and an alternative definition of the pipe dream congruences only using the reflexive and transitive closure of a **relation** on permutations in Section 2.2.5. Finally, we will prove with Proposition 2.3.1 that the image of weak order by the insertion map is exactly the **skeleton of the brick polyhedron** of the associated subword complex, and apply this result to find a polyhedral realization of ν-Tamari lattices in Corollary 2.3.7.

We will then extend these results to **alternating pipe dreams**, a wider class of pipe dreams on non-triangular shapes. They are a graphical representation of subword complexes on alternating words of the symmetric group inspired by a realization of Cambrian lattices. We will first prove with Theorem 3.2.6 that the **pipe dream congruence** defined by linear extension of acyclic pipe dreams is still a **lattice congruence** of a weak order interval. However, the **insertion map** from this interval to the acyclic facets is no longer always surjective, and the image of the weak order is harder to define: we will give a less satisfying characterization of this image in Proposition 3.2.11 than the one obtained in Chapter 2. We will also prove in Section 3.2.5 that the **two algorithms** defined on triangular pipe dreams in Section 2.2.5 still work on alternating pipe dreams. Finally, we will show with Proposition 3.3.1 that the image of the weak order by the insertion map is **part of the skeleton of the brick polytope**, and give a condition for it to be the entirety of this skeleton with Theorem 3.3.4. This final theorem gives a much more satisfying characterization of the image of the weak order.

We end this thesis in Chapter 4 by a discussion of a possible generalization of the previous results to any finite Coxeter group. We start by proving that the **subword congruence** and the **insertion map** are still well-defined by linear extensions of acyclic facets, and give an example in which the subword congruence is **not** a lattice congruence. We use the work we did previously on type A Coxeter groups to suggest **several conjectures** generalizing the results of Chapter 3. In particular, we suggest with Conjecture 4.2.1 and 4.2.3 to 4.2.5 that the subword congruence is a **lattice**

congruence if the word on which the subword complex is defined is **alternating**, and that in this case the image of the weak order by the insertion map is still part of the skeleton of the associated brick polyhedron. These conjectures were tested with numerous computer experiments in various Coxeter groups. Moreover, we use ideas from Chapter 3 to suggest a possible proof for them, starting from the conjectured Lemma 4.2.7. We conclude our work by discussing two concepts defined in type A Coxeter group that seem to be related to our work, **chute moves** and **ν-Tamari lattices**, and the way this link could be used to extend their definition to others finite Coxeter groups.

Chapter 1

Background

This chapter will introduce the various mathematical notions that we will use later. None of the results exposed here come from the author of this thesis; some of the objects are classical notions widely studied for decades, while other were introduced recently.

We will start in Section 1.1 by defining partially ordered sets (Section 1.1.1) and a particular family of them, the lattices (Section 1.1.2). We will define and characterize lattice congruences in Section 1.1.3 and focus in particular on polygonal lattices in Section 1.1.4.

Then in Section 1.2 we will introduce a few geometrical objects: cones in Section 1.2.1, and from them cone fans and hyperplane arrangements in Section 1.2.2. In 1.2.3 we will define the poset of regions of a hyperplane arrangement and give some sufficient conditions for it to be a lattice. We will end this section by giving some tools to study the strucure of this poset.

Next, in Section 1.3, we will give two definitions of the same family of algebraic objects, finite Coxeter groups: one geometric with reflection groups in Section 1.3.1 and one using presentations by Coxeter matrices in 1.3.3. We will also define root systems, that are finite sets of vectors encoding the structure of a Coxeter group, in Section 1.3.2, and finish with a list of all the finite irreducible Coxeter groups in Section 1.3.4.

Section 1.4 will define the weak order on Coxeter groups, using objects introduced in each previous sections. We will first introduce the algebraic notions of length and reduced words in these groups in Section 1.4.1, then see their links with roots in Section 1.4.2, then use this to give two equivalent definitions of the weak order in Section 1.4.3. We will then see in Section 1.4.4 that this order can also be defined as the poset of regions of a specific hyperplane arrangements, and use the properties presented in Section 1.2 to describe the properties of the weak order lattice.

Finally, in Section 1.5 we will introduce the main subject of our works: subword complexes on Coxeter groups. We will start by defining them in Section 1.5.1 as a simplicial complex encoding the subwords of a longer word that have a given product. In Section 1.5.2, we will characterize the nonempty subword complexes by introducing the Bruhat order on Coxeter groups. Then in Section 1.5.3 we will define the root function, defined on indices of the longer word and very useful for studying facets of the subword complex, and in Section 1.5.4 and Section 1.5.5 we will define and study flips, an operation linking some pairs of facets of a subword complex. Finally, in Section 1.5.6, we will define the brick polyhydron of a subword complex and give some of its characteristics.

1.1 Lattices

Partial orders on sets are relations used in every branches of mathematics to compare all kinds of objects. In the first two subsections, we will start with some generalities on posets and lattices; for further details, see for example [DP02]. In the last two subsections we will focus on lattice congruences in general cases and in one specific type of lattice; these subsections come from [Rea16b] and the details of the proofs can be found there.

1.1.1 Generalities on ordered sets

A partial order on a set is a binary relation with some conditions.

Definition 1.1.1. *Let E be a set and $\leqslant$ be a binary relation on the elements of E. We say that $\leqslant$ is a **partial order** on E if it is:*
- ***reflexive:** for all $x \in E$, $x \leqslant x$;*
- ***transitive:** for all $x, y, z \in E$, if $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$;*
- ***antisymmetric:** for all $x, y \in E$, if $x \leqslant y$ and $y \leqslant x$ then $x = y$.*

*In that case, we say that $(E, \leqslant)$ is a **poset** (partially ordered set).*

Remark 1.1.2. If for any $x, y \in E$, $x \leqslant y$ or $y \leqslant x$, then $\leqslant$ is a **total order** on E.

Example 1.1.1. Examples of partial or total orders can be found in every area of mathematics.
- The usual order $\leqslant$ is a total order on $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$.
- For any directed acyclic graph G, the relation on its vertices given by $x \to^* y$ if there is a directed path from x to y is a partial order.
- For any finite set X, the **boolean order** on the subsets of X is $(\mathcal{P}(X), \subseteq)$.

From now on in this section, we consider $(E, \leqslant)$ an ordered set.

Definition 1.1.3. *Let x, y be elements of E, the **interval** $[x, y]$ is the set of all elements $z \in E$ with $x \leqslant z \leqslant y$.*

Definition 1.1.4. *Let $(E, \leqslant)$ be a partially ordered set and $F \subseteq E$. The partial order **induced** on F by $\leqslant$ is the restriction of $\leqslant$ to F.*

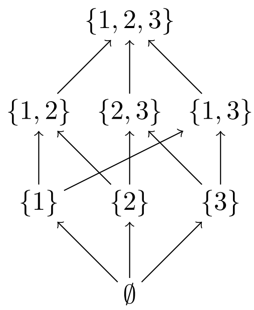
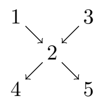


Figure 1.1: The Hasse diagram of the boolean lattice of size 3.

Definition 1.1.5. *Let $(E, \leqslant)$ be a finite partially ordered set with cardinality $n > 0$. A* ***linear extension*** *of E is a bijective map $\phi : [n] \mapsto E$ such that if $\phi(i) \leqslant \phi(j)$, then $i \leqslant j$. For G a directed acyclic graph, a linear extension of G is a linear extension of the order $\to^*$ on the vertices of G.*

Example 1.1.2. The permutation 31254 is a linear extension of this graph.



Definition 1.1.6. *Let $(E, \leqslant)$ be a partially ordered set. For any elements $x, y \in E$, we say that y* ***covers*** *x, denoted by $x \lessdot y$, if $x < y$ and there is no $z \in E$ such that $x < z < y$.*

Example 1.1.3.
- The **cover relations** of $(\mathbb{Z}, \leqslant)$ are all the relations $n \lessdot n + 1$ for $n \in \mathbb{Z}$.
- There are no cover relations in $(\mathbb{R}, \leqslant)$.

Definition 1.1.7. *Let $(E, \leqslant)$ be a finite partially ordered set. The* ***Hasse diagram*** *of this order is the oriented graph with vertex set E and edge set the cover relations of $(E, \leqslant)$. It it generally drawn with all the edges going upward, and the arrow heads are often omitted.*

Example 1.1.4. The Hasse diagram of the boolean lattice of size 3 is given in Fig. 1.1.

Proposition 1.1.8. *If E is finite, then the order $\leqslant$ on E is given by the transitive closure of the covers of $\leqslant$.*

Remark 1.1.9. The order $\leqslant$ on E is given by the order $\to^*$ on the vertices of its Hasse diagram.

Definition 1.1.10. *For $X \subseteq E$, a* ***maximal*** *(resp.* ***minimal****) element of this set is $x \in X$ such that there is no $y \in X$ with $y > x$ (resp. $y < x$). The* ***greatest element*** *or* ***maximum*** *of X (resp.* ***least element*** *or* ***minimum****) is $x \in X$ such that for all $y \in X$, $y \leqslant x$ (resp. $x \leqslant y$).*

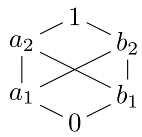


Figure 1.2: A poset that is not a lattice.

Note that while a finite set always has at least one maximal (and minimal) element, it can have no maximum (or minimum).

Definition 1.1.11. *For $X \subseteq E$, we say that X is* ***order convex*** *if for any elements $x \leqslant y$ of X, the interval $[x, y]$ is contained in X. We say that X is a* ***lowerset*** *of E if for any $x \in X$ and $y \in E$, if $y \leqslant x$ then $y \in X$.*

1.1.2 Lattices

Lattices are a subfamily of posets with two binary operations well-defined on them.

Definition 1.1.12. *For $x, y \in E$, the* ***join*** *of x and y, when it exists, is $x \vee y \in E$ the minimum of all elements of E greater than both x and y. Similarly, the* ***meet*** *of x and y, when it exists, is $x \wedge y \in E$ the maximum of all elements of E lower than both x and y. A* ***lattice*** *$(E, \leqslant)$ is a partially ordered set such that all pairs of elements have both a join and a meet.*

Example 1.1.5.

- The boolean order on a set A is a lattice, where the join is the union and the meet is the intersection.
- The divisibility order on $\mathbb{N}$ is a lattice, with the join of two integer being their least common multiple and their meet being their greatest common divisor.

Counterexample 1.1.6. The order defined by the Hasse diagram in Fig. 1.2 is not a lattice: both a_1 and b_1 are smaller than both a_2 and b_2. This means that $a_1 \vee b_1$ and $a_2 \wedge b_2$ are not well-defined.

An alternative, equivalent definition of a lattice consider the meet and join as algebraic operations.

Definition 1.1.13. *A set E with two binary operations $\wedge$ and $\vee$ is a* ***lattice*** *if for all $x, y, z \in E$, they satisfy:*

- *for $* \in \{\vee, \wedge\}$, $x * y = y * x$ (commutative operations);*
- *for $* \in \{\vee, \wedge\}$, $x * (y * z) = (x * y) * z$ (associative operations);*
- *$x \vee (x \wedge y) = x$;*
- *$x \wedge (x \vee y) = x$.*

In that case, the order relation on E that defines a lattice in the sense of Definition 1.1.12 is given by $a \leqslant b$ iff $a \wedge b = a$, or equivalently $a \leqslant b$ iff $a \vee b = b$.

Remark 1.1.14.
- A finite lattice always has a minimum and a maximum.
- If F is an interval of a lattice $(E, \leqslant)$, then the order induced by $\leqslant$ on F is also a lattice.

Finally, the following lemma shows that it is enough for a poset with minimal and maximal elements to satisfy locally the conditions of a lattice, to prove that this poset is a lattice.

Lemma 1.1.15 ([BEZ90, Lemma 2.1])**.** *Suppose that $(E, \leqslant)$ is a finite poset with a minimal and maximal elements. If for all $x, y \in E$ covering a common element, the join $x \vee y$ exists, then E is a lattice.*

1.1.3 Lattice congruences

We will now introduce some equivalence relations that respect the meet and join operations. These relations are useful to create new lattices from a known lattice.

Definition 1.1.16. *Let $\equiv$ be a binary relation on the elements of E. We say that it is an **equivalence relation** if it is:*
- ***reflexive:** for all $x \in E$, $x \equiv x$;*
- ***symmetric:** for all $x, y \in E$, if $x \equiv y$ then $y \equiv x$;*
- ***transitive:** for all $x, y, z \in E$, if $x \equiv y$ and $y \equiv z$ then $x \equiv z$.*

*An equivalence relation on E defines a partition of E into its **equivalence classes**, i.e. the maximal subsets of E such that the elements in each class are all equivalent. We denote by $[x]_\equiv$ the equivalence class of $\equiv$ containing $x \in E$.*

Definition 1.1.17. *An equivalence relation on E is a **lattice congruence** if it is compatible with the meet and join operations, i.e. if $x, x', y, y' \in E$ are such that $x \equiv x'$ and $y \equiv y'$, then $x \wedge y \equiv x' \wedge y'$ and $x \vee y \equiv x' \vee y'$.*

Theorem 1.1.18 ([Rea16b, Theorem 9-5.2])**.** *An equivalence relation $\equiv$ on E is a lattice congruence if and only if:*
1. *every equivalence class of $\equiv$ is an interval;*
2. *the projections $p^\uparrow : E \mapsto E$ and $p^\downarrow : E \mapsto E$, respectively mapping an element to the maximum and minimum of its equivalence class, are order preserving.*

Definition 1.1.19. *For $\equiv$ a lattice congruence on E, we denote by $E/\equiv$ the **lattice quotient** of E by $\equiv$, whose elements are the equivalence classes of $\equiv$ and whose meet and join are given by $[x]_\equiv \vee [y]_\equiv = [x \vee y]_\equiv$ and $[x]_\equiv \wedge [y]_\equiv = [x \wedge y]_\equiv$ (the definition of a lattice congruence guarantees that this is well-defined). The order is then given by $[x]_\equiv \leqslant [y]_\equiv \iff \exists x' \equiv x, y' \equiv y$ such that $x' \leqslant y'$.*

An example of a lattice congruence and the associated lattice quotient it defines is given in Fig. 1.3. The equivalence classes are drawn as blue boxes.

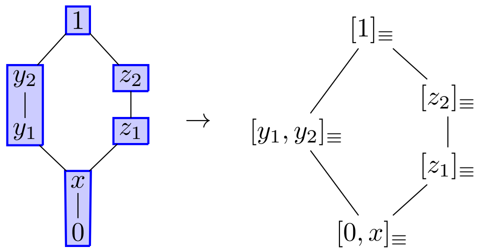


Figure 1.3: A lattice congruence and the associated lattice quotient.

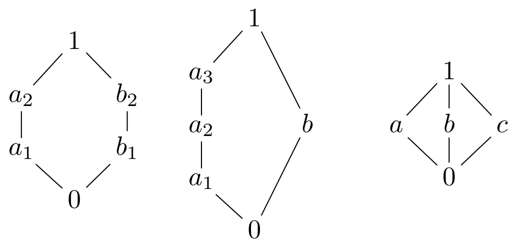


Figure 1.4: Two polygones and one non-polygonal lattice.

A consequence of Theorem 1.1.18 is that the lattice quotient defined by $\equiv$ is isomorphic to the order on E restricted to the top (or bottom) elements of its equivalence classes. However, note that this suborder is not necessarily a sublattice of E, in the sense that if x and y are minimal (or maximal) elements of their respective equivalence classes, then $x \wedge y$ and $x \vee y$ are not necessarily the same. For example, in Fig. 1.3, the elements y_1 and z_1 are the minimum of their classes, but $x = y_1 \wedge z_1$ is not, so $\min([y_1 \wedge z_1]_\equiv) \neq \min([y_1]_\equiv) \wedge \min([z_1]_\equiv)$.

1.1.4 Polygonal lattices

Finally, we will focus on a family of lattices with some strong structural conditions, introduced by N. Reading in Section 9.6 of [Rea16b]: the polygonal lattices.

Definition 1.1.20. *A **polygon** in a lattice is an interval $[x, y]$ that is the union of two maximal chains from x to y, with these chains disjoint except at x and y. A lattice L is **polygonal** if the following two conditions hold:*

- *if two distinct elements y_1 and y_2 both cover an element x, then $[x, y_1 \vee y_2]$ is a polygon;*
- *if two distinct elements y_1 and y_2 are covered by an element x, then $[y_1 \wedge y_2, x]$ is a polygon.*

Intuitively, the polygonal lattices are lattices with as many polygons as possible.

Example 1.1.7. Of the three lattices represented in Fig. 1.4, the first two are polygon (and thus polygonal). The last one is not polygonal: both a and b cover 0, but the interval $[0, a \vee b] = [0, 1]$ also contains c and is thus not a polygon. For a bigger example, the lattice in Fig. 1.1 is also polygonal.

We note that if a lattice is polygonal, then the lattice induced by any of its intervals is also polygonal.

If the interval $[x, y]$ is a polygon, the covers of the order are partitioned by which of the two maximal chains they belong to. Moreover, the polygon has two **bottom edges** corresponding to the two elements of the interval covering x (in Fig. 1.4, those are the covers $0 \lessdot a_1$ and $0 \lessdot b_1$ in the first polygon and the covers $0 \lessdot a_1$ and $0 \lessdot b$ in the second one), and two **top edges** corresponding to the two elements of the interval covered by y (in Fig. 1.4, those are the covers $a_2 \lessdot 1$ and $b_2 \lessdot 1$ in the first polygon and the covers $a_3 \lessdot 1$ and $b \lessdot 1$ in the second one). The **side edges** are all the remaining covers. An edge can be in the same or opposite maximal chain as another edge.

By considering the join and meet of various pairs, it is clear that the structure of a polygon induces certain conditions on lattice congruences.

Proposition 1.1.21. *Let L be a lattice and $[x, y]$ an interval of L that is a polygon. For $\equiv$ a lattice congruence of L, the following implications hold:*

1. *if $x \lessdot a$ is a **bottom edge** of the polygon and $b \lessdot y$ is the **opposite top edge**, then $x \equiv a \iff b \equiv y$;*
2. *if $x \lessdot a$ is a **bottom edge** and $c \lessdot d$ is a **side edge**, then $x \equiv a \Rightarrow c \equiv d$;*
3. *if $b \lessdot y$ is a **top edge** and $c \lessdot d$ is a **side edge**, then $b \equiv y \Rightarrow c \equiv d$.*

The following theorem shows that for polygonal lattices, these conditions characterize completely the lattice congruences.

Theorem 1.1.22 ([Rea16b, Theorem 9-6.5])**.** *If L is a finite polygonal lattice and $\equiv$ is an equivalence relation on L such that the conditions of Proposition 1.1.21 are respected in all polygons of L, then $\equiv$ is a lattice congruence.*

1.2 Poset of regions of a hyperplane arrangement

The following section introduces the geometric objects that we will use all along this thesis, with a focus on hyperplane arrangements and the properties of the poset of regions. The definitions and theorems, along with the complete proofs, are developped in [BEZ90] and [Rea16b].

In all that follows, let us consider $V = \mathbb{R}^n$ for some $n \geqslant 1$ and $(\cdot, \cdot)$ the classical scalar product on $\mathbb{R}^n$ (with $(x, y) = \sum_{1 \leqslant i \leqslant n} x_i y_i$). We note that we could also choose any finite-dimensional Euclidean space for V and do the same work.

1.2.1 Polyhedral cones

Let us start by defining cones and giving a few of their properties.

Definition 1.2.1. *For u a nonzero vector of V, the* ***hyperplane*** *normal to u is the linear subspace $H_u = \{v \in V \mid (u, v) = 0\}$ with dimension $n-1$. It delimitates two* ***(closed) halfspaces*** *$H_u^+ = \{v \mid (u, v) \geqslant 0\}$ and $H_u^- = \{v \mid (u, v) \leqslant 0\}$ that are the closures of the connected components of $V \backslash H_u$. Those connected components are the two* ***open halfspaces*** *defined by H_u and are obtained in the same way by replacing the weak inequalities by straight ones.*

In general, hyperplanes are the subspaces of V with dimension $n-1$, and they can all be defined as the hyperplane normal to a nonzero vector (and since $H_{\lambda u} = H_u$ for any $\lambda \neq 0$, there is an infinity of vectors such that $H = H_u$).

Definition 1.2.2. *For K a subset of V, a hyperplane H* ***supports*** *K if $K \cap H \neq \varnothing$ and K is contained in one of the closed halfspaces delimitated by H. A hyperplane* ***separates*** *K and another subset L of V if $K \subseteq H^+$ and $L \subseteq H^-$ or vice versa.*

Definition 1.2.3. *A* ***cone*** *C of V is a subset of V closed under addition and positive scalar multiplication, i.e. $\alpha x + \beta y \in C$ for all $x, y \in C$ and $\alpha, \beta \in \mathbb{R}_{\geqslant 0}$. For any set of vectors $X \subset V$, the* ***conical hull*** *of X, denoted $\mathrm{Cone}(X)$, is the smallest cone containing X.*

Example 1.2.1. Any linear subspace of V is a cone, and for $u \in V \backslash \{0\}$, the half-spaces H_u^+ and H_u^- are also cones.

Since the intersection of any family of cones is still a cone, the conical hull of a set can be defined as the intersection of all cones containing this set. The following proposition gives an explicit description of the conical hull.

Proposition 1.2.4. *For any subset X of V,*

$$\mathrm{Cone}(X) = \{\sum_{i=1}^{k} \lambda_i x_i \mid k \in \mathbb{N},\ \lambda_i \geqslant 0,\ x_i \in X\}.$$

With these definition, we will introduce a specific family of cones which will be especially useful to us in all that follows.

Definition 1.2.5. *A cone is* ***polyhedral*** *if one of the two following equivalent statements is true:*

- *it is the conical hull of a finite set of vectors;*
- *it is the intersection of finitely many closed halfspaces.*

Remark 1.2.6. For $\dim(V) \leqslant 2$, any closed cone is a polyhedral cone; if $\dim(V) \geqslant 3$, many closed cones in V are not polyhedral.

The equivalence of the two definitions is nontrivial and similar to the two equivalent definitions of polytopes as the convex hull of finitely many points, or as the bounded intersection of finitely many closed affine halfspaces. From now on, we will only consider polyhedral cones, and so any cone described will be polyhedral unless explicitely specified.

Definition 1.2.7. *The* ***dimension*** $\dim(C)$ *of a cone* C *is the dimension of the smallest linear subspace containing it. The cone is* ***fully dimensional*** *if its dimension is the dimension of the vector space that contains it. A* ***face*** *of* C *is the intersection of* C *with one of its supporting hyperplanes. A* ***facet*** *of* C *is a face with dimension* $\dim(C) - 1$*. A* ***ray*** *is a face with dimension* 1*.*

We note that any face of a cone is also a cone.

In general, we will be more interested in cones that do not contain any full line, as a cone containing a full line is the vector sum of this line and a smaller-dimensional cone.

Definition 1.2.8. *A closed cone* C *is* ***pointed*** *if it contains no line, or equivalently if there exists a hyperplane* H *separating* C *from* $-C$ *and with* $(C \cup -C) \cap H = \{0\}$*.*

Of particular interest are the polyhedral pointed cones with a minimal number of facets.

Definition 1.2.9. *A full-dimensional cone* C *is* ***simplicial*** *if one of the three equivalent statements is true:*

1. *it is pointed and has exactly* n *facets;*
2. $C = \mathrm{Cone}(b_1, \dots, b_n)$ *for* (b_i) *a basis of* V*;*
3. $C = \cap_{1 \leqslant i \leqslant n} H^+_{b'_i}$ *for* (b'_i) *a basis of* V*.*

In general, if $\dim(C) < n$*, then* C *is simplicial if it is simplicial as a full-dimensional cone in* $\mathrm{Vect}(C)$*. In that case, each face of* C *has a unique expression as an intersection of facets of* C*, or as a cone generated by a subset of the rays of* C*, and* C *has exactly* $\binom{\dim(C)}{k}$ *faces with dimension* k*.*

A simplicial cone is then the conical hull of its rays and the intersection of the halfspaces defined by its facets. Note that all simplicial cones are pointed but many pointed polyhedral cones are not simplicial.

Example 1.2.2. The cone $C = \{(x, y) \mid 0 \leqslant y \leqslant x\} \subset R^2$ drawn in Fig. 1.5 is polyhedral: it can be defined as $\mathrm{Cone}((1, 0), (1, 1))$ or as $H^+_{(1,0)} \cap H^+_{(-1,1)}$. It has dimension 2, and its faces are itself, the halflines $\mathbb{R}_{\geqslant 0}(1, 0)$ and $\mathbb{R}_{\geqslant 0}(1, 1)$ and the point $\{0\}$. The two halflines are both its rays and its facets, and so it is simplicial.

In this example, it is obvious that $\mathrm{Cone}(X)$ is not equal to $\bigcap_{u \in X} H^+_u$, since the intersection $H^+_{(1,0)} \cap H^+_{(1,1)}$ is $\mathrm{Cone}((0, 1), (1, -1))$. These two cones are however considered dual of each other.

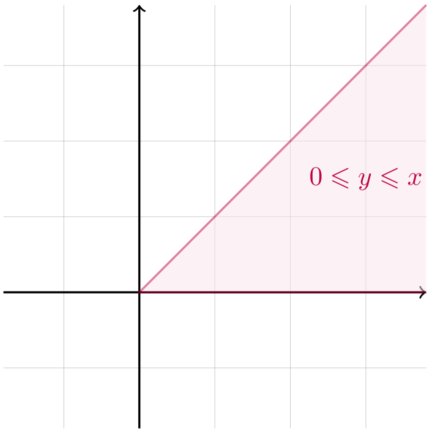


Figure 1.5: A polyhedral cone.

Definition 1.2.10. *For C a cone, its **dual cone** is the set*

$$C^* := \{v \in V \mid \forall u \in C, (u, v) \geqslant 0\}.$$

The hyperplane separation theorem implies that C^{**} is the closure of C.

Lemma 1.2.11. *For X a finite set of vectors:*
- $\mathrm{Cone}(X)^* = \bigcap_{u \in X} H_u^+$*;*
- $(\bigcap_{u \in X} H_u^+)^* = \mathrm{Cone}(X)$.

1.2.2 Fans and hyperplane arrangements

We will now define fans as special collections of polyhedral cones, and hyperplane arrangements as special fans.

Definition 1.2.12. *A **fan** in V is a collection $\mathcal{F}$ of cones of V such that:*
- *if C is in $\mathcal{F}$, any face of C is in $\mathcal{F}$;*
- *if C and D are in $\mathcal{F}$, then $C \cap D$ is a face of C and a face of D.*

*A fan is **complete** if the union of its cones is V and **simplicial** if all its cones are simplicial.*

Example 1.2.3. An example of a complete simplicial fan of $\mathbb{R}^2$ is given in Fig. 1.6: the fan contains
- the 0-dimensional cone $\{0\}$;
- the 1-dimensional cones $\mathbb{R}_{\geqslant 0}u$, $\mathbb{R}_{\geqslant 0}v$ and $\mathbb{R}_{\geqslant 0}w$;
- the 2-dimensional cones $\mathrm{Cone}(u, v)$, $\mathrm{Cone}(v, w)$ and $\mathrm{Cone}(u, w)$.

A simple way of defining a fan is by using hyperplane arrangements.

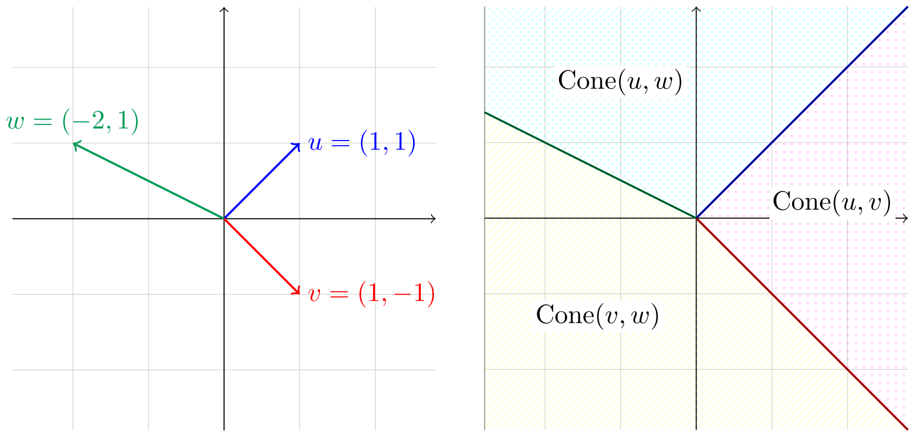


Figure 1.6: A two-dimensional fan.

Definition 1.2.13. *A* ***hyperplane arrangement*** $\mathcal{A}$ *of* V *is a finite collection of hyperplanes. If it is nonempty, its complement* $V \backslash (\cup_{H \in \mathcal{A}} H)$ *is disconnected. The* ***regions*** *of* $\mathcal{A}$*, denoted by* $\mathcal{R}(\mathcal{A})$*, are the closures of the connected components of this complement.*

Since any hyperplane separates the space into two halfspaces, it is easy to define the regions of a hyperplane arrangement as the intersection of a finite set of halfspaces, leading to the following statement.

Proposition 1.2.14. *The regions of a hyperplane arrangement are polyhedral cones, and the intersection of two regions is a face of both.*

In particular, the intersection between two regions A and B of a hyperplane arrangement is the intersection of A (or B) with all the hyperplanes of the arrangement separating A and B. The case were this intersection is $(n-1)$-dimensional is covered by this proposition.

Proposition 1.2.15. *For* Q *a region of* $\mathcal{A}$*, any facet* F *of* Q *is shared by a unique other region* R*, with* $F = Q \cap R$*. In that case, the regions* Q *and* R *are* ***adjacent****.*

N. Reading uses this proposition and a characterization of complete fans to obtain the following result, which explains the link between hyperplane arrangements and fans.

Proposition 1.2.16. *The regions of a hyperplane arrangement are the maximal cones of a complete fan.*

The following lemma gives an interesting property of the graph of adjacent regions.

Lemma 1.2.17. *Given Q and R two regions of $\mathcal{A}$, there exists a sequence of regions $Q = R_0, R_1, \ldots, R_k = R$ such that R_i and R_{i+1} are adjacent for $0 \leqslant i < k$. Moreover, this sequence can be chosen such that while moving from Q to R, no hyperplane of $\mathcal{A}$ is crossed more than once.*

1.2.3 The poset of regions

We will now define an order on the regions of a hyperplane arrangement and discuss some properties of this order.

Definition 1.2.18. *For Q, R regions of $\mathcal{A}$, the* ***separating set*** *of Q and R is the set of hyperplanes separating Q and R.*

We note that the separating set of a region and itself is $\varnothing$, and the separating set of R and $-R$ is $\mathcal{A}$. The cardinality of the separating set between Q and R gives the minimal length of a sequence as described in Lemma 1.2.17. This leads to the definition of a poset on the regions of $\mathcal{A}$, relatively to a base region.

Definition 1.2.19. *Let us consider a* ***base region*** *$B \in \mathcal{R}(\mathcal{A})$, and denote by $\mathcal{S}_B(R)$ the separating set of B and R for any $R \in \mathcal{R}(\mathcal{A})$. The* ***poset of regions*** *$\mathcal{P}(\mathcal{A}, B)$ is the set of regions $\mathcal{R}(\mathcal{A})$ partially ordered with $Q \leqslant R$ iff $\mathcal{S}_B(Q) \subseteq \mathcal{S}_B(R)$.*

This is a valid partial order: reflexivity and transitivity come from the properties of $\subseteq$, and the antisymmetry is easy to obtain by noting that a region R is the intersection of the halfspaces it shares with B (defined by hyperplanes not in $\mathcal{S}_B(R)$), and the halfspaces defined by hyperplanes in $\mathcal{S}_B(R)$ and containing $-B$. In general, the structure of this order depends on our choice of a base region and not just on $\mathcal{A}$.

To find covers of this order, one must find a way to add or remove exactly one separating hyperplane to a region, or exit a region by crossing exactly one hyperplane. Since a k-dimensional face of a region is the intersection of this region with $(n - k)$ hyperplanes of $\mathcal{A}$, crossing exactly one hyperplane means crossing a facet of the region.

Proposition 1.2.20. *The cover relations in $\mathcal{P}(\mathcal{A}, B)$ are $Q \lessdot R$ if and only if Q and R are adjacent and $|\mathcal{S}_B(Q)| < |\mathcal{S}_B(R)|$. In this case $\mathcal{S}_B(R) = \mathcal{S}_B(Q) \cup H$ with $H \in \mathcal{A}$ the hyperplane containing the common facet of Q and R.*

By Proposition 1.2.15, we see that a region Q is involved in exactly one cover relation for each of its facets.

Definition 1.2.21. *For $Q \in \mathcal{R}(\mathcal{A})$, any facet of Q is shared with exactly one region $R \in \mathcal{R}(\mathcal{A})$. The facet is a* ***lower facet*** *regarding the base region B if $R \lessdot Q$ and an* ***upper facet*** *if $Q \lessdot R$. A hyperplane $H \in \mathcal{A}$ is a* ***lower hyperplane*** *of Q regarding B if it defines a lower facet of Q and an* ***upper hyperplane*** *if it defines an upper facet.*

We note that some hyperplanes of $\mathcal{A}$ can be neither lower nor upper hyperplanes of a region, if the face that they define is not a facet.

Some conditions for the poset of regions to be a lattice were first given in [BEZ90].

Theorem 1.2.22 ([BEZ90, Theorems 3.1 and 3.4])**.**
- *If $\mathcal{P}(\mathcal{A}, B)$ is a lattice, then B is a simplicial cone.*
- *If $\mathcal{A}$ is a simplicial arrangement, then $\mathcal{P}(\mathcal{A}, B)$ is a lattice for any $B \in \mathcal{R}(\mathcal{A})$.*

Section 9.3 of [Rea16b] completes this into a characterization of all posets of regions that are lattices. It uses Lemma 1.1.15 and finds a geometric condition for two regions covering a same third one to have a join, and it also proves that in that case, the interval between their meet and their join is polygonal.

Theorem 1.2.23 ([Rea16b, Theorem 9-3.2])**.** *A region $R \in \mathcal{R}(\mathcal{A})$ is tight with respect to B if any two of lower or upper hyperplanes of R regarding B intersect in an $(n-2)$ dimensional face of R. If all the regions of $\mathcal{A}$ are tight with respect to B, then $\mathcal{P}(\mathcal{A}, B)$ is a polygonal lattice.*

Since the intersection of two facets of a simplicial cone is always an $(n-2)$-dimensional face, this leads to the final theorem of this section.

Theorem 1.2.24 ([Rea16b, Corollary 9-3.4])**.** *If all regions of $\mathcal{A}$ are simplicial, then $\mathcal{P}(\mathcal{A}, B)$ is a polygonal lattice for any base region $B \in \mathcal{R}(\mathcal{A})$.*

1.2.4 Separating sets and biconvexity

We will now give a few properties of the separating sets of regions given in section 9-4 of [Rea16b]. We choose $\mathcal{A}$ a hyperplane arrangement and fix $B \in \mathcal{R}(\mathcal{A})$ a base region. We also choose b a nonzero vector in the interior of B (and so $b \notin H$ for each $H \in \mathcal{A}$).

For each hyperplane $H \in \mathcal{A}$, let us define n_H a nonzero vector normal to H and such that $(b, n_H) > 0$. This is possible because b is not in any of the hyperplanes of $\mathcal{A}$ and so b is not normal to any vector normal to one of the hyperplanes.

Remark 1.2.25. Note that the direction of the vector n_H depends on B (but not on the specific vector b, as all vectors inside of B would yield the same result).

Definition 1.2.26. *We define the closure operator $S \mapsto \bar{S}$ on subsets of $\mathcal{A}$ as*

$$\bar{S} = \{H \in \mathcal{A} \mid n_H \in \mathrm{Cone}(\{n_{H'} \mid H' \in S\})\}.$$

*A subset $S \subseteq \mathcal{A}$ is **convex** with respect to B if $\bar{S} = S$, **biconvex** with respect to B if both S and $\mathcal{A} \backslash S$ are convex, and **strongly biconvex** if the intersection of $\mathrm{Cone}(\{n_H \mid H \in S\})$ and $\mathrm{Cone}(\{n_H \mid H \notin S\})$ is $\{0\}$.*

The following proposition comes trivially from these definitions.

Proposition 1.2.27. *For any subset S of $\mathcal{A}$, the following implications hold:*

$$S \text{ is strongly biconvex} \Rightarrow S \text{ is biconvex} \Rightarrow S \text{ is convex.}$$

We note that for any region R and any vector r in the interior of R, a hyperplane $H \in \mathcal{A}$ is in $\mathcal{S}_B(R)$ if and only if $(r, n_H) < 0$ (i.e. r is not in the same halfspace delimitated by H as b), and in $\mathcal{A}\backslash\mathcal{S}_B(R)$ iff $(r, n_H) > 0$. This characterization shows that $\mathcal{S}_B(R)$ must be strongly biconvex, as the scalar product of r with any nonzero positive linear combination of $\{n_H \mid H \in \mathcal{S}_B(R)\}$ must be strictly negative, and strictly positive for $\{n_H \mid H \in \mathcal{A}\backslash\mathcal{S}_B(R)$. The following proposition uses a standard separation theorem to prove the converse implication.

Proposition 1.2.28. *A subset of $\mathcal{A}$ is the separating subset of a region if and only if it is strongly biconvex with respect to B.*

We will now introduce a seemingly much weaker convexity condition.

Definition 1.2.29. *A subset S of $\mathcal{A}$ is* ***rank-two convex*** *with respect to B if for any pair $H_1, H_2 \in S$,*

$$\mathrm{Cone}(n_{H_1}, n_{H_2}) \cap \{n_H \mid H \in \mathcal{A}\} = \mathrm{Vect}(n_{H_1}, n_{H_2}) \cap \{n_H \mid H \in S\}.$$

It is ***rank-two biconvex*** *with respect to B if both S and $\mathcal{A}\backslash S$ are both rank-two convex. The rank-two closure operator takes any subset S of $\mathcal{A}$ and associate ${}^2\overline{S}^2$ the smallest rank-two convex subset of $\mathcal{A}$ containing S.*

Rank-two convexity means that any H such that n_H is a positive linear combination of two normal vectors of hyperplanes in S is also in S. It is obvious that any convex set is rank-two convex (and biconvex set is rank two biconvex) but the reverse is obviously not true. However, the following theorem shows that in an interesting family of arrangements, those implications become equivalents.

Theorem 1.2.30 ([Rea16b, Theorem 9-4.5])**.** *If all regions of $\mathcal{A}$ are simplicial, then for any subset S of $\mathcal{A}$, the following statements are equivalent:*

1. *S is the separating set of a region of $\mathcal{A}$;*
2. *S is strongly biconvex with respect to B;*
3. *S is biconvex with respect to B;*
4. *S is rank-two biconvex with respect to B.*

Moreover, by combining Theorem 1.2.24 and Theorem 1.2.30, we obtain the following result on the join operator of the poset of regions of simplicial arrangements.

Theorem 1.2.31 ([Rea16b, Theorem 9-4.8])**.** *If all the regions of $\mathcal{A}$ are simplicial, then for Q and R regions,*

1. *$Q \vee R$ is the unique region with $\overline{\mathcal{S}_B(Q) \cup \mathcal{S}_B(R)}$ as a separating set;*
2. *$Q \vee R$ is the unique region with ${}^2\overline{\mathcal{S}_B(Q) \cup \mathcal{S}_B(R)}^2$ as a separating set.*

1.3 Coxeter group

This section presents finite Coxeter groups, the algebraic objects on which this thesis is based. The definitions and theorems are taken from [Hum90]; details of all proofs can be found there.

As we did in Section 1.2, we consider V a Euclidean space with dimension $n \geqslant 1$ and $(\cdot,\cdot)$ a scalar product. We denote by $\mathcal{O}(V)$ the orthogonal group on V, i.e. the group of all linear automorphisms $\phi : V \mapsto V$ such that $(\phi(u), \phi(v)) = (u, v)$ for all $u, v \in V$.

1.3.1 Reflection groups

Let us start by defining a family of subgroups of $\mathcal{O}(V)$, the reflection groups.

Definition 1.3.1. *For $\alpha \in V\backslash\{0\}$, we denote by s_α the* ***reflection*** *along α:*

$$\forall u \in V, \quad s_\alpha(u) = u - 2\frac{(u,\alpha)}{(\alpha,\alpha)}\alpha.$$

A ***finite reflection group*** *is a finite subgroup of the orthogonal group $O(V)$ generated by a set of reflections.*

We note that any reflection is an involution of $\mathcal{O}(V)$. The fixed points of s_α are the hyperplane H_α orthogonal to α. Moreover, $s_\alpha(\alpha) = -\alpha$.

Example 1.3.1. For P_n the regular n-gon inscribed in the unit circle, i.e. with vertices $((\cos(\frac{2k\pi}{n}, \sin(\frac{2k\pi}{n}))_{0\leqslant k<n}$, the isometries of P_n are the elements of $\mathcal{O}(\mathbb{R}^2)$ stabilizing its vertices. This set is a reflection group of $\mathbb{R}^2$ for any $k \geqslant 3$.

- The isometries of the equilateral triangle are given in Fig. 1.7: they are the identity, the reflections across its medians and the rotations around its center with angle $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$. We note that by considering the action of isometries on the vertices of the triangle, this group is isomorphic to the symmetric group $\mathfrak{S}_3$.
- The isometries of the square are given in Fig. 1.8: they are the identity, the reflections across the diagonals and the medians, and the rotations around the center with angle $\frac{\pi}{2}$, π and $\frac{3\pi}{2}$.

In general, an isometry of P_n is either the identity, a reflection across a diagonal or a median, or a rotation of an angle multiple of $\frac{2\pi}{n}$, which is the product of two reflections.

Example 1.3.2. For any integer $n \geqslant 2$, the action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ by permutation of the coordinate is generated by the action of the transpositions (i, j) (for $1 \leqslant i < j \leqslant n$), which are reflections along the vectors $e_j - e_i$. It is therefore a finite reflection group.

Definition 1.3.2. *A finite reflection group is* ***essential*** *if its only fixed point is* 0.

Example 1.3.3. The groups of isometries of regular polygons are essentials in $\mathbb{R}^2$.

Counterexample 1.3.4. The action of $\mathfrak{S}_n$ on $\mathbb{R}^n$ by permutations of coordinates is not essential, as the line $\mathbb{R}(1, 1, \ldots, 1)$ is fixed by all the permutations.

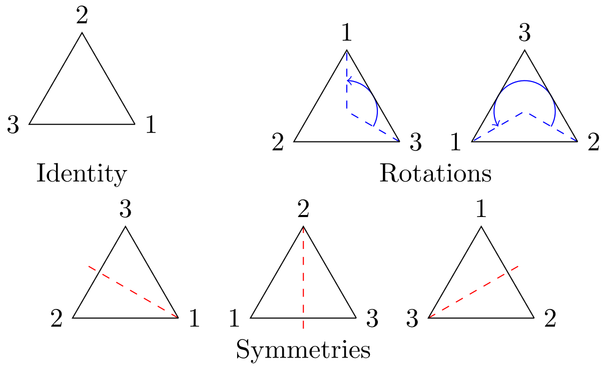


Figure 1.7: Isometries of the equilateral triangle.

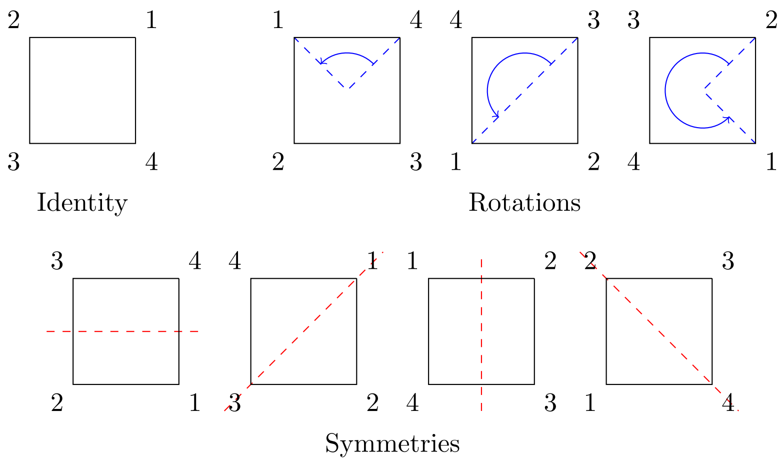


Figure 1.8: Isometries of the square.

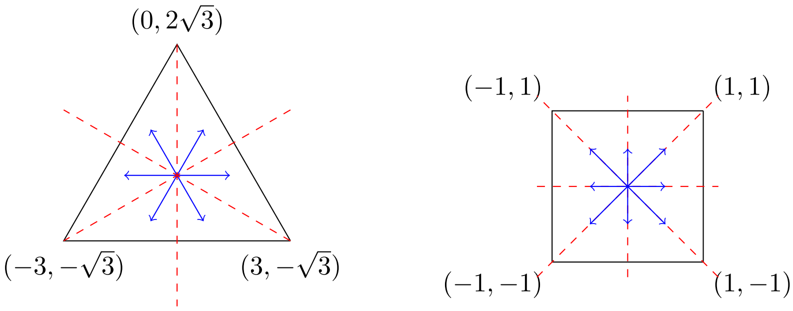


Figure 1.9: A root system for the isometries of the triangle and of the square.

1.3.2 Root systems

Root systems are configurations of vectors in a Euclidean space satisfying certain geometrical properties. They are central in the study of Lie groups and Lie algebras and appear in other areas of mathematics such as singularity theory. A study of root systems in the context of Lie algebra can be found in [Bou07] or [FH04]. Here, we will be interested as root systems as a way to describe finite reflection groups, as they the reflection group is isomorphic to its action on its associated root system.

Definition of root systems

Definition 1.3.3. *A* ***root system*** *of V is a finite set $\Phi \subseteq V$ of nonzero vectors such that for any $\alpha \in \Phi$,*

1. $\mathbb{R}\alpha \cap \Phi = \{\alpha, -\alpha\}$;
2. $s_\alpha(\Phi) = \Phi$.

Its elements are the roots, and the associated reflection group is the one generated by the reflections $(s_\alpha)_{\alpha \in \Phi}$.

For any root system, the associated reflection group is always finite, as it is a subgroup of the permutations of Φ. We note that if W is the reflection group associated to the root system Φ, then for any $\lambda > 0$, it is also the reflection group associated to the root system $\lambda\Phi$.

Example 1.3.5. Some root systems generating the isometries of the triangle and those of the square are given in Fig. 1.9: the direction of the roots are in blue and the axes of the reflections are in red.

Example 1.3.6. The reflection group introduced in Example 1.3.2 is associated to the root system $\{e_i - e_j \mid 1 \leqslant i, j \leqslant n \text{ and } i \neq j\}$.

Definition 1.3.4. *A root system Φ is* ***crystallographic*** *if $\frac{2(\alpha,\beta)}{||\alpha||^2}$ is an integer for all $\alpha, \beta \in \Phi$. A reflection group is crystallographic if it is generated by at least one crystallographic root system.*

A non-crystallographic root system can generate a crystallographic reflection group: for example, the isometries of the square is generated by the root system $\{(0, \pm 1), (\pm 1, 0), (\pm\lambda, \pm\lambda)\}$ for any $\lambda > 0$. This is crystallographic iff $\lambda \in \{1, \frac{1}{2}\}$.

If we consider a reflection group associated to a given root system, the following proposition shows that the reflections in that group are exactly the reflections along roots of the root system. It also shows that the action of the group on the root system is equivalent to the conjugation of reflections of the group.

Proposition 1.3.5. *For Φ a root system and W the reflection group associated to Φ:*

- *if $r \in W$ is a reflection, then $r = s_\phi$ for some $\phi \in \Phi$;*
- *for $\phi \in \Phi$ and $w \in W$, $s_{w(\phi)} = w s_\phi w^{-1}$.*

Positive and simple roots

The root system associated to a finite reflection group can be arbitrarily large compared to the dimension of V: for example, the group of isometries of an n-gon, defined on the 2-dimensional plane, is generated by a root system of cardinality $2n$. It is therefore natural to look for a subset of Φ of size at most $\dim(V)$ that could characterize Φ.

The first step for finding this is partitioning roots into positive and negative ones.

Definition 1.3.6. *A subset Π of Φ is a* ***positive system*** *if $\Phi = \Pi \sqcup -\Pi$ and there exists a linear form $\phi : V \to \mathbb{R}$ such that $\Pi = \Phi \cap \phi^{-1}(\mathbb{R}_{>0})$.*

These positive systems obviously exist, as any linear form ϕ such that $\ker(\phi)$ contains no root gives one. We note that since any positive linear combination of elements of $\phi^{-1}(\mathbb{R}_{>0})$ is obviously still in this half-space, $\mathrm{Cone}(\Pi) \subset \phi^{-1}(\mathbb{R}_{>0}) \cup \{0\}$. In particular, $\mathrm{Cone}(\Pi) \cap \mathrm{Cone}(-\Pi) = \{0\}$.

Definition 1.3.7. *A subset Δ of Φ is a* ***simple system*** *if:*

1. *Δ is a basis of $\mathrm{Vect}(\Phi)$;*
2. *each $\alpha \in \Phi$ is a linear combination of Δ with coefficients all of the same sign.*

The roots in a simple system will be called the **simple roots**, and we will see that they characterize the root system and its associated reflection group.

The existence of simple systems is not trivial, but the following theorem shows that they exist and are in bijection with the positive systems:

Theorem 1.3.8 ([Hum90, Section 1.3])**.**

1. *If Δ is a simple system of Φ, there is a unique positive system containing Δ.*
2. *Any positive system $\Pi \subset \Phi$ contains exactly one simple system.*

The proof of this theorem contains the proof of the following corollary which will be useful later.

Corollary 1.3.9. *If Δ is a simple system in Φ, then $(\alpha, \beta) \leqslant 0$ for all $\alpha \neq \beta$ in Δ.*

In what follows, we will generally want to study the way a root system is generated by a simple system that it contains. The following theorem shows that all simple systems in a root system will have similar geometric configurations:

Theorem 1.3.10 ([Hum90, Sec. 1.4]). *Let Φ be a root system and W the reflection group associated to Φ. Any two positive (resp. simple) systems in Φ are conjugate under W.*

The proof of this theorem uses the following useful proposition:

Proposition 1.3.11 ([Hum90, Sec. 1.4]). *Let Δ be a simple system contained in the positive system Π, and $\alpha \in \Delta$. Then $s_\alpha(\Pi \backslash \{\alpha\}) = \Pi \backslash \{\alpha\}$.*

A root system Φ is generated by any of its simple systems Δ in the sense that $\Phi \subset \mathrm{Cone}(\Delta) \cup -\mathrm{Cone}(\Delta)$. The following theorem shows that the reflection group associated to Φ is also entirely determined by Δ.

Theorem 1.3.12 ([Hum90, Sec. 1.5]). *Let Φ be a root system and W be the reflection group associated to Φ. If Δ is a simple system of Φ, then W is generated by the **simple reflections** $s_\alpha, \alpha \in \Delta$.*

Let us now define a family of vectors defined by the action of generators on them.

Definition 1.3.13. *For Φ a root system associated to an essential reflection group, the **fundamental weights** associated to a simple system Δ are the vectors $(\omega_\alpha)_{\alpha \in \Delta}$ such that $s_\alpha(\omega_\beta) = \omega_\beta$ if $\alpha \neq \beta$ and $s_\alpha(\omega_\alpha) = \omega_\alpha - \alpha$, i.e. $(\alpha, \omega_\beta) = \delta_{\alpha=\beta} \frac{||\alpha||^2}{2}$.*

The weights play a particular role when the reflection group is crystallographic and leads to an alternate definition of crystallographic reflection group.

Definition 1.3.14. *A **geometric lattice** of V is a discrete subset of the space with the form $\{\sum_{1 \leqslant i \leqslant n} a_i v_i \mid a_i \in \mathbb{Z}\}$ with $\{v_1, \ldots, v_n\}$ a basis of V.*

This notion is not linked to the lattices introduced in Section 1.1 and is important in Lie theory.

Proposition 1.3.15. *A reflection group stabilizes a geometric lattice of V if and only if it is crystallographic. In that case, it stabilizes the lattice generated by its fundamental weights, which is then called its **weight lattice**. The elements of the weight lattice are the **weights** of the group.*

1.3.3 Coxeter groups and reflection groups

Coxeter group are groups whose structure can be entirely determined by an integer matrix. They were introduced by H. S. M. Coxeter as an abstraction of reflection groups, and we will see in this section the link between the two notions.

$$\begin{pmatrix} 1 & 3 & 2 & 2 \\ 3 & 1 & 4 & 3 \\ 2 & 4 & 1 & 2 \\ 2 & 3 & 2 & 1 \end{pmatrix} \to$$

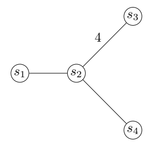


Figure 1.10: A Coxeter matrix and its associated Coxeter graph.

Definition 1.3.16. *Let S be a set, the **free group** F_S over S is the group of words on $S \cup S^{-1}$ such that the operation is the concatenation, and two words are different in F_S unless their equality follows from group axioms (for example $Ass^{-1}B = AB$ for any $A, B \in F_S$ and $s \in S$).*

This group is the universal group generated by S, in the sense that any group generated by S is a quotient of F_S.

Definition 1.3.17. *Let S be a set and R a set of relations (of the type $x = y$) on words on $S \cup S^{-1}$. A group G has **presentation** $\langle S, R\rangle$ if it is generated by S and isomorphic to F_S quotiented by the relations in R. In practice, it means that two products of generators (and generator inverses) $s_1 s_2 \dots s_k$ and $t_1 t_2 \dots t_l$ are equal in G iff we can go from one to the other by a sequence of equalities of the form*

$$AxB = AyB \qquad \text{with } A, B \in F_S \text{ and } \{x = y\} \in R \cup R^{-1}.$$

The **word problem** is the question of whether two products of generators are equal in a group with a given presentation. In general, it is undecidable, as proven by Piotr Novikov [Nov55] in 1955. However, we will consider a class of representations for which the word problem is solved.

Definition 1.3.18. *Let S be a finite set. A matrix $M = (m_{st})_{s,t\in S}$ with its values in $\mathbb{N}_{>0} \cup \{\infty\}$ is a **Coxeter matrix** on S if:*

- *it is symmetric;*
- *$m_{st} = 1 \iff s = t$.*

Equivalently, the content of a Coxeter matrix can be represented by a **Coxeter graph** with vertex set S and whose edges are the (undirected) pairs $s, t \in S$ such that $m_{st} \geqslant 3$. The edges with $m_{st} = 3$ are unlabeled and the ones with $m_{st} \geqslant 4$ are labeled by that number. An example is given in Fig. 1.10.

Definition 1.3.19. *For M a Coxeter matrix on S, the **Coxeter group** W associated to that matrix is the group with the presentation $\langle S, \{(st)^{m_{st}} = e \mid s, t \in S\}\rangle$.*

Since $m_{ss} = 1$ for each generator $s \in S$, we have that $s^2 = e$ and so $s = s^{-1}$. This in turn shows that for any $s, t \in S$, the relation $(st)^{m_{st}} = e$ is equivalent to

$$stst\dots = tsts\dots$$

with both sides of the equality being of length m_{st}. In particular, if $m_{st} = 2$ we know that s and t commute.

Example 1.3.7. The symetric group $\mathfrak{S}_n$ is a Coxeter group with:

1. $S = \{\tau_i = (i, i+1) \mid 1 \leqslant i < n\}$ (simple transpositions);
2. m_{τ_i,τ_j} is 1 if $i = j$, 2 if $|i-j| > 1$ and 3 if $|i-j| = 1$.

The link between finite Coxeter groups and finite reflection groups is given by the following two theorems, proved by H. S. M. Coxeter.

Theorem 1.3.20 ([Cox34, Thm. 8])**.** *A Coxeter group is finite only if it is isomorphic to a finite reflection group.*

Theorem 1.3.21 ([Cox35])**.** *The finite reflection group W generated by a root system Φ with Δ a simple system is isomorphic to the Coxeter group associated to the Coxeter matrix $(m_{s_\alpha s_\beta})_{\alpha,\beta} \in \Delta$, with m_{s_α,s_β} the order of $s_\alpha s_\beta$ in W.*

The value of the coefficients of this matrix can be obtained directly from Δ, as the angle between two distinct simple roots α and β is $\pi(1 - \frac{1}{m_{s_\alpha s_\beta}})$:

$$m_{s_\alpha s_\beta} = -\frac{1}{\arccos(\frac{(\alpha,\beta)}{||\alpha||\cdot||\beta||})}.$$

In everything that follows, we will thus consider finite Coxeter groups with an associated root system generating them.

1.3.4 Classification of finite Coxeter groups

The product of two Coxeter groups is a Coxeter group and its Coxeter graph is the disjoint union of their two Coxeter graphs. As such, the study of Coxeter groups can be limited to the ones with connected Coxeter graphs:

Definition 1.3.22. *A Coxeter group is **irreducible** if its Coxeter graph is connected.*

If a Coxeter graph has several connected components, its roots can be partitioned into two or more orthogonal sets and the generated reflection group is isomorphic to the cartesian product of smaller reflection groups.

A Coxeter graph gives a finite Coxeter group if and only if a bilinear form associated to its Coxeter matrix is positive definite (see [Hum90, Chapter 2] for the full proof). This means that it is possible to characterize all the irreducible finite Coxeter groups as follows:

Theorem 1.3.23 ([Cox34, Thm. 9])**.** *An irreducible Coxeter group is finite if and only if its Coxeter graph is in Fig. 1.11.*

This means that there are only four infinite families of finite Coxeter groups, generally refered to as type A, type B, type D and dihedral groups. The other

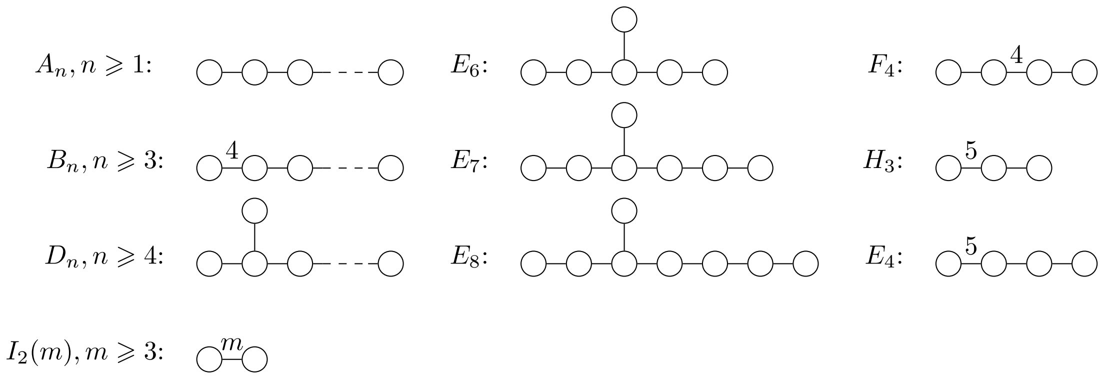


Figure 1.11: Some Coxeter diagrams of finite Coxeter groups.

Coxeter groups (on the middle and right columns in Fig. 1.11) are called exceptional Coxeter groups.

Geometric representations of all the finite Coxeter groups are known; we will give examples for the infinite families, along with some of their properties. For a complete study of finite Coxeter groups and explicit root systems, see [Hum90], Sections 2.10 to 2.13.

Type A

The Coxeter group of type A_n for $n \geqslant 2$ has n generators and its Coxeter graph is an unlabeled line. It is isomorphic to the symetric group $\mathfrak{S}_{n+1}$, or to the group of isometries of the n-dimensional simplex, and its order is $(n+1)!$.

A possible root system is as follows:

- $\Phi = \{e_j - e_i \mid i \neq j\}$;
- $\Pi = \{e_j - e_i \mid 1 \leqslant i < j \leqslant n+1\}$;
- $\Delta = \{e_{i+1} - e_i \mid 1 \leqslant i \leqslant n\}$.

In this case, the associated reflection group is the action of $\mathfrak{S}_{n+1}$ on the coordinates of $\mathbb{R}^{n+1}$ described in Example 1.3.2.

We already said that this root system is not essential, since it is contained in the hyperplan $\{(x_1, \ldots, x_n) \mid \sum x_i = 0\}$. To define its fundamental weight, we thus have to restrict ourselves to this hyperplane (which is the vector span of Φ. This way, for $1 \leqslant i < n$ we obtain

$$\omega_{e_{i+1}-e_i} = \frac{i-n}{n}(e_1 + \ldots + e_i) + \frac{i}{n}(e_{i+1} + \ldots + e_n).$$

We note that this root system is crystallographic.

Type B

The Coxeter group of type B_n for $n \geqslant 3$ has n generators and its Coxeter graph is a line with only the first edge labeled with 4. It is isomorphic to the group of signed permutations of length n, or the isometries of the n-dimensional hypercube, and its order is $2^n n!$.

A possible root system is:

- $\Phi = \{\pm e_j \pm e_i \mid i \neq j\} \cup \{\pm e_i \mid 1 \leqslant i \leqslant n\}$;
- $\Pi = \{e_j \pm e_i \mid 1 \leqslant i < j \leqslant n\} \cup \{e_i \mid 1 \leqslant i \leqslant n\}$;
- $\Delta = \{e_{i+1} - e_i \mid 1 \leqslant i \leqslant n\} \cup \{e_1\}$.

This root system is crystallographic.

Type D

The Coxeter group of type D_n for $n \geqslant 4$ has n generators and its Coxeter graph it an unlabeled line with two branches of length 1 at one end. It is isomorphic to the group of signed permutations of length n with an even number of negative signs, and its order is $2^{n-1}n!$.

A possible root system is:

- $\Phi = \{\pm e_j \pm e_i \mid 1 \leqslant i < j \leqslant n\}$;
- $\Pi = \{e_j \pm e_i \mid 1 \leqslant i < j \leqslant n\}$;
- $\Delta = \{e_{i+1} - e_i \mid 1 \leqslant i < n\} \cup \{e_1 + e_2\}$.

This root system is crystallographic. It is also a subset of the root system associated to type B, just as the signed permutations with an even number of negative signs are a subgroup of all the signed permutations of length n.

Dihedral groups

The dihedral Coxeter group $I_2(n)$ with $n \geqslant 3$ has 2 generators and its Coxeter graph is an edge labeled with n. It is isomorphic to the isometries of the regular n-gon and its order is $2n$.

A possible root system is:

- $\Phi = \{(\cos(\frac{k\pi}{n}), \sin(\frac{k\pi}{n})) \mid 0 \leqslant k < 2n\}$;
- $\Pi = \{(\cos(\frac{k\pi}{n}), \sin(\frac{k\pi}{n})) \mid 0 \leqslant k < n\}$;
- $\Delta = \{(1, 0), (\cos(\frac{(n-1)\pi}{n}), \sin(\frac{(n-1)\pi}{n})\}$.

This root system is not crystallographic for $n > 3$, and in general no crystallographic root system exists for $I_2(n)$. However, we note that for $n = 4$ and $n = 6$, there exists a crystallographic root system generating $I_2(n)$.

A possible crystallographic root system of $I_2(4)$ is:

- $\Phi = \{(1, 0), (1, 1), (0, 1), (-1, 1), (-1, 0), (-1, -1), (0, -1), (1, -1)\}$;
- $\Pi = \{(1, 0), (1, 1), (0, 1), (-1, 1)\}$;
- $\Delta = \{(1, 0), (-1, 1)\}$.

1.4 Weak order

This section defines the weak order on finite Coxeter groups and proves the equivalence of its two definitions. The content of the first three subsections come from [Hum90] and [BB05], while the last two come from [Rea16a].

In this section, we will consider a root system Φ, its associated finite Coxeter group W, and fix Δ a simple system.

1.4.1 Length, reduced words and relations

We know from Theorem 1.3.12 that any element of W can be written as a product of simple reflections.

Definition 1.4.1. *The **length** $\ell(w)$ for $w \in W$ is the minimal integer r such that there exists $\alpha_1, \ldots, \alpha_r$ with $w = s_{\alpha_1} s_{\alpha_2} \ldots s_{\alpha_r}$. Any such product of length $\ell(w)$ is a **reduced expression** of w.*

By convention we say that $\ell(e) = 0$.

Proposition 1.4.2. *For $w \in W$:*
1. $\ell(w) = 1 \iff w = s_\alpha$ *for some* $\alpha \in \Delta$;
2. $\ell(w) = \ell(w^{-1})$;
3. $\ell(ws) = \ell(w) \pm 1$.

The first two items are simple consequences of the definition, with the second one coming from the fact that if $w = s_1 s_2 \ldots s_m$, then $w^{-1} = s_m s_{m-1} \ldots s_1$ since all simple reflections are involutions.

The third item is less obvious: proving that $\ell(w) - 1 \leqslant \ell(ws) \leqslant \ell(w) + 1$ is easy, but proving that $\ell(w) \neq \ell(ws)$ uses the following theorem, which is also a characterization of reduced expressions:

Theorem 1.4.3 ([Hum90, Sec. 1.7])**.** *Consider $w \in W$ and $s_1 s_2 \ldots s_m$ an expression of w as a product of simple reflections, and denote by α_i the simple root such that s_i is s_{α_i}. If $m > \ell(w)$, then there exists $1 \leqslant i < j \leqslant m$ such that:*
1. $\alpha_i = (s_{i+1} s_{i+2} \ldots s_{j-1})(\alpha_j)$;
2. $s_{i+1} \ldots s_{j-1} s_j = s_i s_{i+1} \ldots s_{j-1}$;
3. $w = s_1 \ldots s_{i-1} s_{i+1} \ldots s_{j-1} s_{j+1} \ldots s_m$ *(the expression with s_i and s_j omitted).*

This is the **deletion condition**: if an expression of w is not reduced, there exists a pair of letters that can be removed to obtain a shorter expression of w. Reiterating this operation will yield a reduced expression of w that is a subword of the original expression. This also implies that the lengths of all expressions of an element w have the same parity as $\ell(w)$ (thus proving that $\ell(ws)$ cannot be equal to $\ell(w)$, and thus completing the proof of Proposition 1.4.2).

An equivalent of this previous theorem is the **exchange condition**, which give a way to go from one reduced expression of w to another:

Theorem 1.4.4. *Let $w = s_1 s_2 \dots s_m$ (a product of simple reflections but not necessarily a reduced expression) with $s_i = s_{\alpha_i}$. If for some simple reflection $s = s_\alpha$ we have $\ell(ws) < \ell(w)$, then there exists an index i such that $w = s_1 \dots s_{i-1} s_{i+1} \dots s_m s$. In particular, an element w has a reduced expression ending with s if and only if $\ell(ws) < \ell(w)$.*

This last statements gives a characterization of the pairs $(w, s) \in W \times S$ such that $\ell(ws) = \ell(w) - 1$ (with all other pairs satisfying $\ell(ws) = \ell(w) + 1$). However, this characterization is not very convenient, as it potentially requires finding all reduced expressions of w. The following lemma gives a more direct condition:

Lemma 1.4.5 ([Hum90, Sec. 1.6])**.** *For $w \in W$ and $s = s_\alpha$ a simple reflection:*
1. $w(\alpha) \in \Pi \iff \ell(ws) = \ell(w) + 1$ *(s is a* ***right ascent*** *of w);*
2. $w(\alpha) \in -\Pi \iff \ell(ws) = \ell(w) - 1$ *(s is a* ***right descent*** *of w);*
3. $w^{-1}(\alpha) \in \Pi \iff \ell(sw) = \ell(w) + 1$ *(s is a* ***left ascent*** *of w);*
4. $w^{-1}(\alpha) \in -\Pi \iff \ell(sw) = \ell(w) - 1$ *(s is a* ***left descent*** *of w).*

1.4.2 Inversion set

Let us now consider the action of an element of W on positive roots. We will be especially interested in the sign of their image.

Definition 1.4.6. *For $w \in W$, we define:*
- *the* ***inversions*** *of w as* $\mathrm{inv}(w) := w(-\Pi) \cap \Pi$;
- *the* ***noninversions*** *of w as* $\mathrm{ninv}(w) := w(\Pi) \cap \Pi$.

We note that $\mathrm{inv}(w) \sqcup \mathrm{ninv}(w) = \Pi$.

Example 1.4.1. Let us consider the action of the symmetric group $\mathfrak{S}_n$ on $\mathbb{R}^n$ described in Example 1.3.2. The positive roots of this reflection group are the vectors $e_j - e_i$ for $1 \leqslant i < j \leqslant n$. Then for $\omega \in \mathfrak{S}_n$, we have $\omega^{-1}(e_j - e_i) = e_{\omega^{-1}(j)} - e_{\omega^{-1}(i)}$, which is a positive root if and only if $\omega^{-1}(j) < \omega^{-1}(i)$. This means that the inversion set of ω is $\{e_j - e_i \mid 1 \leqslant i < j \leqslant n \text{ and } \omega^{-1}(i) < \omega^{-1}(j)\}$.

From Proposition 1.3.11 we know that for α a simple root, the only inversion of s_α is α. This means that the composition of m simple transpositions send at most m positive roots on negative roots, or in general that $|\mathrm{inv}(w)| \leqslant \ell(w)$. By considering the conditions given in Lemma 1.4.5 on right ascents and descents of w, we obtain the following theorem:

Theorem 1.4.7 ([BB05, Coro. 1.4.5])**.** *For any $w \in W$, we have $\ell(w) = |\mathrm{inv}(w)|$.*

The most important consequence of this theorem is that the only element w such that $w(\Pi) = \Pi$ is the identity e. This leads to the following statement, which shows that the inversion set of an element characterizes it.

Theorem 1.4.8 ([Hum90, Sec. 1.8]). *For two element $w_1, w_2 \in W$, the following statements are equivalent:*
- $w_1 = w_2$;
- $w_1(\Pi) = w_2(\Pi)$;
- $w_1(\Delta) = w_2(\Delta)$;
- $\mathrm{inv}(w_1) = \mathrm{inv}(w_2)$.

Moreover, from Theorem 1.3.10, we know that any positive system in Φ is the image of Π by some element of W. This means that the inversions sets of elements of W are exactly the $\Pi \cap \Pi'$ for Π' a positive system, which leads to the following characterization of inversion sets:

Theorem 1.4.9. *A subset $S \subseteq \Pi$ is the inversion set of an element of W if and only if it is **biconvex**, i.e:*
- $S = \mathrm{Cone}(S) \cap \Phi$;
- $\Pi \backslash S = \mathrm{Cone}(\Pi \backslash S) \cap \Phi$.

In particular, this theorem implies that a set of roots is the inversion set of an element if and only if it is the noninversion of another element. Applying this reasoning to the identity leads to the following definition.

Definition 1.4.10. *For W a finite Coxeter group, the **longest element** of W, usually denoted by w_0, is the only element such that $\ell(w_0) = |\Pi|$ and $\mathrm{inv}(w_0) = \Pi$.*

Proposition 1.4.11. *For $w \in W$, the inversion set of ww_0 is $\mathrm{ninv}(w)$.*

1.4.3 Two definitions of the weak order

We saw from Theorem 1.4.7 that the cardinality of the inversion set of elements of W is linked to their length, and so to their reduced expressions. Using Proposition 1.3.11, we can see how this link translates when increasing or decreasing the length of an element.

Lemma 1.4.12. *For $w \in W$ and $\alpha \in \Delta$, the inversion set $\mathrm{inv}(ws_\alpha)$ is:*
- $\mathrm{inv}(w) \cup \{w(\alpha)\}$ *if s_α is a right ascent of w;*
- $\mathrm{inv}(w) \backslash \{-w(\alpha)\}$ *if s_α is a right descent of w.*

This leads to a simple way to compute the inversion set of an element of W, given a reduced expression of this element.

Proposition 1.4.13 ([BB05, Corollary 1.4.4]). *For $w \in W$, if $s_1 s_2 \dots s_m$ is a reduced expression of w with α_i the simple root such that $s_{\alpha_i} = s_i$, then*

$$\mathrm{inv}(w) = (\{s_1 s_2 \dots s_{k-1}(\alpha_k) \mid 1 \leqslant k \leqslant m\}.$$

This leads us to the following theorem and definition:

Theorem 1.4.14 ([BB05, Prop. 3.1.2 and 3.1.3])**.** *For $u, w \in W$, the following statements are equivalent:*

1. $\mathrm{inv}(u) \subseteq \mathrm{inv}(w)$*;*
2. $\mathrm{ninv}(w) \subseteq \mathrm{ninv}(u)$*;*
3. *there exists a reduced expression $s_1 s_2 \dots s_m$ of w such that $s_1 s_2 \dots s_{\ell(u)}$ is a reduced expression of u;*
4. *there exists $v \in W$ such that $w = uv$ and $\ell(w) = \ell(u) + \ell(v)$.*

*In that case, we say that $u \leqslant w$ and this defines the **weak order** on W.*

Statements 1 and 2 are obviously equivalent, and from the definition of the length function so are statements 3 and 4. Proposition 1.4.13 shows that statement 3 implies statement 1, and the converse implication is proved by induction on $\ell(u)$.

This weak order is also sometimes called **prefix order** as $u \leqslant w$ when a reduced expression of w has a prefix that writes u.

Corollary 1.4.15. *For $u, w \in W$, if $u \leqslant w$, then* $\mathrm{inv}(u^{-1}w) = u^{-1}(\mathrm{inv}(w) \setminus \mathrm{inv}(u))$.

This is clear by taking the prefix definition of the weak order and the characterization of the inversion set given in Proposition 1.4.13.

The prefix definition of the weak order shows that any cover relation must be of the form $w \lessdot ws$ for some element $w \in W$ and some simple reflection s. In fact, for any such w and s, either s is a right ascent of w and $w \lessdot ws$ is a cover, or it is a right descent and $ws \lessdot w$ is a cover.

Example 1.4.2. As was said in the introduction, in the symetric group, the cover relations of the weak order are of the form $UijV \lessdot UjiV$ with $i < j$.

Example 1.4.3. In Coxeter group of type B seen as signed permutations, the covers of the weak order are of the form $UijV \lessdot UjiV$ with $i < j$ (with the sign taken into account) or $iU \lessdot (-i)U$ with $i > 0$. The Hasse diagram of the weak order on the group B_3 is given in Fig. 1.12, with negative numbers underlined for readability.

Remark 1.4.16. The weak order described here is sometimes called the right weak order; the left weak order is the image of the right weak order by the automorphism $w \mapsto w^{-1}$. We say that u is lower than w in the left weak order if and only if $\Pi \cap u^{-1}(-\Pi) \subseteq \Pi \cap w^{-1}(-\Pi)$, or equivalently u is a suffix of w.

1.4.4 Coxeter arrangements and the weak order

We will now use the objects defined in Section 1.2 to give an alternative definition of the weak order, which will immediately give a lot of interesting properties.

Definition 1.4.17. *The **Coxeter arrangement** associated to W is the hyperplane arrangement $\mathcal{A}_W = \{H_\phi \mid \phi \in \Phi\}$ containing all the hyperplane across which the reflections of W are defined.*

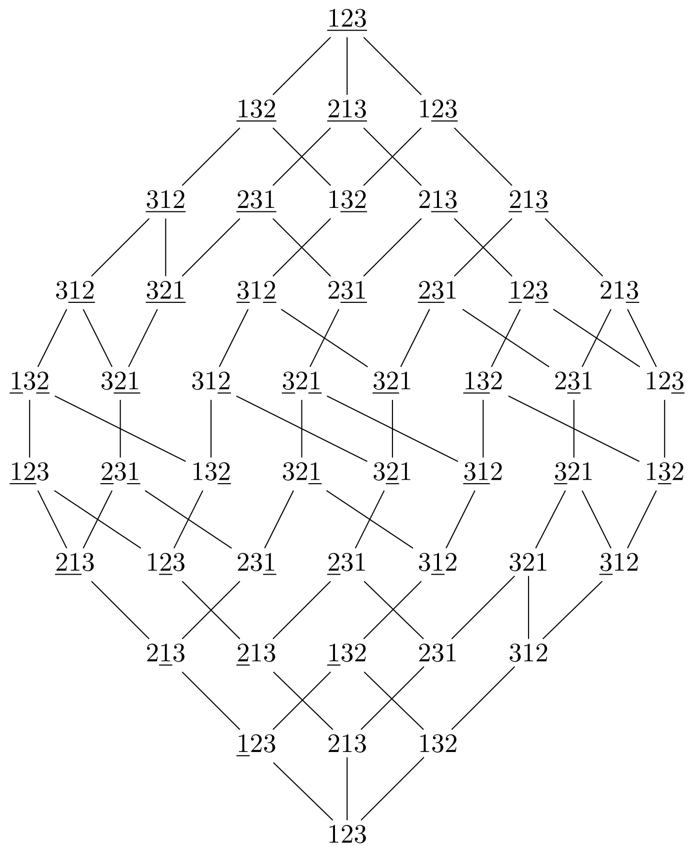


Figure 1.12: The weak order on signed permutation of size 3.

The group W is obviously entirely determined by $\mathcal{A}_W$, and this arrangement contains exactly $\frac{|\Phi|}{2}$ hyperplanes, since $H_\phi = H_{-\phi}$ for any root ϕ.

Example 1.4.4. The hyperplane arrangement of the Coxeter group A_n (isomorphic to $\mathfrak{S}_{n+1}$) contains all the hyperplanes of the form $\{x_i = x_j\}$ for $i < j$. Its regions are the sets $\{x_{\sigma(1)} \leqslant x_{\sigma(2)} \leqslant \ldots \leqslant x_{\sigma(n+1)}\}$ for $\sigma \in \mathfrak{S}_{n+1}$.

Since W is a finite reflection group and stabilizes Φ, it is obvious that W stabilizes $\mathcal{A}_W$, and so the image of a region by an element of W is still a region. Consider now two adjacent regions Q and R of $\mathcal{A}_W$, with H_ϕ the hyperplane separating them. It is obvious that $Q = s_\phi(R)$ (and vice-versa). Then Lemma 1.2.17 naturally gives the following proposition.

Proposition 1.4.18 ([Rea16a, Prop. 10-2.2])**.** *The group W acts transitively on the set of regions $\mathcal{R}(\mathcal{A}_W)$.*

For any pair of regions Q, R, if we choose a sequence $Q = R_0, R_1, \ldots, R_k = R$ as described in Lemma 1.2.17 and denote by s_i the reflection across the hyperplane between R_{i-1} and R_i for $1 \leqslant i \leqslant k$ (with s_i in W by definition of $\mathcal{A}_W$). Then R is equal to $s_k s_{k-1} \ldots s_1(Q)$.

We can then use a general theorem on essential arrangements, that says that any essential arrangement has at least one simplicial region, to naturally obtain the following theorem.

Theorem 1.4.19 ([Rea16a, Thm. 10-2.1])**.** *If W is essential, then all the regions of $\mathcal{A}_W$ are simplicial.*

Let us now consider a linear form $f : V \to \mathbb{R}$ defining the positive system Π in the roots Φ of W (i.e. $\Pi = f^{-1}(\mathbb{R}_{>0}) \cap \Phi$). We know that there exists a unique $b \in V$ such that $f(v) = (b, v)$ for all $v \in V$, and that $f(\phi) \neq 0$ for all $\phi \in \Phi$, so $b \notin \cup_{H \in \mathcal{A}_W}$. This means that b is inside a certain region B_0 of $\mathcal{A}_W$. Moreover, for any other b' in the interior of B_0, since there is no hyperplane of $\mathcal{A}_W$ separating b and b', the linear form $v \mapsto (b', v)$ has the same signs when applied to any root of W and so it defines the same positive system Π.

For each hyperplane $H \in \mathcal{A}_W$, we know that there exists two roots in Φ normal to H (and they are opposite), and the positive root ϕ_H among those two is such that $(b, \phi) > 0$. This means that the roots $(\phi_H)_{h \in \mathcal{A}_W}$ verify the conditions on $(n_H)_{H \in \mathcal{A}}$ defined in Section 1.2.4. Moreover, they do so no matter the choice of b in the interior of B_0, as explained previously.

Consider now the set $X \subseteq \Pi$ of positive roots such that the hyperplanes $(H_\phi)_{\phi \in X}$ define the facets of B_0. Then by Lemma 1.2.11 we know that $\mathrm{Cone}(X)$ is the dual cone of B_0, and so $v \in \mathrm{Cone}(X)$ if and only if $(b', v) > 0$ for all $b' \in B_0$. As such, we get that all the positive roots are positive linear combinations of X, and by the definition of a simplicial cone, that X is a basis of V. This means that X is actually the set of simple roots Δ, and that $B_0 = \mathrm{Cone}(\Delta)^* = \cap_{\alpha \in \Delta} H_\alpha^+$.

This proves that choosing a region of $\mathcal{A}_W$ is equivalent of choosing a simple system Δ in Φ, and if $R = wB_0$, then the set of hyperplanes defining the facets of R is $\{H_{w(\alpha)} \mid \alpha \in \Delta\}$. Then from Theorem 1.4.8, we obtain the following important theorem:

Theorem 1.4.20 ([Rea16a, Thm. 10-2.5]). *For any $B \in \mathcal{R}(\mathcal{A}_W)$, the map $w \mapsto wB$ is a bijection from W to $\mathcal{R}(\mathcal{A}_W)$.*

Combined with Theorem 1.4.19, this means that the poset of regions $\mathcal{P}(\mathcal{A}_W, B_0)$ defines an order on W with many interesting properties; in particular, it is a polygonal lattice and we can characterize separation sets and compute the separation set of a join. The question is, can we find a link between the poset of region and the weak order?

Lemma 1.4.21 ([Rea16a, Prop. 10-3.5]). *For $w \in W$, the separating set $\mathcal{S}_{B_0}(wB_0)$ is exactly $\{H_\phi \mid \phi \in \mathrm{inv}(w)\}$.*

Proof. When discussing the biconvexity of separating sets in Section 1.2.4, we noted that the separating set of a region R contained the hyperplanes such that $(r, n_H) < 0$ for any vector r in the interior of R. By a reasoning of the dual cone of R like when showing that $B_0 = \cap_{\alpha\in\Delta} H_\alpha^+$, we see that for any positive root ϕ (here playing the role of n_H), the hyperplane H_ϕ is in the separating set if and only if the vector ϕ is in $-\mathrm{Cone}(w\Delta) = \mathrm{Cone}(w(-\Delta))$, which is equivalent to saying that $\phi \in w(-\Pi)$ or that $\phi \in \mathrm{inv}(w)$. □

The following theorem is then a direct consequence of this lemma.

Theorem 1.4.22 ([Rea16a, Thm. 10-3.1]). *The map $B \mapsto wB$ is an isomorphism from the weak order on W to the poset of regions $\mathcal{P}(\mathcal{A}_W, B_0)$.*

1.4.5 Properties of the weak order

We can now use the results given in Section 1.2 to study the weak order. All those results will use Theorem 1.4.22.

Theorem 1.4.23 ([Rea16a, Thm. 10-3.7]). *The weak order on a finite Coxeter group is a graded polygonal lattice.*

The proof is a simple combination of Theorem 1.4.19 (we consider the restriction of W to $\mathrm{Vect}(\Phi)$ to obtain an essential Coxeter group) and Theorem 1.2.24. The lattice is graded by the cardinality of the inversion sets.

Theorem 1.4.24 ([Rea16a, Thm. 10-3.24]). *A subset X of Π is the inversion set of some element of W if and only if it is rank-two biconvex.*

This result, which is stronger than Theorem 1.4.9, comes directly from Theorem 1.2.30.

Example 1.4.5. For the Coxeter group A_{n-1} isomorphic to the symmetric group $\mathfrak{S}_n$, a root $e_j - e_i$ is a rank-two positive linear combination of the pairs $(e_j - e_k, e_k - e_i)$ for all $k \neq i, j$. Therefore, a subset X of Π is rank-two biconvex iff for all $i < j < k$ such that $e_j - e_k, e_k - e_i \in X$, the root $e_j - e_i$ is also in X, and for all $i < k < j$ such that $e_j - e_k, e_k - e_i \notin X$, the root $e_j - e_i$ is not in X either. This last condition can be reformulated into: if $i < k < j$ is such that $e_j - e_i \in X$, then $e_j - e_k \in X$ or $e_k - e_i \in X$.

Theorem 1.4.25 ([Rea16a, Thm. 10-3.25])**.** *For any two elements $u, v \in W$, the inversion set of $u \vee v$ is the rank-two closure of* $\mathrm{inv}(u) \cup \mathrm{inv}(v)$*.*

This comes from 1.2.31.

Finally, a remark on intervals of the weak order of W, which will justify that studying intervals of the form $[e, w]$ gives information on any interval of W:

Proposition 1.4.26 ([BB05, Prop. 3.1.6])**.** *For any two elements $u \leqslant v$ in W, the weak order interval $[u, v]$ is isomorphic to the weak order interval $[e, u^{-1}v]$.*

The map $x \mapsto u^{-1}x$ is an isomorphism between those two intervals; this is easy to see by using reduced words of elements in $[u, v]$ with a reduced expression of u as a prefix.

1.5 Subword complexes

This section introduces subword complexes. The first definition of subword complexes was given by A. Knutson and E. Miller in [KM05] for type A Coxeter groups and in [KM04] for general Coxeter groups. This notion is thus much more recent than everything else defined until now, and the definitions and theorems that we will give come from a variety of articles; we will give the relevant reference next to each result.

In the following section, we consider W a Coxeter group with S its generator set, and Φ an associated root system with Δ the simple roots and Π the positive roots.

1.5.1 Definition of subword complexes

Subword complexes are a way of representing reduced expressions of an element of W that are subwords of a word on S. For this purpose, we use an abstract simplicial complex (the combinatorial equivalent of a simplicial complex, with no condition of geometric realisation).

Definition 1.5.1. *An* ***abstract simplicial complex*** *$\mathcal{K}$ on a base set V is a family of finite subsets of V that is closed under taking subsets. Any set $F \in \mathcal{K}$ is a* ***face*** *of $\mathcal{K}$ and the* ***dimension*** *of F is $|F| - 1$. The* ***facets*** *of $\mathcal{K}$ are its maximal faces, and the vertices are the faces of dimension 0. The dimension of $\mathcal{K}$ is the maximal dimension of its faces.*

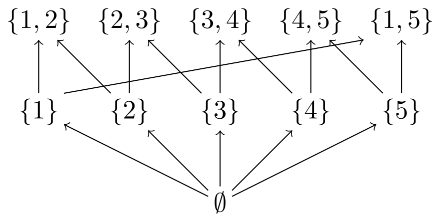


Figure 1.13: The inclusion order on faces of $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_2\tau_1, 321)$.

Example 1.5.1.

- $\{\varnothing, \{1\}, \{1,2\}\}$ is not an abstract simplicial complex, as $\{2\}$ is not in it
- $\{\varnothing, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}\}$ is an abstract simplicial complex with dimension 1 on the vertex set $\{1,2,3,4\}$, and its facets are $\{1,2\}, \{1,3\}$ and $\{4\}$

Here we will only consider simplicial complexes with a finite vertex set, that are thus necessarily finite. We note that an abstract simplicial complex is entirely determined by its facets, as it contains exactly its facets and all their subsets.

Definition 1.5.2. *For $w \in W$ and $Q = q_1q_2\dots q_m$ a word on S, the **subword complex** $\mathrm{SC}(Q, w)$ is the simplicial complex on $[m]$ such that the faces are the sets of positions in Q whose complement contains a reduced expression of w.*

In particular, the facets of this complex are exactly the complements of the subwords of Q that are reduced expressions of w.

Example 1.5.2. Let us choose $W = \mathfrak{S}_3$ with $S = \{\tau_1, \tau_2\}$, and consider the subword complex $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_2\tau_1, 321)$. The reduced expressions of 321 are $\tau_1\tau_2\tau_1$ and $\tau_2\tau_1\tau_2$, and the base set of the simplicial complex is $\{1,2,3,4,5\}$. Therefore, its facets are $\{1,2\}, \{1,5\}, \{2,3\}, \{3,4\}$, and $\{4,5\}$. All the faces and the inclusions order on them are given in Fig. 1.13.

Here we will only care about the combinatorics of facets of a subword complex; the underlying simplicial complex structure will only matter when using previous results on subword complexes.

We will now define a family of words with interesting properties that we will develop later.

Definition 1.5.3. *A word $Q = q_1q_2\dots q_m$ on S is **alternating** if between any two occurrence $i < j$ of a letter s, for any $t \in S$ that does not commute with s, there is an occurrence of t between indices i and j.*

Remark 1.5.4 ([CLS14, Prop 3.8]). For $Q = UstV$ and $Q' = UtsV$ two words differing only by the exchange of two consecutive letters, if s and t commute, then the subword complexes defined on Q and Q' are isomorphic. In general, for $\sim$ the

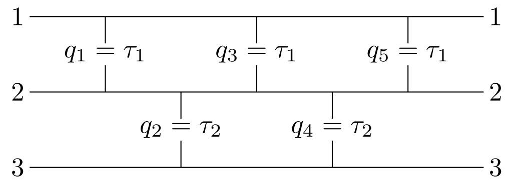


Figure 1.14: The sorting network associated to $Q = \tau_1\tau_2\tau_1\tau_2\tau_1$.

equivalence relationship on words that is the closure of the relations $UstV \sim UtsV$ if s and t commute, if $Q \sim Q'$ the subword complexes on Q and Q' are isomorphic and have the same properties.

Note that the equivalence class of an alternating word only contains other alternating words.

Sorting network representations

In the case where $W = \mathfrak{S}_n$, we represent the facets of a subword complex $\mathrm{SC}(Q, \omega)$ as pipes on a **sorting network**. The sorting network associated to the word Q has n horizontal lines, representing the positions 1 to n from top to bottom, and for each letter τ_i of Q, a vertical line is drawn between the horizontal lines i and $i + 1$, going from left to right. An illustration is given in Fig. 1.14.

Remark 1.5.5. The term "sorting" is generally used to indicate that this network can be used as a sorting algorithm, in the way described in the introduction. This is not the case here, as some of these network would not be able to sort all permutations. It would be more exact to call them "partial sorting networks", but for readability reasons we drop the word "partial" and understand that these objects do not always sort all permutations.

Once the sorting network associated to Q is drawn, a subword of Q is represented by n pipes crossing this network. The pipes starts each on a different horizontal line on the left and are numbered from 1 at the top to n at the bottom. Then, for each vertical line corresponding to a letter of Q:

- if this letter is in the subword, then the two pipes adjacent to the associated vertical line cross each other on this line and exchange their horizontal lines;
- if this letter is not in the subword, then the associated vertical line is ignored by the pipes.

The permutation written by the subword complex is then the order (from top to bottom) in which the pipes exit the network on the right.

As an example, all subwords corresponding to the facets of $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_2\tau_1, 321)$ given in Example 1.5.2 are represented this way in 1.15.

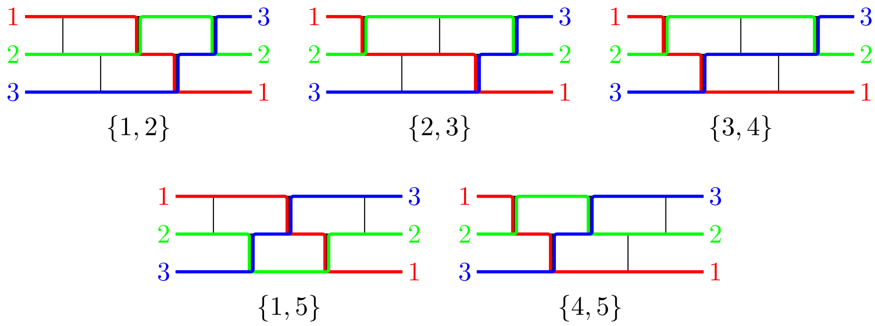


Figure 1.15: The facets of the subword complex $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_2\tau_1, 321)$.

1.5.2 Bruhat order

The definition of a subword complex allows us to choose any word Q on simple transpositions and any element of W, so we note that $\mathrm{SC}(Q, w)$ can easily be empty, for example if $|Q| < \ell(w)$. We do not care about empty subword complexes; we would thus like a necessary and sufficient condition on Q and w so that the subword complex is nonempty. The condition $|Q| \geqslant \ell(w)$ is necessary, but clearly not sufficient: for $w = s$ a simple transposition and Q a word of any length that only uses letters in $S\backslash\{s\}$, we can have $|Q|$ much bigger than $\ell(w) = 1$ with a subword complex $\mathrm{SC}(Q, w)$ still empty. The following section introduces the notions useful to find such a condition.

Definition 1.5.6. *The **Bruhat order** $\leqslant_B$ on W is the partial order defined by the cover relations $w <_B wt$ for $w \in W$ and t a (possibly non-simple) reflection of W, and $\ell(wt) = \ell(w) + 1$.*

It is obvious from this definition that the weak order is a suborder of the Bruhat order, i.e. if $u \leqslant v$ then $u \leqslant_B v$. In general, the Bruhat order has a lot more relations than the weak order, as seen in Fig. 1.16: for example $2143 <_B 4123$ is a cover of the Bruhat order between two permutation that are not comparable in the weak order. See [BB05, Chapter 2] for a complete study of the Bruhat order.

We note that those two examples show that the Bruhat order is generally not a lattice. However, it still has some of the other properties of the weak order.

Proposition 1.5.7. *The Bruhat order on W is graded by the length function, and has a minimum element e and a maximum element w_0.*

Remark 1.5.8. For t a reflection of W, we know from Proposition 1.3.5 that $w^{-1}tw$ is also a reflection for any $w \in W$, so $tw = w(w^{-1}tw)$ and if $\ell(tw) = \ell(w) + 1$ then tw covers w in the Bruhat order. The definition of the Bruhat order is symmetric.

The left weak order described in Remark 1.4.16 is thus also a suborder of the Bruhat order. However, the Bruhat order is stronger than even the union of the

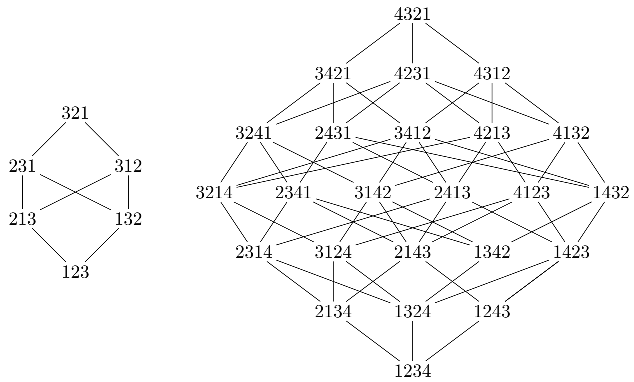


Figure 1.16: The bruhat order on $\mathfrak{S}_3$ and $\mathfrak{S}_4$.

right and left weak order: the cover $2143 \lessdot_B 4123$ is not a relation in either of them.

The following theorem gives an alternate characterization for the Bruhat order.

Theorem 1.5.9 ([BB05, Coro. 2.2.3]). *For u, v two elements of W, the following statements are equivalent:*

1. *$u \leqslant_B v$;*
2. *there is a reduced expression of v that has an expression of u as a subword;*
3. *any reduced expression of v has a reduced expression of u as a subword.*

This definition is obviously relevant to our question on subword complexes: if Q is a reduced word, then $\mathrm{SC}(Q, w)$ is non-empty iff w is below the product of Q in the Bruhat order. However, while this condition is always sufficient, this does not always work if Q is not reduced: if $Q = ss$ (with s a simple reflection) for example, the product of the letters of Q is e, and so s is not lower than this product in the Bruhat order, but $\mathrm{SC}(Q, s)$ contains two facets $\{1\}$ and $\{2\}$ and is not empty. We still need one last definition.

Definition 1.5.10. *The* ***Demazure product*** *of a word $Q = q_1 \dots q_m$ on the simple transpositions of W, denoted by $\mathrm{Dem}(Q)$, is the element of W defined recursively as follows:*

- *if $m = 0$ then $\mathrm{Dem}(Q) = e$;*
- *if $m \geqslant 1$ then*

$$\mathrm{Dem}(Q) = \begin{cases} \mathrm{Dem}(q_1 q_2 \dots q_{m-1}) \cdot q_m & \text{if } \mathrm{Dem}(q_1 q_2 \dots q_{m-1})(\alpha_m) > 0 \\ \mathrm{Dem}(q_1 q_2 \dots q_{m-1}) & \text{if } \mathrm{Dem}(q_1 q_2 \dots q_{m-1})(\alpha_m) < 0. \end{cases}$$

This definition can be understood as follows: the Demazure product is obtained by considering each letter of Q in order, and multiplying the result by this letter if this increases the result in the weak order, and ignoring the letter otherwise. This leads us to the following theorem, which is the characterization that we wanted:

Theorem 1.5.11 ([KM04, Lem. 3.4])**.** *A subword complex* $\mathrm{SC}(Q, w)$ *is nonempty iff* $w \leqslant_B \mathrm{Dem}(Q)$.

Proof. Since Q contains a reduced expression of $\mathrm{Dem}(Q)$, and from Theorem 1.5.9 this reduced expression contains a reduced expression of any $w \leqslant_B \mathrm{Dem}(Q)$, it is obvious that $w \leqslant_B \mathrm{Dem}(Q) \Rightarrow \mathrm{SC}(Q, w) \neq \varnothing$.

Conversely, suppose $\mathrm{SC}(Q, w)$ is nonempty, i.e. that Q contains a reduced expression of w, and let F be a facet. We will proceed by recursion on $|Q|$. If $|Q| = 0$, then necessarily $w = e = \mathrm{Dem}(Q)$ and so $w \leqslant_B \mathrm{Dem}(Q)$. If $|Q| > 0$, suppose that the theorem is true for any word of length at most $|Q|-1$ and denote by Q' the word obtained by removing the last letter of Q, and by s the last letter of Q. Then $\mathrm{Dem}(Q)$ is either $\mathrm{Dem}(Q')$ or $\mathrm{Dem}(Q')s$ and so $\mathrm{Dem}(Q') \leqslant_B \mathrm{Dem}(Q)$ and $\mathrm{Dem}(Q)s < \mathrm{Dem}(Q)$.

- If F contains the last letter of Q, then the reduced expression of w associated to F is contained in Q'. This means that the restriction of F to Q' is a facet of $\mathrm{SC}(Q', w)$, and so by induction hypothesis $w \leqslant_B \mathrm{Dem}(Q') \leqslant_B \mathrm{Dem}(Q)$.

- If F does not contain the last letter of Q, then the reduced expression of w ends with s and so $ws < w$. Then the restriction of F to Q' is a facet of $\mathrm{SC}(Q', ws)$ and by induction hypothesis $ws \leqslant_B \mathrm{Dem}(Q') \leqslant \mathrm{Dem}(Q)$. Since we have $\mathrm{Dem}(Q)s < \mathrm{Dem}(Q)$, we can consider $s_1 s_2 \dots s_l$ a reduced word of $\mathrm{Dem}(Q)$ ending with s. By Theorem 1.5.9, there exists $1 \leqslant i_1 < \ldots < i_k \leqslant l$ such that $s_{i_1} \dots s_{i_k}$ is a reduced word of ws, and since $ws < w = (ws)s$ we know that this subword does not ends with s so it does not contains the last letter and $i_k < l$. Then the word $s_{i_1} \dots s_{i_k} s_l$ is a reduced expression of w and a subword of a reduced expression of $\mathrm{Dem}(Q)$, and so $w \leqslant_B \mathrm{Dem}(Q)$ again. □

1.5.3 Root function

The root function was introduced by C. Ceballos, J.-P. Labbé and C. Stump in [CLS14] as a way to study flips on a subword complex. We will define flip in Section 1.5.4; for now, let us simply define this function and give some first results on its values.

Definition 1.5.12. *For* $Q = q_1 q_2 \dots q_m$ *a word and* $w \in W$*, the* ***root function*** *on the facets of the subword complex* $\mathrm{SC}(Q, w)$ *is defined as follows for* F *a facet and* $1 \leqslant k \leqslant m$*:*

$$\mathbf{r}(F, k) = \Big(\prod_{1 \leqslant i < k,\, i \notin F} q_i\Big)(\alpha_k).$$

The ***root configuration*** *of a facet is then* $\mathbf{R}(F) = \{\mathbf{r}(F, k) \mid k \in F\}$.

We know from Proposition 1.4.13 that the values of the root function on the complement of a facet (i.e. on the set of indices that are a reduced word of w) are exactly the inversions of w. The root configuration contains the values on the other indices.

We note that the root function cannot take any value in Φ, as expressed by the following lemma.

Lemma 1.5.13 ([PS15a, Lem. 3.3])**.** *The root configuration of a facet F of a subword complex $\mathrm{SC}(Q, w)$ is contained in $\Pi \cup -\operatorname{inv}(w)$. Moreover, if $\phi \in \operatorname{inv}(w)$ and i is the only index such that $i \notin F$ and $\mathbf{r}(F, i) = \phi$, then for $1 \leqslant j \leqslant |Q|$:*

1. *if $\mathbf{r}(F, j) = \phi$ then $j \leqslant i$;*
2. *if $\mathbf{r}(F, j) = -\phi$ then $j > i$.*

Proof. Let F be a facet of $\mathrm{SC}(Q, w)$ with $Q = q_1 \ldots q_m$ and consider $1 \leqslant j \leqslant |Q|$. Let us denote by w_j the product $\prod_{1 \leqslant i < j,\, i \notin F} q_i$. This product is by definition a prefix to a reduced word of w, and so $w_j \leqslant w$ and $\operatorname{inv}(w_j) \subseteq \operatorname{inv}(w)$. By definition, we know that $w_j(\Pi) = \operatorname{ninv}(w_j) \cup -\operatorname{inv}(w_j)$, and so since $\mathbf{r}(F, j)$ is the image of a simple root by w_j, we know that $\mathbf{r}(F, j) \in \operatorname{ninv}(w_j) \cup -\operatorname{inv}(w_j)$. Since $\operatorname{ninv}(w_j) \subseteq \Pi$ and $\operatorname{inv}(w_j) \subseteq \operatorname{inv}(w)$, we obtain that $\mathbf{r}(F, j) \in \Pi \cup -\operatorname{inv}(w)$.

Moreover, if $\mathbf{r}(F, j) \in \Pi$, then it is in $\operatorname{ninv}(w_j)$, and so there exists no $i < j$ such that $i \notin F$ and $\mathbf{r}(F, i) = \mathbf{r}(F, j)$. Conversely, if $\mathbf{r}(F, j) \in -\Pi$, then it is in $-\operatorname{inv}(w_j)$ and so there exists $i < j$ such that $\mathbf{r}(F, i) = -\mathbf{r}(F, j)$. This proves the second part of the lemma. $\square$

Contact graph

In the sorting network representation, the root function of a facet on an index is given by the two pipes adjacent to the corresponding letter. For example, for the facet $\{1, 5\}$ in Fig. 1.15, the values of the root function on each letter are (in order) $(1, 2), (2, 3), (1, 3), (1, 2), (3, 2)$. They respectively correspond to the geometric roots $e_2 - e_1, e_3 - e_2, \ldots, e_2 - e_3$. Note that the order is important: it gives the sign of the root. Here, they are all positive except for the last one.

An alternate way to represent the root configuration in this case is the contact graph of a facet.

Definition 1.5.14. *The **contact graph** of a facet F is the directed graph $F^{\#}$ that has the pipes of F as vertices and an arc $p \to q$ if there is an unused letter $\overset{p}{\underset{q}{\mathrm{I}}}$ in the sorting network representation of F.*

An example of the contact graph of the facet $\{1, 5\}$ of Fig. 1.15 is given in Fig. 1.17. Note that only the values of the root function on unused letters are represented. The properties of the contact graph and of the root configuration are linked in the following way.

Proposition 1.5.15. *For F a facet of a type A subword complex $\mathrm{SC}(Q, \omega)$ and for $p \neq q$ two pipes of F:*

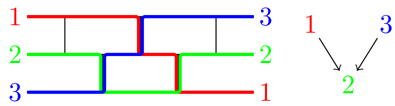


Figure 1.17: A facet of $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_2\tau_2, 321)$ and its contact graph.

1. $e_q - e_p \in \mathbf{R}(F) \iff p \to q$ *is an arc of* $F^\#$*;*
2. $e_q - e_p \in \mathrm{Cone}(\mathbf{R}(F)) \iff$ *there is a path from* p *to* q *in* $F^\#$*;*
3. $\mathrm{Cone}(\mathbf{R}(F))$ *is a pointed cone* $\iff$ $F^\#$ *is acyclic.*

Proof. The first item is an immediate consequence of the definition of the contact graph.

For the second one, suppose first that there is a path $p \to r_1 \to \ldots \to r_k \to q$ in $P^\#$ for some $k \geqslant 0$. We will prove by induction on k that $e_q - e_p \in \mathrm{Cone}(\mathbf{R}(F))$. If $k = 0$, then by the first item we know that $e_q - e_p \in \mathbf{R}(F)$. If $k > 0$, then we know that $p \to r_1$ is an arc of $F^\#$, so $e_{r_1} - e_p \in \mathrm{Cone}(\mathbf{R}(F))$, and there is a path $r_1 \to \ldots \to r_k \to q$ in $F^\#$ so by induction hypothesis $e_q - e_{r_1} \in \mathrm{Cone}(\mathbf{R}(F))$. Therefore, we can add those two roots to obtain that $(e_q - e_{r_1}) + (e_{r_1} - e_p) = e_q - e_p$ is in $\mathrm{Cone}(\mathbf{R}(F))$.

Suppose now that $e_q - e_p \in \mathrm{Cone}(\mathbf{R}(F))$ and write $e_q - e_p = \sum_{j=1}^{k} \lambda_j \alpha_j$ with all $\lambda_j > 0$, all $\alpha_j \in \mathbf{R}(F)$ and k minimal. Consider the nonempty subgraph G of $F^\#$ containing only the arcs corresponding to α_j for some j. Then if G contains a cycle, denote by C the set of roots corresponding to its arcs; we can easily see that $\sum_{\alpha \in C} = 0$, and so for $\lambda = \min\{\lambda_j \mid \alpha_j \in C\}$ we can remove $\lambda \sum_{\alpha \in C} \alpha$ from $e_q - e_p$ to obtain a new expression of this root as a positive linear combination of the α_j with one less term. Thus, by minimality of k, the graph G is acyclic. Moreover, if vertex r is a sink of G, the coefficient in front of e_r in $\sum_{j=1}^{k} \lambda_j \alpha_j$ is strictly positive, and if it is a source, its coefficient is strictly negative. Therefore, graph G has only one source p and one sink q, and so any maximal path in it is from p to q. Since G is nonempty and acyclic such a path exists, and since G is a subgraph of $F^\#$ we obtain that there is a path from p to q in $F^\#$.

For the third item, suppose that $F^\#$ is not acyclic and so that there exists p and q such that there is a path from p to q and a path from q to p for some vertices $p \neq q$. Then from the previous item we know that $e_q - e_p$ and $e_p - e_q$ are both in $\mathrm{Cone}(\mathbf{R}(F))$, and so the line $\mathbb{R}(e_q - e_p)$ is in this cone and the cone is not pointed. Conversely, if $F^\#$ is acyclic, then consider π a linear extension of this graph. For any arc $p \to q$ of $F^\#$ we know that $\pi^{-1}(p) < \pi^{-1}(q)$ and so $e_q - e_p \in \pi(\Pi)$. Therefore $\mathbf{R}(F) \subseteq \pi(\Pi)$ and since $\mathrm{Cone}(\pi(\Pi)) = \mathrm{Cone}(\pi(\Delta))$ is pointed (since Δ is a free family of vectors), so is $\mathrm{Cone}(\mathbf{R}(F))$. $\square$

This representation is very convenient for our purpose and we will use it instead of the geometric root configuration in most of our proofs in type A.

1.5.4 Flips and their properties

To measure proximity between facets of a subword complex, we will define a way to go from facet to facet by changing only one letter.

Definition 1.5.16. *Two facets F_1 and F_2 of a subword complex are linked by a* ***flip*** *if they differ by exactly one index, i.e. there exists $i_1 \neq i_2$ such that $F_1 = (F_2 \cup \{i_1\}) \backslash i_2$. The flip is* ***increasing*** *from F_1 to F_2 if $i_1 < i_2$ and* ***decreasing*** *otherwise.*

Two facets linked by a flip are very similar and so we can determine the values of the root function on one from its values on the other.

Lemma 1.5.17 ([CLS14, Lem. 3.6])**.** *Consider F_1 and F_2 facets of a subword complex linked by a flip and such that $F_1 = (F_2 \cup \{i_1\}) \backslash i_2$, and suppose that $i_1 < i_2$, i.e. the flip is increasing from F_1 to F_2. Then for any index k:*

- *if $k \leqslant i_1$ or $k > i_2$, then $\mathbf{r}(F_1, k) = \mathbf{r}(F_2, k)$;*
- *if $i_1 < k \leqslant i_2$, then $\mathbf{r}(F_1, k) = s_{\mathbf{r}(F_1,i_1)}(\mathbf{r}(F_2, k))$.*

Proof. Let F_1, F_2 facets of $\mathrm{SC}(q_1 q_2 \ldots q_m, w)$ with an increasing flip from F_1 to F_2, and denote by i_1 the only index in $F_1 \backslash F_2$ and i_2 the one in $F_2 \backslash F_1$. For any index i, we denote by α_i the simple root corresponding to the simple reflection q_i.

Since F_1 and F_2 are the same before index i_1, it is obvious that $\mathbf{r}(F_1, k) = \mathbf{r}(F_2, k)$ for any $1 \leqslant k \leqslant i_1$. Similarly, we note that $\prod_{1 \leqslant j < k,\, j \notin F_1} q_j = w(\prod_{k \leqslant j \leqslant m, j \notin F_1} q_j)^{-1}$, and so if F_1 and F_2 are the same after k (index k included) then $\mathbf{r}(F_1, k) = \mathbf{r}(F_2, k)$. It is the case when $k > i_2$.

Consider now $i_1 < k \leqslant i_2$. We will denote by $w_1 = \prod_{1 \leqslant j < i_1,\, j \notin F_1} q_j$ the prefixe of w defined by F_1 and F_2 on $q_1 \ldots q_{i_1 - 1}$ and $w_2 = \prod_{i_1 < j < k,\, j \notin F_1} q_j$ the subword defined by the same two facets on $q_{i_1+1} \ldots q_{k-1}$. Then by definition we know that $\mathbf{r}(F_1, k) = w_1 w_2(\alpha_k)$ and $\mathbf{r}(F_2, k) = w_1 q_{i_1} w_2(\alpha_k)$. Moreover, from Proposition 1.3.5 and since $\mathbf{r}(F_1, i_1) = w_1(\alpha_{i_1})$, we know that $s_{\mathbf{r}(F_1,i_1)} = w_1 q_{i_1} w_1^{-1}$. Therefore:

$$\begin{aligned}
\mathbf{r}(F_1, k) &= w_1 w_2(\alpha_k) \\
&= w_1 q_{i_1}^2 w_2(\alpha_k) \\
&= w_1 q_{i_1} w_1^{-1} w_1 q_{i_1} w_2(\alpha_k) \\
&= s_{\mathbf{r}(F_1,i_1)} w_1 q_{i_1} w_2(\alpha_k) \\
&= s_{\mathbf{r}(F_1,i_1)}\, \mathbf{r}(F_2, k)
\end{aligned}$$

This concludes the proof. □

From this lemma and Theorem 1.4.4, we obtain the following characterization of indices allowing a flip.

Proposition 1.5.18 ([PS15a, Lem. 3.3])**.** *Let F be a facet of a subword complex $\mathrm{SC}(Q, w)$. For $i \in F$, there exists a facet F' linked to F by a flip and such that $F \backslash F' = \{i\}$ if and only if $\mathbf{r}(F, i) \in \mathrm{inv}(w) \cup -\mathrm{inv}(w)$. Moreover, that flip is increasing if $\mathbf{r}(F, i) \in \mathrm{inv}(w)$ and decreasing if $\mathbf{r}(F, i) \in -\mathrm{inv}(w)$.*

Proof. Consider $Q = q_1q_2\dots q_m$ and for each $1 \leqslant k \leqslant m$, denote by α_k the simple root associated to the simple reflection q_k. We choose F a facet of $\mathrm{SC}(Q,w)$ and denote by $w_k = \prod_{1\leqslant j<k,\, j\notin F} q_j$ the prefix of w defined by F on $q_1\dots q_{k-1}$.

Suppose first that for some $i \in F$, we have $\phi := \mathbf{r}(F,i) \in \mathrm{inv}(w)$. Then from Proposition 1.4.13 there exists an index $j \notin F$ such that $\mathbf{r}(F,j) = \phi$, and by Lemma 1.5.13 $i < j$. Since $\phi = w_i(\alpha_i) = w_j(\alpha_j)$, by Proposition 1.3.5 we know that $s_\phi = w_iq_iw_i^{-1} = w_jq_jw_j^{-1}$. Thus $w_i^{-1}q_\phi w_j = q_iw_i^{-1}w_j = w_i^{-1}w_jq_j$. Since $w_i^{-1}w_j = \prod_{i<k<j,\, k\notin F} q_k$, this means that removing i from F and adding j gives a new facet F' of $\mathrm{SC}(Q,w)$ (with $F = (F' \cup \{i\})\backslash\{j\}$) and so there is an ascending flip from F to a facet that does not contain i. The reasoning is the same but with $j < i$ if $\phi \in -\mathrm{inv}(w)$.

Suppose now that there exists a flip from F to F' such that $F\backslash F' = \{i\}$, and denote by j the only index in $F'\backslash F$. If $i < j$ then from Lemma 1.5.17 we know that $\mathbf{r}(F,i) = \mathbf{r}(F',i)$ and since $i \notin F'$, from Proposition 1.4.13 we have $\mathbf{r}(F',i) \in \mathrm{inv}(w)$. If $j < i$, then similarly $\mathbf{r}(F,j) = \mathbf{r}(F',j) \in \mathrm{inv}(w)$ because $j \notin F$. Denote by k the only index such that $k \notin F'$ and $\mathbf{r}(F',k) = \mathbf{r}(F,j)$; from Lemma 1.5.13 we know that $k > j$. We know that $\mathbf{r}(F,k)$ is either $\mathbf{r}(F',k) = \mathbf{r}(F,j)$ or $s_{\mathbf{r}(F,j)}(\mathbf{r}(F',k)) = s_{\mathbf{r}(F,j)}(\mathbf{r}(F,j)) = -\mathbf{r}(F,j)$. Since $k > j$, this means that it must be $-\mathbf{r}(F,j)$ and in particular it is negative and not in $\mathrm{inv}(w)$, which means that $k \in F$. Thus $k \in F\backslash F'$ and so $k = i$, and $\mathbf{r}(F,i) = -\mathbf{r}(F,j) \in -\mathrm{inv}(w)$. This concludes the proof. □

1.5.5 Flip graph and flip poset

We can then use the increasing flips to define a structure on the facets of a subword complex.

Definition 1.5.19. *For $\mathrm{SC}(Q,w)$ a subword complex, its **flip graph** is the graph with its facets as vertices and an edge between two facets iff they are linked by a flip. Its **increasing flip graph** is the orientation of the flip graph such that an arc goes from F to F' if the corresponding flip is increasing from F to F'. The **flip poset** on $\mathrm{SC}(Q,w)$ is the order on its facets such that $F \leqslant F'$ iff there is a path from F to F' in its increasing flip graph.*

An example of the increasing flip graph is given in 1.18. We can see there that the increasing flip graph is not, in general, the Hasse diagram of the flip poset: here, one of the flips is clearly not a cover, since it is equal to the compostion of two other increasing flips.

Remark 1.5.20. We note that in the sorting network representation of a facet, doing a flip means finding an unused letter whose two adjacent pipes cross at some point, and then exchanging the unused letter and the crossing between those two pipes. This operation only changes the trajectory of said two pipes and only between the two positions that were exchanged; the other pipes stay unchanged.

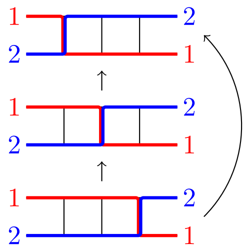


Figure 1.18: The increasing flip graph of $\mathrm{SC}(\tau_1\tau_1\tau_1, 21)$.

We note that we define the flip poset as the $\rightarrow^*$ relation on the increasing flip graph (see Example 1.1.1), which implies that this graph is acyclic. This is easy to prove, as if there is an increasing flip from a facet F to a facet F', then $F \leqslant F'$ in the lexicographic order.

Theorem 1.5.21 ([PS13, Prop. 4.8])**.** *The increasing flip graph of a subword complex is connected and has a minimum and a maximum.*

Proof. We said that the increasing flip graph is acyclic; therefore, it must have at least one minimal and one maximal element.

Suppose that F and F' are two maximal facets and that $F \neq F'$, and denote by i the first index on which they differ. WLOG, we choose $i \in F$ and $i \notin F'$. Since F and F' are maximal, then there is no increasing flip starting from either of them; from Proposition 1.5.18, this means that $\mathbf{r}(F', k) \notin \mathrm{inv}(w)$ for any index $k \in F'$. However, since F and F' are the same before index i, we know that $\mathbf{r}(F, i) = \mathbf{r}(F', i)$ and since $i \notin F$ we know from Proposition 1.4.13 that $\mathbf{r}(F, i) \in \mathrm{inv}(w)$. Thus $\mathbf{r}(F', i)$ must be in $\mathrm{inv}(w)$ but cannot be in it, which is absurd. Therefore, there is only one maximal facet in the increasing flip graph. This also proves that the increasing flip graph is connected, as an acyclic directed graph has at least one maximal element for each of its connected component.

The unicity of the minimal facet is obtained by noticing that for $Q = q_1q_2\dots q_m$ and $w \in W$, any facet F of $\mathrm{SC}(Q, w)$ is the "reverse" of a facet of $\mathrm{SC}(q_m \dots q_2q_1, w^{-1})$. This correspondance is a reversing morphism between the flip order on the two subword complexes. As such, a facet is minimal if and only if its reverse is maximal, and so there is a unique minimum to the increasing flip graph. □

Definition 1.5.22. *We say that the maximum facet in the flip order is the **greedy facet** and the minimum is the **antigreedy facet**.*

These names come from the fact that the greedy facet is obtained by a greedy algorithm: we read the letters of Q from beginning to end, and add an index to the facet iff the root function on this index is in $\mathrm{inv}(w)$. As discussed in the proof of Theorem 1.5.21, this must return the maximum of the flip order.

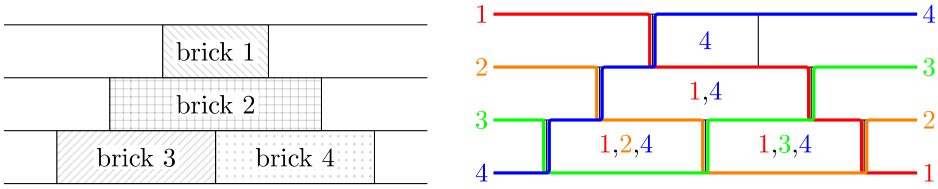


Figure 1.19: Construction of a brick vector with $Q = \tau_3\tau_2\tau_1\tau_3\tau_1\tau_2\tau_3$.

1.5.6 The brick polyhedron

Finally, we will introduce an object that will be central to our work: the brick polyhedron of a subword complex. This object was first introduced by V. Pilaud and F. Santos in [PS12] as the brick polytope of a specific type of subword complexes in type A Coxeter groups using the sorting network representation. This definition was then extended to some subword complexes in all finite Coxeter group by V. Pilaud and C. Stump in [PS15a], and then to any subword complex as the brick polyhedron by D. Jahn and C. Stump in [JS21]. We will give the original definition of the brick polytope and the two extensions.

Brick polytopes on sorting network

In this section we will only consider subword complexes on Coxeter groups of type A, whose elements are the permutations.

Definition 1.5.23. *A word Q on the simple transpositions of $\mathfrak{S}_n$ is **sorting** if it contains an expression of the longest permutation $\omega_0 = n(n-1)\dots21$ as a subword. The sorting network associated to such a word is a **sorting network**.*

All subword complexes on sorting networks are nonempty, but we will now only consider the one representing the longest permutation ω_0.

Let us consider the sorting network associated to the word $\tau_3\tau_2\tau_1\tau_3\tau_1\tau_2\tau_3$ and drawn in Fig. 1.19. We see that the vertical lines representing letters divide the space between horizontal lines in four "bricks". This divide is possible on any sorting network.

Definition 1.5.24. *For Q a sorting word and F a facet of the subword complex $\mathrm{SC}(Q, \omega_0)$, its **brick vector** $\Lambda(F) = (b_1, \dots, b_n) \in \mathbb{R}^n$ is such that b_i is the number of bricks under pipe i in its sorting network representation.*

In the facet represented on the right in Fig. 1.19, we wrote in each brick the number of all the pipes above that brick. We see for example that pipe 1 is above the three bricks 2, 3 and 4. By counting the occurrences of each pipe label in the bricks, we obtain $(3, 1, 1, 4)$ as a brick vector.

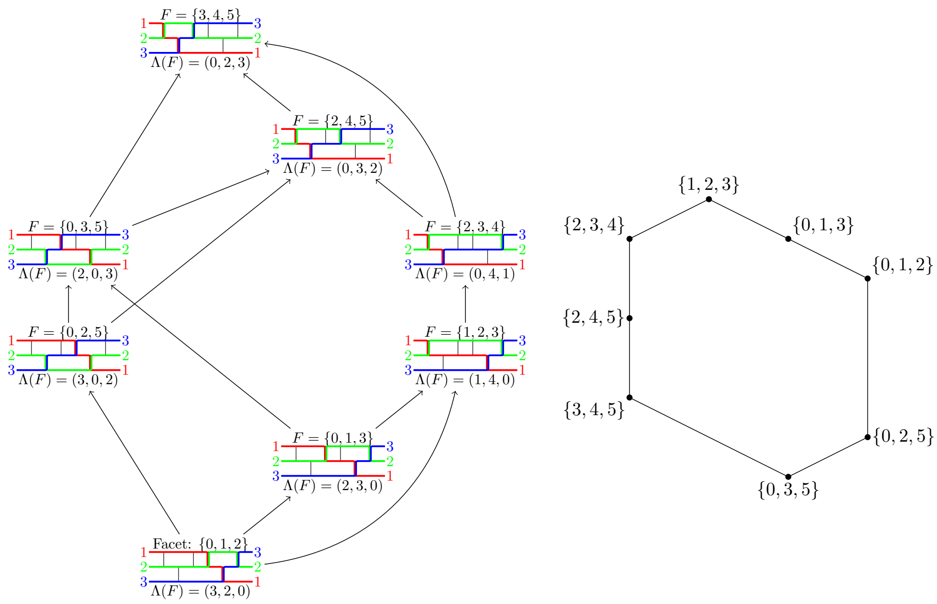


Figure 1.20: The flip graph and brick polytope of $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_1\tau_2\tau_1, \omega_0)$.

Definition 1.5.25. *For Q a sorting network, the* ***brick polytope*** $\Omega(Q)$ *is the convex hull of the brick vectors of all the facets of* $\mathrm{SC}(Q, \omega_0)$.

We note that since each brick is counted the same number of times in each brick vector (as the number of pipes above this brick is constant), the sum of the coordinates is the same for every brick vector on a fixed sorting network Q. The brick polytope is therefore of dimension at most $n-1$.

Example 1.5.3. The flip graph of $\mathrm{SC}(\tau_1\tau_2\tau_1\tau_1\tau_2\tau_1, \omega_0)$ is drawn on the left of Fig. 1.20, with the brick vector of each facet given below its representation. On the left are represented the brick vectors of the facets projected on the plane $x+y+z=5$. We note that not all brick vectors are vertices of the brick polytope and that all edges are flips, but not all flips are along edges.

If we consider an increasing flip between two facets F_1 and F_2, we know from Remark 1.5.20 that only the trajectory of two pipes p and q will be changed (the pipes adjacent to the two positions that were exchanged in the flip). This means that $\Lambda(F_1)$ and $\Lambda(F_2)$ only differ on their p-th and q-th positions. Moreover, if we denote by b the number of bricks between pipes p and q and between the two positions exchanged in the flip, we see that $\Lambda(F_1)_p = \Lambda(F_2) - b$ and $\Lambda(F_1)_q = \Lambda(F_2)_q + b$ if p is below q in the sorting network representation of F_1, and the opposite if it is above q. This means that $\Lambda(F_1) = \Lambda(F_2) \pm b(e_p - e_q)$. Since $e_p - e_q$ is a root of $\mathfrak{S}_n$, and more specifically the value of the root function on the indices of F_1 and F_2 that were flipped, this leads to the following proposition.

Proposition 1.5.26 ([PS12, Thm 3.13])**.** *The incidence cone of $\Omega(Q)$ at the brick vector $\Lambda(F)$ is the cone generated by the root configuration of F. In particular, a facet of $\mathrm{SC}(Q, \omega_0)$ is associated to a vertex of $\Omega(Q)$ if and only if the cone generated by its root configuration is pointed. In that case, we say that the facet is **acyclic**.*

The previous remark proves immediately that $\mathrm{Cone}(\mathbf{R}(F))$ is contained in the incidence cone, since we know from Proposition 1.5.18 that there is an increasing flip on facet F involving any index $i \in F$ (because $\mathrm{inv}(\omega_0)$ contains all the positive roots). The other inclusion is obtained by proving that any facet of $\mathrm{Cone}(\mathbf{R}(F))$ is also a facet of the incidence cone of $\Omega(Q)$ at $\Lambda(F)$.

This term "acyclic" comes from the representation of the root configuration by a contact graph in type A: we saw in Proposition 1.5.15 that in this case, a facet is acyclic if and only if its contact graph is acyclic.

Brick polytopes in general Coxeter groups

In this section we will extend this definition to any finite Coxeter group W, while still limiting ourselves to subword complexes representing the longest element of the group. The goal is to obtain a polytope that is isommorphic to $\Omega(Q)$ when W is a type A Coxeter group and to keep the properties of incidence cones determined previously.

Definition 1.5.27. *A word Q on the simple reflections of W is **sorting** if it has an expression of the longest element w_0 of W as a reduced word.*

To define the brick polytope in that context, we will use an object very similar to the root function using fundamental weights instead of simple roots.

Definition 1.5.28. *For $Q = q_1 q_2 \dots q_m$ a word and $w \in W$, the **weight function** on the facets of the subword complex $\mathrm{SC}(Q, w)$ is defined as follows for F a facet and $1 \leqslant k \leqslant m$:*

$$\mathbf{w}(F, k) = (\prod_{1 \leqslant i < k,\, i \notin F} q_i)(\omega_{q_k}).$$

*The **brick vector** of a facet $F \in \mathrm{SC}(Q, w)$ is then $\mathrm{B}(F) = \sum_{1 \leqslant k \leqslant m} \mathbf{w}(F, k)$.*

*If Q is sorting, the **brick polytope** $\mathcal{B}(Q)$ of $\mathrm{SC}(Q, w_0)$ is then the convex hull of the brick vectors of all its facets.*

A reasoning similar to the one in Lemma 1.5.17 shows that if F_1 and F_2 are facets linked by an increasing flip exchanging the indices i and j, then $\mathbf{w}(F_1, k)$ is either $\mathbf{w}(F_2, k)$ or $s_{\mathbf{r}(F_1,i)}(\mathbf{w}(F_2, k))$. Since for any root α and any vector v we know from the definition of reflections that $v - s_\alpha(v)$ is a multiple of α, this shows that like previously $\mathrm{B}(F_1) - \mathrm{B}(F_2)$ is going to be a multiple of $\mathbf{r}(F_1, i)$. We can then prove the following theorem.

Theorem 1.5.29 ([PS15a, Prop. 4.7]). *The incidence cone of $\mathcal{B}(Q)$ at the brick vector* $\mathrm{B}(F)$ *is the cone generated by the root configuration of* F. *In particular, a facet of* $\mathrm{SC}(Q, w_0)$ *is associated to a vertex of* $\mathcal{B}(Q)$ *if and only if the cone generated by its root configuration is pointed. In that case, we say that the facet is* ***acyclic***.

This shows that the properties of $\mathcal{B}(Q)$ are very similar to those of $\Lambda(Q)$ when W is of type A: the vertices are the same and the incidence cone at each brick vector are the same. However, this does not prove by itself that the graph of those two polytopes are the same, for example. To obtain the following proposition, we must consider closely the definitions of $\Omega(Q)$ and $\mathcal{B}(Q)$.

Proposition 1.5.30. *If* W *is a Coxeter group of type* A *and* Q *a sorting word, then* $\mathrm{B}(Q)$ *and* $\Lambda(Q)$ *are translated of each other.*

Proof. Consider a facet $F \in \mathrm{SC}(Q, \omega_0)$. Let us consider the trajectory of a pipe p. We first note that since each pair of pipes must cross exactly once in F, the direction of the crossing between p and another pipe q depends on whether $p < q$ or $q < p$. In the first case, pipe p starts above pipe q and ends below it so p must go down while crossing q, and in the second case it must go up. We can then express $\Lambda(F)_p$ as follows: it is the total number of bricks below height p in the sorting network, plus the number of bricks after the crossing between p and q at the same level as this crossing for each $q < p$ (when p goes up), minus the number of bricks after the crossing between p and q at the same level as this crossing for each $q > p$ (when p goes down).

Consider now a simple reflection s of W, and denote by $i_0 < \ldots < i_k$ the indices of Q where $q_{i_j} = s$. By definition of the fundamental weight ω_s (chosen such that $t(\omega_s) = \omega_s$ for any simple reflection $t \neq s$), we know that $\mathbf{w}(F, i_0) = \omega_s$, and we can see that $\mathbf{w}(F, i_j) = \mathbf{w}(F, i_{j-1}) - \delta_{i_{j-1} \notin F}\, \mathbf{r}(F, i_{j-1})$ if $j > 0$. This means that in general $\mathbf{w}(F, i_j) = \omega_s - \sum_{l<j,\, i_l \notin F} \mathbf{r}(F, i_l)$ and so by summing this on all these indices $\sum_{0 \leqslant j \leqslant k} \mathbf{w}(F, i_j) = (k+1)\omega_s - \sum_{0 \leqslant j \leqslant k,\, i_j \notin F} (k-j)\, \mathbf{r}(F, i_j)$. By adding these values for each simple reflection of W, we obtain

$$\mathrm{B}(F) = \sum_{i=1}^{|Q|} \omega_{q_i} - \sum_{i \notin F} |\{j > i \mid q_j = q_i\}|\, \mathbf{r}(F, i).$$

However, we note that the number $|\{j > i \mid q_j = q_i\}|$ is exactly the number of bricks at the same level as the letter q_i after the vertical bar corresponding to this letter, and that this number multiplies a root $\mathbf{r}(F, i)$ which is $e_q - e_p$ for p and q the pipes crossing on that vertical bar and $p < q$. Therefore, the p-th coordinate of $\mathrm{B}(F)$ will be the p-th coordinate of $\sum_{1 \leqslant i \leqslant |Q|} \omega_{q_i}$, plus the number of bricks after the crossing between p and q at the same level as this crossing for each $q < p$, minus the number of bricks after the crossing between p and q at the sam elevel as this crossing for each $q > p$.

This means that $\mathrm{B}(F) - \Lambda(F)$ only depends on the number of bricks at each level in the sorting network Q and $\sum_{1 \leqslant i \leqslant |Q|} \omega_{q_i}$, and both are constant for all facets of $\mathrm{SC}(Q, \omega_0)$. This concludes the proof. $\square$

Brick polyhedra

A further generalization of brick polytopes was obtained by noting that any subword complex can be seen as part of a subword complex representing the longest element w_0 as follows: if Q is any word and $w \in W$ such that $w \leqslant \mathrm{Dem}(Q)$, then let Q' be the word obtained by appending a reduced word of $w^{-1}w_0$ to Q. Then Q' is a sorting word, and any facet of $\mathrm{SC}(Q, w)$ is also a facet of $\mathrm{SC}(Q', w_0)$. However, in most cases, some of the facet of $\mathrm{SC}(Q', w_0)$ are not associated to facets of $\mathrm{SC}(Q, w)$, as explained in the following lemma.

Lemma 1.5.31. *The facets of* $\mathrm{SC}(Q', w_0)$ *are exactly the facets of* $\mathrm{SC}(Q, w)$ *if and only if* $\mathrm{Dem}(Q) = w$.

Proof. A facet of $\mathrm{SC}(Q', w_0)$ comes from a facet of $\mathrm{SC}(Q, w)$ iff it only contains indices up to $|Q|$. Since all facets of $\mathrm{SC}(Q', w_0)$ are lexicographically below its greedy facet, this means that all facets come from $\mathrm{SC}(Q, w)$ if the greedy facet does. However, the greedy algorithm giving the maximum facet $F^{\uparrow}$ of $\mathrm{SC}(Q', w_0)$ behaves exactly as computing the Demazure product of a word does: if the root function at an index is negative, then that index is added to the facet, and if it is positive, the letter is used in the subword defined by the facet. This means that the subword defined by $F^{\uparrow}$ restricted to Q is a reduced word of $\mathrm{Dem}(Q)$. Thus, this facet comes from a facet of $\mathrm{SC}(Q, w)$ if and only if $\mathrm{Dem}(Q) = w$. $\square$

We thus want to define an object that contains the structure of the brick polytope $\mathcal{B}(Q')$ on the facets corresponding to facets of $\mathrm{SC}(Q, w)$, and only on them. This object, defined by D. Jahn and C. Stump in [JS21], will be the brick polyhedron.

The first step of this definition is studying the cones generated by the root configurations of facets of $\mathrm{SC}(Q, w)$. For $w = w_0$, we note that the greedy facet has its root configuration entirely contained in $w_0(\Pi) = -\Pi$ and the antigreedy facet has its root configuration in Π, therefore the intersection of the cones generated by root configuration of facets is $\{0\}$.

Definition 1.5.32. *The* ***Bruhat cone*** *of a pair* $u, v \in W$ *such that* $u \leqslant_B v$ *is*

$$\mathcal{C}^+(u, v) = \mathrm{Cone}(\alpha \in \Pi \mid u \lessdot_B s_\alpha u \leqslant_B v).$$

The rays of this cone are the roots $\alpha \in \Pi$ such that $u \lessdot_B s_\alpha u \leqslant_B v$, i.e. such that any reduced expression of v has a reduced subword of length $\ell(u) + 1$ representing $s_\alpha u$, and this subword contains a reduced expression of u by removing exactly one letter.

Lemma 1.5.33 ([JS21, Coro. 3.3]). *For any $w \in W$, $\mathcal{C}^+(w, w_0) \cap \Phi = \mathrm{ninv}(w)$.*

For $F^{\downarrow}$ the antigreedy facet of $SC(Q, w)$ and $F^{\uparrow}$ the greedy facet, we note that since $\mathbf{R}(F^{\downarrow}) \subseteq \Pi$ and $\mathbf{R}(F^{\uparrow}) \subseteq w(\Pi)$, the intersection of all the root cones of facets must be contained in $\mathrm{Cone}(\Pi) \cap \mathrm{Cone}(w(\Pi)) = \mathrm{Cone}(\mathrm{ninv}(w))$. Moreover, if for some facet F and some index k we have $\mathbf{r}(F, k) \in \mathrm{ninv}(w)$, then adding that letter to the subword defined by F gives a reduced subword of Q representing the element $s_{\mathbf{r}(F,k)}w$, and so $s_{\mathbf{r}(F,k)}w \leqslant_B \mathrm{Dem}(Q)$ and $w \lessdot_B s_{\mathbf{r}(F,k)}w$; thus $\mathbf{r}(F, k) \in \mathcal{C}^+(w, \mathrm{Dem}(Q))$.

The following theorem extends this idea to give the exact roots that are in the cone generated by the root configurations of every facet of a subword complex.

Theorem 1.5.34 ([JS21, Thm. 3.1]). *For Q a word and $w \leqslant_B \mathrm{Dem}(Q)$,*

$$\mathcal{C}^+(w, \mathrm{Dem}(Q)) = \bigcap_{F \text{ facet of } \mathrm{SC}(Q,w)} \mathrm{Cone}(\mathbf{R}(F)) \cap \Phi.$$

This powerful theorem allows us to define the brick polyhedron so that the properties that interested us are maintained.

Definition 1.5.35. *The **brick polyhedron** $\mathcal{B}(Q, w)$ of a subword complex $\mathrm{SC}(Q, w)$ is the Minkowski sum of the convex hull of the brick vectors of its facets with the Bruhat cone $\mathcal{C}^+(w, \mathrm{Dem}(Q))$.*

Note that if $w = \mathrm{Dem}(Q)$, then the brick polyhedron is a polytope.

By using results on brick polytopes and Theorem 1.5.34, we obtain the result that we wanted.

Theorem 1.5.36 ([JS21, Thm. 4.4]). *The incidence cone of $\mathcal{B}(Q, w)$ at the brick vector $\mathrm{B}(F)$ is the cone generated by the root configuration of F. In particular, a facet of $\mathrm{SC}(Q, w)$ is associated to a vertex of $\mathcal{B}(Q, w)$ if and only if the cone generated by its root configuration is pointed. In that case, we say that the facet is **acyclic**.*

Chapter 2

Triangular pipe dreams

This chapter will present our results on triangular pipe dreams; they were obtained in collaboration with N. Bergeron, C. Ceballos and V. Pilaud and are presented in Sections 1 to 3 of [BCCP22].

We start Section 2.1 with a definition of triangular pipe dreams and some associated concept like contact graphs in Section 2.1.1. We then explain the link between those pipe dreams and subword complexes in Section 2.1.2, as pipe dreams represent the facets of some subword complexes in type A Coxeter groups, and give some of their properties in Section 2.1.3.

Section 2.2 starts by showing an isomorphism between a certain family of triangular pipe dreams and the Tamari lattice in Section 2.2.1. This isomorphism shows that in that case, the linear extensions of contact graphs of pipe dreams define a lattice congruence of the weak order and a lattice morphism from weak order to flips. We generalize this in Section 2.2.2 to 2.2.4: first we prove in Section 2.2.2 that linear extensions define a partition of a weak order interval, then in Section 2.2.3 we prove that this partition is a lattice quotient of the weak order, and finally in Section 2.2.4 we describe the image of the quotient of the weak order by this congruence on pipe dreams. Finally, in Section 2.2.5 we give two algorithms computing the pipe dream that has a permutation as a linear extension, and a rewriting rule allowing use to determine the equivalence classes of our lattice congruence without drawing the pipe dreams.

We close this chapter with Section 2.3 by giving the link between the previously described congruence and the brick polytope in Section 2.3.1, and we explain in Section 2.3.2 how triangular pipe dreams realize the ν-Tamari lattices.

2.1 Definition and first properties

Triangular pipe dreams were introduced by N. Bergeron and S. Billey in [BB93] as object indexing the monomial of Schubert polynomials, and later revisited by A.

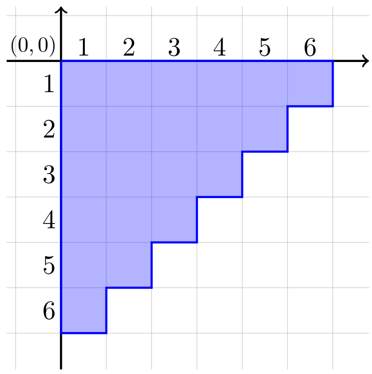


Figure 2.1: The shape $\mathcal{T}_6$ with its rows and columns labeled.

Knutson and E. Miller, who named them "pipe dreams" as a reference to puzzle video game.

2.1.1 Basic definitions

Let us consider the cartesian grid on $\mathbb{R}^2$, whose cells are the 1×1 squares with all their corners in $\mathbb{Z}^2$. For convenience, we will label the columns of this grid with the abscissa of their east side and the rows with the opposite of the ordinate of their south side. The triangular shape of size n, denoted by $\mathcal{T}_n$, is the set of cells (c, r) with $0 < c, r \leqslant n$ and $c + r \leqslant n + 1$. An example for $n = 6$, with its rows and columns labeled, is given in Fig. 2.1.

Definition 2.1.1. *A* ***triangular pipe dream*** *of size n is a filling of the shape $\mathcal{T}_n$ with crossings* $+$ *and contacts* ⌟⌜ *such that n pipes enter from the west side of the shape and exit from its north side. The n pipes are then numbered in order of their starting point from top to bottom on the western side, and the* ***exit permutation*** *of the pipe dream is the order of the exit points of the pipes from west to east along the northern side.*

We note that in order to keep the pipes inside the triangular shape and have them end on the north side, the n cells on the diagonal of the shape must contain a contact ⌟⌜ (and the south-east part of this contact does not count as an elbow of the pipe dream).

Definition 2.1.2. *A pipe dream is* ***reduced*** *if no pair of pipes cross more than once. For any permutation ω, we denote by $\Pi(\omega)$ the set of reduced pipe dreams with exit permutation ω.*

Example 2.1.1. A pipe dream with 5 pipes is given in Fig. 2.2. It is reduced, and its exit permutation is 31542.

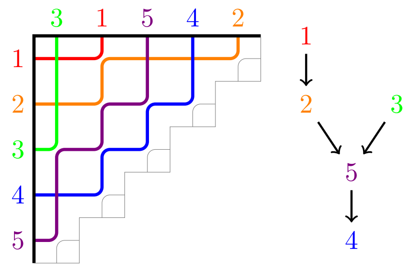


Figure 2.2: A reduced pipe dream with exit permutation 31542 and its contact graph.

Definition 2.1.3. *A contact c in a pipe dream P is **flippable** if the two pipes passing through c have a crossing x. In that case, the **flip** on c exchanges the contact c and the crossing x to obtain a pipe dream P' with the same exit permutation as P. It is **increasing** from P to P' if c is southwest of x and **decreasing** otherwise.*

*The **increasing flip graph** on $\Pi(\omega)$ is the oriented graph with $\Pi(\omega)$ as its vertices set and an edge (P, P') for each pair of pipe dreams in $\Pi(\omega)$ such that there is an increasing flip from P to P'.*

Example 2.1.2. The increasing flip graph on $\Pi(31542)$ is represented in Fig. 2.3.

We note that a flip on a contact between two pipes only changes the trajectory of those two pipes, and only between the cells c and x that were exchanged. Everything else is left unchanged.

Definition 2.1.4. *The **contact graph** of a pipe dream P is the directed graph $P^{\#}$ that has the pipes of P as its vertices and contains the edge $p \to q$ if there is a contact p q in P. The pipe dream P is **acyclic** if its contact graph is acyclic. In that case, we write $p \triangleleft_P q$ to denote that there is a path from p to q in $P^{\#}$.*

Figure 2.2 gives an example of the contact graph associated to a pipe dream.

2.1.2 Link with subword complexes

We note that the vocabulary of the pipe dreams is similar to the vocabulary of words and subword complexes: for example, in both cases we talk about reduction and flips, and the definition of both notions is very similar when comparing the pipes on crossing networks and in pipe dreams. This subsection will explain why pipe dreams are actually equivalent to the facets of some specific subword complexes.

Definition 2.1.5. *For $n \geqslant 2$ an integer, we denote by T_n the **triangular word** on simple transposition of $\mathfrak{S}_n$:*

$$T_n := \prod_{k=1}^{n-1} \tau_{n-1}\, \tau_{n-2}\, \ldots\, \tau_k.$$

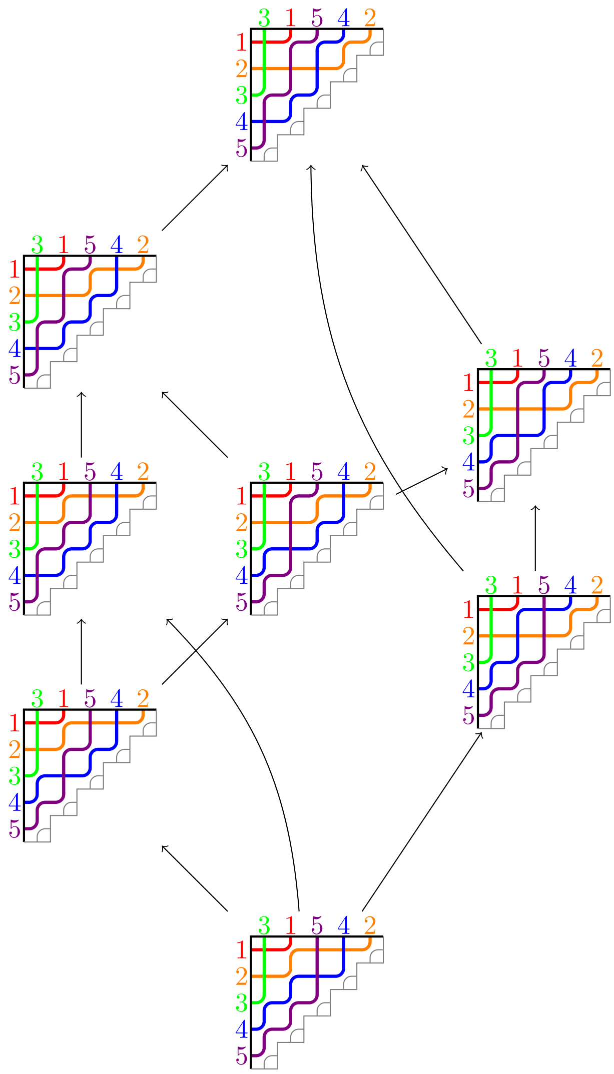


Figure 2.3: The increasing flip graph on $\Pi(31542)$.

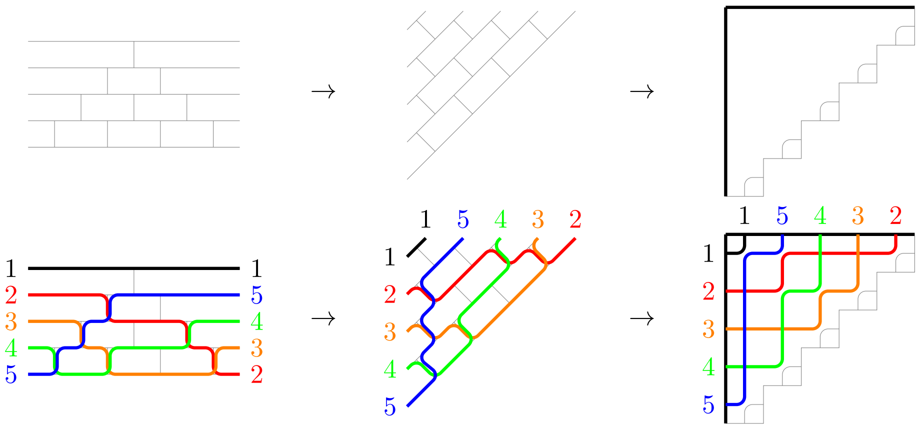


Figure 2.4: The correspondance between facets of $\mathrm{SC}(T_n, \omega)$ and $\Pi(\omega)$.

For example $T_4 = \tau_3\tau_2\tau_1\tau_3\tau_2\tau_3$. In general, the word T_n is a reduced expression of the greatest permutation of $\mathfrak{S}_n$.

Theorem 2.1.6 ([KM05, Lem. 1.4.5])**.** *For any permutation $\omega \in \mathfrak{S}_n$, the increasing flip graph on $\Pi(\omega)$ is isomorphic to the increasing flip graph on the subword complex $\mathrm{SC}(T_n, \omega)$.*

Proof. The correspondance between facets of $\mathrm{SC}(T_n, \omega)$ and pipe dreams in $\Pi(\omega)$ is given in Fig. 2.4: if we rotate the sorting network representation of T_n by an angle of $\frac{\pi}{4}$ counterclockwise, we see that to each letter of T_n corresponds an inside cell of $\mathcal{T}_n$ (the cells on the diagonal can only contain a contact and do not correspond to a letter). The indices are associated to cells by enumerating the cells from left to right and from bottom to top in each column, as illustrated in Fig. 2.5 for $n = 6$.

Note that some of the letters of the sorting network have been shifted to the left to make the representation more compact. While this means that the order of letters in T_n is not exactly the order of the letters from left to right in the sorting network, the facet represented on this network are still in bijection with the facets of the subword complex.

Consider now a facet F of $\mathrm{SC}(T_n, \omega)$: by filling a cell of $\mathcal{T}_n$ with a contact if the associated index is in F and with a cross $+$ otherwise, we obtain a pipe dream whose pipes cross in the same way as the pipes in the crossing network representation, and whose contact are exactly the pairs of pipes touching on unused letters. This means that the exit permutation is the order of the pipes at the end of the crossing network, and no pair of pipes cross more than once, so the resulting pipe dream is in $\Pi(\omega)$. Conversely, from a pipe dream in $\Pi(\omega)$ we obtain a facet of $\mathrm{SC}(T_n, \omega)$. This gives a bijection from the facets of $\mathrm{SC}(T_n, \omega)$ to $\Pi(\omega)$, and the flips on pipe dreams are trivially the image of the flips on facets. $\square$

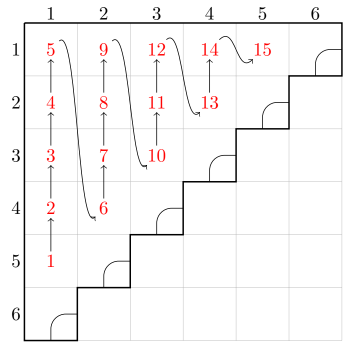


Figure 2.5: The index associated to each cell of $\mathcal{T}_6$.

Corollary 2.1.7. *The increasing flip graph on* $\Pi(\omega)$ *has one source and one sink.*

Proof. This is a simple application of Theorem 1.5.21 to $\mathrm{SC}(T_n, \omega)$. □

We will now see the link between the contact graph of a pipe dream and the root function on the associated facet, thus explaining the use of the term **acyclic** for some facets as discussed in Section 1.5.6.

Remark 2.1.8. Note that if P is a pipe dream and F the associated subword complex facet, the contact graphs $P^{\#}$ and $F^{\#}$ are the same. We can thus deduce many of the characteristics of the root configuration $\mathbf{R}(F)$ from $P^{\#}$ by using Proposition 1.5.15.

2.1.3 Some properties

We will now give a few properties and two characterizations of reduced pipe dreams.

Proposition 2.1.9 ([BCCP22, Lem. 2.3])**.** *A pipe dream P with exit permutation ω is reduced if and only if it contains exactly* $|\operatorname{inv}(\omega)|$ *crosses, and if and only if all crosses have the pipe going horizontally smaller than the pipe going vertically.*

Proof. We note that the first crossing between two pipes must have the pipe going horizontally smaller than the one going vertically, since the smaller pipe starts west of the bigger pipe. If there is a second crossing, that crossing must be in the opposite direction, as the bigger pipe is now west of the smaller one after the first crossing. Since a pipe dream is reduced if and only if there is no second crossing for any pairs of pipes, it is reduced if and only if all crossing are between a smaller pipe going horizontally and a bigger pipe going vertically.

Notice now that each pair of pipe that is an inversion must cross at least once. If the number of crossings is exactly the number of inversions, then there is a bijection between inversions and crossings and no pair can cross more than once. The pipe dream must then be reduced. Conversely, if P is reduced, then any pair (p, q) with $p < q$ crossing must start with p west of q and end after exactly one cross with q west of p, so $\omega^{-1}(q) < \omega^{-1}(p)$ and $(p, q) \in \mathrm{inv}(\omega)$. Thus the number of crosses in P is at exactly $|\,\mathrm{inv}(\omega)|$. □

Definition 2.1.10. *For ω a permutation in $\mathfrak{S}_n$ and $1 \leqslant p \leqslant n$,*

$$\mathrm{ninv}(\omega, p) := \{q < p \mid (q, p) \in \mathrm{ninv}(\omega)\}.$$

Lemma 2.1.11 ([BCCP22, Lem. 2.4])**.** *For any pipe dream $P \in \Pi(\omega)$, the pipe p has exactly*

- $|\,\mathrm{ninv}(\omega, p)|$ *southeast elbows* ╭*;*
- $|\,\mathrm{ninv}(\omega, p)| + 1$ *northwest elbows* ╯*;*
- $p - 1 - |\,\mathrm{ninv}(\omega, p)|$ *vertical crossings* ╂*;*
- $\omega^{-1}(p) - 1 - |\,\mathrm{ninv}(\omega, p)|$ *horizontal crossings* ┿*.*

Proof. Let p be a pipe of P. We will denote by N_v its number of vertical crossings, by N_h its number of horizontal crossings, by N_e its number of southeast elbows and by N'_e its number of northwest elbows. We note that since p starts in the cell $(1, p)$ and ends in the cell $(\omega^{-1}(p), 1)$, it crosses exactly $p + \omega^{-1}(p) - 1$ cells. In each of those, it must behave in one of the four ways described by the lemma, so $N_v + N_h + N_e + N'_e = p + \omega^{-1}(p) - 1$.

Since P is a reduced pipe dream, the pipe p must cross exactly once each pipe q such that $(q, p) \in \mathrm{inv}(\omega)$ and it must do so vertically; conversely, it must cross exactly once each pipe q such that $(p, q) \in \mathrm{inv}(\omega)$ and it must do so horizontally. Those are the only crossing allowed for p. Since any pipe $q < p$ is either such that $(q, p) \in \mathrm{inv}(\omega)$ or in $\mathrm{ninv}(\omega, p)$, we obtain that $N_v = p-1-|\,\mathrm{ninv}(\omega, p)|$. Similarly, since any pipe q such that $\omega^{-1}(q) < \omega^{-1}(p)$ is either such that $(p, q) \in \mathrm{inv}(\omega)$ or in $\mathrm{ninv}(\omega, p)$, we obtain that $N_h = \omega^{-1}(p) - |\,\mathrm{ninv}(\omega, p)|$.

This leaves $2|\,\mathrm{ninv}(\omega, p)|+1$ cells crossed by p containing an elbow. Since p starts horizontally and ends vertically, we can see that it has exactly one more northwest elbow than it has southeast elbows, so $N'_e = N_e+1$, and so $2N_e+1 = 2|\,\mathrm{ninv}(\omega, p)|+1$, and $N_e = |\,\mathrm{ninv}(\omega, p)|$ and $N'_e = |\,\mathrm{ninv}(\omega, p)| + 1$. This concludes the proof. □

Lemma 2.1.12 ([BCCP22, Lem. 2.5])**.** *A collection P of n pipes pairwise disjoint except when crossing such that:*

1. *the pipes are contained in $\mathcal{T}_n$, start on the west side and end on the north side;*
2. *pipe p starts in the p-th row from the top and ends in the $\omega^{-1}(p)$-th column from the west;*
3. *pipe p has exactly $|\,\mathrm{ninv}(\omega, p)|$ southeast elbows* ╭*;*

is a pipe dream of $\Pi(\omega)$.

Proof. We first note that any collection of pipes contained in $\mathcal{T}_n$ such that pipe p starts on the p-th row on the western side and ends on the $\omega^{-1}(p)$ column on the northern side is a pipe dream with exit permutation ω. We only need to prove that this pipe dream is reduced.

Suppose that P is a non-reduced pipe dream with exit permutation ω. By definition, there is a pipe p that cross at least one other pipe at least twice. Moreover, it must cross each pipe q such that (p, q) is an inversion horizontally, and each pipe q such that (q, p) is an inversion vertically. Since there are $p - 1 - |\operatorname{ninv}(\omega, p)|$ pipes in the first category and $\omega^{-1}(p) - 1 - |\operatorname{ninv}(\omega, p)|$ pipes in the second category, pipe p must have at least $p + \omega^{-1}(p) - 2|\operatorname{ninv}(\omega, p)| - 1$ crossings on its trajectory. We already saw in the proof of Lemma 2.1.11 that p crosses exactly $p + \omega^{-1}(p) - 1$ cells, so the total number of elbows of p is at most $2|\operatorname{ninv}(\omega, p)|$. Since it has one more northwest elbows as it does southwest elbows, in that case p has at most $|\operatorname{ninv}(\omega, p)| - 1$ southwest elbows. Thus a pipe dream respecting the third condition of the lemma cannot be non-reduced, and so it must be reduced. $\square$

We will now give three lemmas on the contact graph of reduced pipe dreams.

Lemma 2.1.13 ([BCCP22, Lem. 2.6]). *For P a reduced pipe dream and p, q two pipes of P, if there is an elbow of p weakly northwest of an elbow of q, then $p \lhd_P q$.*

Proof. Let us denote by x and y the cells of the respective elbows of p and q described in the lemma, with by hypothesis x weakly northwest of y. We proceed by induction on the grid distance between those two cells.

If that distance is 0, then the two elbows are in the same cell, so there is an edge between p and q in $P^{\#}$. Since by hypothesis p is northwest of q, this means that $p \to q \in P^{\#}$ and so $p \lhd_P q$. Otherwise, let p' be the pipe with a southeast elbow ╭ in x (which must exist, since y is southeast of x and so x cannot by on the diagonal of $\mathcal{T}_n$), and q' the pipe with a northwest elbow ╯ in y. We know that either $p = p'$ or $p \to p' \in P^{\#}$, and $q = q'$ or $q' \to q \in P^{\#}$. Consider the axis parallel rectangle R with x and y respectively as its northwest and southeast corners; this rectangle must be contained in $\mathcal{T}_n$. Then pipe p' goes along the north and west sides of R until it has an elbow or reaches the northeast and southwest corners; similarly, pipe q' goes along the south and east sides of R until it has an elbow or reaches those same two corners. Since P is reduced, pipes p' and q' cannot cross twice at those two corners, so one of them must have an elbows on one side of R. Suppose for example that p' has an elbow x' along the north or west side of R, then that elbow is still weakly northwest of y strictly closer to y than x, and so by induction $p' \lhd_P q$ and so $p \lhd_P q$. The same can be done if q' has an elbow on a side of R, and so in both case $p \lhd_P q$. $\square$

Lemma 2.1.14 ([BCCP22, Lem. 2.7]). *If $\omega \in \mathfrak{S}_n$ and $(p, q) \in \operatorname{ninv}(\omega)$, then $p \lhd_P q$ for any $P \in \Pi(\omega)$.*

Proof. Since $(p, q) \in \operatorname{ninv}(\omega)$, pipes p and q do not cross, and so pipe p must be northwest of pipe q on their while trajectories. Pipe p has at least one elbow since it starts horizontally and ends vertically; denote by (c, r) the coordinate of a cell containing an elbow of p. Note that $r \leqslant p < q$ and $c \leqslant \omega^{-1}(p) < \omega^{-1}(q)$, so pipe q

must start southwest of (c, r) and end northeast of it, and it must cross column c while going horizontally and row r while going vertically at some points. It must thus have an elbow between those two crossings. Since q stays southeast of p, that elbow must be southeast of (c, r), and so from Lemma 2.1.13 $p \lhd_P q$. $\square$

Lemma 2.1.15 ([BCCP22, Lem. 2.8]). *Let $P \in \Pi(\omega)$ be a pipe dream and $p < q < r$ be pipes of P such that $\omega^{-1}(r) < \omega^{-1}(q) < \omega^{-1}(p)$. If $p \to r$ is an edge of $P^{\#}$, then either $p \lhd_P q \rhd_P r$ or $p \rhd_P q \lhd_P r$.*

Proof. From our choice of p, q, r we know that $(p, q), (q, r), (p, r) \in \mathrm{inv}(\omega)$, therefore those three pipes all cross each other. Let us denote by c a cell containing a contact between p and r. We can divide $\mathcal{T}_n$ into three regions: the region A being the part stricly southwest of c, the region B being all the cells weakly northwest or southeast of c and the region C being strictly northeast of c. Since $q > p$ and $\omega^{-1}(q) > \omega^{-1}(r)$, pipe q must start in region A and end in region C. However, in order to go from A to C it must pass through B and have an elbow in it; from Lemma 2.1.13, if it is northwest of c then $p \rhd_P q \lhd_C r$ and if it is southeast of c then $p \lhd_P q \rhd_P r$. $\square$

2.2 The lattice of acyclic pipe dreams

This section will discuss some interesting links between the weak order and acyclic pipe dreams, starting with a realization of the Tamari lattice, known for being a lattice quotient of the weak order.

2.2.1 Reversing pipe dreams

This section will consider a specific family of pipe dreams and their link with some classical combinatorial objects. This link was first proved by A. Woo in [Woo04] using Dyck path; here we use binary trees, as the sylvester congruence defined by binary search trees on the weak order is more easily described this way. For convenience, we will exceptionnally label our pipes from 0 to n instead of 1 to $n+1$. We will also label the rows and columns of $\mathcal{T}_{n+1}$ this way so that pipe p starts in row p and ends in row $\rho_n^{-1}(p)$.

Definition 2.2.1. *A **reversing pipe dream** is a pipe dream with exit permutation $\rho_n := 0\, n\, (n-1)\, \ldots\, 2\, 1$ (it starts with 0 and then all the other values are in decreasing order).*

We note that in such a pipe dream, like the one represented in Fig. 2.6, pipe 0 always has the same trajectory: a unique northwest elbow.

Theorem 2.2.2. *The flip order on reversing pipe dreams with $n+1$ pipes is isomorphic to the Tamari lattice of size n.*

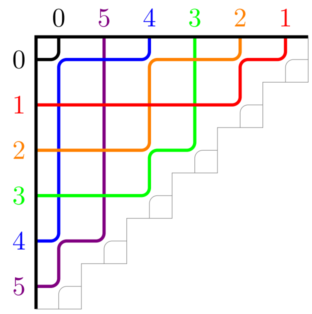


Figure 2.6: A reversing pipe dream with 6 pipes.

The idea is that we can represent a (mirrored) binary tree with n nodes on $\mathcal{T}_{n+1}$, with its $n+1$ leaves on the diagonal and the edges going vertically or horizontally. A reversing pipe dream is then obtained by putting an elbow in each cell containing a node or leaf, and a cross everywhere else. The binary tree we started with will then be isomorphic to the contact graph of the obtained pipe dream. This correspondance is illustrated in Fig. 2.7 for two trees linked by rotations on the bolded edges (a left rotation from top to bottom): the trees are mirrored and placed on $\mathcal{T}_6$, and the two pipe dreams obtained by filling the shape as described are reversing and linked by a flip on the yellow cells (ascending from top to bottom).

Lemma 2.2.3. *The contact graph of a reversing pipe dream is isomorphic to a binary search tree with an added node* 0 *pointing to the root.*

Proof. Consider any pipe $p > 0$ of a reversing pipe dream P. From Lemma 2.1.11 we know that p has exactly $|\operatorname{ninv}(\rho_n, p)|$ southeast elbows and one more northwest elbow. Since the only noninversions of ρ_n are the pairs $(0, k)$ for $1 \leqslant k \leqslant n$, we know that $\operatorname{ninv}(\rho_n, p) = \{0\}$ and so p has exactly one southeast elbows and two northwest elbows. Therefore, the indegree of p in $P^\#$ is exactly one and its outdegree is at most two. Since there are as many contacts between two pipes in P as there are southeast elbows, and pipe 0 has no southeast elbows, there are n contacts in P and n edges in $P^\#$ (as since no pipe has two southeast elbows, no two contacts can be between the same two pipes in the same direction). Lastly, from Lemma 2.1.14 we know that $0 \lhd_P p$ for any pipe $p > 0$ and so $P^\#$ is connected. Since it is a connected graph with $n+1$ vertices and n edges, it is a tree; since the indegree of each node except 0 is one it is a rooted tree, and since the outdegree is at most two it is a binary tree. We can remove 0 to obtain a binary tree rooted in r the only pipe that has a contact with 0.

Let us now choose a left child and a right child for each vertex: the right child of p is the pipe that has a contact with p in its first northwest elbow (if it exists), and its left child is the pipe that has a contact with p in its second northwest elbow. Consider now q such that $p \lhd_P q$ and that the only path from p to q in $P^\#$ is through

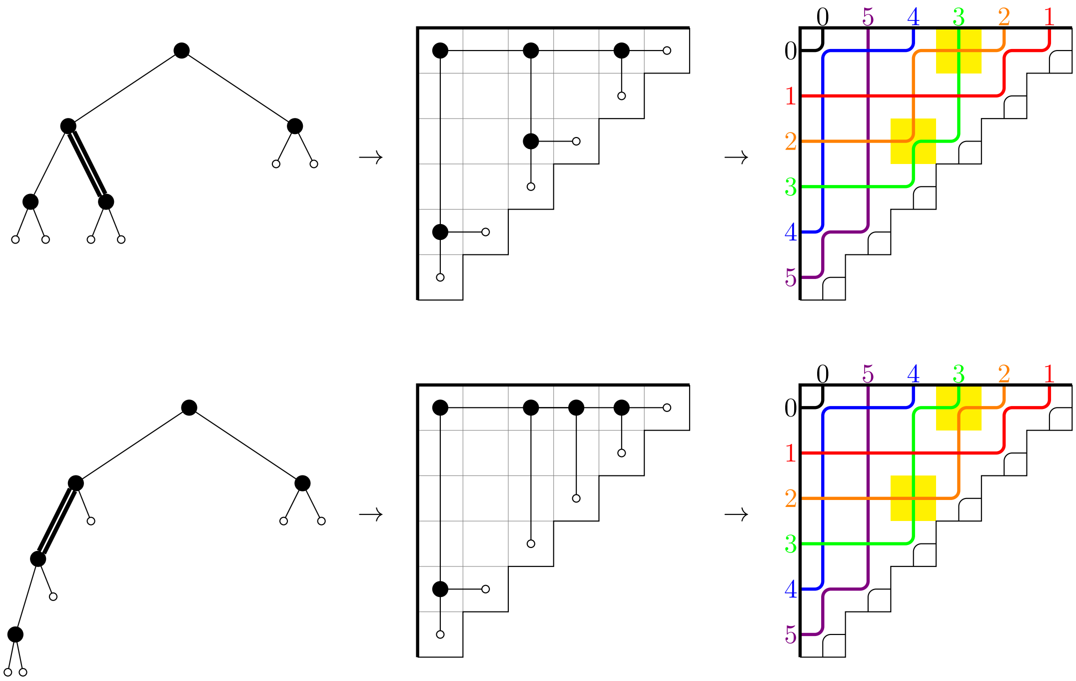


Figure 2.7: Two binary trees linked by a rotation and their associated pipe dreams linked by a flip.

the left child of p. We note that because of the trajectory of each pipe, the only southeast elbow of a pipe in this path is always directly south or east of the southeast elbow of the previous pipe (east if it is its left child, south if it is its right child). In particular, the southeast elbow of q is southeast of the second northwest elbow of p. Moreover, after its second northwest elbow, pipe p goes straight to its ending point on the north side of $\mathcal{T}_{n+1}$: since P is reduced, from Proposition 2.1.9 that means that it can only cross pipes with smaller labels after that elbow. Therefore, if $q > p$, since (p, q) would be an inversion of ρ_n, they would have crossed before that second northwest elbow, and so pipe q would be nortwest of pipe p at that point. This contradict what we just said about the position of the southeast elbow of q and so $q < p$. Conversely, if the only path from p to q is through the right child of p, a similar reasoning proves that $q > p$. This proves that the binary tree $P^{\#} \backslash \{0\}$ has a planar realization that is a binary search tree. $\square$

For T a binary search tree with n nodes and $1 \leqslant k \leqslant n$, we will denote by $l_T(k)$ and $r_T(k)$ the number of nodes in the subtrees rooted respectively in the left and right child of node k in T (i.e. $l_T(k)$ is the number of "left descendants" of k and $r_T(k)$ its number of "right descendants").

Lemma 2.2.4. *Let P be a reversing pipe dream and T the binary search tree isomorphic to $P^{\#} \backslash \{0\}$. For any pipe $p > 0$ of P, the only southeast elbow e_p of P is in*

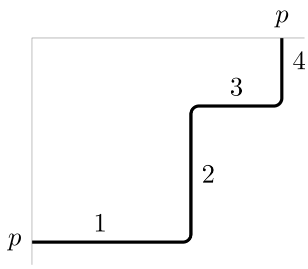


Figure 2.8: The trajectory of pipe p.

cell $(p - l_T(p) - 1, n - p - r_T(p))$.

Proof. We saw in the proof of Lemma 2.2.3 that for $p < q$, the node q is a right descendant of p in T iff pipe q has its only southeast elbow southeast of the first northwest elbow of p. The trajectory of pipe p is as drawn in Fig. 2.8 and can be divided into four parts: before the first northwest elbow, between the first northwest elbow and e_p, between e_p and the second northwest elbow, and after the second northwest elbow. Since $(p, q) \in \mathrm{inv}(\rho_n)$, we know that pipe p must cross pipe q while going horizontally, so either in section 1 or section 3. If it does so in section 1 then pipe q is never southeast of the first northwest elbow of p, so q is not a right descendant of p. Otherwise, since p and q only cross once, pipe q must have an elbow southeast of the first northwest elbow of p, and so from Lemma 2.1.13 we know that $p \lhd_P q$ so q is a descendant of p, and since T is a binary search tree and $q > p$ we know that it can only be a right descendant of p. Therefore, the pipes crossing p in section 3 are exactly its right descendants, so section 3 contains $r_T(p)$ horizontal crossing. Since the last northwest elbow of p is in the column where p ends, i.e. column $n-p+1$, this means that e_p is in column $n-p+1-r_T(p)-1 = n-p-r_T(p)$.

A similar reasoning proves that for $p > q$, pipe p crosses pipe q in section 2 if q is a left descendant of p and section 4 otherwise, so section 2 contains $l_T(p)$ vertical sections and since the first northwest elbow of p is in row p, this means that e_p is in row $p - l_T(p) - 1$. □

This lemma proves that each binary search tree is isomorphic to the contact graph of at most one reversing pipe dream.

Lemma 2.2.5. *For T a binary search tree, the pipe dream P_T containing contacts in the cells $e_p = (n-p-r_T(p), p-l_T(p)-1)$ for $1 \leqslant p \leqslant n$ and $(x, n-x)$ for $0 \leqslant x \leqslant n$ is reversing.*

Proof. We know that the labels of a binary search tree correspond to the in-order traversal of the node; therefore, for q a node, the labels $q+1, \ldots, q+r_T(q)$ belong to the right descendants of q and $q + r_T(q) + 1$ is the first ancestor of q that has q as a left descendant (or is $n+1$ if no such ancestor exist). Similarly, the labels $q-1$ to $q-l_T(q)$ belong to the left descendants of q and $q-l_T(q)-1$ is the first ancestor of q that has q as a right descendant (or is 0 if no such ancestor exist).

We know that pipe p starts horizontally in row p and goes straight until it reaches a contact and has a northwest elbow. Suppose that for some pipe q, the cell e_q is in row p, then by Lemma 2.2.4 we have $p = q - l_T(q) - 1$ so $q > p$ and p is the first ancestor of q in T such that q is a right descendant of p. Such a pipe exists iff p has a right child q_0, and in that case the e_q in row p are q_0 and the descendants of q_0 by going only to the left. Therefore, apart from q_0 they are all the left child of some other right descendant of p. Then the first e_q met by pipe p is the one minimizing $n - q - r_T(q)$, i.e. maximizing $q + r_T(q)$. For $q \neq q_0$, that number is lower than q_0 since q_0 has q as a left descendant, and so the first elbow met by p is e_{q_0}. Since q_0 is the right child of p, the first ancestor r of p that has p as a left descendant is also the first ancestor of q that has q as a left descendant, so $p + r_T(p) = q_0 + r_T(q_0) = r - 1$ (with possibly $r = n + 1$ if p and q are on the rightmost branch of T), and the first elbow of p is in cell $(n - p - r_T(p), p)$. If p has no right child, then no elbow e_q is in row p and so the first contact of p is also in cell $(p, n-p) = (p, n-p-r_T(p))$ since $r_T(p) = 0$.

After that first elbow, pipe p must go north in column $n - p - r_T(p)$ until reaching another contact where it will have a southeast elbow. We note that e_p is in that row. Suppose that there exists e_q strictly between e_{q_0} and e_p, then this means that $n-q-r_T(q) = n-p-r_T(p)$ so $q+r_T(q) = p+r_T(p)$ and so p and q have the same first ancestor with them as a left descendant, and $p > q - l_T(q) - 1 > p - l_T(p) - 1$ so the first ancestor of q with q as a right descendant is between p and the first ancestor of p with p as a right descendant. These two conditions are incompatible, so such a contact e_q does not exist, and the next elbow of p is in e_p.

After that, pipe p must go east in row $p - l_T(p) - 1$ until it meets a contact. A reasoning similar to the first part proves that if p has a left child q_1 in T, then the first contact that p meets is e_{q_1}; in that case, since $q_1 + r_T(q_1) + 1 = p$, the third elbow of p is in cell $(p - l_T(p) - 1, n - p + 1)$. Otherwise, the first contact it meets is in cell $(p - l_T(p) - 1, n - (p - l_T(p) - 1)) = (p - l_T(p) - 1, n - p + 1)$ since $l_T(p) = 0$.

Finally, if for some pipe q we have e_q in column $n - p + 1$ and north of the third elbow of p, then $q + r_T(q) = p - 1$ so p is the first ancestor of q with q as a left descendant, and $q - l_T(q) - 1 < p - l_T(p) - 1$ so the first ancestor of q with q as a right descendant is smaller than the same thing for p. This is not possible, so p ends in column $n - p + 1$.

This leads us to see that each pipe $1 \leqslant p \leqslant n$ ends in column $\rho_n^{-1}(p)$, and for r the root of T we can see that $e_r = (0, 0)$ so pipe 0 ends in column 0 after just one northwest elbow, therefore P_T is a pipe dream with exit permutation ρ_n. Since each pipe $p > 0$ has exactly 1 southeast elbow and $\mathrm{ninv}(\rho_n, p) = \{0\}$, according to Lemma 2.1.12 that pipe dream is reduced, and so P_T is a reversing pipe dream. $\square$

Proof of Theorem 2.2.2. We know from Lemma 2.2.3 that we can define a map ϕ

from reversing pipe dreams to binary search trees, and from Lemma 2.2.4 that this map is injective, since $\phi(P)$ determines the position of all the elbows of P not on the diagonal of $\mathcal{T}_{n+1}$. For T a binary search tree, we consider the reversing pipe dream P_T defined in Lemma 2.2.5. We saw in the proof of that lemma that for each pipe $p > 0$ of P_T, the only southeast elbow of p is in cell $(n - p - r_T(p), p - l_T(p) - 1)$, and from Lemma 2.2.4 we know that for $T' = \phi(P_T)$, that cell is also $(n-p-r_{T'}(p), p-l_{T'}-1)$. Therefore, for each $1 \leqslant p \leqslant n$ we have $r_T(p) = r_{T'}(p)$ and $l_T(p) = l_{T'}(p)$. In particular, for two nodes $p < q$, we know that p is the left child of q in T if and only if $q = p + r_T(p) + 1$ and q is the right child of p iff $p = q - l_T(q) - 1$, and the same goes for T', so $T = T'$ and ϕ is surjective.

We now have a bijection between reversing pipe dreams and the binary search trees with n nodes; all that is left to prove is that the image of the flips are the rotations. Consider an increasing flip from P to P' two reversing pipe dreams, and denote by $p < q$ the two pipes involved in the flip. We know that there is a contact from p to q in P, and this contact must be on the only southeast elbow of q. Since $q > p$, we also saw in the proof of Lemma 2.2.3 that this contact is on the first northwest elbow of p and in the proof of Lemma 2.2.4 that pipes p and q cross on section 3 of the trajectory of p (see Fig. 2.8). The effect of the flip is thus represented in Fig. 2.9; the pipes t_1, t_2, t_3 are respectively the right and left children of q and the left child of p in $\phi(T)$. Since only p and q are changed by the flip, it is clear that the only edges different in $\phi(P)$ and $\phi(P')$ are the ones involving p or q, and the effect on those edges is represented on the bottom of Fig. 2.9: it corresponds to a left rotation on p and q. Conversely, rotating an edge from p to q with $p < q$ (so a left rotation) in $\phi(P)$ gives the binary tree $\phi(P')$ with P' obtained from P by flipping the contact from p to q. This proves that ϕ is an isomorphism between the increasing flip graph on $\Pi(\rho_n)$ and the rotation graph on binary trees with n nodes oriented in the direction of left rotations, and so the flip order on $\Pi(\rho_n)$ is isomorphic to the Tamari lattice of size n. □

2.2.2 Linear extensions of pipe dreams

As the Tamari lattice is a lattice quotient of the weak order, we wanted to know if it was possible to find other relations between flips in pipe dreams and quotients of the weak order.

Definition 2.2.6. *For P an acyclic pipe dream, a* ***linear extension*** *of P is a permutation π such that for any pipes p and q of P, if the edge $p \to q$ is in $P^{\#}$, then $\pi^{-1}(p) < \pi^{-1}(q)$. We denote by $\mathrm{lin}(P)$ the set of linear extensions of P.*

We note that $\pi \in \mathrm{lin}(P)$ is equivalent to saying that π is a linear extension of the partial order $\lhd_P$ on $[n]$.

Theorem 2.2.7 ([BCCP22, Thm 3.2])**.** *For any $\omega \in \mathfrak{S}_n$, the collection of linear extension sets $\{\mathrm{lin}(P) \mid P \in \Sigma(\omega)\}$ is a partition of the weak order interval $[\mathrm{id}, \omega]$.*

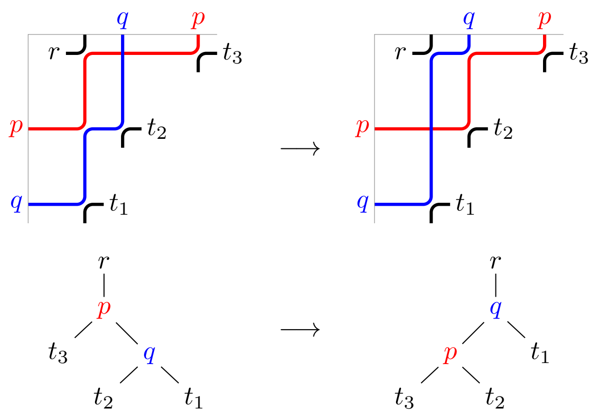


Figure 2.9: An increasing flip between reversing pipe dreams.

Lemma 2.2.8 ([BCCP22, Lem. 3.6])**.** *For any pipe dream $P \in \Pi(\omega)$, all the linear extensions of P are in* $[\mathrm{id}, \omega]$.

Proof. This is a direct consequence of Lemma 2.1.14: for $\pi \in \mathrm{lin}(P)$, any noninversion of ω is also an inversion of π and so $\pi \leqslant \omega$. □

Lemma 2.2.9 ([BCCP22, Lem. 3.4])**.** *Let $\pi' := UpqV \lessdot \pi := UqpV$ be a cover of the weak order (so $p < q$) and suppose that $\pi \in \mathrm{lin}(P)$ for some $P \in \Sigma(\omega)$, then:*

- *if $P^{\#}$ has no arc $q \to p$, then $\pi' \in \mathrm{lin}(P)$;*
- *otherwise, $\pi' \in \mathrm{lin}(P')$ with P' the pipe dream obtained by a descreasing flip on the furthest northeast contact between pipes q and p in P.*

Proof. We note that since π and π' only differ in their position of p and q, for any two pipes i and j, either $\pi^{-1}(i) < \pi^{-1}(j) \Rightarrow \pi'^{-1}(i) < \pi'^{-1}(j)$ or $i = q$ and $j = p$. This means that if $q \to p$ is not an arc of $P^{\#}$, then π' is also a linear extension of P.

Suppose now that there is at least one contact between pipes q and p and denote by c the one that is furthest northeast in P. Since $p < q$ and q is northwest of p in c, we know that pipes p and q must cross in some cell x of P with x southwest of c; we can thus flip c to obtain a new pipe dream $P' \in \Pi(\omega)$, whose pipes have the same trajectory as in P except for p and q. Consider now a contact of P' between pipe i (to the northwest) and j (to the southeast). If i and j are both distinct from p and q, then this contact is the same in P and so $\pi^{-1}(i) < \pi^{-1}(j)$ and so $\pi'^{-1}(i) < \pi'^{-1}(j)$. If only i is either p or q, the same cell in P contained a contact from p or q to j and so $j \in V$ and $\pi'^{-1}(i) < \pi'^{-1}(j)$; the same reasoning holds if only j is either p or q. Finally, if the contact is between q and p, then the cell also contained those same two pipes in P. Since c was the furthest northeast cell containing both those pipes in P, the contact we are considering is southwest of c and so pipes p and q cannot have crossed yet, and so i must be p and j must be q. In every case, we still have $\pi'^{-1}(i) < \pi'^{-1}(j)$, and so π' is a linear extension of P'. □

Lemma 2.2.10 ([BCCP22, Lem. 3.5])**.** *If $\pi \leqslant \omega$, then π is a linear extension of exactly one pipe dream in $\Sigma(\omega)$.*

Proof. We will start by proving that no permutation is a linear extension of more than one pipe dream. Let us start by proving this result for $\pi = \mathrm{id}$. Consider two pipe dreams $P, P' \in \Pi(\omega)$ with id a linear extension of both, and suppose that they differ in at least one cell. Consider x such a cell that is furthest in the northeast direction (there can be more than one such furthest cell), and denote by p the pipe exiting x from the north in P and q the one exiting from the east. Since all cells northeast of x are the same in P and P', we know that the pipes exiting from x in P' have the same trajectory as p and q in P until their end points on the north side of $\mathcal{T}_n$; as P and P' have the same exit permutation ω, this means that p and q also exit the same sides of x in P'. Without loss of generality, we can consider that x contains a contact in P and a cross in P'. This means that $p \to q$ is an arc of $P^{\#}$, and since id is a linear extension of P we obtain that $p < q$; however, this means that p crosses q going vertically in P' with $p < q$, and so by Proposition 2.1.9 this means that P is not reduced. This is a contradiction, and so $P = P'$.

Suppose now that $\pi' := UpqV \lessdot \pi := UqpV$ is a cover of the weak order and $\pi \in \mathrm{lin}(P_1) \cap \mathrm{lin}(P_2)$ with $P_1, P_2 \in \Sigma(\omega)$ distinct. Then by Lemma 2.2.9, three cases are possible:

- if $q \to p$ is not an arc in either $P_1^{\#}$ nor $P_2^{\#}$, then $\pi' \in \mathrm{lin}(P_1) \cap \mathrm{lin}(P_2)$;
- if $q \to p$ is an arc in only one of them, for example WLOG $P_1^{\#}$, then π' is in $\mathrm{lin}(P_1') \cap \mathrm{lin}(P_2)$ with P_1' obtained from P_1 by a flip on pipes p and q, and so P_1' contains a contact between those two pipes and P_2 does not;
- if $q \to p$ is an arc in both of them, then $\pi' \in \mathrm{lin}(P_1') \cap \mathrm{lin}(P_2')$ with P_1' and P_2' respectively obtained by a flip on P_1 and P_2; since P_1 and P_2 can be obtained by flipping the furthest southwest contact between p and q in P_1' and P_2' respectively, we know that $P_1' \neq P_2'$.

In all case, if π is a linear extension of two distinct pipe dreams of $\Sigma(\omega)$, then so is any permutation covered by π. By extension, this is also true for any permutation below π. Since id is the minimum of the weak order and we proved that id can only be a linear extension of one pipe dream in $\Sigma(\omega)$, this proves that no $\pi \in \mathfrak{S}_n$ can be a linear extension of more than one pipe dream in $\Sigma(\omega)$.

Let us now prove that the linear extensions of pipe dream in $\Sigma(\omega)$ cover the interval $[\mathrm{id}, \omega]$. We start by noting that Lemma 2.2.9 proves that the set $\bigcup_{P \in \Sigma(\omega)} \mathrm{lin}(P)$ is a lowerset of the weak order. Consider now a maximal element P_0 of $\Pi(\omega)$ for the flip order (which exists because $\Pi(\omega)$ is finite) and suppose that $P_0^{\#}$ contains an edge $p \to q$ for some pipes p and q. Since P_0 is maximal, a contact between those two pipes is either not flippable, or the corresponding flip is descending. In the first case, it means that (p, q) is a noninversion of ω, and so $p < q$ and $\omega^{-1}(p) < \omega^{-1}(q)$. In the second case, the cross between p and q is southwest of the contact, and so $p > q$ and (q, p) is an inversion of ω; once again $\omega^{-1}(p) < \omega^{-1}(q)$. This means that ω must be a linear extension of P_0, and so $\omega \in \bigcup_{P \in \mathrm{lin}(P)} \mathrm{lin}(P)$; since this set is a lowerset, it must therefore contain all of $[\mathrm{id}, \omega]$ and so any $\pi \in [\mathrm{id}, \omega]$ is a linear extension of some pipe dream in $\Sigma(\omega)$. This concludes the proof. □

Proof of Theorem 2.2.7. The theorem is a direct consequence of Lemma 2.2.8 and Lemma 2.2.10. □

2.2.3 A lattice quotient

Theorem 2.2.7 gives us a partition of $[\mathrm{id}, \omega]$ that we will use to define an equivalence relation.

Definition 2.2.11. *For $\omega \in \mathfrak{S}_n$, the* ***pipe dream congruence*** *is the equivalence relation $\equiv_\omega$ on the weak order interval $[\mathrm{id}, \omega]$ whose equivalence classes are the linear extensions sets $(\mathrm{lin}(P))_{P \in \Sigma(\omega)}$.*

We note that Lemma 2.2.9 shows an interesting link between the flip graph on pipe dreams and the weak order:

Theorem 2.2.12 ([BCCP22, Thm. 3.10])**.** *For any permutation $\omega \in \mathfrak{S}_n$, the relation $\equiv_\omega$ is a lattice congruence of the weak order on $[\mathrm{id}, \omega]$.*

Example 2.2.1. The quotient $\equiv_{32542}$ on the weak order interval $[\mathrm{id}, 31542]$ is represented in Fig. 2.10, along with the associated increasing flip graph on acyclic pipe dreams.

To prove this theorem, we will use Theorem 1.1.18 and the lemmas given in Section 2.1.3.

Lemma 2.2.13. *For any reduced pipe dream P and $p < q < r$ three of its pipes,*
- *if $p \lhd_P r$, then $p \lhd_P q$ or $q \lhd_P r$;*
- *if $r \lhd_P p$, then $r \lhd_P q$ or $q \lhd_P p$.*

Proof. Suppose that $p \lhd_P r$, we will proceed by induction on the length of the shortest path from p to r in $P^{\#}$. Suppose first that this path is of length one, i.e. that $p \to r$ is an arc of $P^{\#}$. Then either $\omega^{-1}(p) < \omega^{-1}(q)$, or $\omega^{-1}(q) < \omega^{-1}(r)$, or $\omega^{-1}(r) < \omega^{-1}(q) < \omega^{-1}(p)$. In the first and second cases, since (p, q) or (q, r) is in $\mathrm{ninv}(\omega)$, from Lemma 2.1.14 we know that $p \lhd_P q$ or $q \lhd_P r$. In the third case, since there exists a contact from pipe p to pipe r, Lemma 2.1.15 tells us that either $p \lhd_P q$ or $q \lhd_P r$.

Suppose now that the lemma is true for any distance strictly lower than $d > 1$ and that the distance from p to r is d, and denote by p' the first pipe on this path after p. Then $p \lhd_P p' \lhd_P r$ and the distance from p' to r is $d - 1$. Then:
- if $p' = q$ then $p \lhd_P q$ and the result holds;
- if $p' < q$ then $p' < q < r$ and $p' \lhd_P r$ so by our induction hypothesis, either $q \lhd_P r$ or $q \rhd_P p' \rhd_P p$ and the result holds;

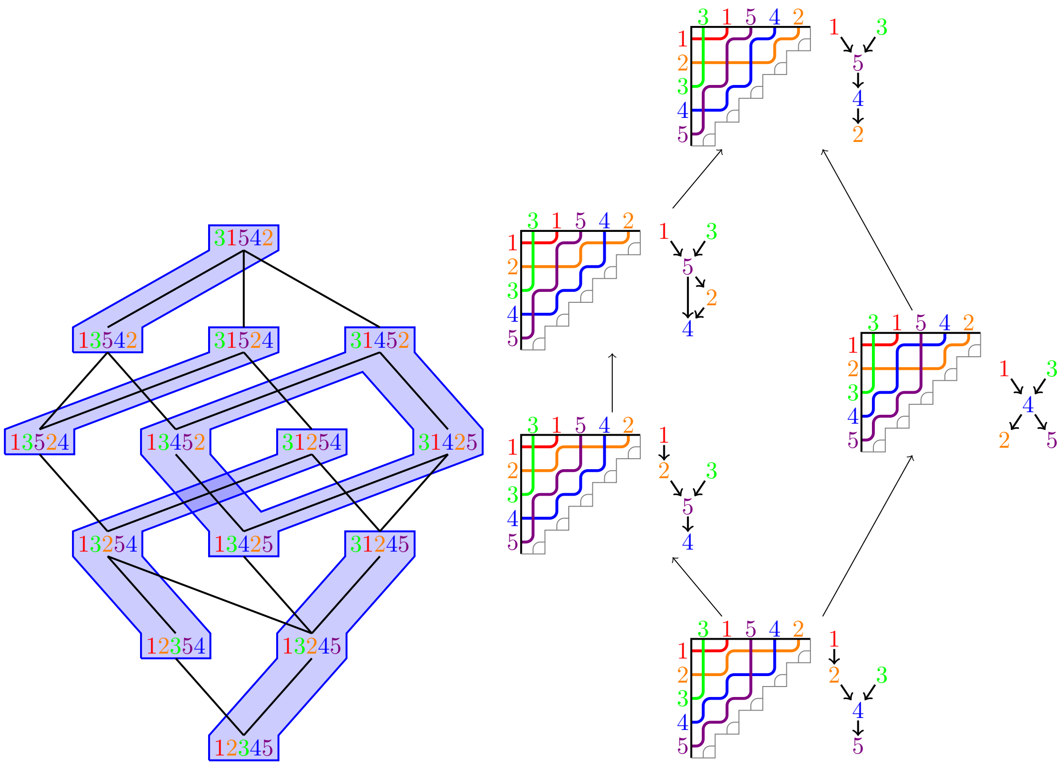


Figure 2.10: The pipe dream congruence $\equiv_{31542}$ and the associated increasing flip graph.

- if $p' > q$, then $p < q < p'$ and $p \lhd_P p'$ with a path of length 1 from p to p' in $P^\#$, so by induction hypothesis either $p \lhd_P q$ or $q \lhd_P p' \lhd_P r$ and the result still holds.

In all cases, the lemma still holds for distance d. By induction, it is thus always true. The case $r \lhd_P p$ is symmetrical. □

Proposition 2.2.14 ([BCCP22, Prop. 3.13]). *For any pipe dream $P \in \Sigma(\omega)$, the set* $\mathrm{lin}(P)$ *is a weak order interval.*

Proof. Let us consider the two sets of pairs of pipes $X^\uparrow = \{(p,q) \mid p < q \text{ and } p \lhd_P q\}$ and $X^\downarrow = \{(p,q) \mid p < q \text{ and } q \lhd_P p\}$. Since P is acyclic, we know that $X^\uparrow \cap X^\downarrow$ is empty. Moreover, by definition of linear extensions we know that $\pi \in \mathrm{lin}(P)$ if and only if $X^\uparrow \subseteq \mathrm{ninv}(\pi)$ and $X^\downarrow \subseteq \mathrm{inv}(\pi)$. Since the relation $\lhd_P$ is transitive, we also know that $X^\uparrow$ and $X^\downarrow$ are convex sets of roots of $\mathfrak{S}_n$ seen as a Coxeter group; similarly, Lemma 2.2.13 tells us that their complementary is convex (voir Example 1.4.5), so $X^\uparrow$ and $X^\downarrow$ are both biconvex. From Theorem 1.4.24, we can then define $\pi^\uparrow$ and $\pi^\downarrow$ such that $\mathrm{ninv}(\pi^\uparrow)$ is $X^\uparrow$ and $\mathrm{inv}(\pi^\downarrow)$ is $X^\downarrow$. By the definition of the weak order given in Theorem 1.4.14, we obtain that $\pi \in \mathrm{lin}(P) \iff \pi^\downarrow \leqslant \pi \leqslant \pi^\uparrow$. Therefore, the set $\mathrm{lin}(P)$ is exactly the interval $[\pi^\downarrow, \pi^\uparrow]$. □

Proposition 2.2.15 ([BCCP22, Prop. 3.14]). *Let C, C' be equivalence classes of $\equiv_\omega$ and consider $\pi \in C$ and $\pi' \in C'$. Then $\pi \leqslant \pi'$ implies that* $\max(C) \leqslant \max(C')$ *and* $\min(C) \leqslant \min(C')$.

Proof. We prove the statement for the maximums, the proof for the minimums is symmetrical. Observe first that we can assume that π is covered by π' in weak order, so that we write $\pi' = \pi\tau_i$ for some simple transposition $\tau_i := (i\ i+1)$. The proof now works by induction on the weak order distance between π and $\max(C)$. If $\pi = \max(C)$, the result is immediate as $\max(C) = \pi < \pi' \leqslant \max(C')$. Otherwise, must be covered π by a permutation σ in the class C, and we write $\sigma = \pi\tau_j$ for some simple transposition τ_j. Let $P, P' \in \Sigma(\omega)$ be such that $C = \mathrm{lin}(P)$ and $C' = \mathrm{lin}(P')$. We now distinguish five cases, according to the relative positions of p and q:

(1) If $i > j+1$, then $\pi = UpqVrsW$, $\pi' = UpqVsrW$ and $\tau = UqpVrsW$ for some $p < q$ and $r < s$. Define $\sigma' := \pi\tau_i\tau_j = \pi\tau_j\tau_i = UqpVsrW$. By Lemma 2.2.9, there is no arc $p \to q$ in $P^\#$ (since π and σ both belong to C), and $P^\#$ and $P'^\#$ can only differ by arcs incident to r or s. Hence, there is no arc $p \to q$ in $P'^\#$. We thus obtain again by Lemma 2.2.9 that $\sigma' \in \mathrm{lin}(P') = C'$.

(2) If $i = j+1$, then $\pi = UpqrV$, $\pi' = UprqV$ and $\sigma = UqprV$ for some $p < q < r$. Define $\sigma' := \pi\tau_i\tau_j\tau_i = \pi\tau_j\tau_i\tau_j = UrqpV$. Since $\pi \in \mathrm{lin}(P)$, we have $p \not\rhd_P q$ and $q \not\rhd_P r$, so that there is no arc $p \to r$ in $P^\#$ by Lemma 2.1.15. By Lemma 2.2.9, there is no arc $p \to q$ in $P^\#$, and $P^\#$ and $P'^\#$ can only differ by arcs incident to q or r. We thus obtain that there is no arc $p \to q$ nor $p \to r$ in $P'^\#$. Consequently, again by Lemma 2.2.9, both $\pi'\tau_j$ and $\sigma' = \pi'\tau_j\tau_i$ belong to $\mathrm{lin}(P') = C'$.

(3) If $i = j$, then $\pi' = \sigma$ is in C, so that $C = C'$ and there is nothing to prove.

(4) If $i = j - 1$, we proceed similarly as in Situation (2).

(5) If $i < j - 1$, we proceed similarly as in Situation (1).

In all cases, we found $\sigma' > \sigma$ with $\sigma' \in C'$. Since $\sigma < \sigma'$ with $\sigma \in C$ and $\sigma' \in C'$, and since σ is closer to $\max(C)$ than π, we obtain that $\max(C) < \max(C')$ by induction hypothesis. □

Proof of Theorem 2.2.12. The theorem is a direct consequence of Proposition 2.2.14 and Proposition 2.2.15 combined with Theorem 1.1.18. □

2.2.4 A lattice morphism

Theorem 2.2.7 allowed us to define the equivalence relation $\equiv_\omega$, but it can also be used to define a map from the permutations below ω to acyclic pipe dreams as follows.

Definition 2.2.16. *We denote by* $\mathrm{ins}_\omega : [\mathrm{id}, \omega] \to \Sigma(\omega)$ *the map that associate to any permutation* $\pi \leqslant \omega$ *the only pipe dream in* $\Sigma(\omega)$ *that has* π *as a linear extension.*

We know that the fibers of ins_ω are the equivalence classes of $\equiv_\omega$, and so we can define ins_ω on the quotient $[\mathrm{id}, \omega]/\equiv_\omega$, with the image of an equivalence class being the common image of all its elements. It is then natural to wonder what the image of the weak order by this map looks like.

Theorem 2.2.17 ([BCCP22, Thm. 3.15])**.** *The map* $\mathrm{ins}_\omega : [\mathrm{id}, \omega]/\equiv_\omega \to \Sigma(\omega)$ *is an isomorphism from the Hasse diagram of this quotient of the weak order to the restriction of the increasing flip graph to reduced pipe dreams. Hence, the transitive closure of the increasing flip graph on* $\Sigma(\omega)$ *is a lattice.*

Remark 2.2.18. The increasing flip poset on $\Pi(\omega)$ is the transitive closure of the increasing flip graph on the same set; as such, the restriction of this poset to $\Sigma(\omega)$ may contain relations that are not in the transitive closure of the restriction of the increasing flip graph to $\Sigma(\omega)$. If $P, P' \in \Sigma(\omega)$ have some paths from P to P' in the increasing flip graph on $\Pi(\omega)$, but that all such paths go through non-acyclic pipe dreams, then there is no path from P to P' in the increasing flip graph on $\Sigma(\omega)$. An example of such a case is given in Fig. 2.11: the acyclic pipe dreams P_1 and P_2 are linked by some increasing flips through two non-acyclic pipe dreams, but there is no such sequence through only acyclic pipe dreams. As a result, their respective only linear extensions 123546 and 126453 are not comparable in the weak order.

On the other hand, in the case where every pipe dream of $\Pi(\omega)$ is acyclic (like in the case of ρ_n as detailed in Section 2.2.1), the image of the weak order is exactly the increasing flip order on $\Pi(\omega)$.

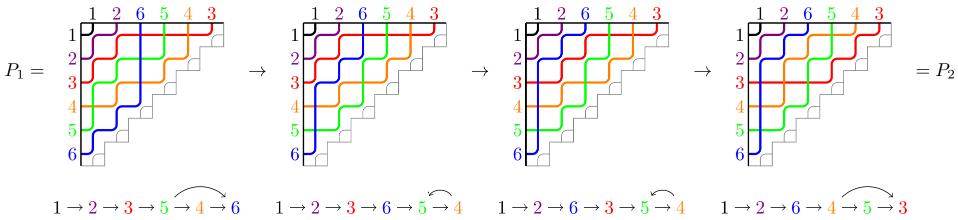


Figure 2.11: Two acyclic pipe dreams in $\Sigma(126543)$ linked by a sequence of increasing flips in $\Pi(126543)$ but not in $\Sigma(126543)$.

Lemma 2.2.19 ([BCCP22, Lem. 3.18])**.** *Consider two acyclic pipe dreams P, P' in $\Sigma(\omega)$ linked by a flip on some pipes p, q. Then any directed path between p and q in $P^{\#}$ or $P'^{\#}$ is an arc.*

Proof. Suppose that $p < q$ and that $p \to q$ is an arc of $P^{\#}$ and $q \to p$ an arc of $P'^{\#}$. Since P is acyclic, there is no path from q to p in $P^{\#}$. Suppose by means of contradiction that there is a path $p \to r_1 \to \ldots \to r_k \to q$ in $P^{\#}$ with $k \geqslant 1$. Since none of the arcs $r_i \to r_{i+1}$ are changed by the flip between P and P', we know that $P'^{\#}$ contains the path $r_1 \to \ldots \to r_k$, as well as at least one of the arcs $p \to r_1$ or $q \to r_1$, and one of the arcs $r_k \to q$ or $r_k \to p$. Since P' is acyclic and contains an arc $q \to p$, it cannot contain $p \to r_1$ or $r_k \to q$ and so it must contain the path $q \to r_1 \to \ldots \to r_k \to p$. In particular, it means that there is a contact from p to r_1 in P that becomes a contact from q to r_1 in P'.

Consider now the cell c containing a contact (from p to q) in P and a cross in P', and c' the cell containing a contact (from q to p) in P' and a cross in P. We know that c is southwest of c', and we denote by R the rectangle delimitated by c in the southwest and c' in the northeast. Then all the cells outside of R are unchanged between P and P', and so pipe r_1 must have a contact inside of R, and so it passes through R. Since $p \lhd_P r_1 \lhd_P q$ and $q \lhd_{P'} r_1 \lhd_{P'} p$ and P and P' are acyclic, we know from Lemma 2.1.13 that r_1 cannot have an elbow northwest or southeast of either c or c', so it must go straight before entering R and after exiting R. This means that its starting point is north of c and its exit point is west of c', so $r_1 < p < q$ and $\omega^{-1}(r_1) < \omega^{-1}(q) < \omega^{-1}(p)$, and so (r_1, p) and (r_1, q) are noninversions of ω. By Lemma 2.1.14, this contradicts $p \lhd_P r_1$ and $q \lhd_{P'} r_1$. $\square$

Proof of Theorem 2.2.17. We need to prove the equivalence of the two statements:
(1) there is an increasing flip from P to P';
(2) there exists $\pi \lessdot \pi'$ a cover of the weak order with $\mathrm{ins}_\omega(\pi) = P$ and $\mathrm{ins}_\omega(\pi') = P'$.

Lemma 2.2.9 proves immediately that (2) $\Rightarrow$ (1). To prove that (1) $\Rightarrow$ (2), let $p < q$ be the two pipes involved in the flip from P to P', so $p \to q$ is an arc of $P^{\#}$ and $q \to p$ is an arc of $P'^{\#}$. We know from Lemma 2.2.19 that there is no other path between those two pipes in $P^{\#}$, so there exists a linear extension of P where p and q are consecutive. We denote by $\pi := UpqV \in \mathrm{lin}(P)$ such a permutation.

Then $\pi' := UqpV \notin \mathrm{lin}(P)$ covers π in the weak order and from Lemma 2.2.9 we know that $\mathrm{ins}_\omega(\pi')$ is obtained from P by a flip on p and q; there is only one such possible flip toward another acyclic pipe dreams, and so $\mathrm{ins}_\omega(\pi') = P'$. $\square$

We can finally give a few equivalent characterization of lattice defined by the increasing flip graph on $\Sigma(\omega)$.

Proposition 2.2.20 ([BCCP22, Prop. 3.19])**.** *For any pipe dreams $P, P' \in \Sigma(\omega)$, the following statements are equivalent:*
(1) there is a path from P to P' in the increasing flip graph on $\Sigma(\omega)$;
(2) there exists $\pi \in \mathrm{lin}(P)$ and $\pi' \in \mathrm{lin}(P')$ such that $\pi \leqslant \pi'$;
(3) the minimal (resp. maximal) linear extensions π of P and π' of P' satisfy $\pi \leqslant \pi'$;
(4) there are no pipes $p < q$ such that $p \rhd_P q$ and $p \lhd_{P'} q$;
(5) for all $p < q$, if $p \lhd_{P'} q$ then $p \lhd_P q$;
(6) for all $p < q$, if $p \rhd_P q$ then $p \rhd_{P'} q$.

Proof. Theorem 2.2.17 tells us that (1) $\iff$ (2), and Proposition 2.2.15 that (2) implies both versions of (3) (the converse are clear). The equivalences between the items from (3) to (6) are consequences of the definition of the weak order by inclusion of inversion or noninversion sets. $\square$

2.2.5 Algorithms and cover relations

While we gave a mathematical definition of the map ins_ω, it does not give an efficient way of computing the image of a given permutation. Similarly, the description of the pipe dream congruence $\equiv_\omega$ uses pipe dreams, and so determining if two permutations are in the same equivalence class requires drawing pipe dreams and their contact graph. This subsection gives two efficient algorithms (quadratic in n) to compute ins_ω and a rule allowing us to know whether a cover of the weak order is contained in an equivalence class of $\equiv_\omega$ simply by observing the relevant permutations and without computing a pipe dream.

Sweeping algorithm

The sweeping algorithm computes $\mathrm{ins}_\omega(\pi)$ by considering the cells of $\mathcal{T}_n$ from southwest to northeast in any order as long as a cell is swept after all the other cells that are southwest of it. Since we know the starting point of each pipe along the west side of the triangular shape, and pipes go from southwest to northeast, this means that when considering a cell we already know which pipes enter it from south and west. We can then choose to fill that cell with a contact or a cross $+$ depending on the values of those pipes and their relative positions in ω and π: with p to the west and q to the south, we know from Proposition 2.1.9 that the pipes p and q can only cross if $(p, q) \in \mathrm{inv}(\omega)$, in which case they cross as soon as they meet if $\pi^{-1}(p) > \pi^{-1}(q)$ (or a contact would induce an arc $p \to q$ in $P^{\#}$, and π would not

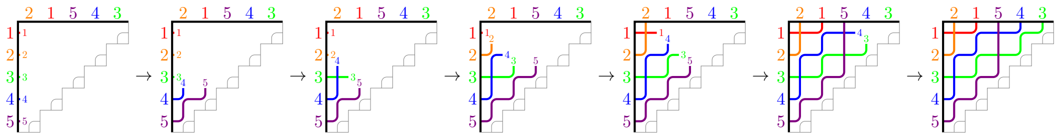


Figure 2.12: Executing the sweeping algorithm for $\omega = 21543$ and $\pi = 21435$.

be a linear extension), or as late as possible otherwise. This gives us the following algorithm.

Algorithm 2.2.21 (Sweeping algorithm, [BCCP22, Prop. 4.1])**.** *For any two permutations $\pi \leqslant \omega$, the pipe dream $\mathrm{ins}_\omega(\pi)$ can be constructed by sweeping the triangular shape $\mathcal{T}_n$ from southwest to northeast. We place a crossing* $+$ *when sweeping cell c if and only if pipe p arriving horizontally and pipe q arriving vertically in c satisfy the following two statements:*

- *(p, q) is an inversion of ω;*
- *$\pi^{-1}(p) > \pi^{-1}(q)$ or the cell c lies in column $\omega^{-1}(q)$.*

Example 2.2.2. The execution of Algorithm 2.2.21 with $\omega = 21543$ and $\pi = 21435$ is drawn in Fig. 2.12. On the first step, the incoming pipes to the first cell are 4 (west) and 5 (south). Since $\pi^{-1}(4) < \pi^{-1}(5)$ and the cell is not in the column where pipe 5 ends, we fill it with a contact. On the second step, the incoming pipes are 3 (west) and 4 (south); since $\pi^{-1}(3) > \pi^{-1}(4)$, we fill this cell with a cross $+$. On the third step, there is a cell with incoming pipes 2 and 4; since $\omega^{-1}(2) < \omega^{-1}(4)$, we fill this cell with a contact. Finally, on the fifth step, we have a cell with incoming pipes 3 and 5 with $\pi^{-1}(3) < \pi^{-1}(5)$, but since this cell lies in the column where pipe 5 ends, we fill this cell with a cross $+$.

Proof. Sweeping $\mathcal{T}_n$ from southwest to northeast allows us to know the two incoming pipes of a cell when considering it. Suppose that this algorithm fills the cells in the same manner as $\mathrm{ins}_\omega(\pi)$ until reaching cell c with incoming pipes p and q. Then:

- if (p, q) is not an inversion of ω, then from Lemma 2.1.12 we know that cell c contains a contact in $\mathrm{ins}_\omega(\pi)$;
- if $(p, q) \in \mathrm{inv}(\omega)$, then:
  - if $\pi^{-1}(p) > \pi^{-1}(q)$, then by definition of $\mathrm{ins}_\omega(\pi)$ its cell c cannot contain a contact or its contact graph would contain the arc $p \to q$;
  - if $\pi^{-1}(p) < \pi^{-1}(q)$, then:
    * if c lies in column $\omega^{-1}(q)$ then pipe q needs to go straight north and so the cell must contain a cross;
    * finally, if cell c is strictly west of column $\omega^{-1}(q)$, then p and q cannot cross in c: otherwise, pipe p would have an elbow east of c and pipe q an elbow north of c, and so by Lemma 2.1.13 we would have a path from q to p in the contact graph; therefore, cell c must contain a contact.

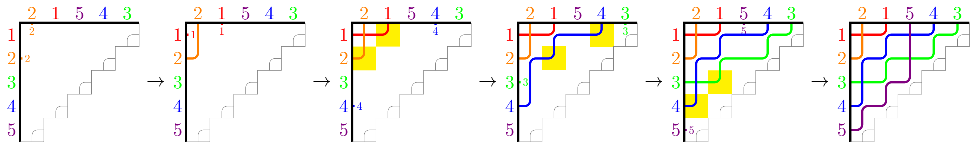


Figure 2.13: Executing the insertion algorithm for $\omega = 21543$ and $\pi = 21435$.

In all cases, the algorithm fills cell c with the same content it has in $\mathrm{ins}_\omega(\pi)$. By induction, this ensure that the final result of the algorithm is $\mathrm{ins}_\omega(\pi)$. $\square$

Insertion algorithm

The insertion algorithm computes $\mathrm{ins}_\omega(\pi)$ by drawing the pipes one by one in the order of π.

Algorithm 2.2.22 (Insertion algorithm, [BCCP22, Prop. 4.3])**.** *For any two permutations $\pi \leqslant \omega$, the pipe dream $\mathrm{ins}_\omega(\pi)$ can be obtained by inserting each pipe in the order given by π. At step i, we insert a pipe starting in row $\pi(i)$, ending in column $\omega^{-1}(\pi(i))$, and whose northeast elbows are precisely completing all the previously created southwest elbows in the rectangle $[\pi(i)] \times [\omega^{-1}(\pi(i))]$.*

Example 2.2.3. The execution of Algorithm 2.2.22 with $\omega = 21543$ and $\pi = 21435$ is given in Fig. 2.13. Before step i, the starting and ending points of $\pi(i)$ are marked and the southwest elbows in the rectangle $[\pi(i)] \times [\omega^{-1}(\pi(i))]$ are highlighted. We note that not all the southwest elbows created before step i are highlighted, as some of them are not in the aforementioned rectangle, and that those highlighted elbows are always placed in a "staircase" from southwest to northeast. This allows the pipe being drawn at that step to complete each of them.

Unlike for the previous algorithm, here we need to prove that there is only one way to insert each pipe as described, that the algorithm returns a pipe dream, and that this pipe dream is reduced and has π as a linear extension. We note that here as long as the result is a pipe dream its exit permutation is ω, since the start and end point of each pipe is fixed by the algorithm. We start with the following essential lemma.

Lemma 2.2.23. *The pipes constructed by the algorithm are disjoint, except at crossings and contacts.*

Proof. Each row r of $\mathcal{T}_n$ is filled with horizontal segments from west to east, with a segment of pipe r placed first and then the only free southeast elbow in that row at the east end of the filled part of r, so any added pipe is completely east of any previously placed horizontal segment in r. This means that there can be no intersection along a horizontal segment (longer than a point) in any row of $\mathcal{T}_n$.

Similarly, each column c of $\mathcal{T}_n$ is filled with vertical segments from north to south, with a segment of pipe $\omega(c)$ placed first and then the only free elbow in that column at the south end of the filled part of c, so any added pipe is completely south of any previously placed vertical segment in c. There can thus be no intersection along a vertical segment in any column of $\mathcal{T}_n$. Therefore, all intersections are points and correspond to contact or crossings. $\square$

We call **staircase** of length k a collection of cells $(c_1, r_1), \ldots, (c_k, r_k)$ such that for any $1 \leqslant i < k$, we have $c_i < c_{i+1}$ and $r_i > r_{i+1}$, i.e. (c_i, r_i) is strictly southwest of (c_{i+1}, r_{i+1}). For the insertion described in the algorithm to be nonambiguous, we need for the free southeast elbows in $[\omega^{-1}(\pi(i))] \times [\pi(i)]$ to be a staircase. The following lemma will prove this as well as count how many such elbows there are.

Lemma 2.2.24. *For any $1 \leqslant r, c \leqslant n$, right before step t, the free elbows in the rectangle $[c] \times [r]$ form a staircase of length*

$$|\{s < t \mid \pi(s) \leqslant r \text{ and } \omega^{-1}(\pi(s)) \leqslant c\}| - |\{s < t \mid \pi(s) > r \text{ and } \omega^{-1}(\pi(s)) > c\}|$$

if that number is positive, and 0 *otherwise.*

Proof. We first note two useful facts: that if (p, q) is a noninversion of ω, then since $\pi \leqslant \omega$ it is also a noninversion of π and so q must be inserted after p, and that we noted in the proof of Lemma 2.2.23 that before inserting pipe p, there is no free elbow in row p or in column $\omega^{-1}(p)$.

We will proceed by induction on t. For $t = 1$, there is no free elbows anywhere in the figure and so the statement is true. Suppose now that it is true for any rectangle before step t, and denote by R the rectangle $[c] \times [r]$. The free elbows in R before step t are a staircase $e_1, \ldots, e_k$ with k given by the formula in the lemma. We denote by i the maximal index such that e_i is south of $\pi(t)$ or 0 if none exist, and by j the minimal index such that e_j is east of $\omega^{-1}(\pi(t))$ or $k + 1$ if none exist. Then:

1. if $\pi(t) \leqslant r$ and $\omega^{-1}(\pi(t)) \leqslant c$, then pipe $\pi(t)$ enters R by row $\pi(t)$ of its west side and then exits it by column $\omega^{-1}(\pi(t))$ of its north side. We know that $i < j$, otherwise there exists $j \leqslant \ell \leqslant i$ and e_ℓ is south of row $\pi(i)$ and east of column $\omega^{-1}(\pi(t))$, so the pipe p creating this elbow is such that $(\pi(t), p)$ is a noninversion of ω; as we noted at the beginning of the proof, pipe p cannot have been inserted before pipe $\pi(t)$. If $j = i + 1$, then pipe $\pi(t)$ cover no free elbow and creates one free elbow e' in cell $(\omega^{-1}(\pi(t)), \pi(t))$ with e_i southwest of e' and $e_j = e_{i+1}$ northeast of it; the free elbows in R now form a staircase $e_1, \ldots, e_i, e', e_j, \ldots, e_k$ of length $k+1$. If $j > i+1$, then pipe $\pi(t)$ covers the free elbows $e_{i+1}, \ldots, e_{j-1}$ and creates new elbows between each of those, as well as one in row $\pi(t)$ and the column of e_{i+1} and one in column $\omega^{-1}(\pi(t))$ and the row of e_{j-1}. Once again, the free elbows in R now form a staircase of length $k + 1$. Since $k + 1$ is the number given by the formula for $t + 1$, by our conditions on $\pi(t)$, the statements still holds.

2. if $\pi(t) \leqslant r$ and $\omega^{-1}(\pi(t)) > c$, then $\pi(t)$ enters R by row $\pi(t)$ of its west side and exits it by its east side, and $j = k+1$. If $i = k$, then pipe $\pi(t)$ covers no free elbow in R and goes straight through it in row $\pi(t)$, and so the free elbows in R are unchanged. If $i < k$, then $\pi(t)$ covers the free elbows $e_{i+1}, \ldots, e_k$ and creates new free elbows $e'_{i+1}, \ldots, e'_k$, with e'_ℓ between $e_{\ell-1}$ and e_ℓ for ℓ between $i+2$ and k and e'_{i+1} in row $\pi(t)$ and the column of e_{i+1}. The created elbows are all northeast of e_i, and so the free elbows in R are still a staircase of length k. Since k is still the number given by the formula for $t+1$, the statement still holds.

3. if $\pi(t) > r$ and $\omega^{-1}(\pi(t)) \leqslant c$, then $\pi(t)$ enters R by its south side and exits it by column $\omega^{-1}(\pi(t)$ of its east side, and $i = 0$. If $j = 1$, then pipe $\pi(t)$ covers no free elbow in R and goes straight through it in column $\omega^{-1}(\pi(t))$, and so the free elbows in R are unchanged. If $j > 1$, then $\pi(t)$ covers the free elbows $e_1, \ldots, e_{j-1}$ and creates new free elbows $e'_1, \ldots, e'_{j-1}$, with e'_ℓ between e_ℓ and $e_{\ell+1}$ for $1 \leqslant \ell < j-1$ and e'_{j-1} in column $\omega^{-1}(\pi(t))$ and the row of e_{j-1}. The created elbows are all southwest of e_j, and so the free elbows in R are still a staircase of length k. Since k is still the number given by the formula for $t+1$, the statement still holds.

4. if $\pi(t) > r$ and $\omega^{-1}(\pi(t)) > c$, then if $k > 0$, pipe $\pi(t)$ enters R by its south side, exits it from its north side, and must cover every single free elbow in R and create a staircase of new free elbows $e'_1, \ldots, e'_{k-1}$ with e'_i placed between e_i and e_{i+1}. Therefore, the free elbows in R now form a staircase of length $k-1$, with $k-1$ the number given by the formula for $t+1$. If $k = 0$, then the pipe doesn't go through R at all and there is still no free elbow in R, so the statement once again holds.

□

This proves that there is never any ambiguity on the trajectory of a pipe when inserting it: it will cover all the free elbows in the rectangle delimitated by its start and end point. Moreover, the following lemma counts those free elbows.

Corollary 2.2.25. *Before step t, the free southeast elbows in $[\omega^{-1}(\pi(t))] \times [\pi(t)]$ form a staircase of length $|\operatorname{ninv}(\omega, \pi(t))|$.*

Proof. We choose $r = \pi(t)$ and $c = \omega^{-1}(\pi(t))$ and apply Lemma 2.2.24. The first remark at the beginning of the proof of that lemma still holds: all pipes in $\operatorname{ninv}(\omega, \pi(t))$ must be inserted before step t and $\{s < t \mid (\pi(t), \pi(s)) \in \operatorname{ninv}(\omega)\}$ must be empty. This proves that the number of free elbows in $[\omega^{-1}(\pi(t))] \times [\pi(t)]$ is exactly $|\operatorname{ninv}(\omega, \pi(t))|$. □

Corollary 2.2.26. *All pipes inserted in Algorithm 2.2.22 stay inside of $\mathcal{T}_n$.*

Proof. Note first that for any pipe p, we have:

- $p-1 = |\operatorname{ninv}(\omega,p)| + \{q < p \mid (q,p) \in \operatorname{ninv}(\omega)\}$ (if $q < p$, either $\omega^{-1}(q) < \omega^{-1}(p)$ or $\omega^{-1}(q) > \omega^{-1}(p)$);
- $\omega^{-1}(p-1) - 1 = |\operatorname{ninv}(\omega,p)| + \{q > p \mid (p,q) \in \operatorname{ninv}(\omega)\}$ (if $\omega^{-1}(q) < \omega^{-1}(p)$, then either $q < p$ or $q > p$).

Therefore $(p-1) + (\omega^{-1}(p) - 1) \geqslant n - 1 + |\operatorname{ninv}(\omega,p)|$, which is equivalent to saying that $p + \omega^{-1}(p) - |\operatorname{ninv}(\omega,p)| \leqslant n+1$.

A pipe p will start on row p and its row will go down at least one for each free elbows it covers. Since by Corollary 2.2.25 it will cover $|\operatorname{ninv}(\omega)|$ free elbows in columns 1 to $\omega^{-1}(p) - 1$, with at most one free elbow in each column, we know that before column c it covers at least $|\operatorname{ninv}(\omega)| - (\omega^{-1}(p) - c)$ free elbows. Therefore, it enters column c in a row $r \leqslant p - |\operatorname{ninv}(\omega,p)| + \omega^{-1}(p) - c$. Since we know that $p + \omega^{-1}(p) - |\operatorname{ninv}(\omega,p)| \leqslant n+1$, we obtain that $r \leqslant n + 1 - c$, and so by definition of $\mathcal{T}_n$, cell (r,c) and any other cell crossed by p in that column are in the triangular shape. $\square$

Proof of the correctness of Algorithm 2.2.22. The insertion algorithm creates a collection of pipes that are disjoint outside of crossings and contacts (Lemma 2.2.23), contained in $\mathcal{T}_n$ (Corollary 2.2.26) and such that each pipe p starts on row p, ends in column $\omega^{-1}(p)$ and has exactly $|\operatorname{ninv}(\omega,p)|$ southeast elbows (Corollary 2.2.25). Therefore, by Lemma 2.1.12, it creates a reduced pipe dream of $\Pi(\omega)$.

Moreover, by construction, any contact between pipe p (northwest) and q (southeast) is created by first inserting pipe p to create a free northwest elbow and then inserting pipe q and completing the contact. This means that if $p \to q$ is an arc of the contact graph, then p was inserted before q and so $\pi^{-1}(p) < \pi^{-1}(q)$. Thus π is a linear extension of the result of the insertion algorithm, and so this result in $\operatorname{ins}_\omega(\pi)$. $\square$

Cover relations

Proposition 2.2.27 ([BCCP22, Prop. 4.10])**.** *The pipe dream congruence $\equiv_\omega$ on the weak order interval $[\mathrm{id},\omega]$ is the reflexive and transitive closure of the relations $UpqV \equiv_\omega UqpV$ with $1 \leqslant p < q \leqslant n$ and U, V possibly empty words on $[n]$ such that*

$$|\{r \in U \mid r > p\}| \geqslant |\{r \in U \mid \omega^{-1}(r) < \omega^{-1}(q)\}|.$$

Proof. As the equivalence classes of $\equiv_\omega$ are intervals, they are connected by simple transpositions. We simply need to prove that for any weak order cover of the form $\pi := UpqV \lessdot \pi' := UqpV$ below ω, $\pi \equiv_\omega \pi'$ iff $|\{r \in U \mid r > p\}|$ is larger than $|\{r \in U \mid \omega^{-1}(r) < \omega^{-1}(q)\}|$. Note that from our choice of cover, we know that $\omega^{-1}(p) > \omega^{-1}(q)$. We proved that two permutations are equivalent for $\equiv_\omega$ if and only if Algorithm 2.2.22 creates the same pipe dream with both as a parameter. Let us consider $t = \pi^{-1}(p) = \pi'^{-1}(q)$; before step p, the insertion algorithm is the same for π and π' as we insert U in both cases. The insertion of p and q commutes if and only if before step t there is no free elbow in the rectangle $R = [\omega^{-1}(q)] \times [p]$. From Lemma 2.2.24, we know that this is true iff

$$|\{s < t \mid \pi(s) > p,\ \omega^{-1}(\pi(s)) > \omega^{-1}(q)\}| \leqslant |\{s < t \mid \pi(s) \leqslant p,\ \omega^{-1}(\pi(s)) \geqslant \omega^{-1}(q)\}|$$

or, written differently,

$$|\{r \in U \mid r > p \text{ and } \omega^{-1}(r) > \omega^{-1}(q)\}| \leqslant |\{r \in U \mid r < p \text{ and } \omega^{-1}(r) < \omega^{-1}(q)\}|.$$

By adding $|\{r \in U \mid r > p \text{ and } \omega^{-1}(r) < \omega^{-1}(q)\}|$ to both sides, we obtain the condition stated in the proposition. □

2.3 Brick polyhedron and an application

The final section of this chapter will first discuss the link between our morphism and the brick polyhedron defined in Section 1.5.6, and then consider the consequences of our results on ν-Tamari lattices, a family of lattices known for being realized by pipe dreams.

2.3.1 Brick polyhedron

We know from Proposition 1.5.15 that the acyclic pipe dreams of $\Pi(\omega)$ are in bijection with the acyclic facets of a subword complex $\mathrm{SC}(T_n, \omega)$. This naturally leads to considering the brick polyhedron of this subword complex; the following

Proposition 2.3.1. *Two acyclic pipe dreams $P, P' \in \Sigma(\omega)$ are linked by a flip if and only if the brick vectors of their associated facets of $\mathrm{SC}(T_n, \omega)$ are linked by an edge of the brick polyhedron of this subword complex.*

Proof. We know from Theorem 1.5.36 that the incidence cone of the brick polyhedron at the vertex associated to an acyclic facet F is exactly $\mathrm{Cone}(\mathbf{R}(F))$. Consider now two acyclic pipe dreams $P, P' \in \Sigma(\omega)$ linked by a flip on the two pipes p and q (with $p \lhd_P q$), and F, F' the associated facets of $\mathrm{SC}(T_n, \omega)$. We know from Lemma 2.2.19 that there is no directed path of length more than 1 from p to q in $P^{\#}$, so if we remove the arc $p \to q$ in this graph there is no longer any directed path between them. As we saw in the proof of Proposition 1.5.15, this means that $e_q - e_p$ is in $\mathbf{R}(F)$ and that there is no positive linear combination of roots in $\mathbf{R}(F)\backslash\{e_q - e_p\}$ that is equal to $e_q - e_p$. Therefore, the line $e_q - e_p$ is a ray of the incidence cone of the brick polyhedron in $\mathrm{B}(F)$, and so there is an edge of $\mathcal{B}(\mathrm{SC}(T_n, \omega)$ incident to $\mathrm{B}(F)$ directed by this root. Since $\mathrm{B}(F') - \mathrm{B}(F) = \sum_{k=1}^{|T_n|}(\mathbf{w}(F', k) - \mathbf{w}(F, k))$, and as seen in the proof of Lemma 1.5.17 for any index we have either $\mathbf{w}(F', k) = \mathbf{w}(F, k)$ or $\mathbf{w}(F', k) = s_{e_q - e_p}\, \mathbf{w}(F, k) = \mathbf{w}(F, k) + \lambda(e_q - e_p)$. Therefore, for some $\lambda \in \mathbb{R}$, we know that $\mathrm{B}(F') = \mathrm{B}(F) + \lambda(e_q - e_p)$ with $\lambda(e_q - e_p) \in \mathrm{Cone}(\mathbf{R}(F))$. Since that cone is pointed and contains $\mathbb{R}_{>0}(e_q - e_p)$, we obtain that $\lambda \geqslant 0$; since the incidence cone of the brick polyhedron is different at $\mathrm{B}(F)$ and $\mathrm{B}(F')$, we know that $\lambda > 0$. Therefore, since $\mathrm{B}(F')$ is also a vertex of the brick polyhedron, there is an edge between $\mathrm{B}(F)$ and $\mathrm{B}(F')$.

Conversely, consider F and F' two acyclic facets of $\mathrm{SC}(T_n, \omega)$ such that there is an edge between $\mathrm{B}(F)$ and $\mathrm{B}(F')$, and let P, P' be their associated acyclic pipe dreams. Since the incidence cone at $\mathrm{B}(F)$ is generated by $\mathbf{R}(F)$, we know that its rays are roots in $\mathbf{R}(F)$ and so $\mathrm{B}(F') - \mathrm{B}(F) = \lambda(e_q - e_p)$ for some $\lambda > 0$ and $p \to q$ an arc of $P^{\#}$. WLOG, we suppose that $q < p$. If there existed a directed path of length more than 1 from p to q in that graph, then $e_q - e_p$ would be a positive linear combination of the roots associated to the arcs in this path and so $e_q - e_p$ would not be a ray of the cone; therefore, such a path does not exist and there exists a linear extension π of $P^{\#}$ such that p and q are consecutive in p, i.e. π is of the form $UpqV$, and covering $\pi' := UqpV$. From Lemma 2.2.9, we know that π' is a linear extension of an acyclic pipe dream P'' of $\Sigma(\omega)$ linked to P by a flip; we saw previously that the brick vector of F and the facet F'' associated to P'' are linked by an edge of the brick polyhedron directed by the root $e_q - e_p$, and so $F' = F''$ and $P' = P''$, and P and P' are linked by a flip. □

Theorem 2.3.2. *The skeleton of the finite part of the brick polyhedron of* $\mathrm{SC}(T_n, \omega)$ *is a lattice quotient of the weak order interval* $[\mathrm{id}, \omega]$.

Proof. We know from Proposition 2.3.1 that the skeleton of the brick polyhedron is isomorphic to the flip graph on $\Sigma(\omega)$, so from Theorem 2.2.17 we obtain that it is a lattice quotient of this interval. □

2.3.2 A realization of ν-Tamari lattices

The ν-Tamari lattices were introduced in [PRV17] by L.-F. Préville-Ratelle and X. Viennot. They are indexed by a lattice path ν composed of a finite number of north and east steps. The elements of the ν-Tamari lattice are the **ν-trees**, which can be represented as **tree-like tableaux**, i.e. sets of points on the part of the integer lattice above ν. The covers of the ν-Tamari lattice are given by the **right rotations** on trees. In particular, for ν_0 the stair path $(NE)^n$, the ν_0-Tamari lattice is the classical Tamari lattice. As in Section 2.2.1, we will once label pipes from 0 to n instead of from 1 to $n+1$.

C. Ceballos, A. Padrol and C. Sarmiento proved in [CPS20, Thm. 5.5] that the ν-Tamari could be obtained as the increasing flip poset on the pipe dreams in $\Pi(0\omega_\nu)$ for an explicit permutation ω_ν associated with ν (the Rothe diagram of ω_ν is the partition with ν as the south east boundary of its Ferrer diagram). For the path ν_0, this correspondance is the same as the one defined in 2.2.1.

Definition 2.3.3. *A permutation ω is **dominant** if it is* 132*-avoidant, i.e. there are no $i < j < k$ such that $\omega(i) < \omega(k) < \omega(j)$.*

By construction, the permutations ω_ν for all possible paths ν are exactly the dominant permutations of any length.

Theorem 2.3.4 ([BCCP22, Thm. 4.18])**.** *All pipe dreams in $\Pi(0\omega)$ are acyclic if and only if ω is a dominant permutation.*

The proof uses the following lemma.

Lemma 2.3.5 ([BCCP22, Lem. 4.17])**.** *For a pipe dream $P \in \Pi(0\omega)$ with a crossing x between two pipes $p < q$, if p has an elbow southwest of x and q an elbow northeast of x, then ω is not dominant.*

Proof. Since p and q cross, we know that $\omega^{-1}(p) > \omega^{-1}(q)$. Denote by r_x the row containing the crossing x and r_c a row containing an elbow of q northeast of x. Since p has an elbow before the crossing x, we know that $p < r_x < r_c$. Suppose that $k < p$ is such that $\omega^{-1}(k) > \omega^{-1}(q)$. Then k must cross q horizontally in a row strictly north of p and different from r_x (in which q crosses p) or r_c (in which q goes horizontally). Since the rows in our shape are labeled from 0 to n, there are p rows strictly north of p and $p-2$ rows in which k can cross q. Therefore, only $p-2$ pipes below p can end after pipe q, and so the set $\{k < p \mid \omega^{-1}(k) < \omega^{-1}(q)\}$ contains at least two pipes, and at least one pipe $k > 0$. Then this pipe satisfies $k < p < q$ and $\omega^{-1}(k) < \omega^{-1}(q) < \omega^{-1}(p)$ so ω does not avoid the pattern 132 and so ω is not dominant. ☐

Proof of Theorem 2.3.4. Suppose first that ω is not dominant and consider $p < q$ such that $\operatorname{ninv}(\omega, p) \cap \operatorname{ninv}(\omega, q) \neq \varnothing$ and $\omega^{-1}(p) > \omega^{-1}(q)$ (with p, q corresponding respectively to 2 and 3 in the pattern). We can find $\pi \leqslant \omega$ such that $\pi = UpqV$ with all the numbers in U below p or in $\operatorname{ninv}(q)$. Then by Proposition 2.2.27 we can see that $UpqV \not\equiv_\omega UqpV$, so $\Pi(\omega)$ contains at least two distinct pipe dreams. In particular, for $P^\uparrow$ the greedy pipe dream of $\Pi(\omega)$, we know that $P^\uparrow$ has at least one contact $p' \to q'$ that is flippable into a decreasing flip, so with $p' > q'$. Consider now the pipe dream P of $\Pi(0\omega)$ obtained by adding a column with $n+1$ contacts to $P^\uparrow$. Then the contacts in that first column are all the $i \to i+1$ for $0 \leqslant i < n$, and so there is a path from q' to p' in $P^\#$. Since $P^\#$ also contains $p' \to q'$, as P has all the contacts of $P^\uparrow$, this means that P is not acyclic.

Suppose now that for some $\omega \in \mathfrak{S}_n$ there exists an acylic pipe dream P in $\Pi(0\omega)$ and consider G the contact graph of P with added edges $p \to q$ if pipe p has an elbow northwest of pipe q. The graph G obviously has at least one cycle; we consider a minimal cycle $C = (c_1, \ldots, c_k)$ (i.e. with no internal edges) and choose c_1 minimal among the c_i. If $k = 2$ then obviously c_1 and c_2 must cross, with an elbow of c_1 northwest of an elbow of c_2 so necessarily before the crossing, and an elbow of c_2 northwest of an elbow of c_1 after the crossing. Then by Lemma 2.3.5 we obtain that ω is not dominant. If $k > 2$, then $c_2 \neq c_k$ and c_k has no elbow northwest of an elbow of c_2 or the cycle would not be minimal. Then by considering the possible relative positions of the elbows of c_2 and c_k that are respectively southeast and northwest of an elbow of a, we obtain a contradiction, which proves that $k = 2$ and that ω is not dominant. $\square$

By noting that the weak order interval $[\mathrm{id}_{[0,n]}, 0\omega]$ is isomorphic to the weak order interval $[\mathrm{id}_{[n]}, \omega]$, we obtain the following consequences:

Corollary 2.3.6 ([BCCP22, Coro 4.19])**.** *The ν-Tamari lattice is a quotient of the interval* $[\mathrm{id}, \omega_\nu]$.

Corollary 2.3.7. *The ν-Tamari lattice is isomorphic to the finite part of the skeleton of the brick polyhedron on* $\mathrm{SC}(T_{n+1}, 0\omega_\nu)$.

Chapter 3

Generalized pipe dreams

The chapter will extend the results of the previous chapter to a much wider family of pipe dreams on nontriangular shapes. The results are presented in [Car22].

We start Section 3.1 with a definition of the alternating shapes on which our new pipe dreams will be defined in Section 3.1.1, as well as pipe dreams and contact graphs in this context. We then give some technical properties on characteristics of pipe dreams on a shape in Section 3.1.2 (coordinates of pipes, number of crossings and contacts...), and in Section 3.1.3 we use these properties to prove an equivalence between pipe dreams on alternating shapes and a family of subword complexes on type A Coxeter groups. Finally, we prove some properties of contact graphs in Section 3.1.4.

Section 3.2 generalizes most of the content of Section 2.2 and discuss the differences between triangular and other pipe dreams. We start in Section 3.2.1 by giving a realization of Cambrian lattices (which are a generalization of Tamari lattices) as pipe dreams on some special shapes, which inspired our definition of alternating shapes. We then prove in Section 3.2.2 that the linear extensions of pipe dreams still define a partition of some set containing a weak order interval, and in Section 3.2.3 that this partition defines a lattice congruence of this interval. We describe in Section 3.2.4 the image of the lattice morphism associated to this congruence and gives two way of computing this morphism in Section 3.2.5.

We close this section by generalizing the results of Section 2.3.1 in Section 3.3: we start in Section 3.3.1 by proving that the image of the morphism defined in Section 3.2.4 is part of the skeleton of the brick polytope, and then in Section 3.3.2 we give a sufficient condition for the image to contain the whole skeleton.

3.1 Definitions

Alternating pipe dreams are a new family of objects generalizing the k-twists defined by V. Pilaud in [Pil18b].

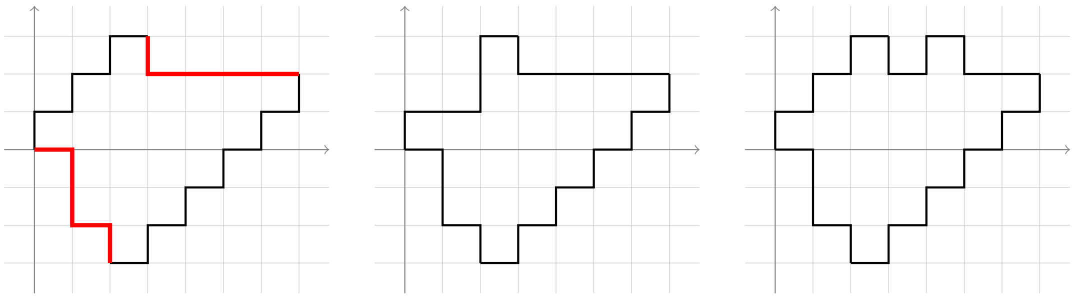


Figure 3.1: An alternating shape and two counter examples.

3.1.1 Alternating shapes and generalized pipe dreams

Consider the cartesian grid on $\mathbb{R}^2$, whose cells are the 1×1 squares with all their corners in $\mathbb{Z}^2$. We index those cells by the coordinates of their lower left corner. We describe paths on this grid, i.e. sequences of points such that two consecutive points are at distance 1, by giving their starting point and the direction of each step: N, S, E and W represent respectively a step north, south, east and west. For a path $\mathcal{P}$, we denote by $|\mathcal{P}|$ its number of steps or length, and for $d \in \{N, S, E, W\}$ a direction we denote by $|\mathcal{P}|_d$ the number of steps in that direction in $\mathcal{P}$.

Definition 3.1.1. *An **alternating shape** F is a connected collection of cells of the cartesian grid whose boundary can be divided in four parts as follows for some integer n:*

- *a **starting path** $\mathcal{S}_F$ from $(0,0)$ to $(|\mathcal{S}_F|_E, -|\mathcal{S}_F|_S)$ with n steps S or E;*
- *a **NW stair path** from $(0,0)$ to (t_F, t_F) with steps $(NE)^{t_F}$ for some $t_F \geqslant 0$;*
- *an **ending path** $\mathcal{E}_F$ from (t_F, t_F) to $(t_F + |\mathcal{E}_F|_E, t_F - |\mathcal{E}_F|_S)$ with n steps S or E;*
- *a **SE stair path** from $(|\mathcal{S}_F|_E, -|\mathcal{S}_F|_S)$ to $(t_F + |\mathcal{E}_F|_E, t_F - |\mathcal{E}_F|_S)$ with $2b_F$ steps $(EN)^{b_F}$ for some $b_F \geqslant 0$.*

For F to be connected, $\mathcal{E}_F$ must stay strictly north and east of $\mathcal{S}_F$ all along.

We note that since $|\mathcal{S}_F|_S + |\mathcal{S}_F|_E = |\mathcal{E}_F|_S + |\mathcal{E}_F|_E = n$, the end points of $\mathcal{S}_F$ at coordinates $(|\mathcal{S}_F|_E, -|\mathcal{S}_F|_S)$ and of $\mathcal{E}_F$ at coordinates $(t + |\mathcal{E}_F|_E, t - |\mathcal{E}_F|_S)$ are on the same diagonal of equation $x - y = n$ and so can be joined by a stair path alternating steps E and N.

Example 3.1.1.
- The first shape in Fig. 3.1 is an alternating shape, with its starting and ending paths bolded and in red.
- The triangular shape $\mathcal{T}_n$ is an alternating shape with a starting path made up of only S steps, a ending path made up of only E steps, a NW stair path of length 0 and a SE stair path of length $2n$ (i.e. $b_{\mathcal{T}_n} = n$).
- The second and third shapes in Fig. 3.1 are not alternating shapes: the NW part of the second shape is not a stair path, and the NE part of the third

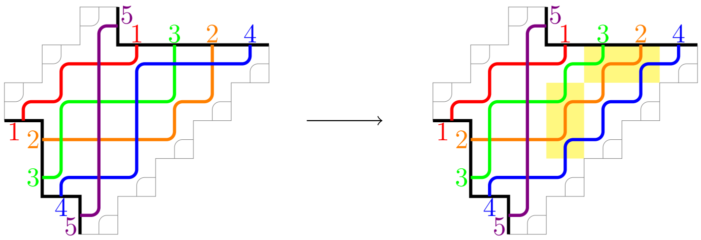


Figure 3.2: A non-reduced pipe dream and its reduced counterpart.

shape has north steps.

Definition 3.1.2. *A **pipe dream** P on an alternating shape F is a filling of the cells in F with crosses* + *and contacts such that each pipe entering on the starting path of F exit on the ending path. We label the pipes from 1 to n following the staring path from north-west to south-east, and we say that the order of their exit points on the ending path (still from north-west to south-east) is the **exit permutation** of P. For examples, the pipe dreams in Fig. 3.2 have exit permutation* 51324*. A permutation $\omega \in \mathfrak{S}_n$ is **sortable** on F if there exists at least one pipe dream on F with exit permutation ω.*

We note that for the pipes to stay inside the alternating shape between the starting and ending path, as for the triangular pipe dreams, the t_F cells right below the NW stair path and the b_F cells right above the SE stair path must contain an elbow.

The notions of reduction and flips introduced for triangular pipe dream is easily extended to generalized pipe dreams.

Definition 3.1.3. *A pipe dream is **reduced** if no pair of pipes cross more than once. For any alternating shape F and permutation $\omega \in \mathfrak{S}_n$ sortable on F, we denote by $\Pi_F(\omega)$ the set of reduced pipe dreams on F with exit permutation ω. The crosses in a reduced pipe dreams are in bijection with the inversions of its exit permutation.*

We note that since we can reduce any non-reduced pipe dream by replacing pairs of crosses between two pipes by pairs of contacts without changing its exit permutation, as illustrated in Fig. 3.2, a permutation ω is sortable on F if and only if $\Pi_F(\omega)$ is nonempty.

Definition 3.1.4. *A contact c in a pipe dream $P \in \Pi_F(\omega)$ is **flippable** if the two pipes passing through c have a crossing x. In that case, the **flip** on c exchanges the contact c and the crossing x to obtain a new pipe dream $P' \in \Pi_F(\omega)$. The flip is **increasing** if c is south-west of x and **decreasing** otherwise. An increasing flip is illustrated in Fig. 3.3.*

*The **increasing flip graph** on $\Pi_F(\omega)$ is the directed graph with $\Pi_F(\omega)$ as its vertices and an arc $P \to P'$ if there is an increasing flip from P to P'.*

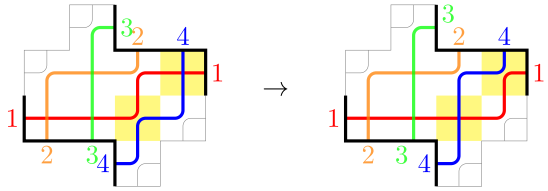


Figure 3.3: An increasing flip between two reduced pipe dreams.

For triangular pipe dreams, the starting and ending points of a pipe was entirely determined by its label and the exit permutation. For generalized pipe dreams, these also depend on the alternating shape on which it is defined; as such, we introduce some notation to facilitate later proofs.

Definition 3.1.5. *Let F be an alternating shape and P a pipe dream on F. For any pipe p of P:*

- *(x_p^s, y_p^s) the **starting coordinates** of p point to the SW corner of its first cell;*
- *(x_p^e, y_p^e) the **ending coordinates** of p point to the NE corner of its last cell.*

*We say that the rectangle that has (x_p^s, y_p^s) as its SW corner and (x_p^e, y_p^e) as its NE corner is the **zone** of p, denoted by $\mathcal{Z}_p$.*

Remark 3.1.6. The zone of p contains the entire trajectory of p: any cell crossed by p must be in it. Since it contains at least the first cell of p, we know that $x_p^s < x_p^e$ and $y_p^s < y_p^e$. We also note that since the pipes start along the starting path of F in increasing order from NW to SE, for any pipes $p < q$ we have $0 \leqslant x_p^s \leqslant x_q^s \leqslant |\mathcal{S}_F|_E$ and $0 \geqslant y_p^s \geqslant y_q^s \geqslant -|\mathcal{S}_F|_S$. Similarly, since the pipes end along the ending path of F from NW to SE in the order of the exit permutation ω, for any positions $i < j$ we have $t_F \leqslant x^e_{\omega(i)} \leqslant x^e_{\omega(j)} \leqslant t_F + |\mathcal{E}_F|_E$ and $t_F \geqslant y^e_{\omega(i)} \geqslant y^e_{\omega(j)} \geqslant t_F - |\mathcal{E}_F|_S$.

Finally, we also extend the notion of the contact graph of a pipe dream, though in this case we will define two different graphs for each pipe dream.

Definition 3.1.7. *The **contact graph** of a pipe dream P is the directed graph $P^{\#}$ that has the pipes of P as its vertices and contains the edge $p \to q$ if there is a contact p q in P. The **extended contact graph** of P is the directed graph $P^{\natural}$ obtained by adding to $P^{\#}$ the missing arc $p \to q$ such that (p, q) is a noninversion of the exit permutation of P.*

*The pipe dream P is **acyclic** if its contact graph is acyclic, and **strongly acyclic** if $P^{\natural}$ is acyclic. In that last case, we write $p \vartriangleleft_P q$ to denote that there is a path from pipe p to pipe q in $P^{\natural}$. We denote by $\Sigma_F(\omega)$ the set of strongly acyclic pipe dreams of $\Pi_F(\omega)$.*

Remark 3.1.8. The linear extensions of $P^{\natural}$ are exactly the linear extensions of $P^{\#}$ that are in the weak order interval $[\mathrm{id}, \omega]$: adding arcs corresponding to the noninversions of ω is the same as adding the condition "the noninversions of linear

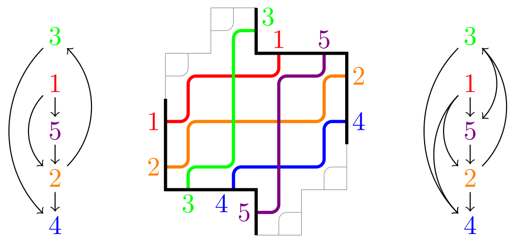


Figure 3.4: The contact graph and extended contact graph of a pipe dream.

extensions must contain the noninversions of ω". In the case of triangular pipe dreams, Lemma 2.1.14 thus guarantees that the linear extensions of $P^\#$ are the same as the linear extensions of $P^\natural$. In particular, any acyclic triangular pipe dream is also strongly acyclic. This is not true in general, as seen in Fig. 3.4: the pipe dream presented is acyclic but not strongly acyclic.

3.1.2 Arithmetics of pipe dreams

We will now discuss a few technical properties on alternating shapes and pipe dreams.

Lemma 3.1.9. *For F an alternating shape and P a pipe dream on F with exit permutation ω,*

- $x^s_p - y^s_p = p - \delta^s_p$ *with* $\delta^s_p = 1$ *if p starts vertically and* 0 *otherwise;*
- $x^e_p - y^e_p = \omega^{-1}(p) - \delta^e_p$ *with* $\delta^e_p = 1$ *if p ends horizontally and* 0 *otherwise.*

Proof. We denote the coordinates of the integer points of $\mathcal{S}_F$ from northwest to southeast by (x_0, y_0), $(x_1, y_1), \dots, (x_n, y_n)$ and know that $x_0 = y_0 = 0$; since this path is made of south and east steps, for each $0 \leqslant i < n$ we have either $x_{i+1} = x_i$ and $y_{i+1} = y_i - 1$, or $x_{i+1} = x_i + 1$ and $y_{i+1} = y_i$. Since $x_0 - y_0 = 0$, this proves that for all i we have $x_i - y_i = i$. Then for $1 \leqslant p \leqslant n$, pipe p starts on the p-th step of $\mathcal{S}_F$ which is between points (x_{i-1}, y_{i-1}) and (x_i, y_i). The cell in which p starts is then indexed by its southwest corner, so (x_{i-1}, y_{i-1}) if the p-th step goes east and (x_i, y_i) if it goes south. The first case corresponds to p starting vertically and the second case to p starting vertically.

The second part of the lemma is similar: if (x'_i, y'_i) is the i-th integer point of $\mathcal{E}_F$ (indexed from 0 to n) then since $x'_0 = y'_0 = t_F$ and $\mathcal{E}_F$ is made of south and east steps, we have $x'_i - y'_i = i$ for each i. Then since pipe p ends on the $\omega^{-1}(p)$-th step of $\mathcal{E}_F$ and its ending coordinates point to the northeast corner of its last cell, depending on the direction of that $\omega^{-1}(p)$-th step (and so the ending direction of pipe p) the ending coordinates of p are either $(x'_{\omega^{-1}(p)-1}, y'_{\omega^{-1}(p)-1})$ or $(x'_{\omega^{-1}(p)}, y'_{\omega^{-1}(p)})$, thus concluding the proof. □

Lemma 3.1.10. *For F an alternating shape and (x, y) one of its cells,*

- *$0 \leqslant x - y \leqslant n$;*
- *if $x - y$ is 0 or n, then (x, y) contains a contact in all pipe dreams on F.*

Proof. Since the NW stair path of F starts on point $(0, 0)$, we note that all the cells just below that path are of the form (x, x); similarly, since the SE stair path of F starts on point $(|\mathcal{S}_F|_E, -|\mathcal{S}_F|_S)$ with $|\mathcal{S}_F|_E + |\mathcal{S}_F|_S = n$, the cells right above the SE stair path are of the form $(x, x - n)$. This means that the cells (x, y) adjacent to one of those paths, and so necessarily containing a contact in any pipe dream on F, satisfy either $x - y = 0$ or $x - y = n$.

Consider now any cell (x, y) in F that is not adjacent to one of the stair paths. If the cell is below the NW stair path then necessarily $x - y > 0$; if it is below the ending path $\mathcal{E}_F$ then it is below the last cell of some pipe p ending vertically, and so from Lemma 3.1.9 we know that $x - y \geqslant p > 0$. Similarly, if the cell is above the SE stair path of F then $x - y < n$; if it is above the starting path $\mathcal{S}_F$ then it is above the first cell of some pipe p starting vertically, and so from Lemma 3.1.9 we know that $x - y \leqslant p - 1 < n$. Thus, in any cases, we have $0 < x - y < n$. □

Proposition 3.1.11. *A pipe dream P with exit permutation ω is reduced if and only if it contains exactly $|\operatorname{inv}(\omega)|$ crosses, and if and only if all crosses have the pipe going horizontally smaller than the pipe going vertically.*

Proof. The proof is the same as for Proposition 2.1.9: each pair of pipes corresponding to an inversion of ω must cross exactly once, and all other pipes must not cross at all. If they cross once, they must do so in the direction given by the proposition. If a pair of pipes cross more than once, the second crossing will be in the opposite direction. □

Lemma 3.1.12. *For any pipe dream $P \in \Pi_F(\omega)$, the pipe p has exactly*

- *$y^e_p - x^s_p + |\operatorname{ninv}(\omega, p)|$ southeast elbows;*
- *$y^e_p - x^s_p + |\operatorname{ninv}(\omega, p)| + 1 - \delta^e_p - \delta^s_p$ northwest elbows;*
- *$p - 1 - |\operatorname{ninv}(\omega, p)|$ vertical crossings;*
- *$\omega^{-1}(p) - 1 - |\operatorname{ninv}(\omega, p)|$ horizontal crossings;*

with δ^e_p and δ^s_p as defined in Lemma 3.1.9.

Proof. Let us denote by N_v the number of vertical crossings of p, by N_h its number of horizontal crossings, by N_e its number of southeast elbows and by N'_e its number of northwest elbows. We first note that since p starts in cell (x^s_p, y^s_p) and ends in cell $(x^e_p - 1, y^e_p - 1)$, the total number of cells crossed by p is $x^e_p - x^s_p + y^e_p - y^s_p - 1$. Therefore, the sum $N_v + N_h + N_e + N'_e$ is equal to that number. Moreover, from Lemma 3.1.9, we know that $y^s_p = x^s_p - p + \delta^s_p$ and $x^e_p = y^e_p + \omega^{-1}(p) - \delta^e_p$, so we have $N_h + N_v + N_e + N'_e = 2y^e_p - 2x^s_p + \omega^{-1}(p) + p - \delta^e_p - \delta^s_p - 1$.

Since P is reduced, pipe p must cross once each pipe q such that $(q, p) \in \operatorname{inv}(\omega)$ and do so vertically, and each pipe q such that $(p, q) \in \operatorname{inv}(\omega)$ and do so horizontally. As we saw in the proof of Lemma 2.1.11, this gives us that $N_h = p - 1 - |\operatorname{ninv}(\omega, p)|$ and $N_v = \omega^{-1}(p) - 1 - |\operatorname{ninv}(\omega, p)|$.

This leaves us with $N_e+N'_e = 2(y_p^e-x_p^s+|\operatorname{ninv}(\omega,p)|)-\delta_p^e-\delta_p^s+1$. A consideration of the possible trajectories of pipes shows that $N'_e = N_e + 1 - \delta_p^e - \delta_p^s$, so we obtain that $N_e = y_p^e - x_p^s + |\operatorname{ninv}(\omega,p)|$ and $N'_e = y_p^e - x_p^s + |\operatorname{ninv}(\omega,p)| + 1 - \delta_p^e - \delta_p^s$. $\square$

Lemma 3.1.13. *Any collection P of n pipes pairwise disjoint except when crossing such that*

1. *the pipes are contained in an alternating shape F, start on its starting path and end on its ending path;*
2. *pipe p starts in the p-th step of $\mathcal{S}_F$ and ends in the $\omega^{-1}(p)$-th step of $\mathcal{E}_F$;*
3. *pipe p has at least $y_p^e - x_p^s + |\operatorname{ninv}(\omega,p)|$ southeast elbows;*

is a pipe dream of $\Pi_F(\omega)$.

Proof. As for triangular pipe dreams, a collection of pipes as described in the frist two items is necessarily a pipe dream on F with exit permutation ω. We only need to prove that it is reduced.

Suppose that P is a non-reduced pipe dream on F with exit permutation ω. By definition, there is a pipe p that cross at least one other pipe at least twice. Moreover, it must cross each pipe q such that (p,q) is an inversion horizontally, and each pipe q such that (q,p) is an inversion vertically. Since we have $p-1-|\operatorname{ninv}(\omega,p)|$ pipes in the first category and $\omega^{-1}(p)-1-|\operatorname{ninv}(\omega,p)|$ pipes in the second category, pipe p must have at least $p+\omega^{-1}(p)-2|\operatorname{ninv}(\omega,p)|-1$ crossings on its trajectory. We already saw in the proof of Lemma 3.1.12 that p crosses exactly $2y_p^e-2x_p^s+\omega^{-1}(p)+p-\delta_p^e-\delta_p^s-1$ cells, so the total number of elbows of p is at most $2(y_p^e-x_p^s+|\operatorname{ninv}(\omega,p)|)-\delta_p^e-\delta_p^s$. With the link between the number of southeast elbows and northwest elbows given previously, in that case p has at most $y_p^e-x_p^s+|\operatorname{ninv}(\omega,p)|-1$ southwest elbows. Thus a pipe dream respecting the third condition of the lemma cannot be non-reduced, and so it must be reduced. $\square$

3.1.3 Link to subword complexes

Triangular pipe dreams were shown in Section 2.1.2 to be a graphical representation of facets of subword complexes on triangular words (see Fig. 2.4). This subsection details a similar link between pipe dreams on alternating shapes and facets on alternating words.

Definition 3.1.14. *Let F be an alternating shape, we denote by Q_F the word on simple transpositions of $\mathfrak{S}_n$ associated to F and created as follows:*

- *the cells (x,y) of F such that $1 \leqslant x-y \leqslant n-1$ are enumerated in lexicographical order (as seen in Fig. 3.5);*
- *for the cell (x,y), the letter τ_{x-y} is added to Q.*

Example 3.1.2. The enumeration of cells of the 5-shape drawn in Fig. 3.1 is given in Fig. 3.5; we can deduce that its associated word is $\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_1\tau_4\tau_3\tau_2\tau_4\tau_3\tau_4$.

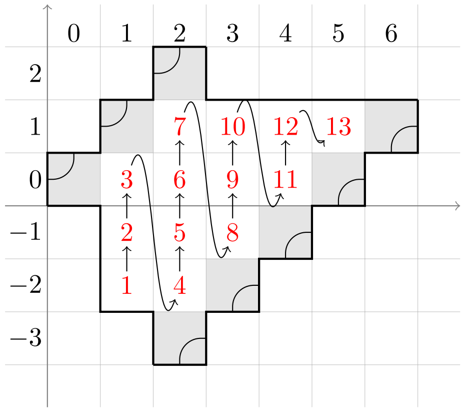


Figure 3.5: The enumeration of cells in the 5-shape of Fig. 3.1.

Lemma 3.1.15. *If F is an alternating shape, then Q_F is alternating.*

Proof. Let (x_i, y_i) be the coordinates of the i-th cell of F as enumerated in Definition 3.1.14 for any $1 \leqslant i \leqslant |Q_F|$. Suppose that $1 \leqslant i < j \leqslant |Q_F|$ are such that $x_i - y_i = x_j - y_j =: k$ (i.e. the i-th and j-th letters of Q_F are the same). Since (x_i, y_i) is smaller than (x_j, y_j) in the lexicographical order, we know that $x_i < x_j$ and so $y_j = y_i + (x_j - x_i) > y_i$. The upper boundary of F is made of its NW stair path and its ending path, so (x_i, y_i) is below one of those. Then:

- if it is below the NW stair path, then since (x_i, y_i) is not right below that path (or we would have $x_i - y_i = 0$), we know that $(x_i, y_i + 1) \in F$, and so either $x_i - y_i = 1$ or $(x_{i+1}, y_{i+1}) = (x_i, y_i + 1)$ and the $(i + 1)$-th letter of Q_F is $\tau_{x_i - y_i - 1}$;
- if it is below the ending path, then so is (x_j, y_j) (because it is further east) and since the ending path goes from NW to SE, we know that the cell furthest north in column x_i is at least in row y_j. Therefore, cell $(x_i, y_i + 1)$ is once again in F.

Therefore, either $x_i - y_i = 1$, or $(x_i, y_i + 1)$ is the cell of F right after (x_i, y_i) in the enumeration of F and the $(i + 1) - th$ letter of Q_F (with $i < i + 1 \leqslant j$) is $\tau_{x_i - y_i - 1} = \tau_{k-1}$.

Similarly, the lower boundary of F is made of its starting path and its SE stair path, so (x_j, y_j) is above one of those. Then:

- if it is above the starting path, then so is (x_i, y_i) (because it is further west) and since the starting path goes from NW to SE, we know that the cell furthest south in column x_j is at most in row $y_i < y_j$, and so cell $(x_j, y_j - 1)$ is in F;
- if it is above the SE stair path, then since (x_j, y_j) is not right above that path (or we would have $x_j - y_j = n$), once again cell $(x_j, y_j - 1)$ is in F.

Therefore, either $x_j - y_j = n - 1$ or $(x_j, y_j - 1)$ is the cell of F enumerated right before (x_j, y_j) and the $j - 1$-th letter of Q_F (with $i \leqslant j - 1 < j$) is $\tau_{x_j - y_j + 1} = \tau_{k+1}$.

Thus, all the letters that do not commute with τ_k appear at least once between indices i and j, and so Q_F is alternating. $\square$

Lemma 3.1.16. *Any alternating word Q on the simple transpositions of $\mathfrak{S}_n$ in which each simple transposition appears at least once is equivalent, up to commutation of commuting letters, to a word Q_F for some alternating shape F.*

Proof. The case $n = 2$ is obvious: the word associated to any 2-shape is of the form $\tau_1\tau_1 \dots \tau_1$, with τ_1 the only simple transposition of $\mathfrak{S}_1$, and so any word on this alphabet is associated to a 2-shape. We can thus suppose that $n > 2$ and so that two consecutive letters of an alternating word are never the same.

For Q an alternating word of length m containing each simple transposition, consider the set $\mathcal{C}_Q$ of all words equivalent to Q up to commutation of commutative letters. This set is finite, as it only contains words of length m on a finite alphabet. Therefore, it has a minimum $Q' = \tau_{k_1} \dots \tau_{k_m}$ for the lexicographical order. This word is alternating and for any $1 \leqslant j < m$, either $k_j < k_{j+1}$ or k_j does not commute with k_{j+1}, otherwise we could obtain a word in $\mathcal{C}_Q$ lexicographically smaller than Q' by exchanging its j-th and $(j+1)$-th letters. This means that either $k_j < k_{j+1}$ or $k_j = k_{j+1} + 1$. We can thus divide Q' as the product $Q_0Q_1 \dots Q_M$ with the factors Q_i of the form $\tau_{a_i}\tau_{a_{i-1}} \dots \tau_{b_i}$, by cutting Q' between the indices j and $j+1$ such that $k_j < k_{j+1}$. Then:

1. by choice of decomposition we know that $a_i \geqslant b_i < a_{i+1}$;
2. suppose that $a_{i+1} \leqslant a_i$, then $b_i < a_{i+1} \leqslant a_i$ so the letter $\tau_{a_{i+1}}$ appears in Q_i. Then no letter $\tau_{a_{i+1}+1}$ appears after that occurrence in Q_i and before the first letter of Q_{i+1}, which is also $\tau_{a_{i+1}}$. Since Q' is alternating, this means that $a_{i+1} = n - 1$. Therefore, either $a_i < a_{i+1}$ or $a_i = a_{i+1} = n - 1$;
3. since τ_{n+1} appears at least once in Q', the previous item gives $a_M = n - 1$;
4. suppose that $b_i \geqslant b_{i+1}$, then $b_{i+1} \leqslant b_i < a_{i+1}$ and so the letter τ_{b_i} appears in Q_{i+1}. Then no letter τ_{b_i-1} appears before that occurrence in Q_{i+1} and after the last letter of Q_i, which is also τ_{b_i}. Since Q' is alternating, this means that $b_i = 1$. Therefore, either $b_i < b_{i+1}$ or $b_i = b_{i+1} = 1$;
5. since τ_1 appears at least once in Q', the previous item implies that $b_1 = 1$;
6. for any $0 \leqslant i < M$, all the letters in $Q_0, \dots, Q_i$ are in $\tau_1, \dots, \tau_{a_i}$ and the letters in $Q_{i+1}, \dots, Q_M$ are in $\tau_{b_{i+1}}, \dots, \tau_{n-1}$, so since if $a_i < n - 1$ the letter τ_{a_i+1} must appear somewhere in Q' we obtain that $b_{i+1} \leqslant a_i + 1$.

Denote by $T = \max\{i \mid b_i = 1\}$ and $B = \min\{i \mid a_i = n - 1\}$. We know from item 4 that $i \leqslant T \iff b_i = 1$ and from item 2 that $i \geqslant B \iff a_i = n - 1$, and that (a_i) is strictly increasing before index B and (b_i) strictly decreasing after index T.

Consider now the collection of cells

$$\begin{aligned} F = &\{(i, y) \mid 0 \leqslant i \leqslant M \text{ and } i - a_i \leqslant y \leqslant i - b_i\} \\ &\cup \{(i, i) \mid 0 \leqslant i < T\} \\ &\cup \{(i, i - n) \mid B < i \leqslant M\} \end{aligned}$$

If (x, y) is in F then $0 \leqslant x \leqslant M$ and $0 \leqslant x - y \leqslant n$. Let us study the boundaries of F:

1. for $0 \leqslant x < T$, the cell furthest north of F in column x is (x, x), so the upper boundary of F in those columns is a stair path $(NE)^T$ going from $(0, 0)$ to $(T - 1, T - 1)$;
2. for $B < x \leqslant M$, the cell furthest south of F in column x is $(x, x - n)$ so the lower boundary of F in those columns is a stair path $(EN)^{M-B}$ going from $(B + 1, B + 1 - n)$ to $(M + 1, M + 1 - n)$;
3. for $T \leqslant x \leqslant M$, the cell furthest north of F in column x is $(x, x - b_x)$; if $x < M$ then $b_x < b_{x+1}$ so $x - b_x \geqslant (x + 1) - b_{x+1}$, so the upper boundary of F in those column is a path going from $(T - 1, T - 1)$ to $(M + 1, M + 1 - n)$ with only east and south steps. That path has $M + 1 - (T - 1)$ east steps as well as $(T - 1) - (M + 1 - n)$ south steps, so its total length is n steps;
4. for $0 \leqslant x \leqslant B$, the cell furthest south of F in column x is $(x, x - a_x)$; if $x < B$ then $a_x < a_{x+1}$ so $x - a_x \geqslant (x + 1) - a_{x+1}$, so the lower boundary of F in those columns is a path going from $(0, 0)$ to $(B + 1, B + 1 - n)$ with only east and south steps. That path has $B + 1$ east steps and $n - B - 1$ south steps, so its length is n.

This is exactly the description the boundaries of an alternating shape, and so F is one. □

Theorem 3.1.17. *For F an alternating shape and $\omega \in \mathfrak{S}_n$ sortable on F, the increasing flip graph on $\Pi_F(\omega)$ is isomorphic to the increasing flip graph on the subword complex* $\mathrm{SC}(Q_F, \omega)$*.*

Proof. The correspondance is given in Fig. 3.6: by rotating the sorting network representing Q_F by an angle of $\frac{\pi}{4}$, we can superpose each letter of Q_F with the corresponding inside cell of F (this correspondance is the same as the enumeration described in Definition 3.1.14). Like in the case of Fig. 2.4, we note that some letters have been shifted to make the sorting network more compact, but that this does not alter the correspondance with facets of the subword complex. A cell of the pipe dream then contains a cross $+$ iff the associated letter is not in the facet and a contact iff that letter is in the facet. We can check that the pipes entering a cell of the pipe dream are the same as the pipes adjacent to a letter of a facet, which means that the pipe dream is reduced iff the subword is reduced, and both have the same exit permutation. It is clear that the flips on pipe dreams are the images of the flips on facets. □

Corollary 3.1.18. *The increasing flip graph on $\Pi_F(\omega)$ has one source and one sink.*

Proof. This is a direct application of Theorem 1.5.21 to $\mathrm{SC}(Q_F, \omega)$. □

Like in the triangular case, we can then characterize the link between the root configuration of a facet and the contact graph of the associated pipe dream.

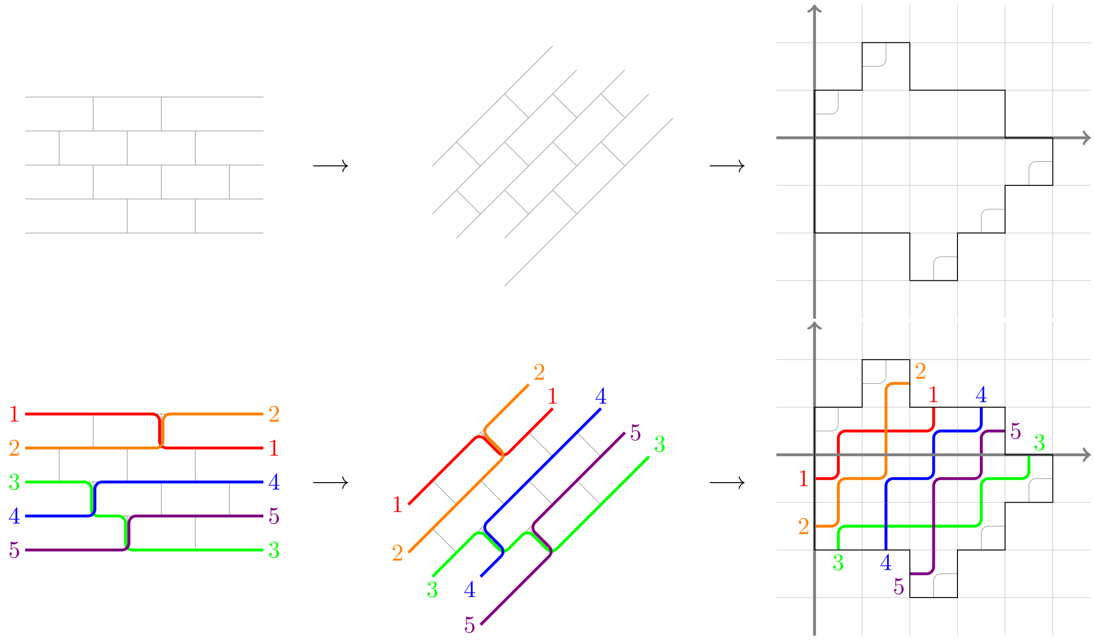


Figure 3.6: The correspondance between facets of $\mathrm{SC}(Q_F, \omega)$ and $\Pi_F(\omega)$.

Remark 3.1.19. As was the case for triangular pipe dreams, the contact graph of a generalized pipe dream P is the same as the contact graph of its associated subword complex facet F, and so Proposition 1.5.15 allows us to deduce many properties of the root configuration $\mathbf{R}(F)$ from $P^{\#}$.

Remark 3.1.20. In Section 2.2.1 and 2.3.2 of Chapter 2, we worked on pipe dreams with $n+1$ pipes labeled from 0 to n and such that the trajectory of pipe 0 is only one southeast elbow. We note that these pipe dreams are equivalent to pipe dreams with n pipes on a shape $\mathcal{T}'_n$, with its starting path made of n south steps and its ending path made of n east steps, and with $T_{\mathcal{T}'_n} = 1$ and $b_{\mathcal{T}'_n} = n+1$ (the only "pseudopipe" below the NW stair path of this shape replaces pipe 0). In general, the pipe dreams in $\Pi(0\omega)$ are equivalent to those in $\Pi_{\mathcal{T}'_n}(\omega)$. In terms of subword complexes, this shape is associated to the word cT_n on simple transpositions, with c the product of all simple transpositions in decreasing order from τ_{n-1} to τ_1. We note that as a product of all simple transpositions of $\mathfrak{S}_n$, the element c is what is often called a Coxeter element of this group; moreover, the word T_n is actually the reduced word of ω_0 denoted by $\omega_0(c)$, i.e. the one obtained by considering the simple transpositions in the order given by reduced words of c and adding the ones that keep the word reduced.

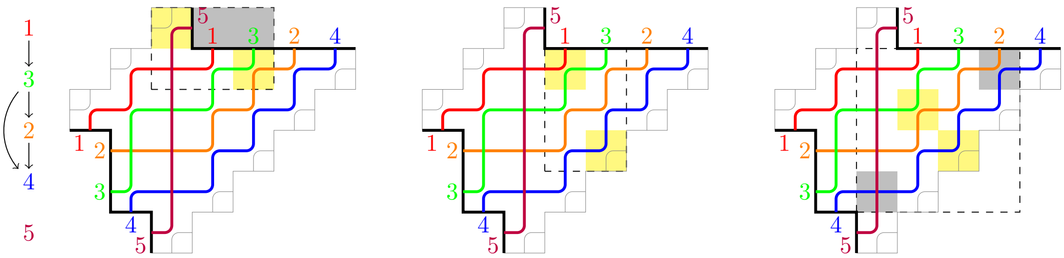


Figure 3.7: Illustrations of the cases of Lemma 3.1.21.

3.1.4 Some properties of the contact graphs

As we were for triangular pipe dreams, we will be interested in the linear extensions of contact graphs; therefore, we will give a few of their properties in this section. We start with a lemma similar to Lemma 2.1.13 but weaker, as the argument used in its proof cannot always be applied to alternating shapes.

Lemma 3.1.21. *Let F be an alternating shape, P a reduced pipe dream on F and p, q two pipes of P. Suppose that pipe p has an elbow in a cell (x_p, y_p) weakly northwest of an elbow of q in a cell (x_q, y_q). If any of the following three conditions is met, then there is a directed path from p to q in $P^\#$:*

1. *the cells (x_p, y_q) and (x_q, y_p) (the other corners of the rectangle between the two elbows) are in F;*
2. *there exists (x_1, y_1) and (x_2, y_2) cells of F such that the two elbows are both north-east of (x_1, y_1) and south-west of (x_2, y_2);*
3. *$(x_p, y_p) \in \mathcal{Z}_q$ (see Definition 3.1.5).*

Some cases of this lemma are illustrated in Fig. 3.7. The first pair of elbows has the rectangle between them partially outside of the shape, so the lemma cannot be applied, and there is no path from 5 to 3 in the contact graph. The second pair of elbows has the rectangle between them completely in the shape, so we can apply case one of the lemma, and we can check that there is a path $1 \to 3 \to 2 \to 4$ in the contact graph. Finally, the third pair of elbows is between the two gray cells, both inside the shape, and so we can apply case 2 of the lemma; once again, there is a path $3 \to 2 \to 4$ in the contact graph.

Proof. We will start by proving the first case, and the second and third cases are natural consequences of it. Suppose thus that the situation is as described in the lemma, and that (x_p, y_q) and (x_q, y_p) are in F. The proof is exactly the same as the one for Lemma 2.1.13: we proceed by induction on the grid distance between (x_p, y_p) and (x_q, y_q), and we follow the pipes southeast of the first cell and northwest of the second cell along the edges of the rectangle. Either one of them has an elbow strictly closer, to the other, and we conclude by induction, or both pipes proceed until the cells (x_p, y_q) and (x_q, y_p). In that last case, since P is reduced, at least one of those cells must contain a contact, which concludes the proof.

Suppose now that we are in case 2 with (x_1, y_1) and (x_2, y_2) as described. From the positions of the cells, we know that $x_1 \leqslant x_p \leqslant x_q \leqslant x_2$ and $y_1 \leqslant y_q \leqslant y_p \leqslant y_2$. Since the upper boundary of F is made up of the NW stair path and of the ending path, it goes north then south when sweeping F from west to east; in particular, since column x_q is between column x_p and column x_2, that boundary cannot be lower in column x_q than in both column x_p and column x_2. Therefore, since the cells (x_p, y_p) and (x_2, y_2) are both in F, the boundary in column x_q is at least on row $\min(y_p, y_2) = y_p$. The cell (x_q, y_p) is therefore south of the upper boundary of F and north of its cell (x_q, y_q), and so it is in F. Similarly, since the lower boundary of F goes south (on the starting path) then north (on the SE stair path) when sweeping F from west to east, it cannot be higher in column x_p than in both column x_1 and x_q. It must thus be below row $\max(y_1, y_q) = y_q$, and so cell (x_p, y_q) is below (x_p, y_p) and above the lower boundary, so it is in F. Since both (x_p, y_q) and (x_q, y_p) are in F, we can then apply case 1.

Finally, case 3 is a direct consequence of case 2 with the first and last cell of pipe q used for (x_1, y_1) and (x_2, y_2). $\square$

We use this lemma to prove the following statement, which is a direct extension of Lemma 2.1.15 to generalized pipe dreams.

Lemma 3.1.22. *Let F be an alternating shape and P a reduced pipe dream on F with exit permutation $\omega \in \mathfrak{S}_n$, and some pipes $1 \leqslant p < q < r \leqslant n$ of P such that $\omega^{-1}(r) < \omega^{-1}(q) < \omega^{-1}(p)$. If there is a contact of P involving the pipes p and r, then $p \lhd_P q \rhd_P r$ or $p \rhd_P q \lhd_P r$.*

Proof. We know that $(p, q), (q, r), (p, r) \in \mathrm{inv}(\omega)$ so the three pairs must all cross at some point in P. Denote by x the cell where p and q cross, and x' the cell where q and r cross. Since pipe q goes through both of those cells, one must be southwest of the other; for now, we suppose that x is southwest of x'. Moreover, since pipe q starts and ends between pipes p and r, any contact between p and r must be between x and x'; suppose that such a contact exist and denote it by c (x is SW of c which is SW of x').

Since pipe q goes through x vertically (as $q > p$) and through x' horizontally (as $q < r$), it must have at least one elbow between those two points; we denote by e the first elbow of q after x, which must therefore be in the same column as x and north of it, and also be southwest of x'. Similarly, since pipe p has a contact c after x, we can consider its first elbow e_p after x; it must be in the same row as x and east of it, and also be weakly southwest of c. Then e is strictly northwest of e_p and both elbows are northeast of x and southeast of x', so by case 2 of Lemma 3.1.21 we obtain that there is a path from q to p in $P^{\#}$.

Similarly, the last elbow e' of q before x' is directly west of x' and northeast of x, and the last elbow e_r of r before x' is directly south of x' and northeast of c, so also

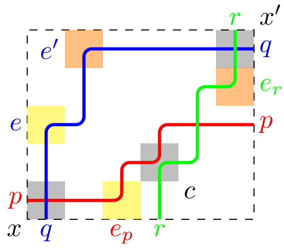


Figure 3.8: Illustration of the proof of lemma Lemma 3.1.22.

northeast of x. Since e' is northwest of e_r and both are between x and x', we apply case 2 of Lemma 3.1.21 and obtain that there is a path from q to r in $P^{\#}$.

See Fig. 3.8 for an illustration of the placement of the various cells that we used. The reasoning is similar for x' southwest of x, and in that case we obtain paths from p and r to q in $P^{\#}$. □

3.2 Lattice properties on acyclic pipe dreams

We will now study the linear extensions of contact graphs and extended contact graphs on generalized pipe dream, and see to what extent the results on triangular pipe dreams can be generalized.

3.2.1 Cambrian lattices

Cambrian lattices were introduced by N. Reading in [Rea06] as congruences of the weak order of finite Coxeter groups. A Cambrian lattice is characterized by an orientation of the Coxeter diagram associated to the group; it is the smallest lattice congruence containing the relations $t \equiv tsts\dots$ for each edge $s \to t$ of the Coxeter diagram. Therefore, by definition, it is a lattice congruence of the weak order. In particular, for Coxeter groups of type A and the orientation of the Coxeter diagram given by $\tau_i \to \tau_{i+1}$, we find the sylvester congruence, whose quotient is the Tamari lattice.

In general, a Cambrian lattice on the Coxeter group of type A_n (or the symmetric group $\mathfrak{S}_{n+1}$) can be parametrize by a sequence of $n-1$ signs $+$ or $-$ depending on whether the edge $\tau_i - \tau_{i+1}$ of the Coxeter diagram is directed from τ_i to τ_{i+1} or vice versa. For such a sequence ϵ, G. Chatel and V. Pilaud defined **Cambrian trees** in [CP17] as trees with $n+1$ internal nodes, in which each node has either one parent and two children (if $\epsilon_{i-1} = +$) or two parents and one child (if $\epsilon_{i-1} = -$), and such that any node linked to node i through its left child or left parent has a smaller label, and through its right child or right parent has a larger label (note that since the left subtree of 1 and the right subtree of n are always empty, we can consider them to have one child and one parent WLOG). A **linear extension** of a cambrian tree is a permutation such that for any edge of the tree, the position of the parent in the permutation is before the position of the child. The rotation order on Cambrian trees parametrized by ϵ is isomorphic to the Cambrian lattice parametrized by the same sequence, with equivalence classes of the Cambrian congruence corresponding to linear extensions of each tree.

Using a realization of Cambrian lattices as subword complexes given in [PS15b], in [Pil18b] V. Pilaud defined k-twists as a pipe arrangement similar to triangular pipe dreams but contained in a shape F_ϵ whose boundaries are determined by ϵ. This boundary is made of a starting and ending path, containing n south or east steps (the direction of step i is given by ϵ_i), and of two stair paths in the northwest and the southeast. The Cambrian lattice parametrized by ϵ is then isomorphic to the flip graph on $\Sigma_{F_\epsilon}(\omega_0)$, and the contact graph of each pipe dream in that set is a Cambrian tree.

3.2.2 Linear extensions

As the Cambrian lattices are by definition congruences of the weak order, we knew from the results described in Section 3.2.1 that the linear extensions of pipe dreams in some sets $\Sigma_F(\omega)$ have properties similar to the linear extensions of triangular pipe dreams, it is natural to wonder the results in Section 2.2 are still true for generalized pipe dreams. We start with a generalization of Theorem 2.2.7.

Definition 3.2.1. *For P a generalized pipe dream, a **linear extension** of P is a linear extension of its extended contact graph. We denote by* $\mathrm{lin}(P)$ *the set of linear extensions of P.*

Theorem 3.2.2. *For F an alternating shape and $\omega \in \mathfrak{S}_n$ sortable on F, the collection $\{\mathrm{lin}(P) \mid P \in \Sigma_F(\omega)\}$ is a partition of the weak order interval $[\mathrm{id}, \omega]$.*

We start by a few results on linear extensions of contact graph, which we will then use to prove the theorem on extended contact graphs.

Lemma 3.2.3. *If $\pi' := UpqV \lessdot \pi := UqpV$ is a cover of the weak order and π is a linear extension of $P^\#$ for some pipe dream $P \in \Sigma_F(\omega)$, then:*

- *if $P^\#$ has no arc $q \to p$ then π' is a linear extension of $P^\#$;*
- *otherwise π' is a linear extension of $P'^\#$, with P' the pipe dream obtained by a decreasing flip on the furthest northeast contact between pipes q and p in P.*

Proof. The proof is very similar as the one for Lemma 2.2.9. If there is no arc $q \to p$ in $P^\#$, then π' is obviously also a linear extension of that same graph; otherwise there is at least one contact between q and p in P. Since q starts southeast of p and is northwest of it on that contact, pipes p and q must cross at some point in P, so any contact between them is flippable. If we flip the furthest northeast one, then the changes this makes to $P^\#$ make sure that the result has π' as a linear extension. $\square$

Lemma 3.2.4. *Any permutation is a linear extension of at most one contact graph of a pipe dream in $\Pi_F(\omega)$.*

Proof. Suppose that some pipe dream $P \in \Pi_F(\omega)$ has the identity as a linear extension of $P^\#$. Then any arc $p \to q$ of $P^\#$ is such that $p < q$, and so any contact c in P has the pipe with the smaller label at the northwest and the one with the larger label on the southeast. Since these are the relative positions in which p and q start, this means that if those two pipes cross in some cell x of F, then x is after c along pipes p and q, and so a flip on the contact c would be increasing. Therefore P must be a source of the increasing flip graph on $\Pi_F(\omega)$. Since there is exactly one such source, this means that the identity is a linear extension of exactly one contact graph of a pipe dream in $\Pi_F(\omega)$.

Suppose now that π is a linear extension of the contact graphs of two pipe dreams $P_1, P_2 \in \Pi_F(\omega)$ and consider π' a permutation covered by π. Then from Lemma 3.2.3 we know that π' is a linear extensions of the contact graph of two pipe dreams $P'_1, P'_2 \in \Pi_F(\omega)$, with $P'_i = P_i$ or P'_i obtained from P_i with a decreasing flip. A study of all the possible cases proves that if $P'_1 = P'_2$, then $P_1 = P_2$. Since the identity is only a linear extension of one contact graph and $\pi \geqslant \mathrm{id}$, this means that P_1 and P_2 must be equal, thus proving the statement. □

Proof of Theorem 3.2.2. For any pipe dream, the linear extensions of the extended contact graph are all linear extensions of the contact graph, since the former has all the arcs of the latter. Therefore, Lemma 3.2.4 tells us that any permutation is a linear extensions of none or one pipe dream in $\Pi_F(\omega)$, and the linear extension sets $\{\mathrm{lin}(P) \mid P \in \Sigma_F(\omega)\}$ are disjoint. If $\pi \notin [\mathrm{id}, \omega]$, then there is some noninversion (p, q) of ω that is not a noninversion of π, therefore $p \to q$ is an arc of all the extended contact graphs of pipe dreams in $\Sigma_F(\omega)$ and $\pi^{-1}(q) < \pi^{-1}(p)$, and so π is not a linear extension of any of them, so $\cup_{P\in\Sigma_F(\omega)} \mathrm{lin}(P) \subseteq [\mathrm{id}, \omega]$.

Consider now the only sink $P^\uparrow$ of the increasing flip graph on $\Pi_F(\omega)$, and $p \to q$ an arc of $P^{\uparrow\natural}$. If $(p, q) \in \mathrm{ninv}(\omega)$, then by definition $\omega^{-1}(p) < \omega^{-1}(q)$; otherwise, there is a contact c between p (at the northwest) and q (at the southeast) in $P^\uparrow$. In that last case, if $p < q$ then since (p, q) is not a noninversion of ω, it must be an inversion and p and q must cross in some cell x of $P^\uparrow$, and c is flippable. Since the respective positions of the two pipes at c are the same as when they start, the cross must happen after that contatc so x is northeast of c, and so the flip on c is increasing; this contradict our choice of $P^\uparrow$, so this is not possible. Therefore $p > q$ and since their relative position is not the same in c as it is at the start, pipes p and q must have crossed at some point before, so $(q, p) \in \mathrm{inv}(\omega)$ and so once again $\omega^{-1}(p) < \omega^{-1}(q)$. This proves that ω is a linear extension of $P^{\uparrow\#}$ and so by Remark 3.1.8 $\omega \in \mathrm{lin}(P^\uparrow)$. Therefore, from Lemma 3.2.3 we conclude that for any $\pi \leqslant \omega$ there exists $P \in \Pi_F(\omega)$ such that π is a linear extension of $P^\#$, and since $\mathrm{ninv}(\omega) \subseteq \mathrm{ninv}(\pi)$ then $\pi \in \mathrm{lin}(P)$, and since $\mathrm{lin}(P) \neq \varnothing$ then P must be strongly acyclic. We thus proved that if $\pi \in [\mathrm{id}, \omega]$ then it is a linear extension of at least one pipe dream in $\Sigma_F(\omega)$. Therefore, the sets $\mathrm{lin}(P)$ cover the weak order interval $[\mathrm{id}, \omega]$ and complete the proof that they form a partition of this interval. $\square$

3.2.3 A lattice congruence

Like for triangular pipe dream, we use Theorem 3.2.2 to define an equivalence relation on permutations. We will use Lemma 3.1.22 to prove that this relation is a lattice congruence.

Definition 3.2.5. *For F an alternating shape and $\omega \in \mathfrak{S}_n$ sortable on F, the pipe dream congruence $\equiv_{F,\omega}$ on $[\mathrm{id}, \omega]$ is the equivalence relation whose equivalence classes are the linear extension sets $(\mathrm{lin}(P))_{P \in \Sigma_F(\omega)}$.*

Theorem 3.2.6. *The relation $\equiv_{F,\omega}$ is a lattice congruence of the weak order on the interval $[\mathrm{id}, \omega]$.*

Proposition 3.2.7. *For any $P \in \Sigma_F(\omega)$, the set $\mathrm{lin}(P)$ is a weak order interval.*

Proof. The proof is the same as for Proposition 2.2.14, but using Lemma 3.1.22 and the definition of the extended contact graph rather than Lemma 2.1.15 and Lemma 2.1.14.

We start by proving that if for some $p < q < r$ pipes of P, if $p \lhd_P r$ then $p \lhd_P q$ or $q \lhd_P r$, and that if $r \lhd_P p$, then $r \lhd_P q$ or $q \lhd_P r$. If $p \to r$ or $r \to p$ is an arc of $P^\natural$, then either (p, q) or (q, r) is a noninversion of ω and so the corresponding arc is in $P^\natural$, or we can apply Lemma 3.1.22 to obtain that one of the relations is true. We can then proceed by induction on the length of the path from p to r or from r to p in $P^\natural$.

Consider now the sets of pairs of pipes $X^\uparrow := \{(p, q) \mid p < q \text{ and } p \lhd_P q\}$ and $X^\downarrow := \{(p, q) \mid p < q \text{ and } q \lhd_P p\}$. Since the relation $\lhd_P$ is transitive, we know that these to sets are closed if we see them as root set, and the result obtained in the previous paragraph shows that they are also biconvex (see Example 1.4.5). Thus $X^\uparrow$ is the noninversion set of some permutation $\pi^\uparrow$ and $X^\downarrow$ is the inversion set of some permutation $\pi^\downarrow$, and since $X^\downarrow \cap X^\uparrow = \varnothing$ (because $P^\natural$ is acyclic), from Theorem 1.4.14 we have $\pi^\downarrow \leqslant \pi^\uparrow$. Finally, it is obvious that a permutation π is a linear extension of $P^\natural$ if and only if $\pi^\downarrow \leqslant \pi \leqslant \pi^\uparrow$ and so $\mathrm{lin}(P) = [\pi^\downarrow, \pi^\uparrow]$ is a weak order interval. $\square$

Proposition 3.2.8. *Let C, C' be equivalence classes of $\equiv_{F,\omega}$ and consider $\pi \in C$ and $\pi' \in C'$. Then $\pi \leqslant \pi'$ implies that $\max(C) \leqslant \max(C')$ and $\min(C) \leqslant \min(C')$.*

Proof. Once again, the proof is the same as for 2.2.15: we reason on weak order covers and proceed by induction on the distance in the Hasse diagram of the weak order from π to $\max(C)$ and π' to $\min(C')$. We will give the steps to prove that $\max(C) \leqslant \max(C')$ and the steps for $\min(C) \leqslant \min(C')$ are symmetrical. If $\pi = \max(C)$, then $\max(C) = \pi < \pi' \leqslant \max(C')$ so the statement is obviously

$$\begin{tikzcd}[column sep=small]
 & \pi\tau_i\tau_j\tau_i = \pi\tau_j\tau_i\tau_j = UrqpV \arrow[dl, dash] \arrow[dr, dash] & \\
\pi\tau_i\tau_j = UrpqV \arrow[d, dash] & & \pi'\tau_i = \pi\tau_j\tau_i = UqrpV \arrow[d, dash] \\
\pi\tau_i = UprqV \arrow[dr, dash] & & \pi' = \pi\tau_j = UqprV \arrow[dl, dash] \\
 & \pi = UpqrV &
\end{tikzcd}$$

Figure 3.9: The weak order interval $[\pi, \pi\tau_i \vee \pi']$.

true. If $\pi < \max(C)$, then there exists τ_i, τ_j such that $\pi \lessdot \pi\tau_i \leqslant \max(C)$ (and so $\pi \equiv_{F,\omega} \pi\tau_i$) and $\pi' = \pi\tau_j$. Denote by $P, P' \in \Sigma_F(\omega)$ the pipe dreams such that $\mathrm{lin}(P) = C$ and $\mathrm{lin}(P') = C'$. Then we consider the possible values of $i - j$:

1. if $i = j$ then $\pi' \in C$ and so $\max(C) = \max(C')$;

2. if $|i - j| \geqslant 2$, then $\tau_i\tau_j = \tau_j\tau_i$; since $\pi \equiv_{F,\omega} \pi\tau_i$, we know from Lemma 3.2.3 that the two pipes $\pi(i)$ and $\pi(i+1)$ have no contact in P. Then P' is either P or obtained from P by flipping pipes $\pi(j)$ and $\pi(j+1)$; since $\pi(i)$ and $\pi(i+1)$ are distinct from $\pi(j)$ and $\pi(j+1)$, that flip cannot create a contact between pipes $\pi(i)(= \pi'(i))$ and $\pi(i+1)(= \pi'(i+1))$. Thus there is no arc $\pi(i) \to \pi(i+1)$ in $P'^{\natural}$, and so from Lemma 3.2.3 we know that $\pi'\tau_i \equiv_{F,\omega} \pi'$. Thus $\pi\tau_j\tau_i = \pi\tau_i\tau_j$ is in C' and we can conclude by induction hypothesis on the cover $\pi\tau_i \lessdot \pi\tau_i\tau_j$;

3. if $i - j = 1$, then the weak order interval $[\pi, \pi\tau_i \vee \pi']$ is represented in Fig. 3.9. We note that since $\pi', \pi\tau_i \leqslant \omega$, the whole interval is contained in $[\mathrm{id}, \omega]$ and $\omega^{-1}(p) > \omega^{-1}(q) > \omega^{-1}(r)$. Then since $\pi \in \mathrm{lin}(P)$, we know that neither $p \rhd_P q$ nor $q \rhd_P r$ can be true, and so from Fig. 3.8 we know that $p \to r$ is not an arc of $P^{\natural}$, and from $\pi \equiv_{F,\omega} \pi\tau_i$ we know that neither is $q \to r$. Therefore, since P' is either P or obtained by flipping pipes p and q in P, we obtain that $P'^{\natural}$ has no arc $p \to r$ nor $q \to r$ either. Therefore, by Lemma 3.2.3 we get that $\pi\tau_i\tau_j \in C$ and $\pi\tau_j\tau_i, \pi\tau_j\tau_i\tau_j \in C'$. We can then apply the induction hypothesis to the weak order cover $\pi\tau_i\tau_j \lessdot \pi\tau_i\tau_j\tau_i$;

4. if $j - i = 1$ then the proof is similar to the one in the previous item but with $\pi\tau_i = UqprV$ and $\pi\tau_j = UprqV$. □

Proof of Theorem 3.2.6. The theorem is a direct consequence of Proposition 3.2.7 and Proposition 3.2.8 combined with Theorem 1.1.18. □

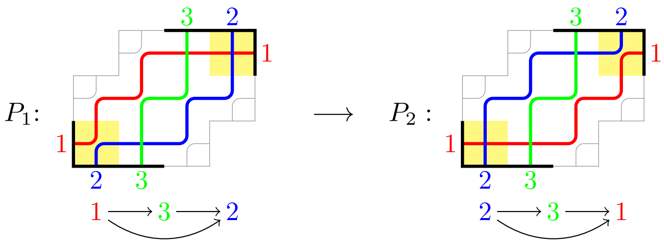


Figure 3.10: An increasing flip between two strongly acyclic pipe dreams that is not the image of a weak order relation.

3.2.4 A lattice morphism

We already used Theorem 3.2.2 to define the equivalence relation $\equiv_{F,\omega}$, but we can also use it to define a map from permutations to pipe dreams.

Definition 3.2.9. *We denote by* $\mathrm{ins}_{F,\omega} : [\mathrm{id}, \omega] \to \Sigma_F(\omega)$ *the map that associates to any permutation* $\pi \leqslant \omega$ *the only pipe dream in* $\Sigma_F(\omega)$ *that has* π *as a linear extension.*

We know that the fibers of $\mathrm{ins}_{F,\omega}$ are the equivalence classes of $\equiv_{F,\omega}$, and so as we did for triangular pipe dreams we can define $\mathrm{ins}_{F,\omega}$ on the quotient $[\mathrm{id}, \omega]/\equiv_{F,\omega}$. For triangular pipe dreams, any increasing flip between two acyclic pipe dreams was then the image of a cover of this quotient of the weak order, which allowed us to describe the image of the weak order by $\mathrm{ins}_{F,\omega}$ in simple terms. However, that simple description no longer holds for generalized pipe dreams: Fig. 3.10 shows a flip between two strongly acyclic pipe dreams P_1 and P_2, with $\mathrm{lin}(P_1) = \{132\}$ and $\mathrm{lin}(P_2) = \{231\}$. Since 132 and 231 are not comparable in the weak order, this flip is not the image of a relation of the weak order by $\mathrm{ins}_{F,\omega}$. We thus have to define a new order relation on $\Sigma_F(\omega)$.

Definition 3.2.10. *For two pipe dreams* $P, P' \in \Sigma_F(\omega)$*, we say that* $P \leqslant P'$ *in the* ***acyclic order*** *if there exists* $\pi \in \mathrm{lin}(P)$ *and* $\pi' \in \mathrm{lin}(P')$ *such that* $\pi \leqslant \pi'$ *in the weak order.*

This is obviously the image of the weak order by $\mathrm{ins}_{F,\omega}$. We will use the following proposition, close to Proposition 2.2.20, to give a characterization of the acyclic order that does not use linear extensions.

Proposition 3.2.11. *For any pipe dreams* $P, P' \in \Sigma_F(\omega)$*, the following statements are equivalent:*
(1) there exists $\pi \in \mathrm{lin}(P)$ *and* $\pi' \in \mathrm{lin}(P')$ *such that* $\pi \leqslant \pi'$*;*
(2) the minimal (resp. maximal) linear extensions π *of* P *and* π' *of* P' *satisfy* $\pi \leqslant \pi'$*;*
(3) there are no pipes $p < q$ *such that* $p \rhd_P q$ *and* $p \lhd_{P'} q$*;*

(4) for all pipes $p < q$, if $p \rhd_P q$ then $p \rhd_{P'} q$;
(5) for all pipes $p < q$, if $p \lhd_{P'} q$ then $p \lhd_P q$.

Proof. Both versions of item (2) obviously imply item (1), and from Proposition 3.2.8 the converse is true, so (1) $\iff$ (2). The equivalence (2) $\iff$ (3) $\iff$ (4) $\iff$ (5) comes from the description of the inversions of the bottom elements and noninversions of the top elements of $\mathrm{lin}(P)$ and $\mathrm{lin}(P')$ given in the proof of Proposition 3.2.8, and the definition of the weak order in term of inclusion of inversion sets. $\square$

Proposition 3.2.12. *For any two distinct pipe dreams $P_1, P_2 \in \Sigma_F(\omega)$, P_2 covers P_1 in the acyclic order if and only if there exists $\pi_1 \in \mathrm{lin}(P_1)$ and $\pi_2 \in \mathrm{lin}(P_2)$ such that $\pi_1 \lessdot \pi_2$ is a cover of the weak order.*

Proof. Suppose that P_2 covers P_1 and choose two permutations $\pi_1 < \pi_2$ respectively in $\mathrm{lin}(\pi_1)$ and $\mathrm{lin}(P_2)$, then there exist $\pi_1 = \sigma_0 \lessdot \sigma_1 \lessdot \ldots \lessdot \sigma_k = \pi_2$ a maximal chain from π_1 to π_2 for some $k \geqslant 1$. If for any $0 \leqslant j \leqslant k$ we have $\mathrm{ins}_{F,\omega}(\sigma_j) = P \notin \{P_1, P_2\}$, then by definition of the acyclic order $P_1 < P < P_2$ and P_2 does not cover P_1; therefore, the whole chain is in $\mathrm{lin}(P_1) \cup \mathrm{lin}(P_2)$. Since it starts in $\mathrm{lin}(P_1)$ and ends in $\mathrm{lin}(P_2)$, there exists j such that $\sigma_j \in \mathrm{lin}(P_1)$ and $\sigma_{j+1} \in \mathrm{lin}(P_2)$, and σ_{j+1} covers σ_j.

Suppose now that $\pi_1 \lessdot \pi_2$ is a cover of the weak order and $P_1 := \mathrm{ins}_{F,\omega}(\pi_1)$ is distinct from $P_2 := \mathrm{ins}_{F,\omega}(\pi_2)$. By definition of the acyclic order $P_1 < P_2$; consider $P \in \Sigma_F(\omega)$ such that $P_1 \leqslant P \leqslant P_2$ and denote by $\pi^\downarrow, \pi^\uparrow$ the minimal and maximal elements of $\mathrm{lin}(P)$. Then since $P \leqslant P_2$ we know that $\pi^\downarrow \leqslant \pi_2$ and we choose $\pi_1 \leqslant \pi_2$, so $\pi_1 \leqslant \pi_1 \vee \pi^\downarrow \leqslant \pi_2$; since π_2 covers π_1, this means that $\pi_1 \vee \pi^\downarrow$ is either π_1 or π_2. In the first case, this means that $\pi^\downarrow \leqslant \pi_1$ and so $\pi_1 \in [\pi^\downarrow, \pi^\uparrow] = \mathrm{lin}(P)$, so from Theorem 3.2.2 this means that $P = P_1$. In the second case, since $P_1 \leqslant P$ we know that $\pi_1 \leqslant \pi^\uparrow$ and by definition $\pi^\downarrow \leqslant \pi^\uparrow$ so $\pi_2 = \pi_1 \vee \pi^\downarrow \leqslant \pi^\uparrow$, thus $\pi_2 \in [\pi^\downarrow, \pi^\uparrow] = \mathrm{lin}(P)$ and so in the same way $P = P_2$. Thus $P_1 < P < P_2$ is not possible, and so P_2 covers P_1 in the acyclic order. $\square$

Theorem 3.2.13. *For F an alternating shape and $\omega \in \mathfrak{S}_n$ sortable on F, the acyclic order defines a lattice on $\Sigma_F(\omega)$.*

Proof. Consider $P_1, P_2 \in \Sigma_F(\omega)$ and write $\mathrm{lin}(P_1) = [\pi_1^\downarrow, \pi_1^\uparrow]$ and $\mathrm{lin}(P_2) = [\pi_2^\downarrow, \pi_2^\uparrow]$. We call $P^\downarrow = \mathrm{ins}_{F,\omega}(\pi_1^\uparrow \wedge \pi_2^\uparrow)$ and $P^\uparrow = \mathrm{ins}_{F,\omega}(\pi_1^\downarrow \vee \pi_2^\downarrow)$; by definition, it is clear that $P^\downarrow \leqslant P_1, P_2$ and $P^\uparrow \geqslant P_1, P_2$. Then for $P \in \Sigma_F(\omega)$, if $P \leqslant P_1$ and $P \leqslant P_2$ then for $\pi = \min(\mathrm{lin}(P))$ we know that $\pi \leqslant \pi_1^\uparrow$ and $\pi \leqslant \pi_2^\uparrow$, so $\pi \leqslant \pi_1^\uparrow \wedge \pi_2^\uparrow$ and so $P \leqslant P^\downarrow$; therefore $P^\downarrow$ is the maximum of all pipe dreams below both P_1 and P_2, and so $P^\downarrow = P_1 \wedge P_2$. A similar reasoning proves that $P^\uparrow = P_1 \vee P_2$, and so any pair of elements of $\Sigma_F(\omega)$ has both a join and a meet and the acyclic order defines a lattice on $\Sigma_F(\omega)$. $\square$

3.2.5 Two algorithms

We will now give two algorithms computing $\mathrm{ins}_{F,\omega}(\pi)$. These two algorithms are direct extensions of Algorithm 2.2.21 and Algorithm 2.2.22, although we will see that general shapes introduce some additional difficulties into the definition of the sweeping algorithm and the proof of the insertion algorithm.

Sweeping algorithm

As we did in Algorithm 2.2.21 for triangular pipe dreams, the sweeping algorithm creates a pipe dream by sweeping the shape from southwest to northeast; this allows us to know which pipes are entering a cell from the west and the south when considering that cell, as the trajectory of those pipes from their starting point to that cell is already fully determined. We can then choose to fill that cell with a contact ╯╭ or a cross $+$ depending on the entering pipes and their relative positions in ω and π: with p to the west and q to the south, the two pipes can only cross if $(p, q) \in \mathrm{inv}(\omega)$, in which case they cross as soon as they meet if $\pi^{-1}(p) > \pi^{-1}(q)$ and as late as possible otherwise. However, finding the latest point at which two pipes can cross is harder than in the triangular case, where it was simply a matter of finding the column in which q ended. We must thus first find a way to determine what the last possible crossing point of two pipes is.

To that end, we will define the greedy completion of a partial execution of the sweeping algorithm. A partially filled alternating shape will then be a filling of some cells of the shape with crosses and contacts such that if a cell is filled, every cell weakly southwest is also filled (and so we know the entering pipes of any cell that can be filled next).

Algorithm 3.2.14. *For F a partially filled alternating shape and $\omega \in \mathfrak{S}_n$, the* ***greedy completion*** *of F under ω is the pipe dream obtained by sweeping the remaining cells from southwest to northeast and when considering a cell with pipe p entering from the west and pipe q from the south, filling it with:*

1. *a cross $+$ if $\omega^{-1}(q) < \omega^{-1}(p)$;*
2. *a contact ╯╭ if $\omega^{-1}(p) < \omega^{-1}(q)$.*

Lemma 3.2.15. *A partially filed alternating shape F can be completed into a reduced pipe dream with exit permutation $\omega \in \mathfrak{S}_n$ iff the greedy completion of F under ω is a reduced pipe dream with exit permutation ω.*

Proof. It is clear that if the greedy completion of F under ω returns a reduced pipe dream with exit permutation ω, then this pipe dream complete the already filled cells of F. Suppose now that such a completion P of F exists, and proceed by descending induction on the remaining number of cells in F. If there is no cell left, then the greedy completion does nothing and returns P (as the partially filled shape is already a pipe dream); otherwise, consider the first cell swept by the greedy completion, with p the pipe entering it from the west and q the one entering it from the south. We note that the same pipes enter the same cell in P. Then:

- if $\omega^{-1}(p) < \omega^{-1}(q)$, then either $p < q$ and so $(p, q) \in \mathrm{ninv}(\omega)$ and the two pipes cannot cross, or $p > q$ and from Proposition 3.1.11 they cannot cross in that direction, so in both cases that cell contains a contact in P and is filled by a contact by the greedy completion;

- if $\omega^{-1}(p) > \omega^{-1}(q)$, then pipe p is currently northwest of pipe q and ends southeast of it, so pipes p and q must cross at some point after the already filled cells in P. Either they cross in the current cell, or the current cell is a flippable contact and we denote by P' the pipe dream obtained by flipping it. Then since we flip the current cell with one further northeast, all the cells whose contents are changed by this flip are northeast of the current cell, and so are not filled yet, so P' also completes the partially filled cells into a reduced pipe dream with exit permutation ω and has a cross in the current cell.

Thus in both cases there still exists a reduced pipe dream with exit permutation ω completing the filled cell with the addition of the current cell filled by the greedy completion (and so there is one less remaining cell to be filled). Therefore, by induction, the greedy completion will return a reduced pipe dream with exit permutation ω. $\square$

We note that the greedy completion of an empty alternating shape under ω gives a pipe dream with exit permutation ω iff ω is sortable on that shape.

Algorithm 3.2.16 (Sweeping algorithm)**.** *For any alternating shape F and permutations $\pi \leqslant \omega$ with ω sortable on F, the pipe dream $\mathrm{ins}_{F,\omega}(\pi)$ can be constructed by sweeping F from southwest to northeast. We place a crossing* $+$ *when sweeping cell c if and only if pipe p arriving horizontally and pipe q arriving vertically in c satisfy the following two statements:*
- *(p, q) is an inversion of ω;*
- *$\pi^{-1}(p) > \pi^{-1}(q)$, or the greedy completion under ω of the previously filled cells with an added contact* ╯╭ *in c is not in $\Pi_F(\omega)$.*

Example 3.2.1. The execution of Algorithm 3.2.16 with $\omega = 23145$ and $\pi = 21345$ is drawn in Fig. 3.11. On the first step, the incoming pipes of the bottom cell are 4 (west) and 5 (south); since $\omega^{-1}(4) < \omega^{-1}(5)$, we fill this cell with a contact ╯╭. On the second step, the incoming pipes of the considered are 1 and 2; since $\omega^{-1}(2) < \omega^{-1}(1)$ and $\pi^{-1}(1) > \pi^{-1}(2)$, we fill this cell with a cross $+$. On the third step, the incoming pipes are 1 and 3; since $\pi^{-1}(1) < \pi^{-1}(3)$ and those two pipes can still cross later, we fill this cell with a contact ╯╭. Finally, on the fifth step, the incoming pipes are once again 1 and 3 but this time it is the last possible crossing point for them, so we fill this cell with a cross $+$.

Proof of correctness. Suppose that the sweeping algorithm returns a reduced pipe dream $P \in \Pi_F(\omega)$. We note that since the pairs (a, b) with $1 \leqslant a < b \leqslant n$ are all either in $\mathrm{ninv}(\omega)$ or $\mathrm{inv}(\omega)$, and also either in $\mathrm{ninv}(\pi)$ or $\mathrm{inv}(\pi)$, and that by definition of the weak order $\mathrm{inv}(\pi) \subseteq \mathrm{inv}(\omega)$ and $\mathrm{ninv}(\omega) \subseteq \mathrm{ninv}(\pi)$. Suppose now

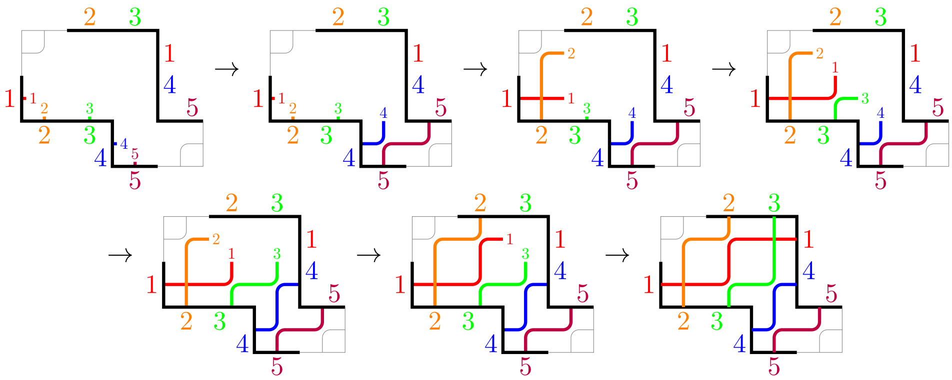


Figure 3.11: Executing the sweeping algorithm for $\omega = 23145$ and $\pi = 21345$.

that for some pipe p and q, the arc $p \to q$ is in $P^\natural$. Either $(p,q) \in \mathrm{ninv}(\omega)$, or there exists a cell c containing a contact with p at the northwest and q at the southeast. Then:

- if $p < q$:
  - if $(p,q) \in \mathrm{ninv}(\pi)$, then by definition $\pi^{-1}(p) < \pi^{-1}(q)$;
  - if $(p,q) \in \mathrm{inv}(\pi) \subseteq \mathrm{inv}(\omega)$, then cell c exists; however, when sweeping that cell, the sweeping algorithm can only fill it with a cross, so no such cell (and no such arc) exists.
- if $p > q$:
  - if $(q,p) \in \mathrm{inv}(\pi)$, then by definition $\pi^{-1}(q) > \pi^{-1}(p)$;
  - if $(q,p) \in \mathrm{ninv}(\omega)$, then cell c exists and p is northwest of q in that cell, so p and q must have crossed at some point in P. From Proposition 3.1.11 we know that this is not possible, and so no such arc exists;
  - if $(q,p) \in \mathrm{inv}(\omega) \cap \mathrm{ninv}(\pi)$, then cell c exists and pipe p and q must cross in some cell x with x southwest of c. This means that c is flippable in P and that the pipe dream P' obtained with this flip is identical to P southwest of cell x, and contains a contact in x. Therefore, when sweeping cell x, the partially filled shape at that point with an added contact in x can be completed into $P' \in \Pi_F(\omega)$, and so by Lemma 3.2.15 the greedy completion of those cell under ω is in $\Pi_F(\omega)$, so the sweeping algorithm should have filled x with a contact; therefore, no such cell and no such arc exists.

Thus we proved that all the arcs $p \to q$ of $P^\natural$ satisfy $\pi^{-1}(p) < \pi^{-1}(q)$, and so π is a linear extension of P and $P = \mathrm{ins}_{F,\omega}(\pi)$.

We will now prove that the sweeping algorithm returns a reduced pipe dream in $\Pi_F(\omega)$ as long as ω is sortable on F and $\pi \leqslant \omega$. We will prove by induction that before and after any step of the sweeping algorithm, the partially filled shape can be completed into a pipe dream of $\Pi_F(\omega)$; this will complete the proof as if it is true after the last step, then since the shape is already complete, it must contain such a pipe dream and so the proof is done. That statement is obviously true before the first step if and only if ω is sortable on F. Suppose now that it is true before step t, and denote by P a pipe dream of $\Pi_F(\omega)$ completing the already filled cells and by c the cell swept at step t, with p, q the pipes entering c respectively from the west and the south. By our choice of P, those two pipes also enter c from the same directions in P. Then:

- if $q < p$ or $\omega^{-1}(p) < \omega^{-1}(q)$, then the sweeping algorithm fills c with a contact and by Proposition 3.1.11 we know that c can only contain a contact in P, so P still completes the partially filled cells after step t;
- if $p < q$ and $\omega^{-1}(p) > \omega^{-1}(q)$, then $(p, q) \in \mathrm{inv}(\omega)$. Then:
  - if cell c is filled with a contact by the sweeping algorithm, then the greedy completion under ω of the partially filled cells with a contact added in c returns a pipe dream $P' \in \Pi_F(\omega)$ and from Lemma 3.2.15 this pipe dream completes the partially filled cells after step t;
  - if it is filled with a cross, then since $(p, q) \in \mathrm{inv}(\omega)$, pipes p and q must cross in some cell x of P, and since p is northeast of q when entering c, we know that x is weakly northeast of c. Then either $x = c$, and P still completes the partially filled cells after step t, or c is a contact and can be flipped with x to obtain $P' \in \Pi_F(\omega)$, with P' identical to P in all cells not northeast of c. In particular P' is identical to P in all cells filled before step t and contains a cross in c, so it completes the cell filled after step t.

In all cases, if a good completion exists before step t, then one also exists after step t. By induction, this means that if ω is sortable on F, then the sweeping algorithm returns a pipe dream in $\Pi_F(\omega)$. This concludes the proof. □

Insertion algorithm

Once again, the insertion algorithm determines $\mathrm{ins}_{F,\omega}(\pi)$ by finding the trajectory of each pipe in the order given by π, and it does it by keeping track of free northwest elbows at each step and completing them with the next pipes.

Algorithm 3.2.17 (Insertion algorithm)**.** *For any alternating shape F and permutations $\pi \leqslant \omega$ with ω sortable on F, the pipe dream $\mathrm{ins}_{F,\omega}(\pi)$ can be obtained by inserting each pipe in the order given by π. At step i, we insert a pipe starting* *on step $\pi(i)$ of $\mathcal{S}_F$, ending on step $\omega^{-1}(\pi(i))$ of $\mathcal{E}_F$, and whose northeast elbows are precisely completing all the previously created southwest elbows in the zone of pipe $\pi(i)$.*

We note that in the triangular case, there were no free elbows before the first step when the shape was still empty; in the general case, all the cells right below the NW staircase of F originally contain one.

Like in the triangular case, we need to prove first that this description of the algorithm applies to all encountered situations and is nonambiguous; this means proving that the free elbows in the area of a pipe being inserted form a staircase from southwest to northwest, but also that if a pipe starts vertically (resp. ends horizontally), there will be a free elbow in its zone in its starting column (resp. in its ending row). The rest of the proof is similar to the triangular case: we need to check that the result is a pipe dream and is reduced; the exit permutation and the fact that π is a linear extension will follow from the way the pipes are inserted.

In all that follows, we will denote by $\Delta^{\updownarrow}(x)$ the vertical line with abscissa x and $\Delta^{\leftrightarrow}(y)$ the horizontal line with ordinate y.

Lemma 3.2.18. *If $\pi(t)$ starts vertically (resp. ends horizontally) and does not ends vertically in the same column (resp. starts horizontally in the same row), then right before step t the column $x^s_{\pi(t)}$ (resp. the row $y^e_{\pi(t)}-1$) contains a free elbows in $\mathcal{Z}_{\pi(t)}$.*

Proof. We will prove the statement for $\pi(t)$ starting vertically; the case where it ends horizontally is similar. We note that since $\pi(t)$ starts vertically, it is the last pipe starting in column $x^s_{\pi(t)}$ or west of it, so for any p, if $x^s_p \leqslant x^s_{\pi(t)}$ then $p \leqslant \pi(t)$. Let us first prove that column $x^s_{\pi(t)}$ contains a free elbow. If $x^s_{\pi(t)} < t_F$, then that column contained a free elbow at the beginning of the algorithm; otherwise, the top boundary of F in that column is a horizontal step of $\mathcal{E}_F$ on which a pipe p ends (with p determined by ω). Suppose that such a pipe p exists; by hypothesis $p \neq \pi(t)$, and $x^e_p = x^s_{\pi(t)} + 1$, so since $x^s_p < x^e_p$ we know that $x^s_p \leqslant x^s_{\pi(t)}$ and so $p < \pi(t)$. Moreover, since $\pi(t)$ does not go straight up from start to end, it must end east of p, so $\omega^{-1}(p) < \omega^{-1}(\pi(t))$ and $(p, \pi(t)) \in \mathrm{ninv}(\omega)$. Since $\pi \leqslant \omega$, this means that $\pi^{-1}(p) < t$ and so p has already been inserted, and column $x^s_{\pi(t)}$ contained a free elbow created by p right after that insertion. Then if a step $s < t$ completed an elbow in that column, the pipe $\pi(s)$ would either create another free elbow or start vertically in that column; since the only pipe starting vertically in that column is $\pi(t)$, that second option is not possible, so the column $x^s_{\pi(t)}$ must still contain a free elbow right before step t.

Denote by y the ordinate of the cell in which the free elbow is. This elbow is either right below the NW stair path or it was created by some pipe q inserted before step t; in the first case $y^s_{\pi(t)} \leqslant 0 \leqslant y$ and in the second case since $x^s_q \leqslant x^s_{\pi(t)}$ (because q goes through that column) we know that $q \leqslant \pi(t)$ and $y \geqslant y^s_q \geqslant y^s_{\pi(t)}$. Therefore, in order to have the elbow in $\mathcal{Z}_{\pi(t)}$, we only need $y < y^e_{\pi(t)}$. We note that no pipe enters the cell $(x^s_{\pi(t)}, y)$ by its south side at this point. Suppose that $y \geqslant y^e_{\pi(t)}$, then by counting pipes crossing the horizontal line $\Delta^{\leftrightarrow}(y)$ west of abscissa $x^s_{\pi(t)} + 1$ in any pipe dream of $\Pi_F(\omega)$ (which by hypothesis is nonempty), we see that at least one of them, denoted by p, does not cross $\Delta^{\leftrightarrow}(y)$ west of that point at this point of time. Since $x^s_p \leqslant x^s_{\pi(t)}$ and $y^e_p > y \geqslant y^s_{\pi(t)}$, we know that $(p, \pi(t)) \in \mathrm{ninv}(\omega)$ and so p was already inserted; therefore it must cross $\Delta^{\leftrightarrow}(y)$ strictly east of $x^s_{\pi(t)} + 1$. This means that this pipe starts southwest of the free elbow and ends northeast of it, and so it has this free elbow in its zone. Since it did not complete this free elbow, this elbow did not exist when p was inserted, and is thus made up of a pipe q inserted after p. We can then follow q going north and west from the free elbow; then either we meet a southeast elbow of q in one of those directions, or we reach the vertical line $\Delta^{\updownarrow}(x^s_p)$ going west and the horizontal line $\Delta^{\leftrightarrow}(y^e_p)$ going north (we cannot reach an end of q before those lines, because of the position of the start and end of p). If we meet another elbow, then the northwest part of that elbow cannot have been inserted before p either or p would have completed it and crossed $\Delta^{\leftrightarrow}(y)$ west of where it does, so we can repeat the reasoning on that elbow which is strictly west or north of the original one. Since this elbow must be southeast of the point (x^s_p, y^e_p), we can only repeat this operation a finite number of times before reaching a northwest elbow with a pipe q' that crosses both $\Delta^{\updownarrow}(x^s_p)$ and $\Delta^{\leftrightarrow}(y^e_p)$, and thus pipe q' starts west of p and ends north of p, so $(q', p) \in \mathrm{ninv}(\omega)$ and so q' must have been inserted before p. Thus p should have completed the free elbow of q' (or one of the previously considered ones) and so p should not cross $\Delta^{\leftrightarrow}(y)$ east of $\Delta^{\updownarrow}(x^s_p + 1)$. Therefore, the free elbow (x^s_p, y) must be in $\mathcal{Z}_{\pi(t)}$. □

This lemma proves that pipes can be inserted in a way that only creates free northwest elbows. We can then give a characterization of the columns and rows of F containing a free elbow at any point of the algorithm.

Lemma 3.2.19. *At some point of the algorithm, a column x of F contains one free elbow if either $x < t_F$ or the pipe ending vertically in column x has already been inserted, and either $x \geqslant |\mathcal{S}_F|_E$ or the pipe starting vertically in column x has not been inserted yet; it contains no free elbow otherwise. Similarly, a row y of F contains a free elbow if either $y \geqslant 0$ or the pipe starting horizontally in row y has already been inserted, and either $y < t_F - |\mathcal{E}_F|_S$ or the pipe ending horizontally in row y has not been inserted yet; it contains no free elbow otherwise.*

Proof. We will prove the lemma for columns and rows behave similarly. Column x contains a free elbow at the beginning of the algorithm iff $x < t_F$, i.e. its upper border is part fo the NW stair path, otherwise the upper border is a step of $\mathcal{E}_F$ on which some pipe must end vertically. We know from Lemma 3.2.18 that the first free elbow appearing in x must be created by inserting the pipe ending on that step. That pipe must create such an elbow except if it is also the pipe starting vertically in column x (with a straight trajectory from south to north). Moreover, the only way a step can remove the elbow in column x is by starting vertically directly south of that elbow, and such a pipe only exists if $x < |\mathcal{S}_F|_E$, i.e. the south boundary of F in that column is a step of $\mathcal{S}_F$. This confirms that the conditions given are necessary and sufficient for a free elbow to be present in column x. □

Corollary 3.2.20. *The pipes created by the insertion algorithm are disjoint except at crossings and contacts.*

Proof. Suppose that pipe p is added and has a vertical segment in column x. Then either p is the pipe ending vertically in x, in which case no other pipe had a vertical segment in that column, or it completes a free elbow, in which case the vertical segment will be completely south of the free elbow it completes, and all the vertical segments of previously inserted pipes in column x will be completely north of that free elbow. Thus, pipe p has no intersection along a (non punctual) vertical segment with a previously inserted pipe. A similar analysis shows that it has no intersection along a horizontal segment with a previously inserted pipe, and so the intersections are limited to crossings and contacts. □

Corollary 3.2.21. *No pipe created by the insertion algorithm passes through a cell (x, y) with $y > x$.*

Proof. We know that if a pipe p goes through cell (x, y), then $0 \leqslant x_p^s \leqslant x < x_p^e$ and $y_p^s \leqslant y < y_p^e \leqslant t_F$, so if $y > x$ then $t_F > y > x \geqslant 0$, and so the cell (x, y) is above a cell of the NW stair path of F. Moreover, if we choose p the first such inserted pipe and x minimal so that such a cell exist, we see that p must cross the horizontal line $\Delta^{\leftrightarrow}(x + 1)$ vertically and it ends east of column x (since $x < t_F - 1$), so it must have a southeast elbow in column x, and complete a free elbow in a cell (x, y') with $y' \geqslant y$. Since $y' > x$, that free elbow did not exist at the beginning of the algorithm, so p is not the first pipe going through (x, y') and so p is not the first such inserted pipe, which contradict our hypothesis. Therefore, such a pipe p does not exist. □

Lemma 3.2.22. *For any $0 \leqslant i, j \leqslant n$, right before step t, the free elbows in the rectangle between (x^s, y^s) the i-th integer point of $\mathcal{S}_F$ and (x^e, y^e) the j-th integer point of $\mathcal{E}_F$, provided that (x^s, y^s) is southwest of (x^e, y^e), form a staircase of length at least*

$$y^e - x^s + |\{s < t \mid \pi(s) \leqslant i,\ \omega^{-1}(\pi(s)) \leqslant j\}| - |\{s < t \mid \pi(s) > i,\ \omega^{-1}(\pi(s)) > j\}|.$$

Proof. Denote by R the rectangle $[x^s, x^e] \times [y^s, y^e]$. We note that if a pipe passes through R, the side it enters through depends on the comparison of p and i (west side if $p \leqslant i$, south side otherwise); and the side it exits through depends on the comparison between $\omega^{-1}(p)$ and j (north side if $\omega^{-1}(p) \leqslant j$, east side otherwise). We denote by $X_t^{\downarrow} := \{s < t \mid \pi(s) \leqslant i \text{ and } \omega^{-1}(\pi(s)) \leqslant j\}$ (pipes entering from the west and exiting through the north) and $X_t^{\uparrow} := \{s < t \mid \pi(s) > i \text{ and } \omega^{-1}(\pi(s)) > j\}$

(pipes entering from the south and exiting through the east) the two sets in the statement.

We will treat separately the cases $y^e \geqslant x^s$ and $y^e < x^s$. The first case is very similar to the triangular case; we simply note that at the beginning of the algorithm, the free elbows in the figure are in the cells (x, x) for $0 \leqslant x < t_F$, and such a cell is in R iff $x^s \leqslant x < x^e$ and $y^e \leqslant x < y^s$. However, from the placement of the points on the starting and ending path, we know that $x^e \geqslant t_F$ and $y^s \leqslant 0$, and $0 \leqslant x^s \leqslant y^e \leqslant t_F$, therefore the free elbows in R at that point are the cells (x, x) with $x^s \leqslant x < y^e$, and there are $y^e - x^s$ of them. We then proceed by induction, as in the triangular case:

- if pipe $\pi(t)$ is neither in $X^{\downarrow}_{t+1}$ nor in $X^{\uparrow}_{t+1}$, then it cannot change the number of free elbows in R and if it changes some free elbows, the new free elbows still form a staircase;

- if $\pi(t) \in X^{\downarrow}_{t+1}$ then it must go through R to go from its starting point to its exit point, as the point (x^s, y^e) is northwest of the NW stair path of F so $\pi(t)$ must pass it from the southeast. Then pipe $\pi(t)$ creates one more free elbow than it completes, and by the same reasoning as case 1 of the proof of Lemma 2.2.24 the free elbows after step t still form a staircase with the length increased by one;

- finally, if $\pi(t) \in X^{\uparrow}_{t+1}$, then pipe $\pi(t)$ either does not enter R at all or it completes some free elbows and creates exactly one less free elbow than it completes, so the number of free elbows in R either stays the same or decreases by one.

This proves that before any step t, the free elbows in rectangle R form a staircase of length at least $y^e - x^s + |X^{\downarrow}_t| - |X^{\uparrow}_t|$.

Suppose now that $x^s > y^e$, then the point (x^s, y^e) is inside of F and some pipes can pass it from the northwest. We note that if a pipe p does that, then it goes through a cell (x, y) with $x < x^s$ and $y \geqslant y^e$, so $x^s_p < x^s$ and $y^e_p > y^e$ and so $p < i$ and $\omega^{-1}(p) < \omega^{-1}(j)$. Moreover, since this pipe starts west of the vertical line $\Delta^{\updownarrow}(x^s)$, either it also ends west of that line or it crosses that line north of (x^s, y^e). If $x^s \leqslant t_F$ then no pipe ends west of $\Delta^{\updownarrow}(x^s)$ and the pipes crossing that line north of (x^s, y^e) must do so between ordinates x^s (the top boundary of F) and y^e, and so there are at most $x^s - y^e$ such pipes. If $x > t_F$, then some point (x^s, y') is the east extremity of a horizontal step of $\mathcal{E}_F$; by Lemma 3.1.9 this is the $(x^s - y')$-th step of $\mathcal{E}_F$ and all the following steps of $\mathcal{E}_F$ are east of that point, so only $x^s - y'$ pipes end west of $\Delta^{\updownarrow}(x^s)$ and at most $y' - y^e$ pipes cross $\Delta^{\updownarrow}(x^s)$ north of (x^s, y^e); in total, at most $(x^s - y') + (y' - y^e) = x^s - y^e$ pipes pass northwest of (x^s, y^e).

We can then once again proceed by induction: before step 1, there are no free elbows in R, and when inserting pipe $\pi(t)$:

- if $\pi(t) \notin X^{\downarrow}_{t+1}, X^{\uparrow}_{t+1}$ then once again the number of free elbows does not change and any existing free elbows are still in a staircase;
- if $\pi(t) \in X^{\downarrow}_{t+1}$ then either $\pi(t)$ passes northwest of (x^s, y^e) or it goes through R; in that last case, it creates one more free elbow than it completes and the resulting free elbows are still in a staircase;
- if $\pi(t) \in X^{\uparrow}_{t+1}$ then either $\pi(t)$ does not pass through R or it completes one more free elbow than it creates.

Thus the free elbows in R before step t once again form a staircase of length at least $|X^{\downarrow}_t| - |X^{\uparrow}_t| - \#\{$ pipes passing northwest of $(x^s, y^e)\}$; since that last set contains at most $x^s - y^e$ pipes, this gives the expected result. □

Corollary 3.2.23. *Before step t, the free southeast elbows in $\mathcal{Z}_{\pi(t)}$ form a staircase of length at least $y^s_{\pi(t)} - x^e_{\pi(t)} + |\operatorname{ninv}(\omega, \pi(t))|$.*

Proof. We choose the points $(x^s, y^s) = (x^s_{\pi(t)}, y^s_{\pi(t)})$ and $(x^e, y^e) = (x^e_{\pi(t)}, y^e_{\pi(t)})$ to apply Lemma 3.2.22. Then depending on the starting and ending directions of $\pi(t)$, either $i = \pi(t)$ or $i = \pi(t) - 1$ and either $j = \omega^{-1}(\pi(t))$ or $j = \omega^{-1}(\pi(t))$. Since $\pi(t)$ has not been inserted yet, in all cases this means that the first set of the lemma is $\{s < t \mid \pi(s) \in \operatorname{ninv}(\omega, \pi(t))\}$ and the second one is $\{s < t \mid \pi(t) \in \operatorname{ninv}(\omega, \pi(s))\}$. Since $\pi \leqslant \omega$, the second set is empty and the first one has cardinality $|\operatorname{ninv}(\omega, \pi(t))|$, and so we obtain the desired result. □

Corollary 3.2.24. *No pipe created by the insertion algorithm passes through a cell (x, y) with $x - y > n$.*

Proof. Consider a pipe p inserted by the algorithm; from Corollary 3.2.23 we know that p has at least $x^s_p - y^e_p + |\operatorname{ninv}(\omega, p)|$ southeast elbows. Consider now a column $x^s_p \leqslant x < x^e_p$ and denote by y the ordinate of the cell furthest south crossed by p in that column. We know that $y^s_p \leqslant y < y^e_p$; moreover, if e_x is the number of elbows of p strictly west of column x, we note that $y \geqslant y^s_p + e_x - \delta^s_p$ (with $\delta^s_p = 1$ if p starts vertically and 0 otherwise), since any southeast elbow except the first if p starts vertically means that p goes up at least one row. Lastly, from Lemma 3.1.9 we know that $x^s_p - y^s_p = p - \delta_p$, which gives $x - y = x - x^s_p + x^s_p - y^s_p - e_x + \delta_p = p + (x - x^s_p) - e_x$.

However, since p has at most one southeast elbow in each column and none in its last column if p ends vertically, we have $e_x \geqslant y^e_p - x^s_p + |\operatorname{ninv}(\omega, p)| - (x^e_p - x - 1 + \delta^e_p)$, so $x - y \leqslant p + x^e_p - y^e_p - |\operatorname{ninv}(\omega, p)| - 1 + \delta^e_p$ and since according to Lemma 3.1.9 we have $x^e_p - y^e_p + \delta^e_p = \omega^{-1}(p)$, so $x - y \leqslant p + \omega^{-1}(p) - 1 - |\operatorname{ninv}(\omega, p)|$. This formula is equivalent to counting pipes (weakly) smaller than p, adding pipes strictly before p in ω and substracting pipes that are both smaller than p and before p in ω; the total can therefore not be greater than n. This concludes the proof. □

Corollary 3.2.25. *The pipes created by the insertion algorithm are contained in F.*

Proof. The trajectory of an inserted pipe p is inside the rectangle $\mathcal{Z}_p$, so entirely northeast of $\mathcal{S}_F$ and southwest of $\mathcal{E}_F$. From Corollary 3.2.21, we know that it is also southeast of the NW stair path of F, and from $\mathcal{E}_F$ that it is northwest of the SE stair path of F, so it is entirely inside F. $\square$

Proof of the correctnes of Algorithm 3.2.17. The algorithm creates a collection of pipes that are disjoint outside of crossings and contacts according to Corollary 3.2.20, and stay inside of F according to Corollary 3.2.25. By construction, any pipe p starts on the p-th step of $\mathcal{S}_F$ and ends on the $\omega^{-1}(p)$-th step of $\mathcal{E}_F$. Moreover, we know from Corollary 3.2.23 that pipe p has at least $y_p^s - x_p^s + |\operatorname{ninv}(\omega, p)|$ southeast elbows. Therefore, we know from Lemma 3.1.13 that the result is a pipe dream $P \in \Pi_F(\omega)$.

Moreover, since by construction any contact between p at the northwest and q at the southeast was first a free elbow created by inserting p, and then was completed by inserting q, we know that if $p \to q$ is an arc of $P^{\#}$ then $\pi^{-1}(p) < \pi^{-1}(q)$. Since by hypothesis $\pi \leqslant \omega$, this means that π is a linear extension of $P^{\natural}$ and so $P = \operatorname{ins}_{F,\omega}(\pi)$. $\square$

3.3 Links with the brick polyhedron

The image of the weak order by $\operatorname{ins}_{F,\omega}$ is less easily defined for generalized pipe dreams than it was for triangular pipe dreams if we only consider the flip graph on $\Pi_F(\omega)$, as discussed in Section 3.2.4. However, we will see in this section that this description becomes much more natural when considering the brick polytope.

3.3.1 Acyclic order and brick polyhedron

Once again, we know from Theorem 1.5.36 that the vertices of the brick polyhedron are in correspondance with the acyclic pipe dreams, and that the rays of the cone generated by the root configuration (and so the contact graph of the pipe dream) determine edges. The following result is thus a logical consequence of the definition of $\operatorname{ins}_{F,\omega}$.

Proposition 3.3.1. *The Hasse diagram of the acyclic order on $\Sigma_F(\omega)$ is an orientation of the part of the skeleton of the brick polyhedron of $\mathrm{SC}(Q_F, \omega)$ corresponding to strongly acyclic pipe dreams.*

Proof. Consider a cover $P \lessdot P'$ of the acyclic order on $\Sigma_F(\omega)$ with F, F' the facets of $\mathrm{SC}(Q_F, \omega)$ associated to P and P'; we know from 3.2.12 that there exists $\pi \lessdot \pi'$ a cover of the weak order on $[\mathrm{id}, \omega]$ with $\pi \in \operatorname{lin}(P)$ and $\pi \in \operatorname{lin}(P')$. We can write $\pi = UpqV$ and $\pi' = UqpV$ with $p < q$ and, according to Lemma 3.2.3, an arc $p \to q$ in $P^{\#}$. According to Proposition 3.2.12, this arc is the only directed path from p to q in $P^{\#}$. As we saw in the proof of Proposition 2.3.1, this means that a ray of the incidence cone of the brick polyhedron at $\mathrm{B}(F)$ is directed by $e_q - e_p$, and we can prove that $\mathrm{B}(F') - \mathrm{B}(F)$ is directed by the same vector. Since both $\mathrm{B}(F)$ and $\mathrm{B}(F')$ are vertices of $\mathcal{B}(\mathrm{SC}(Q_F, \omega))$, this means that they are linked by an edge.

Conversely, for F, F' acyclic facets of $\mathrm{SC}(Q_F, \omega)$ associated to two strongly acyclic pipe dreams P and P' and linked by an edge of the brick polyhedron, we know that this edge is directed by a ray of the incidence cone of the brick polyhedron at $\mathrm{B}(F)$. Since that cone is generated by $\mathbf{R}(F)$, we know that such a ray is directed by a root $e_q - e_p$ for some arc $p \to q$ of $P^{\#}$ and $q \to p$ of $P'^{\#}$; moreover, such an arc must be the only path from p to q in $P^{\#}$. WLOG we suppose that $q < p$. Since $P^{\#}$ is acyclic, there must therefore be a linear extension π of that graph such that p and q are consecutive, i.e. π is of the form $UpqV$ (but π is not necessarily in $[\mathrm{id}, \omega]$). Then for $\pi' = UqpV$, we know from Lemma 3.2.3 that π' is a linear extension of the contact graph of some acyclic pipe dream $P'' \in \Pi_F(\omega)$ linked to P by a flip on pipes p and q, and so for F'' the acyclic facet associated to P'' we know that $\mathrm{B}(F) - \mathrm{B}(F'')$ must be directed by $e_q - e_p$ and both brick vectors are vertices of the brick polyhedron; therefore F'' must be F' and P'' is P'.

Suppose now that there is a path $p \to r_1 \to \ldots \to r_k \to q$ with $k \geqslant 1$ in $P^{\natural}$. Then since P' is obtained by flipping pipes p and q in P, the graph $P'^{\natural}$ contains a path $r_0 \to r_1 \to \ldots \to r_k \to r_{k+1}$ with r_0 and r_{k+1} both in $\{p, q\}$. Since P' is strongly acyclic, we know that $r_0 \neq r_{k+1}$, and since $q \to p$ is an arc of $P'^{\#}$ then necessarily $r_0 = q$ and $r_{k+1} = p$. This means that there is a contact $p \to r_1$ in a cell e_1 of P between the two cells c and x that are flipped to obtain Q. Since p is northwest of Q between those two cells, and q exits the first flipped cell in a different direction than it enters the second one, there must be an elbow of pipe q in a cell e_2 southeast of e_1 and between c and x. We can then apply case 2 of Lemma 3.1.21 to obtain a path from r_1 to q in $P^{\#}$, which means that there is a path of length at least two from p to q in $P^{\#}$. This contradicts what we said previously, so such a path does not exist in $P^{\natural}$. We can thus find $\pi_1 \in \mathrm{lin}(P)$ with p and q consecutive, and exchanging p and q gives π_1', which is covered by π_1 and also a linear extension of P'. Thus, by Proposition 3.2.12, P covers P' in the acyclic order. $\square$

Theorem 3.3.2. *For F an alternating shape and $\omega \in \mathfrak{S}_n$ sortable on F, the strongly acyclic part of the brick polyhedron of $\mathrm{SC}(Q_F, \omega)$ is isomorphic to a lattice quotient of the weak order interval $[\mathrm{id}, \omega]$.*

3.3.2 Acyclicity, strong acyclicity and complete shapes

Proposition 3.3.1 is not very satisfying in the sense that it only applies to the vertices of the brick polyhedron corresponding to strongly acyclic facets. We would like to have a sufficient condition guaranteeing that all the acyclic pipe dreams in some $\Pi_F(\omega)$ are also strongly acyclic.

Definition 3.3.3. *An alternating shape is* ***complete*** *if $\omega_0 = n\,(n-1)\,\ldots\,2\,1$ the longest permutation of $\mathfrak{S}_n$ is sortable on it.*

Theorem 3.3.4. *If F is complete, then for any $\omega \in \mathfrak{S}_n$ all the acyclic pipe dreams in $\Pi_F(\omega)$ are also strongly acyclic.*

This theorem is a direct consequence of the following stronger result that we will prove first.

Theorem 3.3.5. *Let F be a complete alternating shape and P be a reduced pipe dream on F with ω its exit permutation. Then for any $1 \leqslant p < q \leqslant n$, if (p, q) is a noninversion of ω, there is a path from p to q in the contact graph $P^{\#}$.*

We note that this theorem is a type A equivalent of Theorem 1.5.34; we discovered this result after writing the following proof that uses the properties of alternating shapes.

We will start by giving a few properties of complete shapes in Lemma 3.3.6, then use them to see how pipes behave in pipe dreams on those shapes in Lemmas 3.3.7 and 3.3.8. We will then use these last two lemmas to construct a path from p to q in the contact graph of a pipe dream on a complete shape, provided that (p, q) is a noninversion of the exit permutation.

Lemma 3.3.6. *Let F be a complete shape and P be any pipe dream on F with ω its exit permutation, then:*

1. *for any pipe p of P, $x^s_p \leqslant x^s_n < x^e_{\omega(1)} \leqslant x^e_p$;*
2. *for any pipe p of P, $y^s_p \leqslant y^s_1 < y^e_{\omega(n)} \leqslant y^e_p$;*
3. *$x^s_n \leqslant |\mathcal{S}_F|_E \leqslant x^e_{\omega(1)}$;*
4. *$y^s_1 \leqslant 0 \leqslant y^e_{\omega(n)}$;*
5. *$|\mathcal{S}_F|_E \leqslant t_F + 1$.*

Proof. By Remark 3.1.6 we know that $x^s_p \leqslant x^s_n \leqslant |\mathcal{S}_F|_E$ and $t_F \leqslant x^e_{\omega(1)} \leqslant x^e_p$, as well as $y^s_p \leqslant y^s_1 \leqslant 0$ and $y^e_{\omega(n)} \leqslant y^e_p$. We then note that for any pipe dream $P_0 \in \Pi_F(\omega_0)$, the starting coordinates of pipes are the same in P and P_0, and since $\omega_0(1) = n$ and $\omega_0(n) = 1$, the ending coordinates of 1 in P_0 are the same as $\omega(n)$ in P and the ones of n in P_0 are the same as ω_1 in P. We thus obtains that $x^s_n < x_\omega(1)^e$ and $y^s_1 < y^e_{\omega(n)}$. The last relations are proved by noting that $|\mathcal{S}_F|_E - 1 \leqslant x^s_n \leqslant |\mathcal{S}_F|_E$, that $-1 \leqslant y^s_1 \leqslant 0$ and that $t_F \leqslant y^e_{\omega(1)} \leqslant t_F + 1$. □

Lemma 3.3.7. *Let F be a complete alternating shape and P be a reduced pipe dream on F with exit permutation ω. For q any pipe of P, if there exists a cell c of F strictly northwest of $\mathcal{Z}_q$, then there exists a pipe p with an elbow in a cell strictly southeast of c and such that $c \in \mathcal{Z}_p$, and (p, q) is a noninversion of ω.*

Proof. We will proceed by counting the number of pipes that have the cell c in their area, and then we will see that not all of those pipe can pass to the NW of c; therefore, at least one of them must have an elbow SE of that cell and thus be as described by the lemma.

Let us consider a pipe q of F and suppose that a cell c of F with coordinates (x, y) is strictly northwest of $\mathcal{Z}_q$, i.e. $x_q^s > x \geqslant y \geqslant y_q^e$. From Lemma 3.3.6 we deduce that any pipe p satisfies $x < x_p^e$ and $y > y_p^s$, so $(x, y) \in \mathcal{Z}_p$ if and only if $x_p^s \leqslant x$ and $y_p^e > y$. If we call $X := \{p \mid x_p^s \leqslant x\}$ and $Y := \{p \mid y_p^e > y\}$, this means that $c \in \mathcal{Z}_p$ if and only if $p \in X \cap Y$. Moreover, if $p \in X \cap Y$, then p starts strictly west of q and ends strictly north of q, then (p, q) is necessarily a noninversion of ω.

We know that $0 \leqslant y_q^e < y \leqslant x < x_p^s$ so the south boundary of F in columns x and y are respectively the i-th and j-th step of $\mathcal{S}_F$ for some $1 \leqslant j \leqslant i < p$. Since step i is the $(x + 1)$-th east step of $\mathcal{S}_F$ and step j the $(y + 1)$-th east step, we note that $i - j \geqslant x - y$. Let us now consider $P_0 \in \Pi_F(\omega_0)$, according to Lemma 3.1.9 the starting coordinates of pipe j are $(y, y-j+1)$; moreover, we know from Lemma 3.1.12 that pipe j has $j - 1$ vertical crossings (since $\mathrm{ninv}(\omega_0) = \varnothing$), and at least one southeast elbow since it does not end in column y (because $y < t_F$). Therefore, it passes through at least j different rows of F, and so its ending ordinate is at least $y+1$. Thus, for any pipe p of P, if $\omega^{-1}(p) \leqslant \omega_0^{-1}(j) = n+1-j$, then $y_p^e \geqslant y+1$.

We just proved that $p \leqslant i \Rightarrow p \in X$ and $\omega^{-1}(p) \leqslant n + 1 - j \Rightarrow p \in Y$. This means that $|X| \geqslant i$ and $|Y| \geqslant n + 1 - j$, and since $p \notin X \cup Y$, we also know that $|X \cup Y| \leqslant n - 1$. We can then apply the inclusion-exclusion principle to get

$$\begin{aligned}
|X \cap Y| &\geqslant |X| + |Y| - |X \cup Y| \\
&\geqslant i + (n + 1 - j) - (n - 1) \\
&\geqslant i - j + 2 \\
|X \cap Y| &\geqslant x - y + 2
\end{aligned}$$

Then each pipe in X starts west of the vertical line $\Delta^{\updownarrow}(x + 1)$ and ends east of it (since $x + 1 \leqslant x_q^s < x_p^e$ for any p), and each pipe in Y starts south of the horizontal line $\Delta^{\leftrightarrow}(y)$ (because $y \geqslant y_q^e > y_p^s$ for any p) and ends north of it. It can thus cross either first the horizontal line then the vertical one, or the reverse. In the first case, it crosses $\Delta^{\leftrightarrow}(y)$ west of point $(x + 1, y)$ and inside F, so east of point (y, y); with one pipe crossing the line in each column, there are at most $x + 1 - y$ such pipes. In the second case, the pipe crosses $\Delta^{\updownarrow}(x + 1)$ going horizontally then $\Delta^{\leftrightarrow}(y)$ going vertically, so there is at least one elbow between those two crossing, and that elbow is southeast of the cell c. Since $|X \cap Y| > x - y + 1$, there is at least one pipe $p \in X \cap Y$ in the second case, and that pipe satisfies the requirements of the lemma. □

Lemma 3.3.8. *Let F be a complete alternating shape and $P \in \Pi_F(\omega)$, and (p, q) be a noninversion of ω. Then:*
1. *pipe p has an elbow northwest of an elbow of pipe q;*
2. *either one such elbow is strictly northwest of $\mathcal{Z}_q$, or it is in $\mathcal{Z}_q$ and there is a path from p to q in $P^{\#}$.*

Proof. From Lemma 3.3.6 and our choice of p and q, we know that $x_p^s \leqslant x_q^s < x_p^e \leqslant x_q^e$ and $y_q^s \leqslant y_p^s < y_q^e \leqslant y_p^e$. Pipe p must therefore cross the vertical line $\Delta^{\updownarrow}(x_q^s)$ and the horizontal line $\Delta^{\leftrightarrow}(y_q^e)$ (we note that p cannot start vertically in column x_q^s or end horizontally in row y_q^e); since it changes direction between those two crossings, it has an elbow in a cell (x, y) between them. Then:

- if pipe p crosses $\Delta^{\leftrightarrow}(y_q^e)$ first and $\Delta^{\updownarrow}(x_q^s)$ second, then $x < x_q^s$ and $y \geqslant y_q^e$, so this elbow is strictly NW of $\mathcal{Z}_q$. Moreover, pipe q must cross $\Delta^{\updownarrow}(x_q^e)$ and $\Delta^{\leftrightarrow}(y_q^s)$ and it changes direction between these crossings, so it has at least one elbow in $\mathcal{Z}_q$, and so that elbow is southwest of (x, y);

- if pipe p crosses $\Delta^{\updownarrow}(x_q^s)$ first and $\Delta^{\leftrightarrow}(y_q^e)$ second, then $x_q^s \leqslant x$ and $y_q^e > y$ so $(x, y) \in \mathcal{Z}_q$. Moreover, since $x < x_p^e \leqslant x_q^e$ and $y \geqslant y_q^s \geqslant y_p^s$, pipe p must cross the line $\Delta^{\updownarrow}(x+1)$ and $\Delta^{\leftrightarrow}(y)$ at some point, so it has an elbow between thoese two crossing; since $(p, q) \in \mathrm{ninv}(\omega)$, pipe q is southeast of pipe p on its whole trajectory, and so that elbow is southeast of the elbow of q in cell (x, y). In that case, we can apply case 3 of Lemma 3.1.21 to obtain a path from p to q in $P^{\#}$.

□

Proof of Theorem 3.3.5. We will find a path from p to q in $P^{\#}$ by using Lemma 3.3.7 to prove that as long as p passes northwest of $\mathcal{Z}_q$, we can find p' such that there is a path from p to p' and (p', q) is also a noninversion of ω. This will create a sequence with its last element in $\mathcal{Z}_q$, allowing us to apply the second part of 3.3.8.

Let P be a reduced pipe dream of F a complete shape with $\omega \in \mathfrak{S}_n$ its exit permutation. We choose (p, q) a noninversion of ω and we suppose that for any pipes $1 \leqslant p' < q' < q \leqslant n$ with $(p', q') \in \mathrm{ninv}(\omega)$, there is a path from p' to q' in $P^{\#}$. We will prove that there is also a path from p to q in $P^{\#}$. To do so, we will build a non-repeating sequence of pipes $p = p_0, p_1, \ldots, p_N$ and a sequence of cells $(x_0, y_0), \ldots, (x_{N-1}, y_{N-1})$ of F such that for all $0 \leqslant k < N$:

1. cell (x_k, y_k) contains an elbow of p_k;
2. $x_k < x_q^s$ and $y_k \geqslant y_q^s$;
3. the pipe p_{k+1} has an elbow strictly SE of (x_k, y_k);
4. there is a path in $P^{\#}$ from p to p_{k+1};
5. there exists $l \leqslant k$ such that $x_{p_{k+1}}^s \leqslant x_l$;
6. there exists $l \leqslant k$ such that $y_{p_{k+1}}^s > x_l$;

and pipe p_N has an elbow in $\mathcal{Z}_q$.

We already chose $p_0 = p$. Suppose now that we have the pipes $p_0, \ldots, p_k$ and the cells $(x_0, y_0), \ldots, (x_{k-1}, y_{k-1})$ as described. We already know that either $k = 0$ and by hypothesis $(p, q) \in \mathrm{ninv}(\omega)$, or there exists $l_1, l_2 < k$ such that $x_{p_k}^s \leqslant x_{l_1} < x_q^s$ and $y_{p_k}^e > y_{l_2} \geqslant y_q^e$, so pipe p_k starts strictly west and ends strictly north of pipe q, and so $(p_k, q) \in \mathrm{ninv}(\omega)$. Therefore, by Lemma 3.3.8, pipe p_k either has an elbow in $\mathcal{Z}_q$ and in that case we decide that $k = N$ and stop the sequence, or it has at least one elbow strictly NW of $\mathcal{Z}_q$ in a cell that we denote by (x_k, y_k) (and thus statements 1 and 2 of our conditions on the sequences are still true).

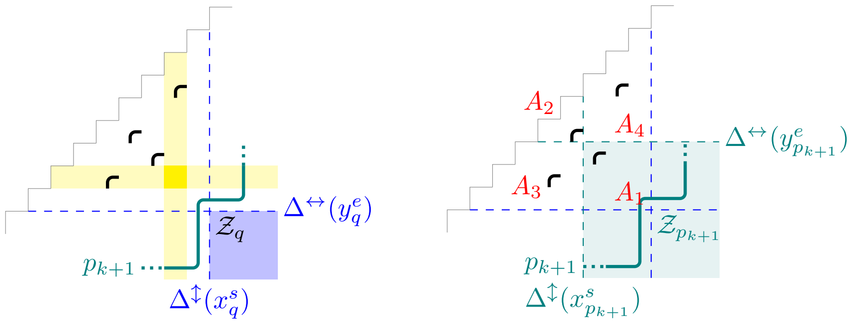


Figure 3.12: Illustration of the placement of the cells (x_l, y_l).

In that case, let us now denote by $x = \max_{0 \leqslant l \leqslant k}(x_l)$ and $y = \min_{0 \leqslant l \leqslant k}(y_k)$, meaning that the cell (x, y) is the furthest northwest cell that is southeast of every (x_l, y_l) for $0 \leqslant l \leqslant k$, as depicted in yellow in Fig. 3.12. We know that $x < x_q^s$ and $y \geqslant y_q^s$, so we can apply Lemma 3.3.7 to obtain p_{k+1} with $(x, y) \in \mathcal{Z}_{p_{k+1}}$ and that has an elbow strictly southeast of (x, y) (and so strictly southeast of every (x_l, y_l) for $0 \leqslant l \leqslant k$). This last proposition tells us that $p_{k+1} \neq p_l$ for all $0 \leqslant l \leqslant k$, since pipes go east and north and thus no pipe could go through two cells with one strictly SE of the other. Moreover, this choice of p_{k+1} guarantees that conditions 3, 5 and 6 on our sequences are true. A possible trajectory of pipe p_{k+1} relative to the elbows in cells (x_l, y_l) (in black) and to cell (x, y) (in yellow) is illustrated on the left side of Fig. 3.12.

Let us now divide the indices $0 \leqslant l \leqslant k$ in four sets A_1, A_2, A_3, A_4 depending on the placement of the cell (x_l, y_l) relative to $\mathcal{Z}_{p_{k+1}}$ as depicted on the right side of Fig. 3.12: we say that l is in A_1 if $(x_l, y_l) \in \mathcal{Z}_{p_{k+1}}$, in A_2 if it is strictly NE of $\mathcal{Z}_{p_{k+1}}$, in A_3 is it is strictly west but not north of $\mathcal{Z}_{p_{k+1}}$ and in A_4 if it is strictly north but not west of $\mathcal{Z}_{p_{k+1}}$. At least one of those sets cannot be empty; one of the following cases must then be true:

- If $A_1 \neq \varnothing$, then for any $l \in A_1$ we have $(x_l, y_l) \in \mathcal{Z}_{p_{k+1}}$, and pipe p_{k+1} has an elbow strictly southeast of (x, y) and so strictly SE of the elbow of p_l in cell (x_l, y_l). We can thus apply statement 3 of Lemma 3.1.21 to obtain a path from p_l to p_{k+1} in $P^{\#}$, and since by induction hypothesis there is a path from p to p_l in $P^{\#}$, this guarantees that condition 4 is true.

- If $A_2 \neq \varnothing$, then for any $l \in A_2$ pipe p_l starts strictly west and ends strictly north of pipe p_{k+1} and thus $(p_l, p_{k+1}) \in \mathrm{ninv}(\omega)$. Since we supposed at the begining that for any noninversion (p', q') of ω such that $q' < q$ there is a path from p' to q' in $P^{\#}$, we know that there is a path from p_l to p_{k+1} in $P^{\#}$, and so since by induction hypothesis there is a path from p to p_l condition 4 on our sequences is once again true.

- Suppose now that $A_1 = A_2 = \varnothing$. In that case, since there exist $0 \leqslant l_3, l_4 \leqslant k$ with $x_{l_4} = x \geqslant x_{p_{k+1}}^s$ and $y_{l_3} = y < y_{p_{k+1}}^e$, we know that $l_3 \in A_3$ and $l_4 \in A_4$.

- Suppose now that $0 \in A_3$ and consider $m = \min(A_4) > 0$. Since m is in A_4, we know that $y^e_{p_{k+1}} \leqslant x_m < x^e_{p_m}$, so $\omega^{-1}(p_m) < \omega^{-1}(p_{k+1})$. Moreover, by our choice of m, any $l < m$ is in A_3, so $x_l < x_{p_{k+1}}$; by condition 5 on our sequences, this means that $x^s_{p_m} < x^s_{p_{k+1}}$ so $p_m < p_{k+1}$.
- Similarly, if $0 \in A_4$, we choose $m = \min(A_3) > 0$. The same reasoning tells us that because $m \in A_3$ then $a_m < a_{k+1}$ and because of condition 6 applied to m then $\omega^{-1}(a_m) < \omega^{-1}(a_{k+1})$.

In both cases, we have $m \leqslant k$ such that (p_m, p_{k+1}) is a non-inversion of ω. As in the case where $A_2 \neq \varnothing$, this means that there is a path from p_m to p_{k+1} in $P^\#$ and so condition 4 is still true.

We have thus proved the existence of such a sequence, and since it is non-repeating and all elements are pipes of P, it is finite. Therefore, this sequence has a last element p_N such that there is a path from p to p_N in $P^\#$, and by our choice of an endpoint, we know that $(p_N, q) \in \mathrm{ninv}(\omega)$ and p_N has an elbow in $\mathcal{Z}_q$. By Lemma 3.3.8, this tells us that there is a path from p_N to q in $P^\#$, and by transitivity there is a path from p to q in $P^\#$.

This tells us that the set $S = \{(p,q) \in \mathrm{ninv}(\omega) \mid \text{there is no path } p \to^* q \text{ in } P^\#\}$ must be empty, since $\min_{(p,q)\in S}(q)$ cannot exist. This conclude the proof. $\square$

Proof of Theorem 3.3.4. A directed graph is acyclic if and only if its transitive closure is acyclic. For F a complete shape and $\omega \in \mathfrak{S}_n$ a permutation, let us consider $P \in \Pi_F(\omega)$; if it is acyclic, then by definition $P^\#$ is acyclic and so is its transitive closure $\overline{P^\#}$. We know from Theorem 3.3.5 that for any $(p,q) \in \mathrm{ninv}(\omega)$, the edge (p,q) is in $\overline{P^\#}$, and so is any edge of $P^\#$; therefore, the graph $P^\natural$ is a subgraph of $\overline{P^\#}$ and is also acyclic. Therefore, if P is acyclic, then it is also strongly acyclic. $\square$

We have thus proven the following theorem, a specialization of Proposition 3.3.1 that is a true generalization of Theorem 2.3.2.

Theorem 3.3.9. *For F a complete alternating shape and $\omega \in \mathfrak{S}_n$, the skeleton of the finite part of the brick polyhedron is isomorphic to a lattice quotient of the weak order interval* $[\mathrm{id}, \omega]$.

Chapter 4

Extension to Coxeter groups

In previous chapters, we proved that the linear extensions of a family of pipe dreams defined a lattice quotient on an interval of the weak order, and that the associated maps from this interval to the subword complex represented by these pipe dreams is a lattice morphism from the weak order to part of the brick polyhedron. This result can be expressed in the vocabulary of Coxeter groups; as such, it is natural to ask to what extent the same property is true for any finite Coxeter group. This chapter will discuss the terms of that generalization, state our conjectures on the subject, and give some ideas of how those conjectures could be proved. The results and conjectures are presented in Section 5 of [BCCP22].

We start Section 4.1 by defining what a linear extensions of a subword complex facet is in Section 4.1.1. We prove in Section 4.1.2 that they still define partitions of weak order intervals, and so we define subword equivalences and an insertion map in the same way as pipe dream equivalences and pipe dream insertion in previous chapters. We also give in Section 4.1.3 an algorithm computing the insertion map on any subword complex.

Section 4.2 discusses the links between subword equivalences and insertions, and the weak order on the intervals they are defined on. We propose some conjectures generalizing the results of Chapter 3 to Coxeter groups of any types in Section 4.2.1; these conjectures were extensively tested by computers experiments. While we are not yet able to prove them, Section 4.2.2 gives an idea of how to proceed, starting from a conjectured lemma inspired by Lemma 3.1.22. In particular, in Section 4.2.3 we prove this lemma for type B Coxeter groups and dihedral Coxeter groups.

Finally, in Section 4.3, we propose a possible generalization of two definitions on type A Coxeter groups to any finite Coxeter group. The first, chute moves, are a subset of ascending flips on pipe dreams. As they are linked to the image of the weak order by $\mathrm{ins}_{F,\omega}$, in Section 4.3.1 we use characteristics of this image in any Coxeter groups to give a tentative general definition of chute moves. The second, in Section 4.3.2, is about ν-Tamari lattices: we use our characterization of dominant permutations given in Section 2.3.2 to generalize the definition of ν-Tamari lattices as the increasing flip poset of a subword complex to any Coxeter group.

4.1 The insertion map in subword complexes

In this chapter, we consider W a Coxeter group with S its generator set, and Φ an associated root system with Δ the simple roots and Π the positive roots.

We will start by defining the terms of the problem: what is a linear extension of a subword complex facet? And can we generalize the definition of pipe dream congruences, or of insertion maps, to any subword complex?

4.1.1 Linear extensions

We start with this simple definition generalizing the idea of a linear extension.

Definition 4.1.1. *For Q a word and $w \in W$ sortable on Q, a **linear extension** of a facet F of $\mathrm{SC}(Q, w)$ is an element $u \in W$ such that $\mathbf{R}(F) \subseteq u(\Pi)$. We denote by $\mathrm{lin}(F)$ the set of linear extensions of F, and by $\mathrm{lin}(Q, w)$ the set of all linear extensions of all facets of $\mathrm{SC}(Q, w)$.*

For P a pipe dream, we know that the root configuration of the associated subword complex facet F is $\{e_q - e_p \mid p \to q \text{ is an arc of } P^{\#}\}$. Then if $\pi \in \mathrm{lin}(P)$, for any arc $p \to q$ of $P^{\#}$ we know that $\pi^{-1}(p) < \pi^{-1}(q)$, i.e. $e_{\pi^{-1}(q)} - e_{\pi^{-1}(p)}$ is a positive root. Since $e_q - e_p = \pi(e_{\pi^{-1}(q)} - e_{\pi^{-1}(p)})$, this means that $e_q - e_p \in \pi(\Pi)$, and so the linear extensions of P are exactly the linear extensions of F.

Once again, as for pipe dreams, we find a link between the existence of linear extensions of a facet and its acyclicity.

Proposition 4.1.2 ([BCCP22, Lem. 5.9])**.** *A facet F is acyclic, as defined in Theorem 1.5.36, if and only if $\mathrm{lin}(F) \neq \varnothing$.*

Proof. Note first that if some $u \in W$ is a linear extension of F, then by definition $\mathrm{Cone}(\mathbf{R}(F)) \subseteq \mathrm{Cone}(u(\Pi)) = \mathrm{Cone}(u(\Delta))$; since $u(\Delta)$ is a free family of vectors, this means that $\mathrm{Cone}(u(\Delta))$ is simplicial, so it is pointed, and so $\mathrm{Cone}(\mathbf{R}(F))$ is also pointed. Suppose now that $\mathrm{Cone}(\mathbf{R}(F))$ is pointed and choose H a hyperplane separating $\mathrm{Cone}(\mathbf{R}(F))$ and its opposite strictly, i.e. with $\mathbf{R}(F) \subseteq H^+ \cup \{0\}$. Since $\mathbf{R}(F)$ and $-\mathbf{R}(F)$ are closed, we can choose H such that $H \cap \Phi = \varnothing$, as we can pertubate H slightly. Then by definition $\Phi \cap H^+$ is a positive system of Φ, and so from Theorem 1.3.10 we know that $\Phi \cap H^+ = u(\Pi)$ for some $u \in W$. Then $u \in \mathrm{lin}(F)$ and so $\mathrm{lin}(F) \neq \varnothing$. □

The following theorem, which is a Coxeter version of Theorem 2.2.7 and Theorem 3.2.2, will allow us to generalize the definition of pipe dream congruences to any subword complex.

Theorem 4.1.3 ([BCCP22, Thm. 5.11]). *For* $\mathrm{SC}(Q, w)$ *a nonempty subword complex, then:*
1. *for any facet* F *of* $\mathrm{SC}(Q, w)$*, the set* $\mathrm{lin}(F)$ *is order convex;*
2. *the set* $\mathrm{lin}(Q, w)$ *is a lower set of the weak order;*
3. $[e, w] \subseteq \mathrm{lin}(Q, w)$*;*
4. *if* $F_1 \neq F_2$ *are both facets of* $\mathrm{SC}(Q, w)$*, then* $\mathrm{lin}(F_1) \cap \mathrm{lin}(F_2) = \varnothing$*.*

Note that this theorem is actually a little different from Theorem 3.2.2 as it consider linear extensions of the contact graph, and not the extended contact graph, but we can see easily from Remark 3.1.8 that Theorem 3.2.2can be directly deduced from it.

To prove this theorem, we will once again use the following generalization of Lemma 2.2.9 and Lemma 3.2.3.

Lemma 4.1.4. *For* $us_\alpha \lessdot u$ *a cover of the weak order, if* u *is a linear extension of some facet* F *of* $\mathrm{SC}(Q, w)$*, then:*
- *if* $u(\alpha) \notin \mathbf{R}(F)$ *then* us_α *is also a linear extension of* F*;*
- *if* $u(\alpha) \in \mathbf{R}(F)$ *then* us_α *is a linear extension of* F' *the facet of* $\mathrm{SC}(Q, w)$ *obtained by flipping the last index* i *of* F *such that* $\mathbf{r}(F, i) = u(\alpha)$*.*

Proof. Note first that since s_α is a descent of u, according to Lemma 1.4.12 we know that $us_\alpha(\Pi) = (u(\Pi) \cup \{-u(\alpha)\})\backslash\{u(\alpha)\}$; this directly implies the first item of the lemma. Suppose now that $u(\alpha) \in \mathbf{R}(F)$, i.e. that for some $i \in F$ we have $\mathbf{r}(F, i) = u(\alpha)$, and denote by i_2 the last such index. Since $u(\alpha) \in -\Pi$ (as seen in Lemma 1.4.5), we know from Lemma 1.5.13 that $u(\alpha) \in -\mathrm{inv}(w)$ and so from Proposition 1.5.18 that there exists a descending flip from F to some facet F' on index i_2. Denote by i_1 the other flipped index, i.e. the only element of $F'\backslash F$; then since the flip is descending we know that $i_1 < i_2$. Then from Lemma 1.5.17, for $j \in F'$, we have:
- if $j < i_1$ or $j > i_2$ then $\mathbf{r}(F', j) = \mathbf{r}(F, i) \in \mathbf{R}(F)\backslash\{u(\alpha)\} \subseteq us_\alpha(\Pi)$;
- if $j = i_1$ then $\mathbf{r}(F', j) = \mathbf{r}(F, j) = -u(\alpha) = us_\alpha(\alpha) \in us_\alpha(\Pi)$;
- if $i_1 < j \leqslant i_2$ then $\mathbf{r}(F', j) = s_{u(\alpha)}\,\mathbf{r}(F, j) \in s_{u(\alpha)}u(\Pi)$. Since $s_{u(\alpha)} = us_\alpha u^{-1}$, we have $s_{u(\alpha)}u = us_\alpha$ and so $\mathbf{r}(F', j) \in us_\alpha(\Pi)$.

This proves that $us_\alpha \in \mathrm{lin}(F')$ and concludes the proof. □

We will also use the following lemma discussing facets that have e or w as a linear extension.

Lemma 4.1.5 ([BCCP22, Lem. 5.10]). *For* F *a facet of* $\mathrm{SC}(Q, w)$*:*
- $e \in \mathrm{lin}(F)$ *if and only if* F *is the antigreedy facet of* $\mathrm{SC}(Q, w)$*;*
- $w \in \mathrm{lin}(F)$ *if and only if* F *is the greedy facet of* $\mathrm{SC}(Q, w)$*.*

Proof. By definition, a facet F has e as a linear extension if and only if $\mathbf{R}(F) \in \Pi$. From Proposition 1.5.18, this means that any flip from F is increasing, so F is the antigreedy facet of $\mathrm{SC}(Q, w)$. Conversely, if F is the antigreedy facet, then from

Proposition 1.5.18 we know that $\mathbf{R}(F) \cap \mathrm{inv}(w) = \varnothing$; then from Lemma 1.5.13 this means that $\mathbf{R}(F) \subseteq \Pi$ and so $e \in \mathrm{lin}(F)$.

Similarly, by definition a facet F has the element w as a linear extension if and only if $\mathbf{R}(F) \subseteq w(\Pi) = \mathrm{ninv}(w) \cup -\mathrm{inv}(w)$. In that case, we know from Proposition 1.5.18 that any flip from F is decreasing, so F is the greedy facet of $\mathrm{SC}(Q, w)$. Conversely, if F is the greedy facet, then from Proposition 1.5.18 we know that $\mathbf{R}(F) \cap \mathrm{inv}(w) = \varnothing$, and from Lemma 1.5.13 that $\mathbf{R}(F) \cap -\mathrm{ninv}(w) = \varnothing$, so $\mathbf{R}(F) \subseteq \mathrm{ninv}(w) \cup -\mathrm{inv}(w)$ and so $w \in \mathrm{lin}(F)$. □

We can now prove the theorem:

Proof of Theorem 4.1.3.

1. For F a facet of $\mathrm{SC}(Q, w)$, suppose that $u_1 \leqslant u_2 \leqslant u_3$ and $u_1, u_3 \in \mathrm{lin}(F)$. Then for $\phi \in \mathrm{lin}(F)$, we know that $\phi \in u_1(\Pi) \cap u_3(\Pi)$. If $\phi \in \Pi$ this implies that $\phi \in \mathrm{ninv}(u_3) \subseteq \mathrm{ninv}(u_2) \subseteq u_2(\Pi)$. Conversely, if $\phi \in -\Pi$ this implies that $\phi \in -\mathrm{inv}(u_1) \subseteq -\mathrm{inv}(u_2) \subseteq u_2(\Pi)$. In both cases we have $\phi \in u_2(\Pi)$ and so u_2 is a linear extensions of F; therefore $\mathrm{lin}(F)$ is order convex.

2. This item is a direct consequence of Lemma 4.1.4: if u covers v and u is in $\mathrm{lin}(Q, w)$ then $v \in \mathrm{lin}(Q, w)$, so by induction if $v \leqslant u$ and $u \in \mathrm{lin}(Q, w)$ then $v \in \mathrm{lin}(Q, w)$, so $\mathrm{lin}(Q, w)$ is a lowerset.

3. This is a consequence of the previous item and the second part of Lemma 4.1.5: since $w \in \mathrm{lin}(Q, w)$ and $\mathrm{lin}(Q, w)$ is a lowerset, then $[e, w] \subseteq \mathrm{lin}(Q, w)$.

4. Suppose that for some facets F_1, F_2, some element u is a linear extension of both. Then for us_α covered by u in the weak order, the element us_α is a linear extension of F_1' and F_2' such that F_i' is either F_i or obtained from F_i by flipping the last index of F_i where its root function is equal to $u(\alpha)$. Since we can find F_i from F_i' in the reverse way, i.e. if $-u(\alpha) \notin \mathbf{R}(F_i')$ then $F_i = F_i'$ and otherwise F_i is obtained by flipping the first index of F_i' where its root function is equal to $-u(\alpha)$, we know that $F_1 \neq F_2 \Rightarrow F_1' \neq F_2'$. Thus the set of elements that are linear extensions of more than one facet is also a lowerset. However, we know from Lemma 4.1.5 that e is a linear extension of exactly one facet, the antigreedy one; since a lowerset that does not contain e is necessarily empty, this means that no element is a linear extension of more than one facet so for any two facets $F_1 \neq F_2$ we have $\mathrm{lin}(F_1) \cap \mathrm{lin}(F_2)$.

□

Definition 4.1.6. *We denote by $\Sigma_Q(w)$ the set of acyclic facets of* $\mathrm{SC}(Q, w)$*, and by $\Sigma_Q'(w)$ the set of* ***strongly acyclic facets*** *with a linear extension in $[e, w]$.*

In general, the set $\mathrm{lin}(Q, w)$ can be larger than $[e, w]$; we saw in Fig. 3.4 that some acyclic facets even have no linear extension in that interval, and so are not strongly acyclic. However, as we did in Section 3.3.2, we can find a sufficient condition guaranteeing that this set is equal to $[e, w]$, and in particular that all acyclic facets are strongly acyclic.

Theorem 4.1.7 ([BCCP22, Thm. 5.16])**.** *For Q a word sorting w_0 the longest element of W and $w \in W$, the sets $(\mathrm{lin}(F))_{F \in \Sigma_Q(w)}$ define a partition of the weak order interval $[e, w]$.*

Proof. Since w_0 is the maximum of W for the Bruhat order, we know from Theorem 1.5.11 that Q sorts w_0 iff $\mathrm{Dem}(Q) = w_0$. Then in that case, we know from Theorem 1.5.34 that for any $w \in W$ and any facet F of $\mathrm{SC}(Q, w)$, the bruhat cone $\mathcal{C}^+(w, w_0)$ is contained in $\mathrm{Cone}(\mathbf{R}(F))$; according to Lemma 1.5.33 this means that $\mathrm{ninv}(w) \subseteq \mathrm{Cone}(\mathbf{R}(F))$. Then for any $u \in \mathrm{lin}(F)$, we know that $\mathrm{Cone}(\mathbf{R}(F)) \subseteq \mathrm{Cone}(u(\Pi))$ so if $\alpha \in \mathrm{ninv}(w)$ then $\alpha \in \mathrm{Cone}(u(\Pi)) \cap \Pi$ and so $\alpha \in u(\Pi) \cap \Pi = \mathrm{ninv}(u)$. Thus $\mathrm{ninv}(w) \subseteq \mathrm{ninv}(u)$ and so $u \leqslant w$ in the weak order. This means that $\mathrm{lin}(Q, w) \subseteq [e, w]$; since from Theorem 4.1.3 we know that $[e, w] \subseteq \mathrm{lin}(Q, w)$, we get that $[e, w] = \mathrm{lin}(Q, w)$ and that $(\mathrm{lin}(F))_{F \in \Sigma_Q(w)}$ is a partition of that interval. □

4.1.2 Subword equivalence and insertion map

We can now use Theorem 4.1.3 to define both an equivalence class on $\mathrm{lin}(Q, w)$ a map from $\mathrm{lin}(Q, w)$ to acyclic facets of $\mathrm{SC}(Q, w)$. Since we are more interested in the link between these objects and lattice properties, we want to define them on a lattice; the biggest lattice that we can guarantee is contained in $\mathrm{lin}(Q, w)$ is the weak order interval $[e, w]$, so that is the definition set that we will choose.

Definition 4.1.8. *For Q a word and $w \in W$ sortable on Q, the* ***subword complex congruence*** *$\equiv_{Q,w}$ on $[e, w]$ is the equivalence relation whose equivalence classes are the linear extension sets $(\mathrm{lin}(F))_{F \in \Sigma_Q(w)}$. The map $\mathrm{ins}_{Q,w} : [e, w] \mapsto \Sigma'_Q(w)$ associates to an element $u \leqslant w$ the only facet of $\mathrm{SC}(Q, w)$ that has u as a linear extension.*

We know from Lemma 4.1.4 that the image of the weak order by $\mathrm{ins}_{Q,w}$ is still always a subset of the increasing flip graph on acyclic facets. However, the lattice properties of the weak order are not always respected by this map: Fig. 4.1 gives an example of the Q congruence on the interval $[e, w]$ for $w = 3421$ an element of the Coxeter group A_3 and $Q = \tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_2\tau_3\tau_1$, and in this case, unlike the congruences studied in Chapters 2 and 3, not all the equivalence classes are intervals. This implies in particular that this equivalence relation is not a lattice congruence.

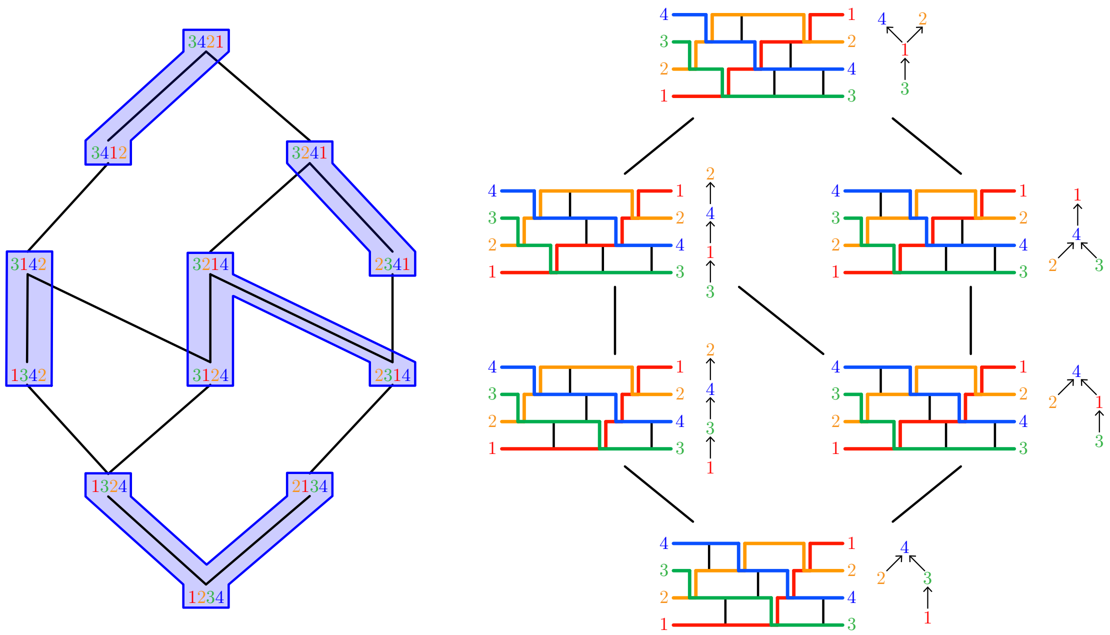


Figure 4.1: A subword complex congruence $\equiv_{Q,w}$ for W the Coxeter group A_3, and $Q = tau_2\tau_3\tau_1\tau_3\tau_2\tau_1\tau_2\tau_3\tau_1$ and $w = 3421$.

4.1.3 The sweeping algorithm

We have defined $\text{ins}_{Q,w}$ but we have no simple way of computing it yet; a natural question is whether the insertion and sweeping algorithms on pipe dreams can be generalized on any Coxeter group for that purpose. While the insertion algorithm does not have any clear equivalent in general Coxeter groups, as there are no single "pipes" in that context, the sweeping algorithm does. We replace the pair of pipes entering a cell by the root function at an index, and this leads to the following algorithm.

Algorithm 4.1.9 (Sweeping algorithm)**.** *For $Q = s_{\alpha_1}s_{\alpha_2}\dots s_{\alpha_m}$ a word on S and $u, w \in W$, the sweeping algorithm computes $\texttt{Sweeping}_{Q,w}(u)$ as follows. We set $w_0 = e$ and $F_0 = \varnothing$, and for $t = 1, \dots, m$:*

1. *if $w_{t-1}(\alpha_t) \in \text{inv}(w)$ and either $w_{t-1}(\alpha_t) \in \text{inv}(u)$ or $w_{t-1}^{-1}w$ is not sortable on $Q_{[t+1,m]}$, then $F_t = F_{t-1}$ and $w_t = w_{t-1}s_{\alpha_t}$;*
2. *otherwise $F_t = F_{t-1} \cup \{t\}$ and $w_t = w_{t-1}$.*

The algorithm returns F_m.

Theorem 4.1.10 ([BCCP22, Thm. 5.28])**.** *For Q a word on S and $u, w \in W$:*

- *if w is sortable on Q then $\texttt{Sweeping}_{Q,w}(u)$ is a facet of $\text{SC}(Q, w)$;*
- *if $u \in \text{lin}(Q, w)$ then $u \in \text{lin}(\texttt{Sweeping}_{Q,w}(u))$.*

We note that this result is slightly stronger than the one we proved for Algorithm 3.2.16, since it guarantees that the sweeping algorithm works for any $u \in \mathrm{lin}(Q, w)$ and not simply for $u \leqslant w$.

Proof. Note that F_t is the restriction of the result returned by the algorithm to $[t]$, and w_t the product of the letters of $Q_{[1,t]}$ not in F_t; therefore, the result is a facet of $\mathrm{SC}(Q, w)$ iff $w_m = w$ and the subword defined by F_m is reduced, i.e. for all $t > 0$ we have $w_{t-1} \leqslant w_t$. For the second part, if $w_{t-1} = w_t$ then it is true, and otherwise necessarily $w_{t-1}(\alpha_t) \in \mathrm{inv}(w) \subseteq \Pi$, so according to Lemma 1.4.5 we know that for any $t > 0$, we have $w_t = w_{t-1}s_{\alpha_t} > w_{t-1}$.

We will now prove by induction that after any step t, the element $v_t := w_t^{-1}w$ is sortable on $Q_{[t+1,m]}$ and $w_t \leqslant w$. For $t = 0$, this is clearly true iff w is sortable on Q, which is our hypothesis. Suppose now for $t > 0$ that this is true for $t-1$, i.e. $w_{t-1} \leqslant w$ and $\mathrm{SC}(Q_{[t,m]}, v_{t-1})$ contains at least one facet F (equivalent to v_{t-1} sortable on that word according to Theorem 1.5.11). Then from Corollary 1.4.15 we know that $\mathrm{inv}(v_{t-1}) = w_{t-1}^{-1}(\mathrm{inv}(w)\backslash \mathrm{inv}(w_{t-1}))$, so:

- if $w_{t-1}(\alpha_t) \notin \mathrm{inv}(w)$ (and so $w_t = w_{t-1}$), then since $w_{t-1}(\mathrm{inv}(v_{t-1})) \subseteq \mathrm{inv}(w)$ we know that $\alpha_t \notin \mathrm{inv}(v_{t-1})$, so from Proposition 1.4.13 we know that $1 \in F$, so F also defines a reduced word of $v_{t-1} = v_t$ on $Q_{[t+1,m]}$ and so v_t is also sortable on $Q_{[t+1,m]}$ and $w_t = w_{t-1} \leqslant w$;

- otherwise, from Lemma 1.4.12 we have $\mathrm{inv}(w_{t-1}s_{\alpha_t}) = \mathrm{inv}(w_{t-1}) \cup w_{t-1}(\alpha_t)$ and that is included in $\mathrm{inv}(w)$, so $w_t \in \{w_{t-1}, w_{t-1}s_{\alpha_t}\}$ is necessarily below w in the weak order. Moreover, since $w_{t-1}(\alpha_t) \in w_{t-1}(\Pi)$, we know that this root is in $\mathrm{inv}(w)\backslash \mathrm{inv}(w_{t-1})$ and so $\alpha_t \in \mathrm{inv}(v_{t-1})$. Then:
  - for F the maximal facet of $\mathrm{SC}(Q_{[t,m]}, v_{t-1})$, since $\mathbf{r}(F, 1) = \alpha_t \in \mathrm{inv}(v_{t-1})$, we know that $1 \notin F$ or according to Proposition 1.5.18 we could do an increasing flip on F on index 1, so F also defines a reduced subword of $s_{\alpha_t}v_{t-1}$ on $\mathrm{SC}(Q_{[t+1,m]})$, and so if $w_t = w_{t-1}s_{\alpha_t}$ then $v_t = s_{\alpha_t}v_{t-1}$ is sortable on $Q_{[t+1,m]}$;
  - if $w_t = w_{t-1}$ then the conditions on the algorithm guarantee that $v_t = v_{t-1}$ is stills sortable on $Q_{[t+1,m]}$.

Thus by induction we obtain that v_m is sortable on the empty word, so $v_m = e$ and $w_m = wv_m^{-1} = w$. This concludes the proof of the first item.

Suppose now that $F \in \mathrm{SC}(Q, w)$ such that $u \in \mathrm{lin}(F)$ exists. We will prove by induction that after step t the set F_t is always the restriction of F to $[t]$. This is clearly true for $t = 0$. Suppose that it is true for $t-1$ with $t > 0$, and note that $w_{t-1}(\alpha_t) = \mathbf{r}(F, t)$. Then:

- if $w_{t-1}(\alpha_t) \notin \mathrm{inv}(w)$ then $t \in F_t$ and from Proposition 1.4.13 we have $t \in F$;

- if $w_{t-1}(\alpha_t) \in \mathrm{inv}(u)$ then $t \notin F_t$ and since $u \in \mathrm{lin}(F)$ we have $t \notin F$;

- if $w_{t-1}(\alpha_t) \in \mathrm{inv}(w) \cap \mathrm{ninv}(u)$, then:
  - if v_{t-1} is not sortable on $Q_{[t+1,m]}$, then $t \notin F_T$ and F does not define a reduced word of $w_{t-1}^{-1}w$ on $Q_{[t+1,m]}$ so $t \notin F$;
  - if v_{t-1} is sortable on $Q_{[t+1,m]}$, then $t \in F_t$. Suppose that $t \notin F$ and let F' be the facet of $\mathrm{SC}(Q_{[t+1,m]}, s_{\alpha_t}v_{t-1})$ defined by the restriction of F to the end of Q. Then from Theorem 1.5.11 we know that $\mathrm{Dem}(Q_{[t+1,m]}) \geqslant_B v_{t-1}$, and $s_{\alpha_t}v_{t-1}$ is covered by v_{t-1} in the Bruhat order (see Remark 1.5.8), so α_t is by definition in the Bruhat cone $\mathcal{C}^+(s_{\alpha_t}v_{t-1}, \mathrm{Dem}(Q_{[t+1,m]}$. We can thus apply Theorem 1.5.34 to obtain that $\alpha_t \in \mathrm{Cone}(\mathbf{R}(F'))$. However, since F' correspond to a suffix of F representing $s_{\alpha_t}v_t$, with the associated prefix of F representing $w_{t-1}s_{\alpha_t}$, we know that $w_{t-1}s_{\alpha_t}(\mathbf{R}(F')) \subseteq \mathbf{R}(F)$ and so $w_{t-1}s_{\alpha_t}(\alpha_t) = -w_{t-1}(\alpha_t) \in \mathrm{Cone}(\mathbf{R}(F))$. Since by hypothesis $-w_{t-1}(\alpha_t) \in u(-\Pi)$, this contradicts $u \in \mathrm{lin}(F)$, so $t \in F$.

Thus F_t is the restriction of F to $[t]$, and by induction this is true for all t. In particular, the result of the algorithm is $F_m = F$. □

4.2 Linear extensions and lattice properties

As we said in Section 4.1.2, not all subword complex congruences are lattice congruences. Our study of Coxeter group in type A proved that they were in the case where Q is alternating (see Theorem 3.1.17 and Lemma 3.1.16). We will start from this idea to propose a few conjectures on the subword complex equivalences and insertion maps.

4.2.1 Five conjectures on alternating words

We start with this conjecture, which is a direct generalization of our work in type A in Chapter 3.

Conjecture 4.2.1 ([BCCP22, Conj. 5.20])**.** *For Q an alternating word on S and w sortable on Q, the subword complex equivalence $\equiv_{Q,w}$ is a lattice congruence of the weak order interval $[e, w]$.*

This conjecture was tested extensively with computer experiments: we verified it for all alternating words of length at most $\ell(w_0)$ in the Coxeter groups B_2, B_3, D_4 and H_3.

We also noted in Fig. 3.10 that the image of the weak order by $\mathrm{ins}_{Q,w}$ was not always the increasing flip graph on acyclic facets, even for alternating words. In pipe dreams, a flip on a pair of pipes (p, q) between acyclic facets can only be in the image of the weak order if the only path from p to q in the contact graph is the edge $p \to q$. This condition on the pair (p, q) can be translated as follows in a condition on the root $e_q - e_p$.

Definition 4.2.2. *A flip between two facets F and F' of some subword complex, with $i \in F$ and $i' \in F'$ the indices flipped, is* ***extremal*** *if the root $\mathbf{r}(F, i)$ is a ray of the cone generated by $\mathbf{R}(F)$ (or equivalently $\mathbf{r}(F', i')$ is a ray of the cone generated by $\mathbf{R}(F')$).*

This leads to the following conjecture on the image of the weak order by the insertion map.

Conjecture 4.2.3 ([BCCP22, Conj. 5.22])**.** *For Q an alternating word on S and w sortable on Q, the insertion map $\mathrm{ins}_{Q,w}$ is a lattice isomorphism from $[e, w]/\equiv_{Q,w}$ to the graph of extremal increasing flips between strongly acyclic facets of $\mathrm{SC}(Q, w)$.*

In the case where Q is both alternating and sorts w_0, we can use Theorem 4.1.7 to specialize Conjecture 4.2.3:

Conjecture 4.2.4 ([BCCP22, Conj. 5.23])**.** *For Q an alternating word on S sorting w_0 and $w \in W$, the insertion map $\mathrm{ins}_{Q,w}$ is a lattice morphism from $[e, w]/\equiv_{Q,w}$ to the graph of extremal increasing flips between acyclic facets of $\mathrm{SC}(Q, w)$.*

Since the cone generated by $\mathbf{R}(F)$ is also the incidence cone of the brick polyhedron $\mathcal{B}(Q, w)$ at the brick vector of F, if F is acyclic, the rays of this cone correspond to the edges of the polyhedron incident to $\mathrm{B}(F)$. This leads to

Conjecture 4.2.5 ([BCCP22, Conj. 5.24])**.** *For Q an alternating word on S sorting w_0 and $w \in W$, the bounded oriented graph of the brick polytope $\mathcal{B}(Q, w)$ is isomorphic to the Hasse diagram of the lattice quotient $[e, w]/\equiv_{Q,w}$.*

Finally, since the brick polyhedron $\mathcal{B}(Q, w)$ is a polytope when $w = \mathrm{Dem}(Q)$, we can specialize 4.2.5 to the case $w = w_0$ (and in that case the interval $[e, w]$ is the whole Coxeter group W).

Conjecture 4.2.6 ([BCCP22, Conj. 5.25])**.** *For Q an alternating word on S sorting w_0, the oriented graph of the brick polytope $\mathcal{B}(Q, w_0)$ is isomorphic to the Hasse diagram of a quotient of the weak order.*

4.2.2 Sketch of some possible proofs

In Chapter 3, the most important lemma to prove our results on pipe dream congruences and insertion maps was Lemma 3.1.22. This lemma can be reformulated in the vocabulary of Coxeter groups as follows:

Lemma 4.2.7 (Conjectured)**.** *Let Q be an alternating word sorting $w \in W$ and F a facet of $\mathrm{SC}(Q, F)$. For $\alpha, \gamma \in \mathrm{inv}(w)$, if $\beta \in \mathbf{R}(F)$ is in the interior of $\pm\,\mathrm{Cone}(\alpha, \gamma)$, then either $\alpha, -\gamma \in \mathrm{Cone}(\mathbf{R}(F))$ or $-\alpha, \gamma \in \mathrm{Cone}(\mathbf{R}(F))$.*

Suppose that this lemma is true, we can then prove Conjecture 4.2.1 using the properties of polygonal lattices.

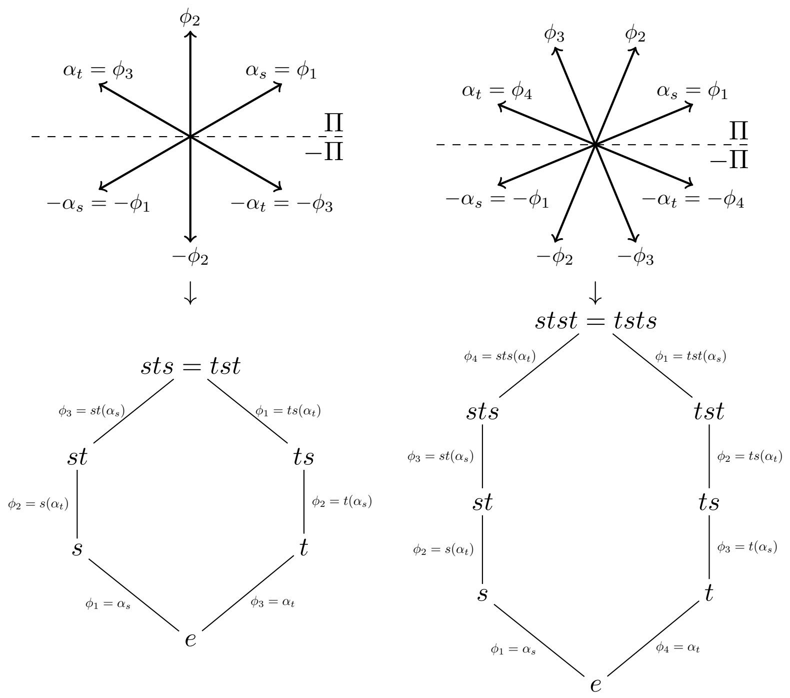


Figure 4.2: The root system Φ' and the weak order on W' for $m_{st} = 3$ and 4.

Proof of Conjecture 4.2.1. Consider two simple transpositions $s = s_{\alpha_s}$ and $t = s_{\alpha_t}$ of W. We define $\Delta' = \{\alpha_s, \alpha_t\}$, and $\Phi' = \Phi \cap \mathrm{Vect}(\Delta')$ and $\Pi' = \Pi \cap \mathrm{Vect}(\Delta')$. Then Φ' is clearly a root system with Π' a positive system and Δ' the associated root system, and so W' the reflection group generated by s and t is a two-dimensional Coxeter group with Φ' as a root system. From Theorem 1.3.23, this means that either α_s and α_t are perpendicular (and $\Phi' = \{\pm\alpha_s, \pm\alpha_t\}$) or W' is the dihedral group $I_2(k)$ with $k = m_{st}$ the order of st.

Then there exists a labeling $\alpha_s = \phi_1, \phi_2, \ldots, \phi_k = \alpha_t$ of the roots in Π' such that for each $1 \leqslant i < k$, the angle between ϕ_i and ϕ_{i+1} is $\frac{\pi}{n}$. Such labelings are shown on the top row of Fig. 4.2 for $k = 3$ and $k = 4$. We can then see that the weak order on W' is made of two chains of length k from e to $s \wedge t$, one starting with s and the other starting with t, and note that the inversion set of an element u is $\{\phi_1, \ldots, \phi_{\ell(u)}\}$ if u is on the first chain and $\{\phi_{k+1-\ell(u)}, \ldots, \phi_k\}$ if u is on the second chain. We represented this order on the bottom row of Fig. 4.2 with each cover $u \lessdot u'$ labeled with the only positive root in $\mathrm{inv}(u') \setminus \mathrm{inv}(u)$.

Suppose that Lemma 4.2.7 is true and consider a polygone $[u, v]$ of the weak order with $v \leqslant w$. We denote by s, t the two simple transposition such that $u \lessdot us, ut \lessdot v$. We know that $u = us \wedge ut$ and $v = us \vee ut$, and that $\ell(v) - \ell(u)$ is the order of st, or the element m_{st} in the Coxeter matrix of W. The polygon is then the image by u of the two-dimensional Coxeter group W' generated by s and t as described previously, with $v = u(s \wedge t)$. The bottom edges of the polygon are $u \lessdot us$ and $u \lessdot ut$, and the top edges are $vs \lessdot v$ and $vt \lessdot v$ (with us in the same chain as vs if m_{st} is odd and in the opposite chain if m_{st} is even). We note that since by hypothesis $us, ut > u$, the roots in $u(\Pi')$ are all positive and the roots in $u(-\Pi')$ are all negative. Then:

- if $u \equiv_{Q,w} us$, then from Lemma 4.1.4 we know that $u(\alpha_s)$ is not in the root configuration of $F = \mathrm{ins}_{Q,w}(u)$; moreover, since $u \in \mathrm{lin}(F)$ and $u(\alpha_s) \in u(\Pi)$, we also know that $-u(\alpha_s)$ is not in that root configuration either. Then for $1 < i < k$, since $u(\phi_i)$ is in the interior of $\mathrm{Cone}(u(\alpha_s), u(\alpha_t))$ with $u(\alpha_s)$ and $u(\alpha_t)$ positive, we know from Lemma 4.2.7 that $u(\phi_i) \notin \mathbf{R}(F)$ (or either $u(\alpha_s)$ or $-u(\alpha_s)$ would be in the root configuration). This means that for any element $us \leqslant x < v$ we have $x \equiv_{Q,w} u$, and so each side edge $x \lessdot x'$ of the polygon on the chain containing us satisfies $x \equiv_{Q,w} x'$. Consider now $F' = \mathrm{ins}_{Q,w}(ut)$; we know from Lemma 4.1.4 that F' is either F or obtained from F by flipping some index i such that $\mathbf{r}(F, i) = u(\alpha_t)$. As such, for $\phi \in \mathbf{R}(F')$, either $\phi \in \mathbf{R}(F)$ or $s_{u(\alpha_t)}(\phi) \in \mathbf{R}(F)$. For $1 \leqslant i < k$, we have $s_{u(\alpha_t)}(u(\phi_i)) = utu^{-1}(u(\phi_i)) = ut(\phi_i)$, and since we know from Proposition 1.3.11 that $t(\Delta' \backslash \{\alpha_t\}) = \Delta' \backslash \{\alpha_t\}$ this means that there exists $1 \leqslant j < k$ such that $s_{u(\alpha_t)}(u(\phi_i)) = u(\phi_j)$. Therefore, since no $u(\phi_j)$ with $1 \leqslant j < k$ is in $\mathbf{R}(F)$, this means that no $u(\phi_i)$ with $1 \leqslant i < k$ is in $\mathbf{R}(F')$. Thus, for any element $ut \leqslant x \leqslant v$ we have $x \equiv_{Q,w} ut$ and so all the side or top edges $x \lessdot x'$ of the polygon on the chain containing ut satisfy $x \equiv_{Q,w} x'$.

- if $v \equiv_{Q,w} vs$, then by a similar reasoning we know that neither $v(\alpha_s)$ nor $-v(\alpha_s)$ are in the root configuration of $F = \mathrm{ins}_{Q,w}(v)$. Since $v(\alpha_s)$ is either $-u(\alpha_s)$ (if k is even) or $-u(\alpha_t)$ (if k is odd), from Lemma 4.2.7 we deduce that for any $1 < i < k$ the root $-\phi_i$ is not in $\mathbf{R}(F)$. Similarly, since $F' = \mathrm{ins}_{Q,w}(vt)$ is either F or obtained from F by flipping an index i such that $\mathbf{r}(F, i) = v(\alpha_t)$ with $v(\alpha_t)$ either $-u(\alpha_s)$ (if k is odd) or $-u(\alpha_t)$ (if k is even), we can prove that none of the other $-\phi_i$ appear in $\mathbf{R}(F')$. This proves that for $x \lessdot x'$ any side edge of the polygon, or the bottom edge opposite to $vs \lessdot v$, we have $x \equiv_{Q,w} x'$.

These implications allow us to apply Theorem 1.1.22: since we know from Theorem 1.4.23 that the weak order lattice on W is polygonal, and so that the weak order on $[e, w]$ is also polygonal, and the equivalence relation $\equiv_{Q,w}$ satisfies the conditions described in Proposition 1.1.21, then this equivalence relation is a lattice congruence. $\square$

To study the nature of the flips in the image of the weak order, we need the following lemma that is a consequence of Lemma 4.2.7.

Lemma 4.2.8. *Let Q be an alternating word on S and F, F' two strongly acyclic facets of the subword complex $\mathrm{SC}(Q, w)$ linked by a flip on indices $i < i'$. Suppose that for some cover $u \lessdot us$ of the weak order, we have $u \in \mathrm{lin}(F)$ and $us \in \mathrm{lin}(F')$. Then for any $j \in F$ such that $i < j < i'$, either $\mathbf{r}(F, j) = \mathbf{r}(F, i)$ or $(\mathbf{r}(F, j), \mathbf{r}(F, i)) \leqslant 0$.*

Proof. For $1 \leqslant t \leqslant |Q|+1$, denote by $w_t = \prod_{j<t, j\notin F} q_j$ the prefix of w defined by F on Q before index t. We have $e = w_1 \leqslant w_1 \leqslant \ldots \leqslant w_{|Q|+1} = w$. We can suppose that $i = 1$ by "cutting" the beginning of Q and multiplying everything (the values of the root function and w and u) by w_i^{-1}. We thus suppose that $\mathbf{r}(F,i) = u(\alpha_s)$ is a simple root that we denote by α. Since there exists a flip on index i of F we know that $\alpha \in \mathrm{inv}(w)$. Suppose that for some $j \in F$ between i and i' we have $(\mathbf{r}(F,j), \alpha) > 0$ with $\beta = \mathbf{r}(F,j) \neq \alpha$. The set $\Phi' = \Phi \cap \mathrm{Vect}(\alpha,\beta)$ is a two-dimensional root system, with $\Pi' = \Pi \cap \mathrm{Vect}(\alpha,\beta)$ a positive system and the associated simple sytem is $\{\alpha,\gamma\}$ for some $\gamma \neq \beta$. As we discussed in the proof of Conjecture 4.2.1, this means that Π' can be enumerated as $\alpha = \phi_1, \phi_2, \ldots, \phi_k = \gamma$ such that the angle between two consecutive roots is always $\frac{\pi}{k}$. For $0 \leqslant \ell \leqslant k$, denote by $A_\ell = \{\phi_1 \ldots, \phi_\ell\}$ and $B_\ell = \{\phi_{\ell+1}, \ldots, \phi_k\}$; then the positive systems of Φ' are all the $A_\ell \cup -B_\ell$ and $-A_\ell \cup B_\ell$; therefore for any $i < t < i'$ we know that $w_t(\Pi) \cap \Phi'$ is one of these sets. Moreover, since $\mathbf{r}(F,i') = \alpha$ we know that α is in $w_{i'}(\Pi)$; since $\mathrm{ninv}(w_t) = w_t(\Pi) \cap \Pi$ decreases for the inclusion relation when t increases (because w_t is a prefix of $w_{t'}$ for $t < t'$, and so $w_t \leqslant w_{t'}$ in the weak order), this means that $\alpha \in w_t(\Pi)$ for all $t \leqslant i'$, and so $w_t(\Pi) \cap \Phi'$ is $A_\ell \cap -B_\ell$ for some ℓ if $t \leqslant i'$. Lastly, since $\alpha \in w_{i'}(\Delta)$, we know that α is a ray of the cone generated by $w_{i'}(\Pi)$ and so $w_{i'}(\Pi)$ is either $A_k = \Pi'$ or $A_1 \cup -B_1$.

- If $w_{i'} = \Pi$ then $\Pi \subseteq \mathrm{ninv}(w_{i'}) \subseteq \mathrm{ninv}(w_j)$; since $\beta \in w_j(\Delta)$, it must direct a ray of the cone generated by $w_j(\Pi)$ so $\beta = \alpha$ or $\beta = \gamma$. Since we choose $\beta \neq \alpha$ and by Corollary 1.3.9 we know that $(\alpha,\gamma) \leqslant 0$, this is not possible.

- If $w_{i'} = A_1 \cup -B_1$ then $B_1 \in \mathrm{inv}(w_{i'}) \subseteq \mathrm{inv}(w)$. Denote by $\lambda = \frac{2(\alpha,\beta)}{(\alpha,\alpha)} > 0$, such that $s_\alpha(\beta) = \beta - \lambda\alpha$.
  - If $\beta \in \Pi'$, then by Proposition 1.3.11 we have $s_\alpha(\beta) \in \Pi' \backslash \{\alpha\} = B_1$ so $\beta, s_\alpha(\beta) \in \mathrm{inv}(w)$. Since $\beta = s_\alpha(\beta) + \lambda\alpha$, we can apply Lemma 4.2.7 to obtain that either $-\alpha$ or $-s_\alpha(\beta) \in \mathrm{Cone}(\mathbf{R}(F))$; in the second case since $-\lambda\alpha = \beta - s_\alpha(\beta)$ we still have $-\alpha \in \mathrm{Cone}(\mathbf{R}(F))$. Since by hypothesis $\alpha \in \mathrm{ninv}(u)$ we know that $-\alpha \notin u(\Pi)$ and so this contradicts $u \in \mathrm{lin}(F)$.
  - If $\beta \in -\Pi'$ then by Proposition 1.3.11 we have $s_\alpha(\beta) \in -\Pi' \backslash \{-\alpha\} = -B_1$ so $-\beta, -s_{\alpha(\beta)} \in \mathrm{inv}(w)$. We also know that $\mathbf{r}(F',j) = s_\alpha(\mathbf{r}(F,j)) = s_\alpha(\beta)$ and $-s_\alpha(\beta) = -\beta + \lambda\alpha$, so we can apply Lemma 4.2.7 to obtain that either α or $-\beta \in \mathrm{Cone}(\mathbf{R}(F'))$. Since $us(\Pi) = u(\Pi) \cup \{-\alpha\} \backslash \{\alpha\}$ and by hypothesis $\beta \in u(\Pi)$, the first option contradict $us \in \mathrm{lin}(F')$ and the second contradicts $u \in \mathrm{lin}(F)$.

We thus get contradictions no matter what, and so such an index j cannot exist. $\square$

We can now prove the second conjecture.

Proof of Conjecture 4.2.3. Note first that if $u \lessdot us_\alpha$ is a cover of the weak order, then $u(\alpha)$ directs a ray of $\mathrm{Cone}(u(\Pi))$ and so any cone $C \subseteq \mathrm{Cone}(u(\Pi))$ containing $u(\alpha)$ has a ray also directed by $u(\alpha)$; therefore, any flip that is the image by $\mathrm{ins}_{Q,w}$ of a weak order cover is extremal.

Consider now a decreasing flip from F to F' two strongly acyclic facets of the subword complex $\mathrm{SC}(Q,w)$ with $\alpha \in -\Pi$ the value of the root function on the flipped index of F. Suppose that α directs a ray of the cone C generated by $\mathbf{R}(F)$. By definition of a ray of a cone, there is a generic hyperplane H such that $H \cap C = \mathbb{R}_{\geqslant 0}\alpha$; since that hyperplane is generic we can choose it such that $H \cap \Phi = \{\pm\alpha\}$. Then by perturbating it a little we can get H' such that $H' \cap \Phi = \varnothing$ and $\mathbf{R}(F) \in H'^{+}$, and for any two roots $\beta, \gamma \in \Phi$ such that $\alpha = \lambda\beta + \mu\gamma$ with $\lambda, \mu > 0$, either β or γ is in H'^{-}. Then $H'^{+} \cap \Phi$ is a positive system and so by Theorem 1.3.10 there exists $u \in W$ such that $H'^{+} \cap \Phi = u(\Pi)$, and necessarily $\alpha \in u(\Delta)$, so there exists $s_\beta \in S$ such that $u(\beta) = \alpha$. Since $\mathbf{R}(F) \subseteq u(\Pi)$ we know that u is a linear extension of F; then by Lemma 4.1.4 we know that us_β is a linear extension of an acyclic facet obtained by a flip on F on the root α. There is only one such facet, and it is F', so $us_\beta \in \mathrm{lin}(F')$.

We will now prove that we can choose u such that $w \geqslant u$. It is clear that we can do so if α is a ray of the cone C' generated by $\mathbf{R}(F) \cup \mathrm{ninv}(w)$. Suppose that it is not, then $\alpha = \sum_{\phi \in \mathrm{ninv}(w)} \lambda_\phi(\phi) + \sum_{j \in F, \mathbf{r}(F,j) \neq \alpha} \mu_j\, \mathbf{r}(F,j)$ with all $\lambda_\phi, \mu_j \geqslant 0$. Then $\sum_{\phi \in \mathrm{ninv}(w)} \lambda_\phi(\phi) + \sum_{j \in F, \mathbf{r}(F,j) \neq \alpha} \mu_j\, \mathbf{r}(F',j)$ is by definition in the cone generated by $\mathbf{R}(F') \cup \mathrm{ninv}(w)$, which is pointed since F' is supposed strongly acyclic. However, since $\mathbf{r}(F',j) = \mathbf{r}(F,j)$ if $j < i'$ or $j > i$, and $\mathbf{r}(F',j) = \mathbf{r}(F,j) - \frac{2(\alpha, \mathbf{r}(F,j))}{(\alpha,\alpha)}\alpha$, by Lemma 4.2.8 we get that $\sum_{j \in F, \mathbf{r}(F,j) \neq \alpha} \mu_j\, \mathbf{r}(F',j) = \sum_{j \in F, \mathbf{r}(F,j) \neq \alpha} \mu_j\, \mathbf{r}(F,j) + \mu\alpha$ for some $\mu \geqslant 0$. Therefore $(1+\mu)\alpha \in \mathrm{Cone}(\mathbf{R}(F') \cup \mathrm{ninv}(w))$, and since $\mathbf{r}(F',i') = -\alpha$ this means that $\mathbb{R}\alpha$ is contained in this cone, which is thus not pointed. This is a contradiction, so α must be a ray of $\mathrm{Cone}(\mathbf{R}(F) \cup \mathrm{ninv}(w))$ and so we could choose $u \leqslant w$, thus the flip from F' to F is the image by $\mathrm{ins}_{Q,w}$ of a weak order cover $us_\beta \lessdot u$ inside of the weak order interval $[e, w]$.

Lastly, we have left to check that the covers of the image of the weak order by $\mathrm{ins}_{Q,w}$ are exactly the images of covers of the weak order. This is done in the exact same way as in the proof of Proposition 3.2.12. □

Conjecture 4.2.4 is an immediate consequence of this proof, and Conjecture 4.2.5 and Conjecture 4.2.6 are obtained simply by noting that the extremal flips between acyclic facets correspond exactly to the edges of the brick polyhedron.

4.2.3 Some special cases

Since the classification discussed in Section 1.3.4 shows that there are few infinite families of finite Coxeter groups, and that we already proved the conjectures in one of those families, it is natural to wonder if some other families may be similarily treated.

Type A Coxeter groups

We discussed extensively the case of type A alternating words in 3, and proved every conjecture by reasoning on pipes.

Dihedral groups

In the dihedral groups $I_2(n)$, alternating words are all of the form $ststst\dots$ with s and t the two simple reflections. Moreover, the structure of root systems is very rigid, as discussed in the conditional proofs of Conjecture 4.2.1 and Conjecture 4.2.3.

Proof of Lemma 4.2.7. Consider $Q = ststs\dots$ an alternating word on S and denote by $\phi_1, \phi_2, \dots, \phi_n$ the positive roots of $I_2(n)$ with an angle of $\frac{\pi}{n}$ between two consecutive roots, with $s = s_{\phi_1}$ and $t = s_{\phi_n}$. Note that $\phi_k = \lambda\phi_j + \mu\phi_\ell$ with $\lambda, \mu > 0$ if and only if either $j < k < \ell$ or $\ell < k < j$. For F a facet of some subword complex $\mathrm{SC}(Q, w)$, we suppose that for some $j < k < \ell$ the roots ϕ_j, ϕ_ℓ are inversions of w and that for some index $i \in F$ we have $\mathbf{r}(F, i) = \pm\phi_k$. Since $1 \leqslant j < k$ we know that $i > 1$, as $\mathbf{r}(F, i) = \phi_1 \neq \phi_k$. Similarly, if $i = |Q|$ then $\phi_j \in w(\Delta)$ or $-\phi_j \in w(\Delta)$. Since $\phi_i, \phi_k \in \mathrm{inv}(w) \subseteq w(-\Pi)$, we know that neither ϕ_j nor $-\phi_j$ can direct a ray of the cone generated by $w(\Pi)$, so $i < |Q|$. Then both $i - 1$ and $i + 1$ are indices of Q, and since Q is alternating we know that $q_{i-1} = q_{i+1} \neq q_i$; since $i \in F$ we also have $\mathbf{r}(F, i - 1) = \pm\,\mathbf{r}(F, i + 1)$ (the sign is $+$ if $i - 1 \in F$ and $-$ otherwise). Therefore, since the complement of F is a reduced subword of Q, we know that $i-1$ or $i + 1$ is in F. We denote by j one of these two index that is in F.

Denote now by w_i the prefix of w defined by F on Q before index i. It is also the prefix defined on Q before index j, so $\{\mathbf{r}(F, i), \mathbf{r}(F, j)\} = \{w_i(\phi_1), w_i(\phi_n)\}$, and $w_i(\Pi) \subseteq \mathrm{Cone}(\mathbf{R}(F))$. Since $w_i(\Pi)$ is a positive system of Φ, any root of Φ is either in $w_i(\Pi)$ or in $-w_i(\Pi)$. Since $\phi_k \in w_i(\Pi)$ we know that ϕ_j or ϕ_ℓ must be in $w_i(\Pi)$; since ϕ_k directs a ray of the cone generated by $w_i(\Pi)$ we also know that they cannot both be in $w_i(\Pi)$ so $-\phi_j$ or $-\phi_\ell$ must also be in $w_i(\Pi)$. Therefore, we obtain that ϕ_j and $-\phi_\ell$ are in $w_i(\Pi)$ or $-\phi_j$ and ϕ_ℓ are in $w_i(\Pi)$. Since $w_i(\Pi)$ is contained by $\mathrm{Cone}(\mathbf{R}(F))$, this concludes the proof. $\square$

Type B Coxeter groups

As noted in Section 1.3.4, the Coxeter group B_n is isomorphic to signed permutations of length n. A signed permutation can be seen as a permutation of $[n]$ with a plus or minus sign in front of each number (or, as represented in Fig. 1.12, an overline on some of the numbers), or as an antisymmetric permutation of $-[n] \cup [n]$: for example, the signed permutation $2\underline{1}4\underline{3}$ is also the antisymmetric permutation $3\underline{4}1\underline{2}2\underline{1}4\underline{3}$. In that case, the simple reflections of this Coxeter group are the simple transpositions $\tau_i = (i, i + 1)$ (for $1 \leqslant i < n$) and $\tau_0 = (1, \underline{1})$.

With this idea, a sorting network representation exists for type B. For Q a word on the simple reflections of B_n, we draw $2n$ horizontal line representing the

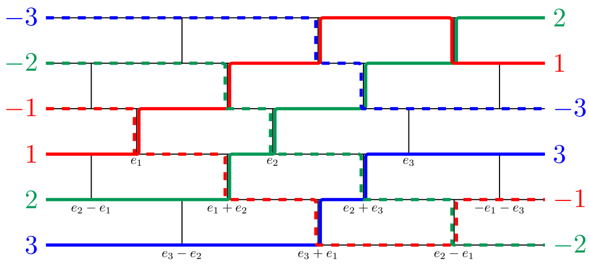


Figure 4.3: The sorting network representation of facet $\{1, 2, 9, 10\}$ of the subword complex $\mathrm{SC}(\tau_1\tau_0\tau_2\tau_1\tau_0\tau_2\tau_1\tau_0\tau_2\tau_1, 3\underline{1}\underline{2})$.

positions $\underline{n}$ to $\underline{1}$ then 1 to n, from top to bottom. Then, for each letter τ_0 of Q, a vertical line is drawn between the horizontal lines $\underline{1}$ and 1; and for each letter τ_i with $i > 0$, two vertical lines are drawn: one between the horizontal lines i and $i+1$ and one between the horizontal lines $\underline{i}$ and $\underline{i+1}$. The subwords of Q are then represented by $2n$ pipes labeled by their starting position, that cross the letters of Q in the subword and ignore the letters not in the subword. An example is given in Fig. 4.3 with the positive pipes drawn continuously and the negative ones drawn dashed; the exit signed permutation is $3\underline{1}\underline{2}$.

Like in the case of type A Coxeter groups, the value of the root function at each index of Q can be read by considering the pair(s) of pipes adjacent to the vertical line(s) corresponding to that index. Note that since the figure is completely antisymmetric, either the letter is τ_0 and the pair adjacent to the only vertical line is of the form $(\underline{i}, i)$ (with i possibly negative), or the letter is some other simple reflection and the two pairs adjacent to the two vertical lines are of the form (i, j) and $(\underline{i}, \underline{j})$. In the first case, the associated value of the root function is e_i (with the convention $e_i = -e_{\underline{i}}$ if $i < 0$); in the second case, the associated root is $e_j - e_i$. As an example, the values of the root function are given below each letter of Q in Fig. 4.3. The root configuration of a facet F can then be represented as its contact graph $F^{\#}$ on $[-n] \cup [n]$ with an edge $i \to j$ if $e_j - e_i \in \mathbf{R}(F)$ for $i \neq j$ (note that $e_{\underline{i}} - e_{\underline{j}} = e_j - e_i$, so there is an edge $i \to j$ iff there is an edge $\underline{j} \to \underline{i}$) and an edge $\underline{i} \to i$ iff $e_i \in \mathbf{R}(F)$; a study of the paths, sources and sinks of $F^{\#}$ proves that the root $e_j - e_i$ is in $\mathrm{Cone}(\mathbf{R}(F))$ iff there is a directed path from i to j (and equivalently from $\underline{j}$ to $\underline{i}$) in $F^{\#}$, and the root e_i is in the same cone iff there is a path from $\underline{i}$ to i in that same graph.

For Q an alternating word, we can find the word Q' equivalent to Q up to commutativity of letters minimal for the lexicographic order, and since the commutation relations in type B are the same as in type A, everything we said in Section 3.1.3 about the form of Q' still applies (with τ_0 replacing τ_1). We can thus divide Q' into a product $Q_0Q_1 \ldots Q_M$ with the Q_i products of decreasing consecutive simple reflections. Then with the notations of Lemma 3.1.16, the following shape can be

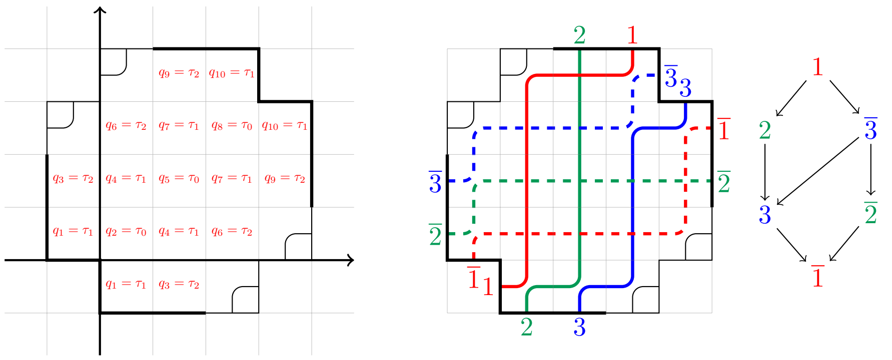


Figure 4.4: The symmetric alternating shape and pipe dream associated to the word and facet represented in Fig. 4.3.

filled with crosses $+$ and contacts $\asymp$ to create type B pipe dreams representing the subwords of Q':

$$\begin{aligned} F = &\{(i, i-j) \mid 0 \leqslant i \leqslant M \text{ and } \tau_j \in Q_i\} \\ &\cup \{(i-j, i) \mid 0 \leqslant i \leqslant M \text{ and } \tau_j \in Q_i\} \\ &\cup \{(i-n, i) \mid B < i \leqslant M\} \\ &\cup \{(i, i-n) \mid B < i \leqslant M\} \end{aligned}$$

The antisymmetric alternating shape and type B pipe dream associated to the word and facet of Fig. 4.3 are given in Fig. 4.4. The contact graph of the facet is also represented; it is acyclic and has $12\underline{3}3\underline{2}\underline{1}$ and $1\underline{3}2\underline{2}3\underline{1}$ as linear extensions.

This shape is actually an alternating shape with $2n$ pipes translated by the vector $(B-n, B)$, and the type B pipe dreams are translated of symmetric type A pipe dreams. For $1 \leqslant i \leqslant n$, the pipe i of the type B_n pipe dream is a translated of the pipe $n+i$ of the type A_{2n-1} pipe dream, and the pipe $\underline{i}$ in type B_n is a translated of the pipe $n+1-i$ in type A_{2n-1}. We can thus use our remarks on the link between the root configuration and the contact graph in type B, and everything we know about type A pipe dreams from Chapter 3, to prove that Lemma 4.2.7 is true in type B.

4.3 Link to other objects

The following two subsections were inspired by discussions with Cesar Ceballos and other researchers: Martin Rubey for Section 4.3.1 and Eva Philippe and Daniel Tamayo for Section 4.3.2.

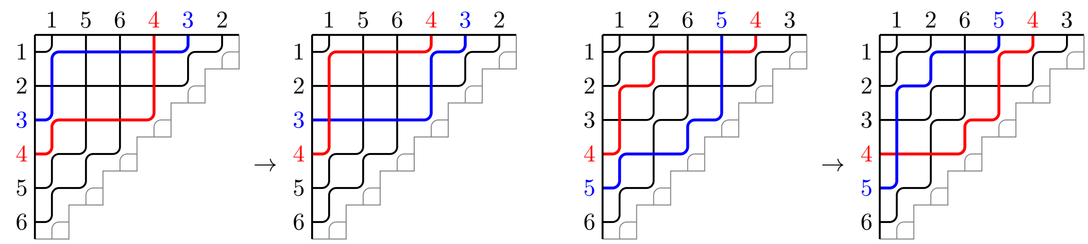


Figure 4.5: An example of a chute move and a flip that is not a chute move.

4.3.1 Chute moves

Chute moves were introduced in a restricted form by N. Bergeron and S. Billey in [BB93], and later their definition was extended by M. Rubey in [Rub12]. They are a subfamily of the increasing flips on triangular pipe dreams.

Definition 4.3.1. *An increasing flip between cells c and x of a triangular pipe dream P is a* ***chute move*** *if the cells between c and x all contain crosses* $+$*, except for the other two corners of the rectangle which contain elbows.*

An example of a chute move (on the left) and of a flip that is not a chute move (on the right) are given in Fig. 4.5. In general, an increasing flip on pipes $p < q$ is a chute move if and only if the trajectories of p and q between the two flipped cells only has one elbow (southeast for p and northwest for q). N. Bergeron and S. Billey proved in [BB93] that the graph defined by chute moves on $\Pi(\omega)$ was connected for any $\omega \in \mathfrak{S}_n$. Moreover, M. Rubey conjectured in [Rub12, Conj. 2.8] that the poset defined by chute moves on $\Pi(\omega)$ was generally a lattice.

While this poset is very different from the one we studied, since it is not even defined on the same set, there still seems to be a link: an increasing flip in the image of the weak order by ins_ω is always between two facets linked by a sequence of increasing chute moves on the same two pipes. Consider a flip on an acyclic pipe dream P: if the contact being flipped is p q, then any contact with pipe p at the northwest, or pipe q at the southeast, between the two cells being flipped is also between p and q. Otherwise some pipe r has an elbow between pipes p and q and between the two cells being flipped, so by Lemma 2.1.13 we have $p \triangleleft_P r \triangleleft_P q$, and so there is a path from p to q that is not an edge in $P^{\#}$; by Lemma 2.2.19 this means that the flip is not between two acyclic pipe dreams. This is also true on general, non-triangular pipe dreams where the definition of chute moves can be extended: a flip like the one in Fig. 3.10 is not equivalent to a sequence of increasing chute moves, and it is not in the image of the weak order by $\mathrm{ins}_{Q,\omega}$.

This condition on flips in the image of the weak order is actually the specification in type A of Lemma 4.2.8, since two roots $\phi = e_j - e_i$ and $\phi' = e_{j'} - e_{i'}$ of the Coxeter group A_n satisfy the condition $(\phi, \phi') > 0$ if and only if $i = i'$ or $j = j'$. This leads us to this possible definition of generalized chute moves in finite Coxeter groups.

Definition 4.3.2. *An increasing flip between two facets F and F' of a subword complex* $\mathrm{SC}(Q, w)$ *with Q alternating is a* ***chute move*** *if for $i \in F$ and $i' \in F'$ the indices flipped, if $i < j < i'$ then* $(\mathbf{r}(F, i), \mathbf{r}(F, j)) \leqslant 0$.

This generalization corresponds to other definitions considered for type B Coxeter groups. In type A as well as for generalized Coxeter groups, we then know that if Lemma 4.2.7 is true, the image of the weak order by $\mathrm{ins}_{Q,w}$ is contained in the order defined by chute move on facets, but it is not clear if it is equal to the restriction of this chute move order to strongly acyclic facets. A counter-example in the style of Fig. 2.11 might exist: there could be a sequence of chute moves from F to F' two strongly acyclic facets going through non-acyclic facets, but with F not below F' in the image of the weak order.

This leave us with the following questions on these generalized chute moves. Is the image of the weak order by $\mathrm{ins}_{Q,w}$ the restriction of the chute move order to strongly acyclic facets? And is the chute move order on a subword complex defined on an alternating word always a lattice?

We did some preliminary computer experiments on the subject and found no counter-example to either of those questions.

4.3.2 A generalization of υ-Tamari lattices

As discussed in Section 2.3.2, the ν-Tamari lattices are exactly isomorphic to the increasing flip order on pipe dream sets $\Pi(0\omega_\nu)$ such that all the pipe dreams are acyclic. We noted in Remark 3.1.20 that this is equivalent to the facets of a subword complex $\mathrm{SC}(c\omega_0(c), \omega)$ such that all facets are acyclic, with c a specific Coxeter element of the Coxeter group A_n.

We propose to define the ν-Tamari lattices in a Coxeter group W as the increasing flip order on the facets of the subword complexes $\mathrm{SC}(cw_0(c), w)$ such that all the facets are acyclic. By Conjecture 4.2.5, this would immediately give a polyhedron realizing this lattice; moreover, projecting this polyhedron along its infinite edges could give a standard triangulation of a polytope. This would extend ideas developped in [GDMP$^+$23], where the authors realize the s-permutahedron as a standard triangulation of a flow polytope.

Note that we only studied this set for $c = \tau_{n-1}\tau_{n-2}\ldots\tau_1$ a specific Coxeter element in type A. In general, the alternating shape associated to the word $c\omega_0(c)$ is one on which the Cambrian twists defined in [Pil18b] are drawn. While for $\omega = \omega_0$ we saw in Section 3.2.1 that we still only obtain acyclic facets, it is not clear whether this generalizes to any dominant permutation. Studying this would be an interesting starting point for this proposed generalization.

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