ECSE 446/546

IMAGE SYNTHESIS

Monte Carlo Integration I

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(slides in part by W. Jarosz)

McGill

image credit: feelgrafix

Directly Sampling a Sphere

- Uniformly sample (directions on the surface of) a unit sphere
  - pick two uniform random variables ξ_1, ξ_2
  - select point at (θ, ϕ) with $\theta = \pi\xi_1$ and $\phi = 2\pi\xi_2$
  - problem: not uniform with respect to area!
  - correct solution: $\theta = \cos^{-1}(2\xi_1 - 1)$ and $\phi = 2\pi\xi_2$

Algorithm

$$\begin{aligned}
\theta &= \cos^{-1}(2\xi_1 - 1) \\
\phi &= 2\pi\xi_2 \\
\vec{\omega}_x &= \sin\theta\cos\phi \\
\vec{\omega}_y &= \sin\theta\sin\phi \\
\vec{\omega}_z &= \cos\theta
\end{aligned}$$

$\rightarrow$

Better

$$\begin{aligned}
\vec{\omega}_z &= 2\xi_1 - 1 \\
r &= \sqrt{1 - \vec{\omega}_z^2} \\
\phi &= 2\pi\xi_2 \\
\vec{\omega}_x &= r\cos\phi \\
\vec{\omega}_y &= r\sin\phi
\end{aligned}$$

Ambient Occlusion

Ambient Occlusion

- Consider diffuse objects illuminated by an ambient overcast sky

$$L_r(\mathbf{x}, \vec{\omega}_r) = \int_{H^2} f_r(\mathbf{x}, \vec{\omega}_i, \vec{\omega}_r) L_i(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

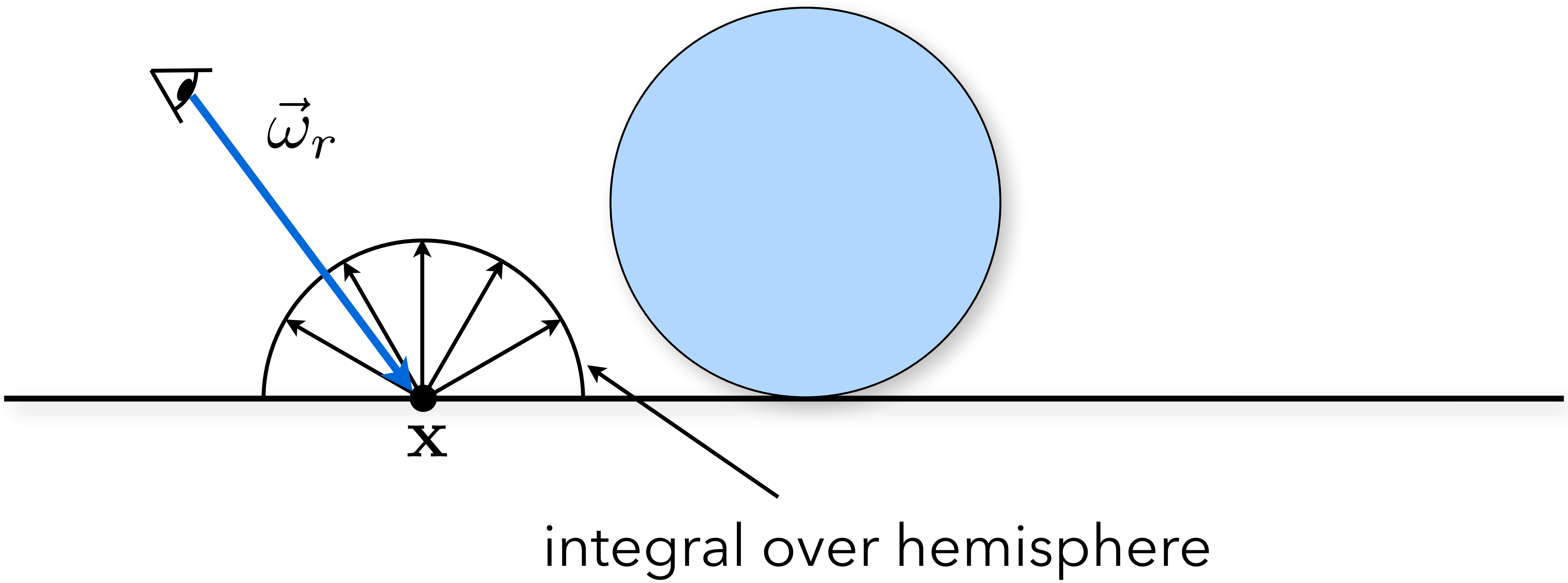
$$\approx \frac{\rho}{\pi N} \sum_{i=1}^{N} \frac{V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i}{p(\vec{\omega}_i)}$$

- Uniform hemispherical sampling, $p(\vec{\omega}_i) = 1/2\pi$:

$$L_r(x) \approx \frac{2\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i$$

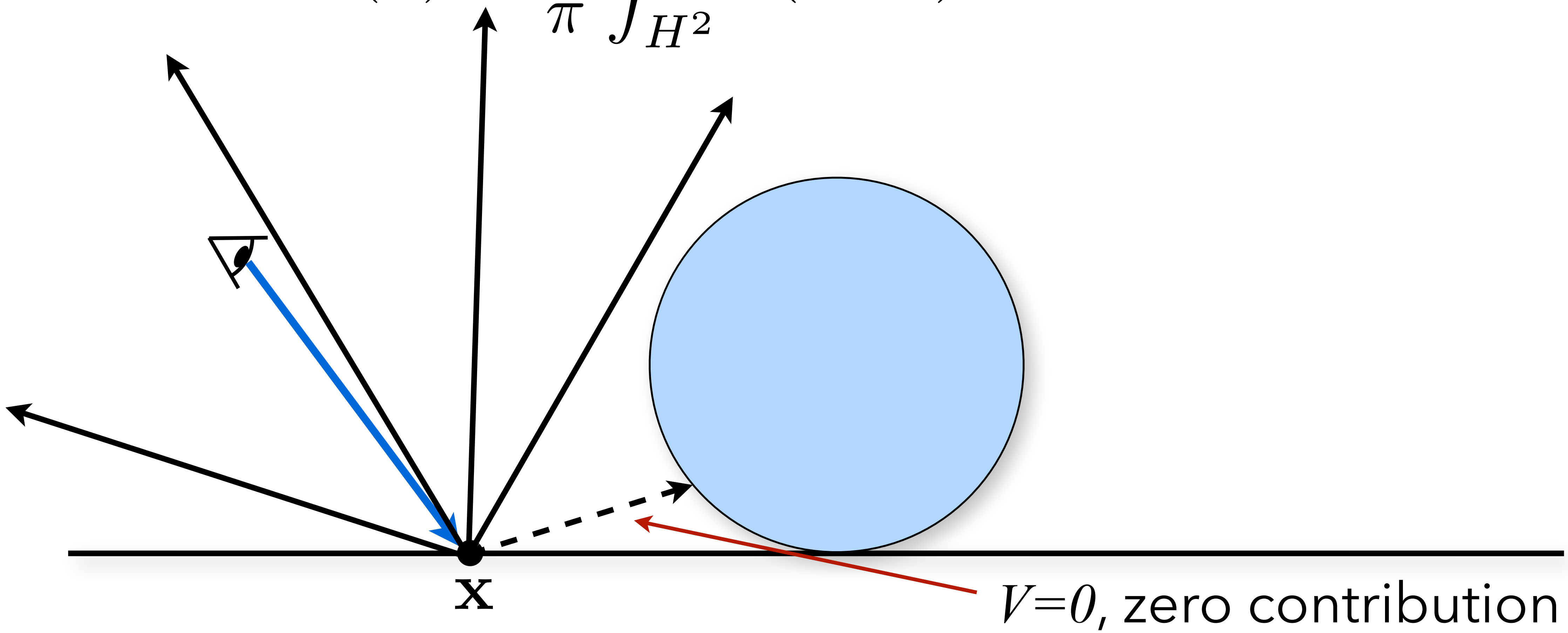
Ambient Occlusion

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$



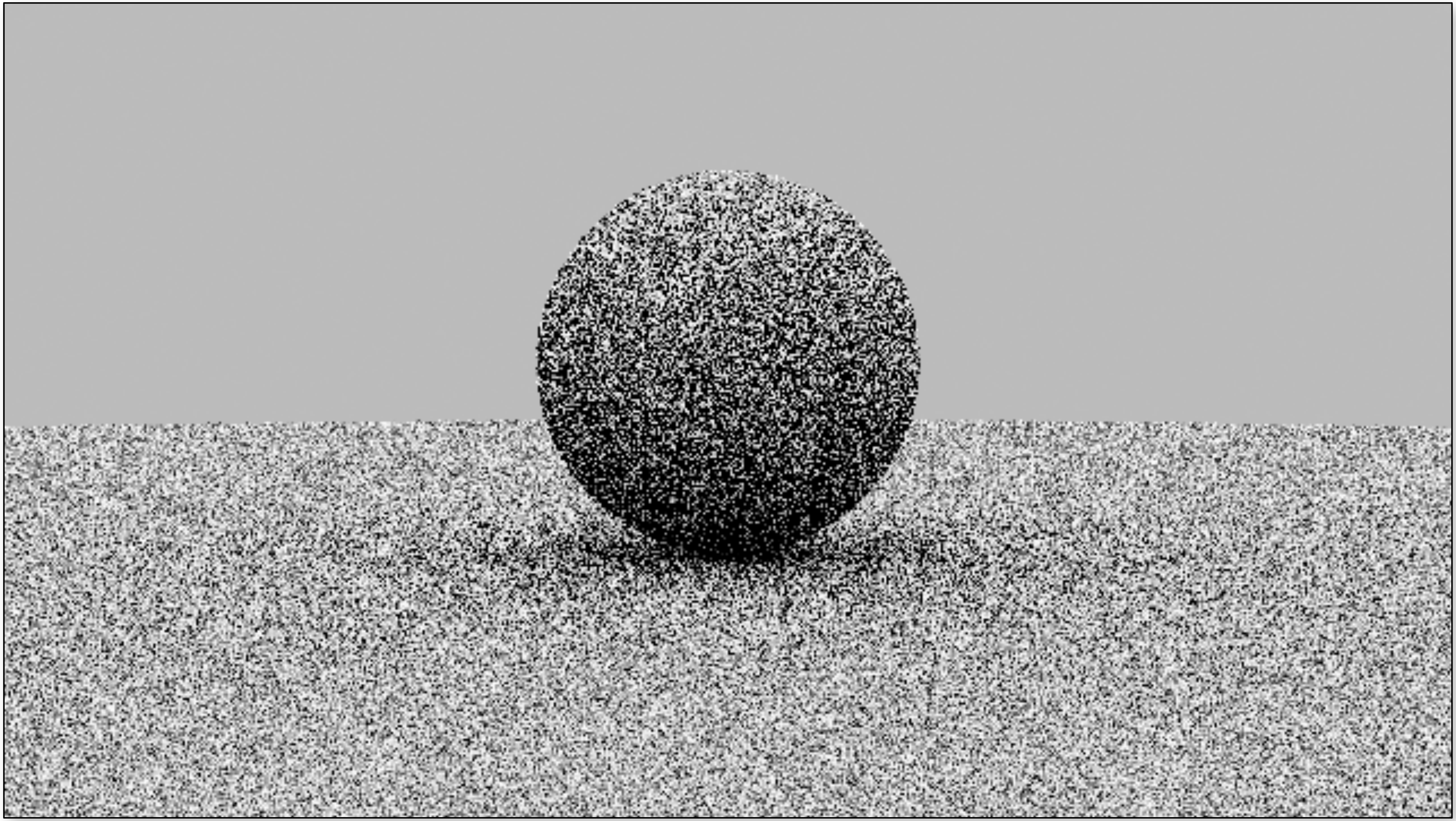
Ambient Occlusion

$$L_r(\mathbf{x}) = \frac{\rho}{\pi} \int_{H^2} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i \, d\vec{\omega}_i$$

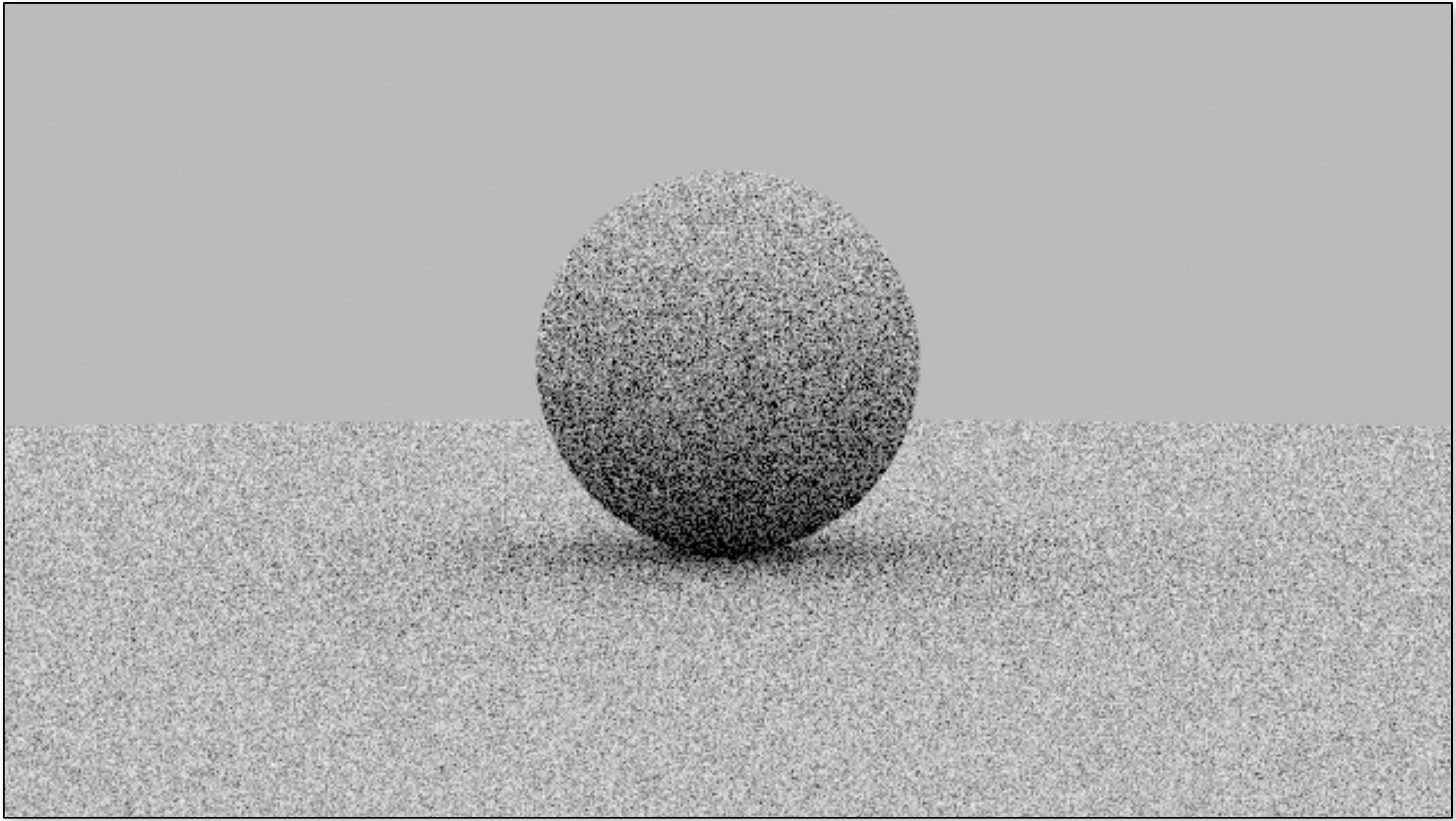


$$L_r(x) \approx \frac{2\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i$$

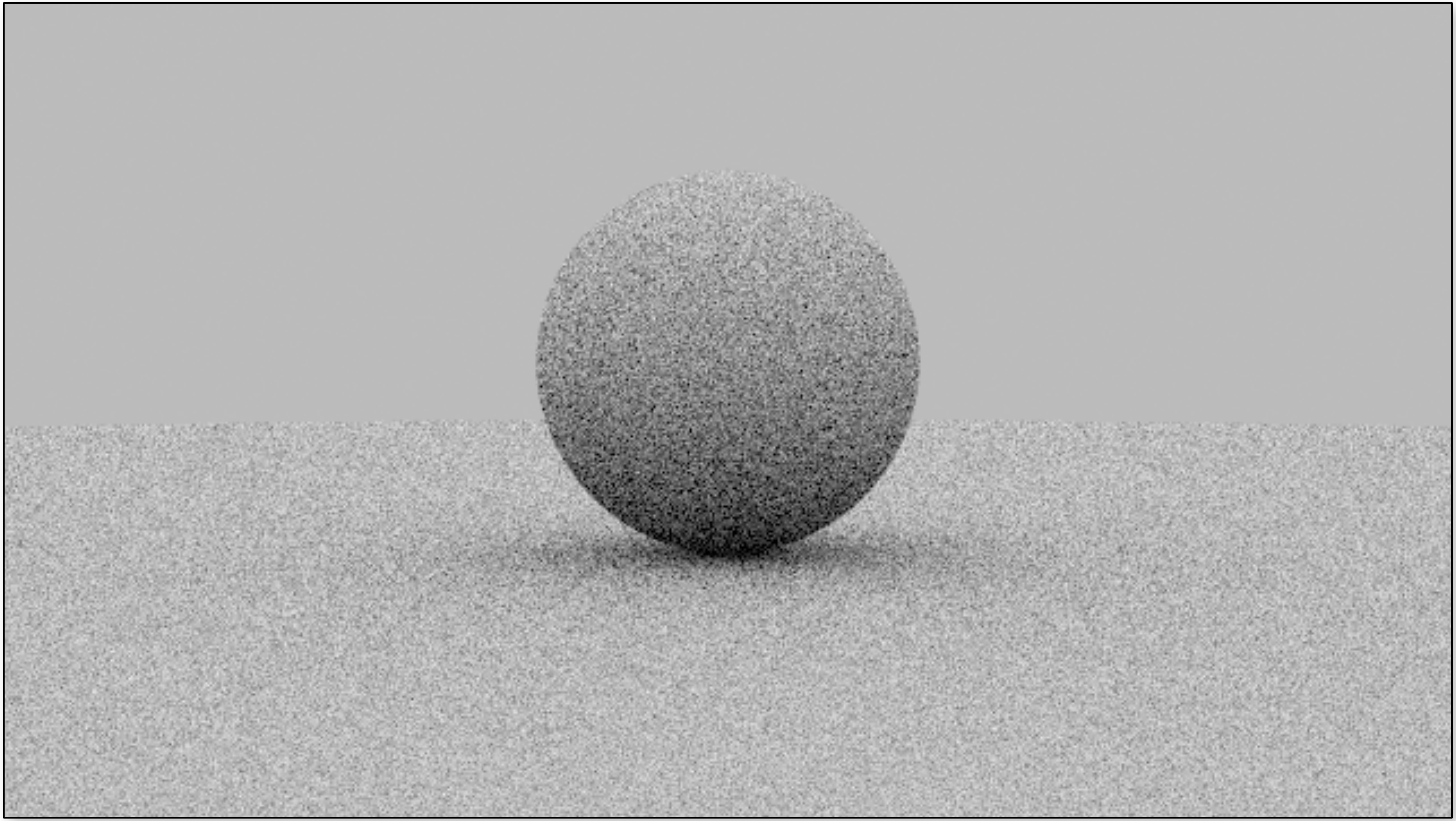
Hemispherical Sampling (1 Sample)



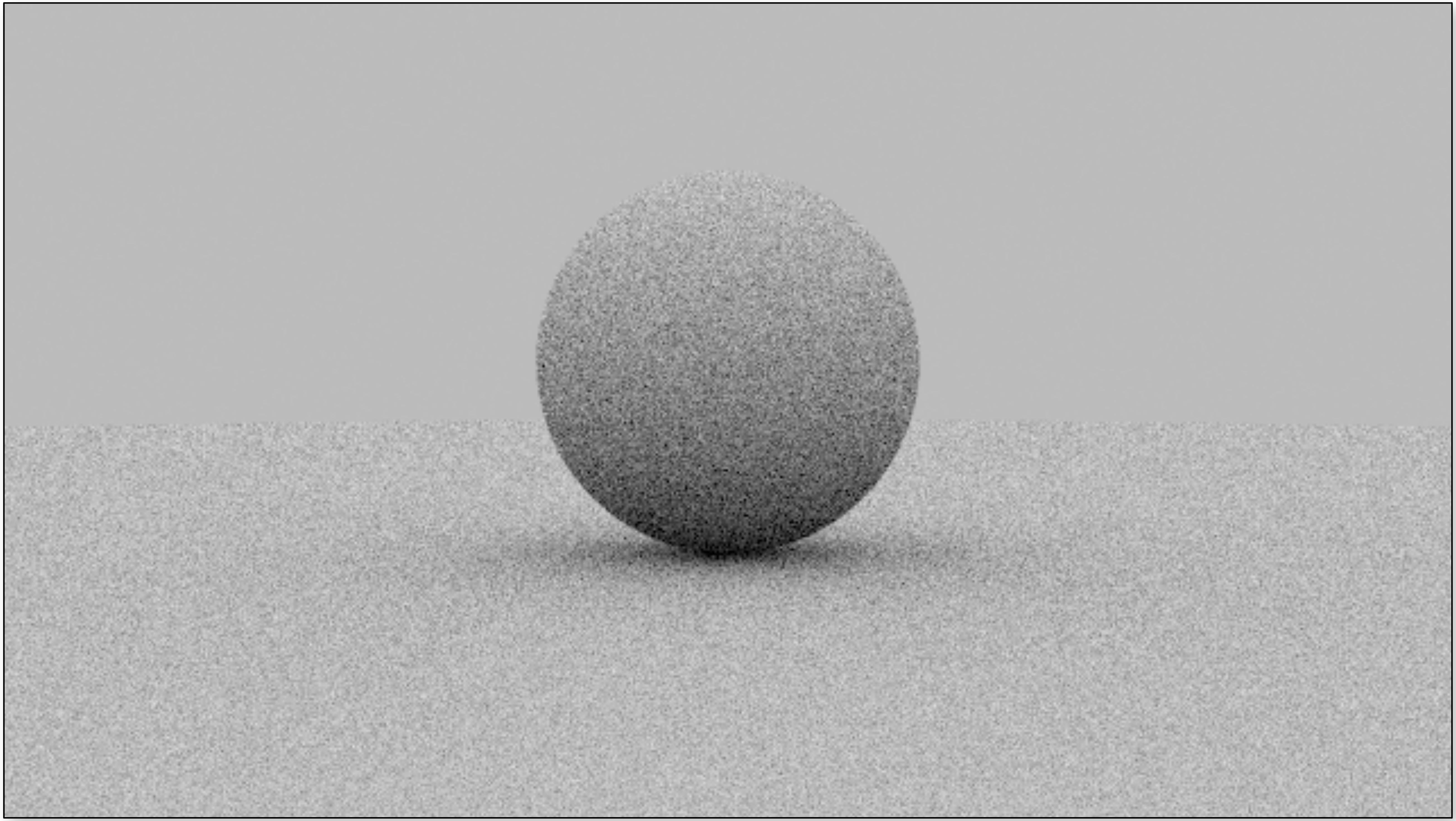
Hemispherical Sampling (4 Samples)



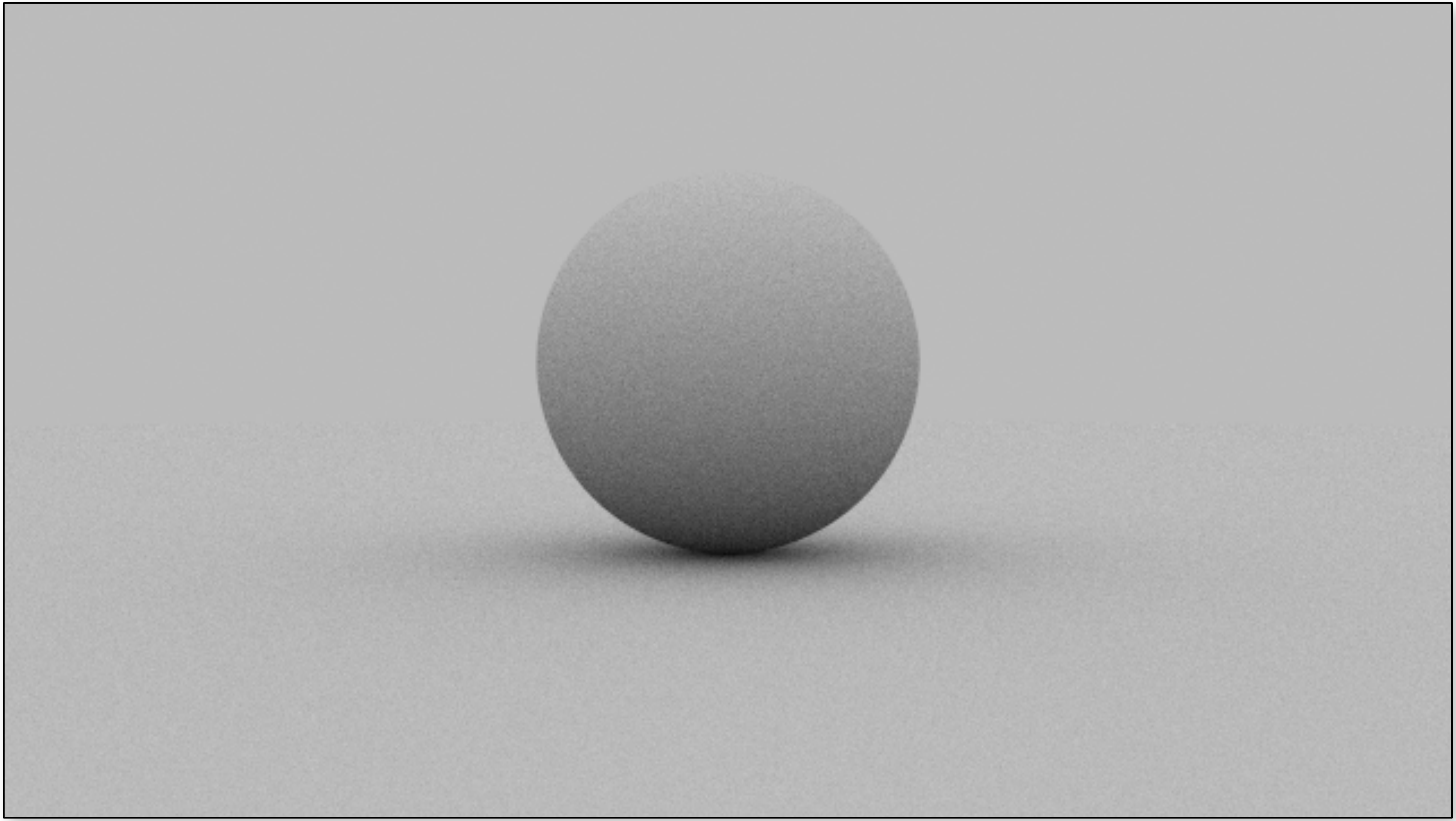
Hemispherical Sampling (9 Samples)



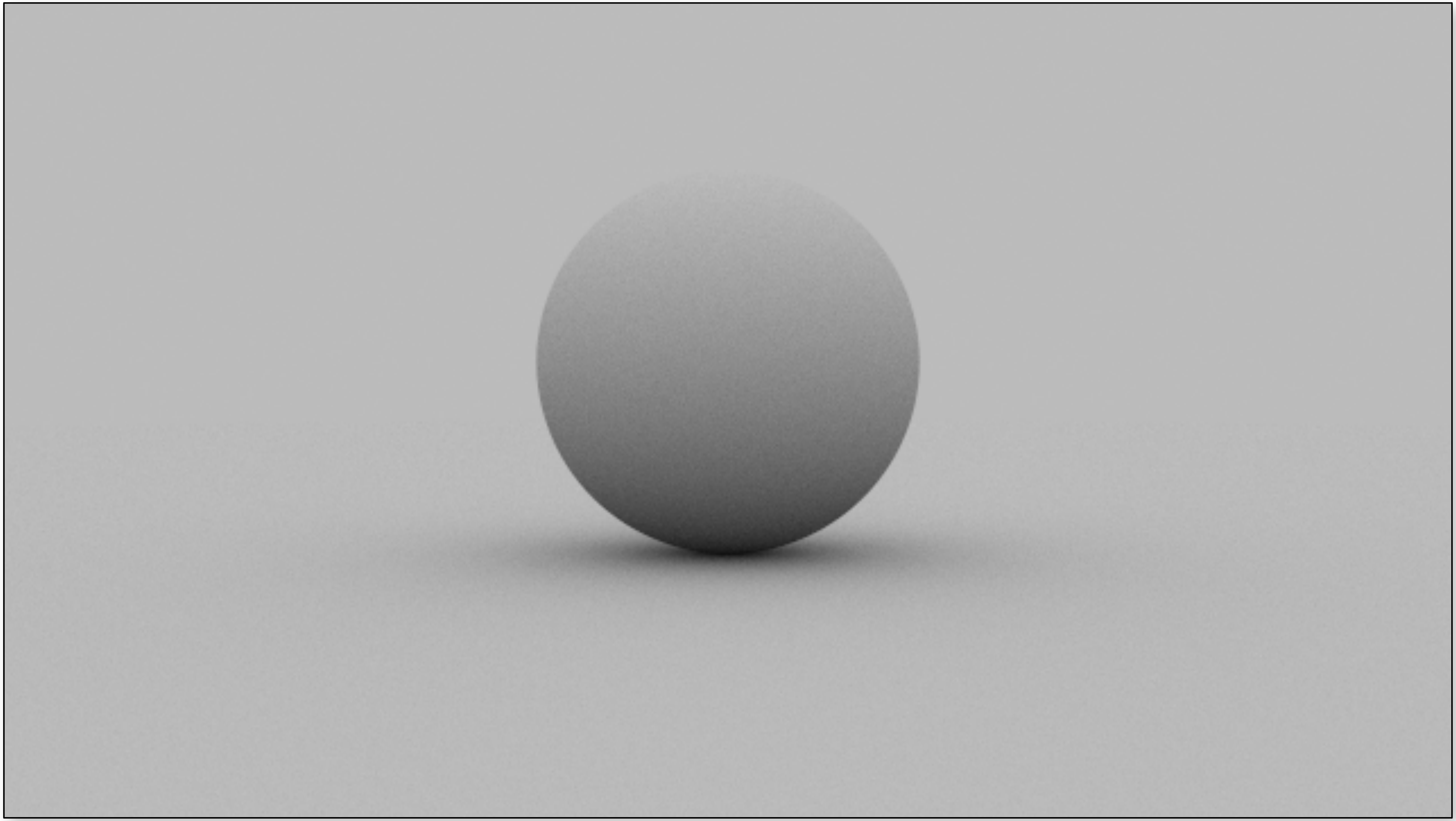
Hemispherical Sampling (16 Samples)



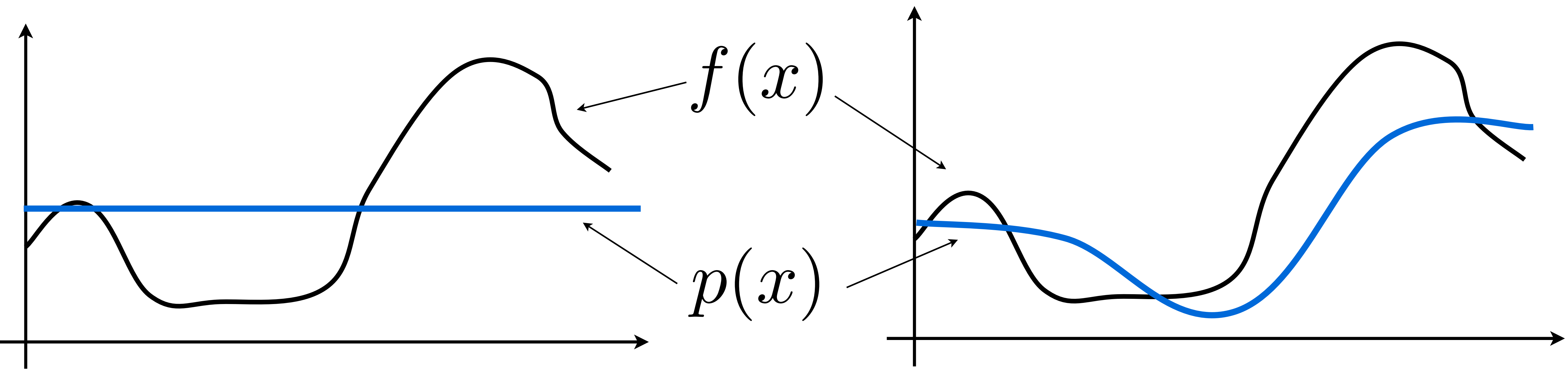
Hemispherical Sampling (256 Samples)



Hemispherical Sampling (1024 Samples)



Reducing Variance: Importance Sampling



uniform sampling

importance sampling

Reducing Variance: Importance Sampling

- Importance sampling

$$\int f(x)dx \qquad\qquad F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)}$$

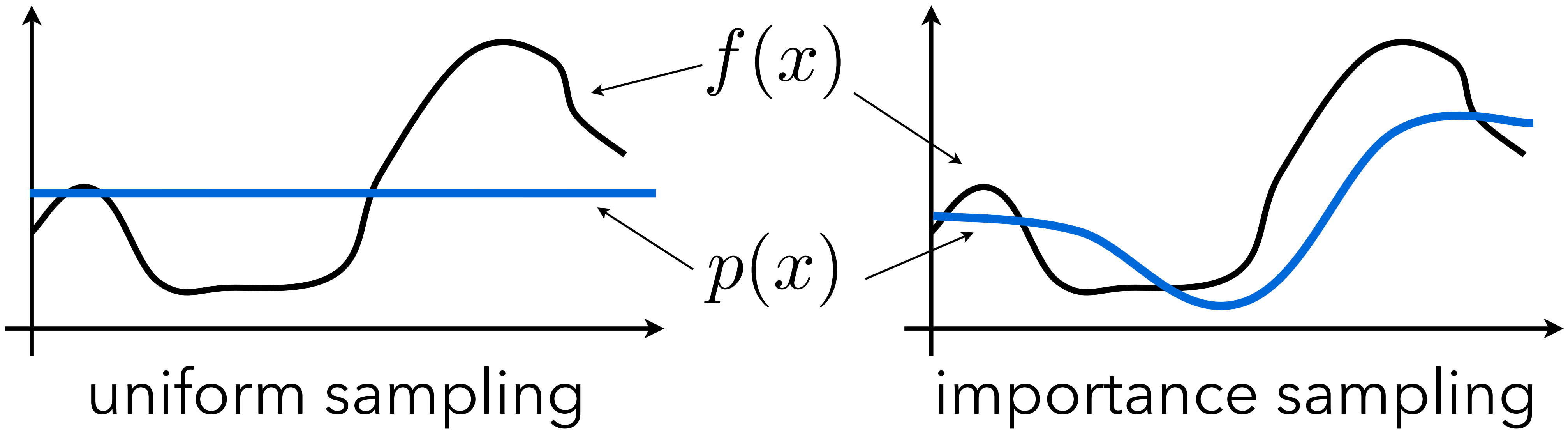
- assume $p(x) = cf(x)$

$$\int p(x)dx = 1 \quad \rightarrow \quad c = \frac{1}{\int f(x)dx}$$

- estimator $\frac{f(X_i)}{p(X_i)} = \frac{1}{c} = \int f(x)dx$ zero variance!

Reducing Variance: Importance Sampling

- $p(x) = cf(x)$ requires knowledge of integral, which is what we are trying to solve!
- But: If PDF is similar to integrand, variance can be significantly reduced
- **Common strategy: sample according to part of the integrand**



Ambient Occlusion

- Ambient occlusion estimator:

$$L_r(\mathbf{x}) \approx \frac{\rho}{\pi N} \sum_{i=1}^{N} \frac{V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i}{p(\vec{\omega}_i)}$$

- Uniform hemispherical sampling, $p(\vec{\omega}_i) = 1/2\pi$:

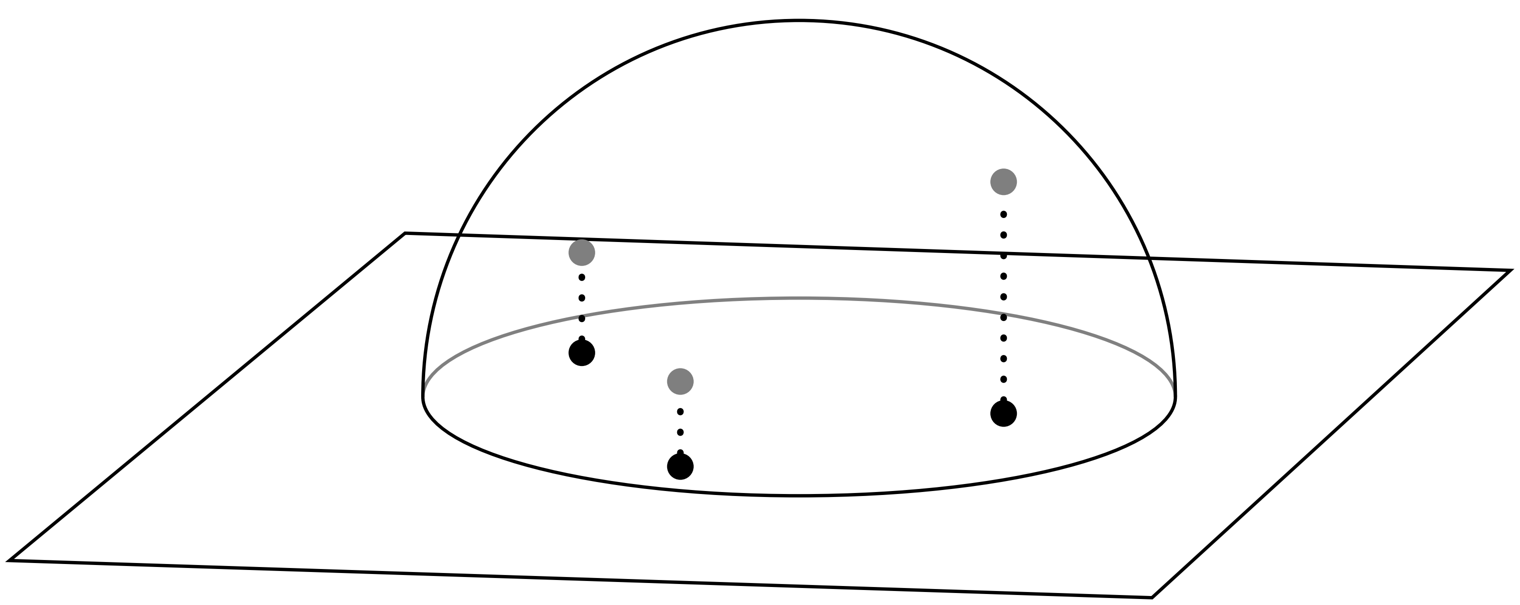
$$L_r(x) \approx \frac{2\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i) \cos\theta_i$$

- Using a PDF proportional to cosine, $p(\vec{\omega}_i) = \cos\theta_i/\pi$:

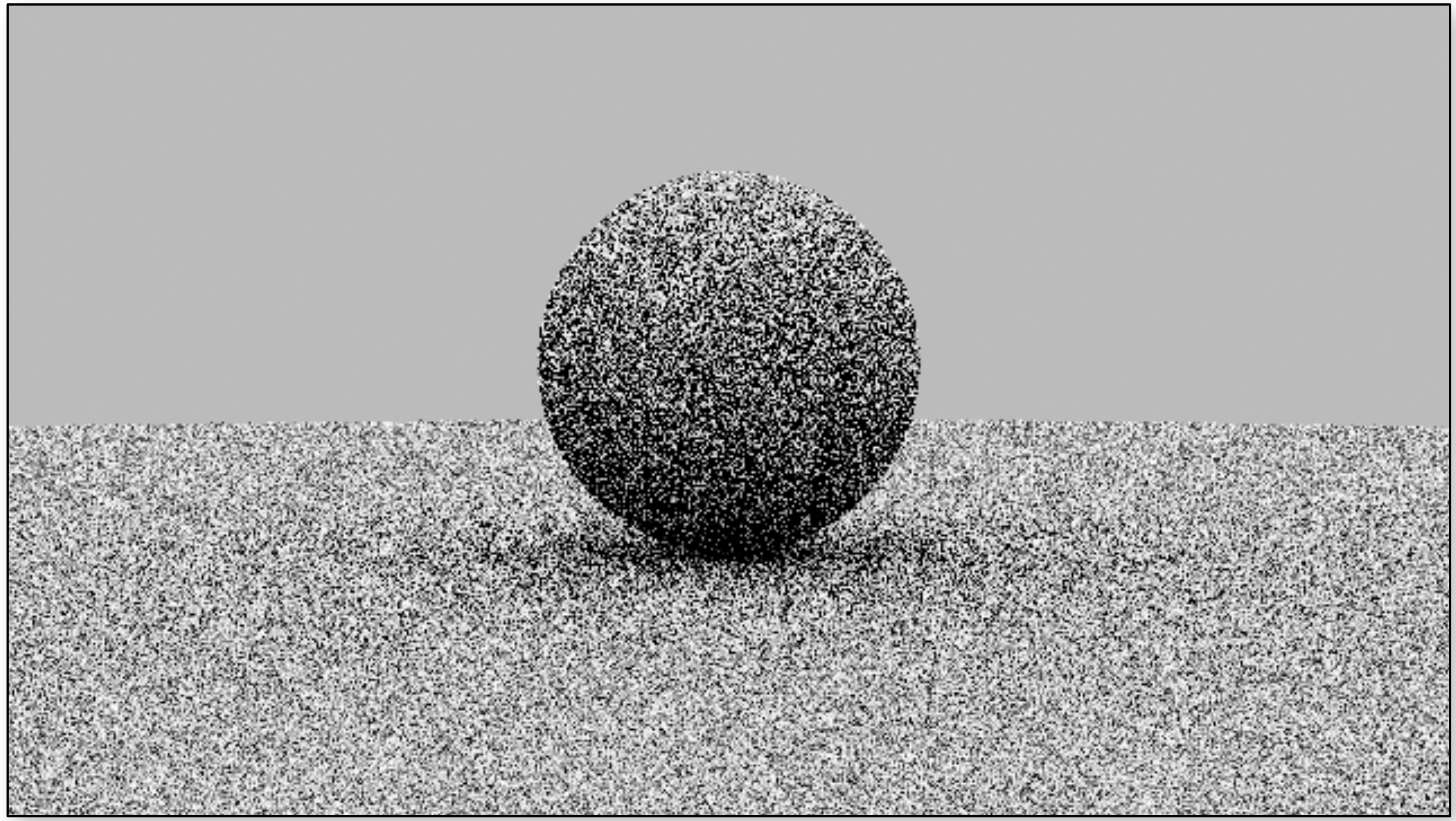
$$L_r(\mathbf{x}) \approx \frac{\rho}{N} \sum_{i=1}^{N} V(\mathbf{x}, \vec{\omega}_i)$$

Cosine-weighted Hemispherical Sampling

- Could proceed as before: compute marginal and conditional densities, then use inversion method
- It turns out that:
  - generating points uniformly on the disc, and then projecting these points to the surface of the hemisphere produces the desired distribution

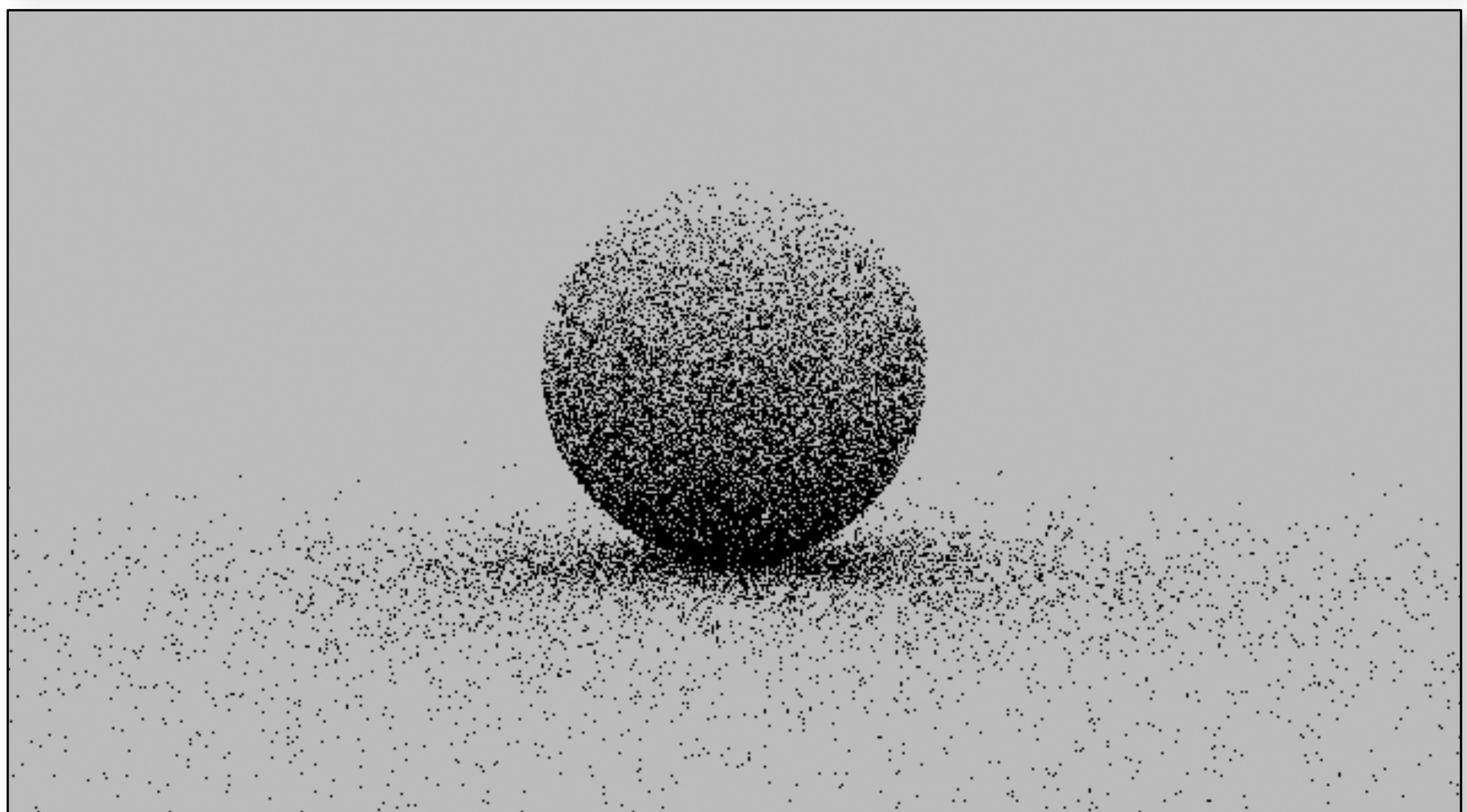


Uniform Hemispherical Sampling

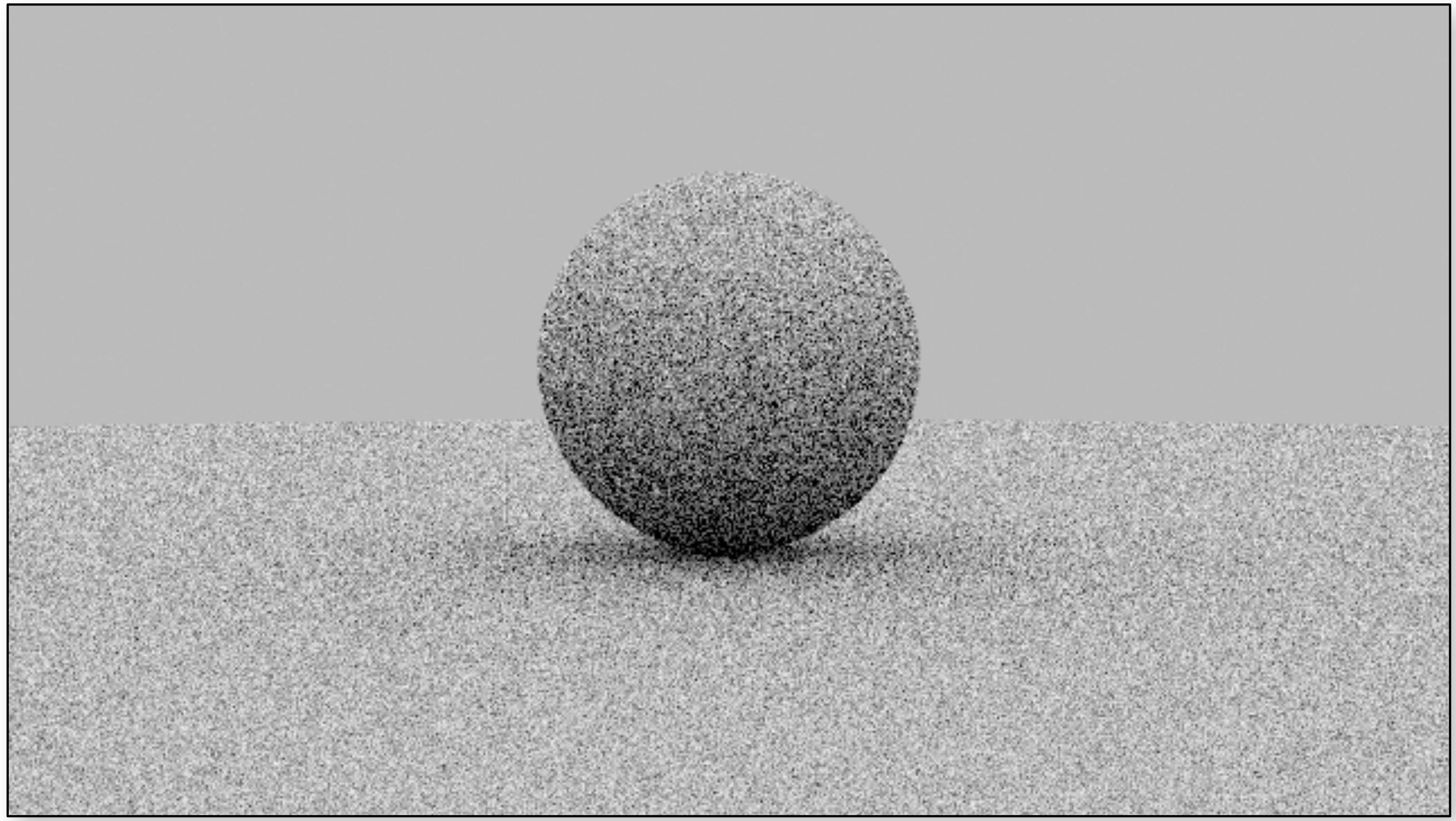


1 sample

Cosine-weighted Hemispherical Sampling

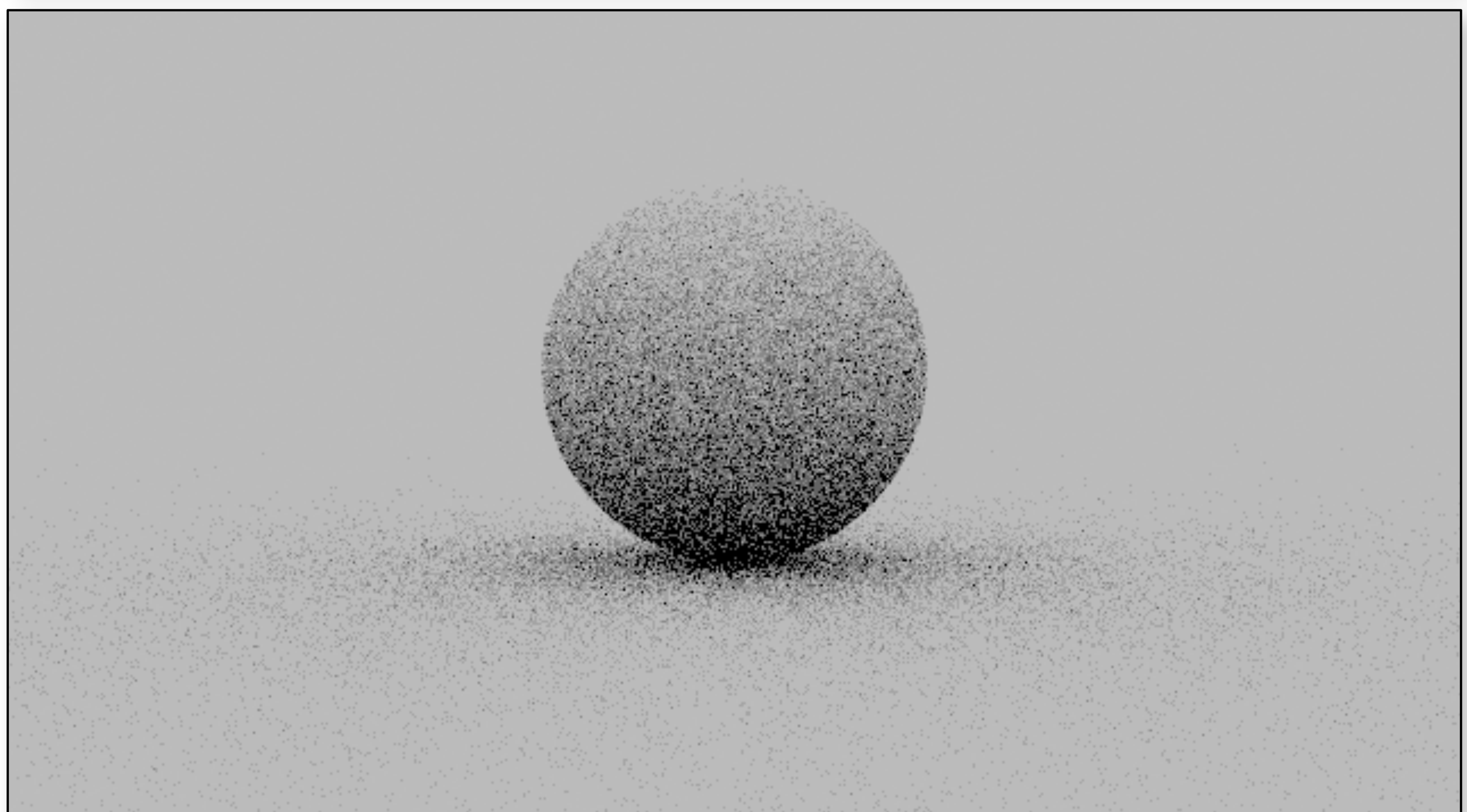


Uniform Hemispherical Sampling

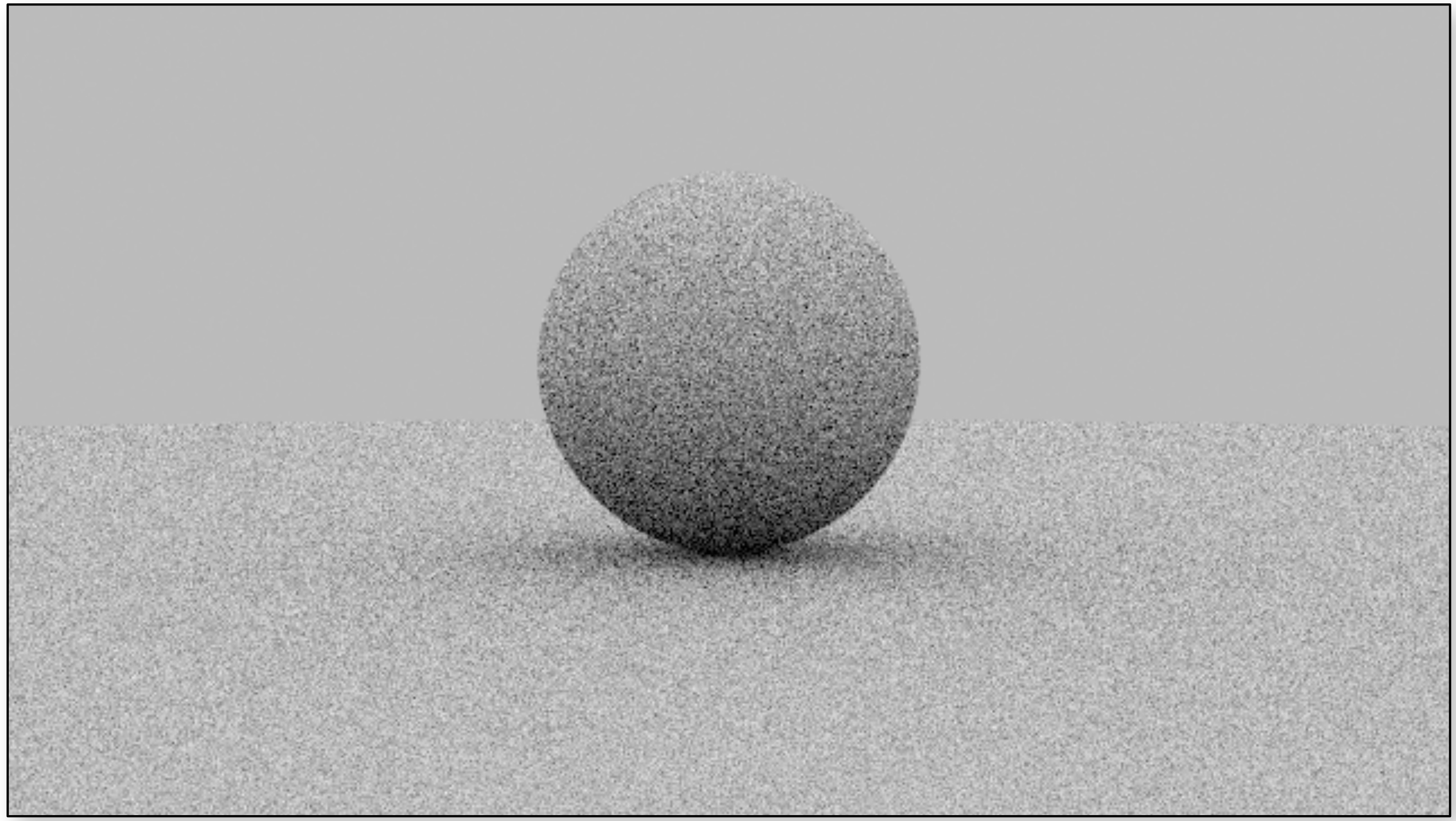


4 samples

Cosine-weighted Hemispherical Sampling

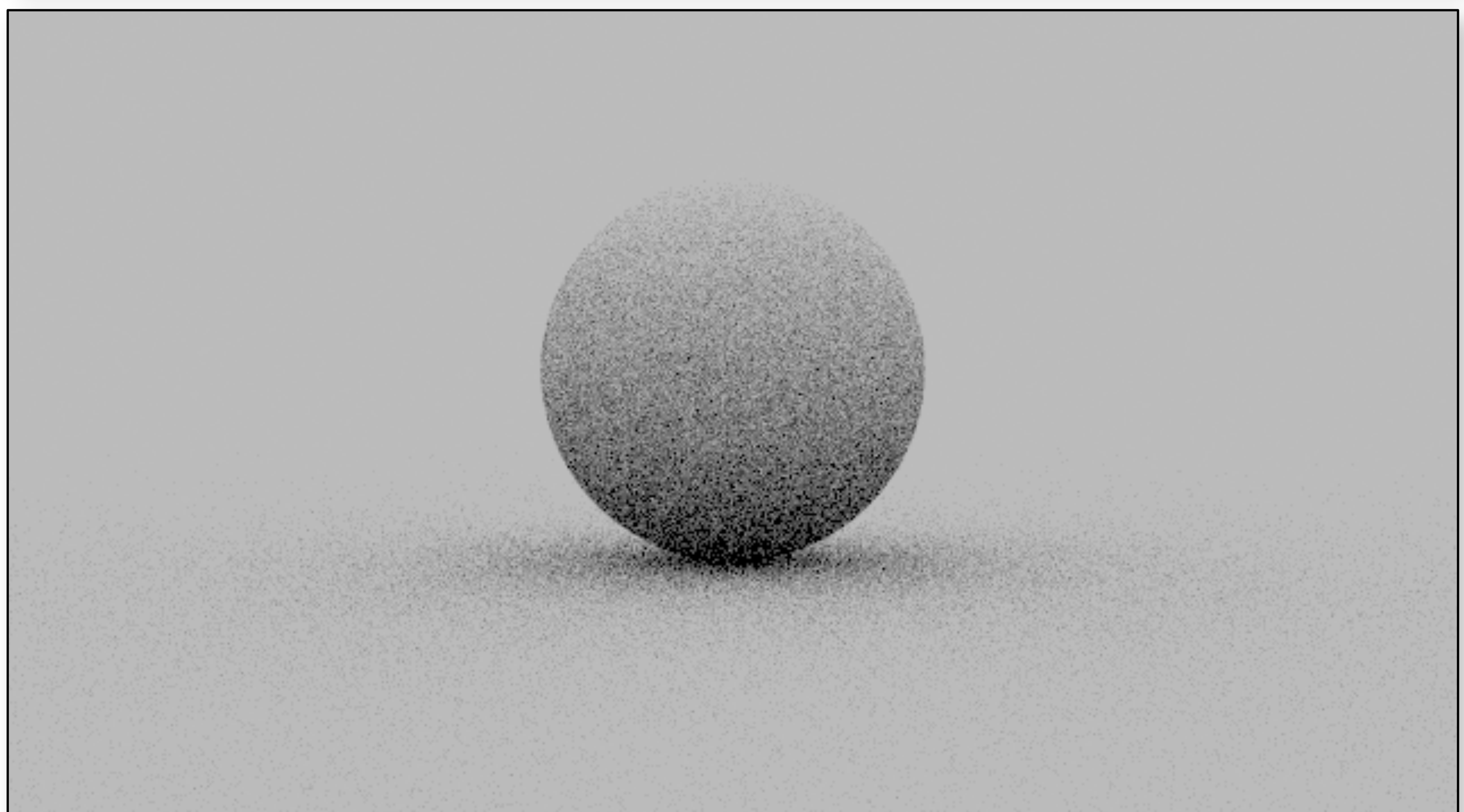


Uniform Hemispherical Sampling

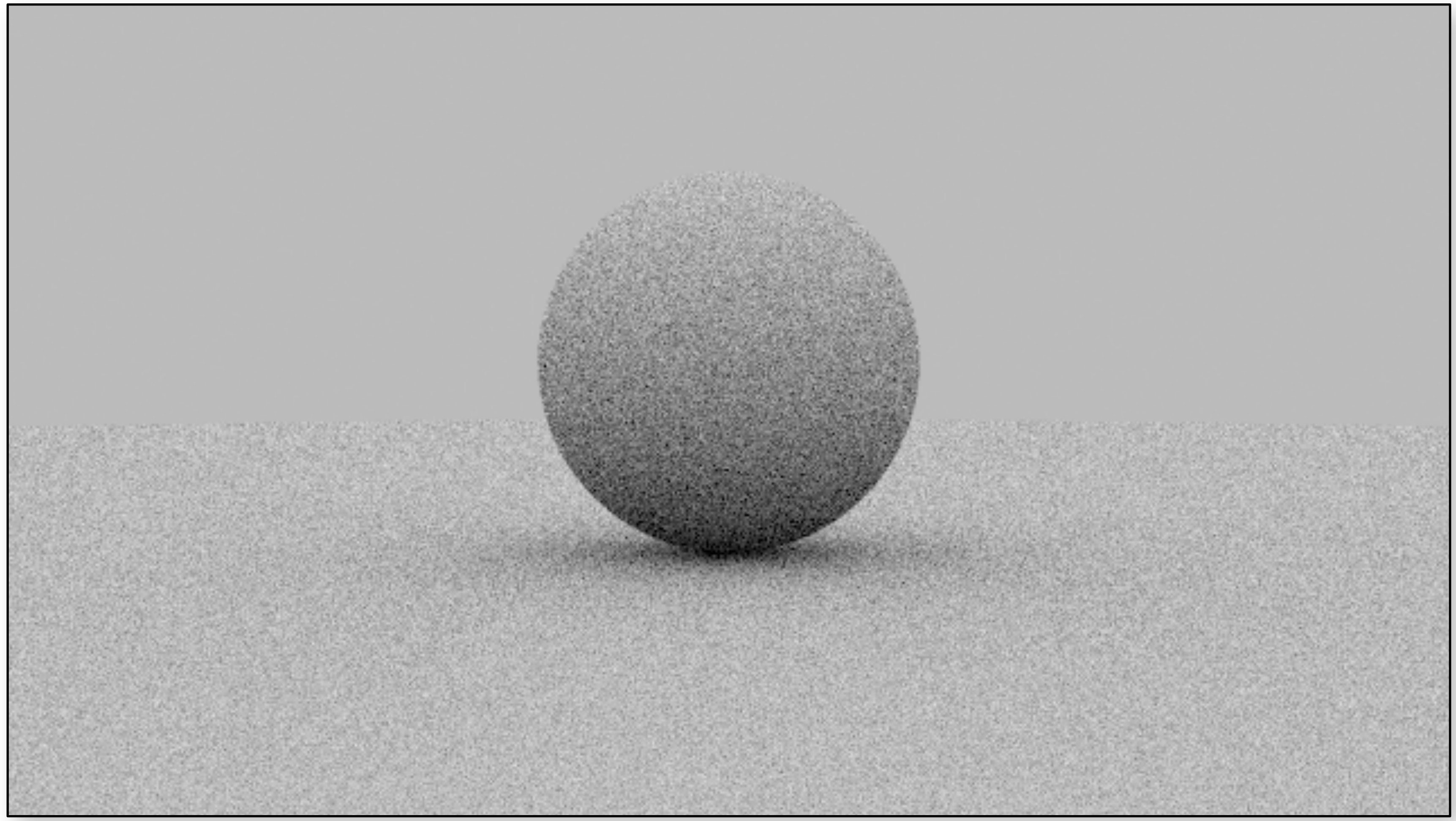


9 samples

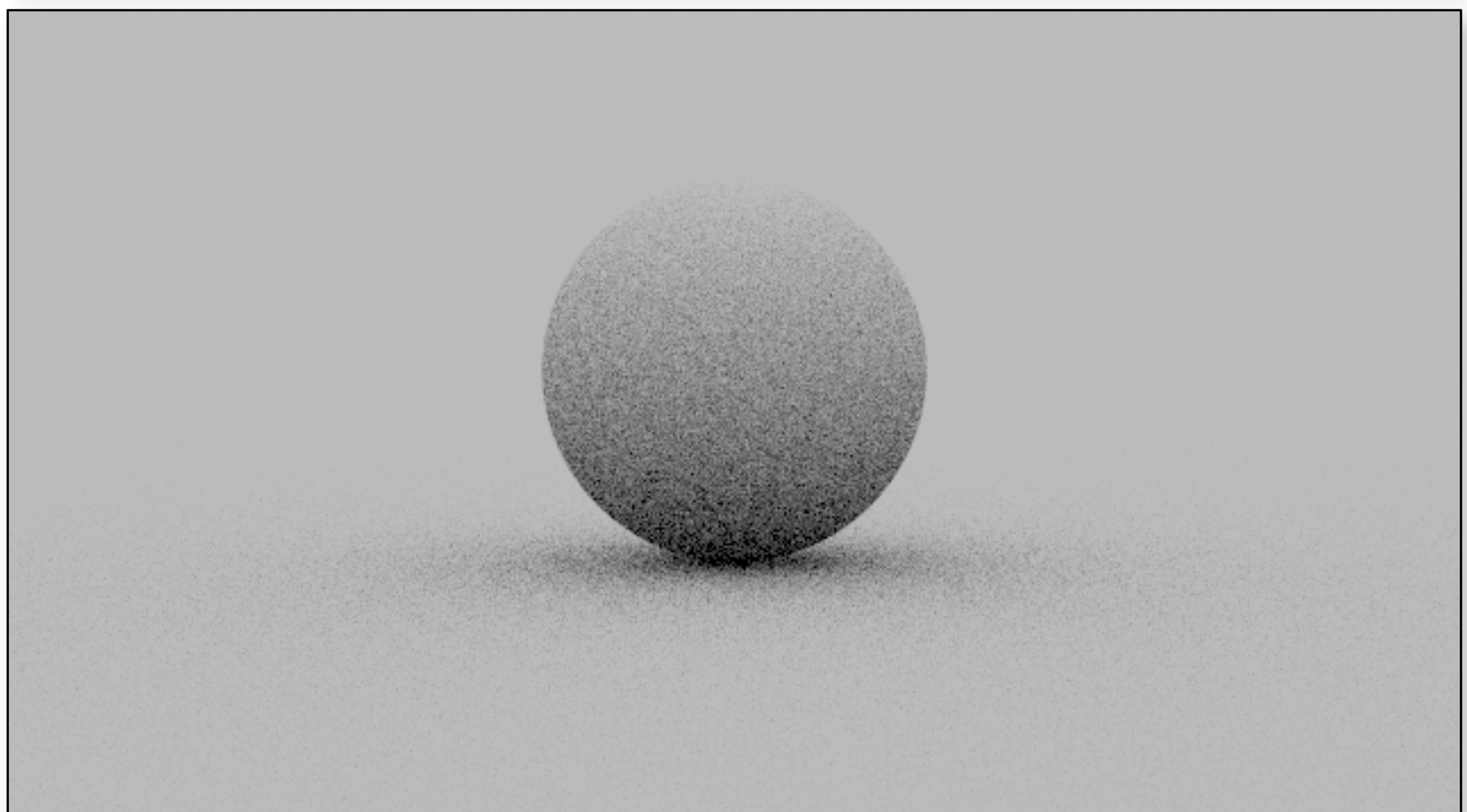
Cosine-weighted Hemispherical Sampling



Uniform Hemispherical Sampling

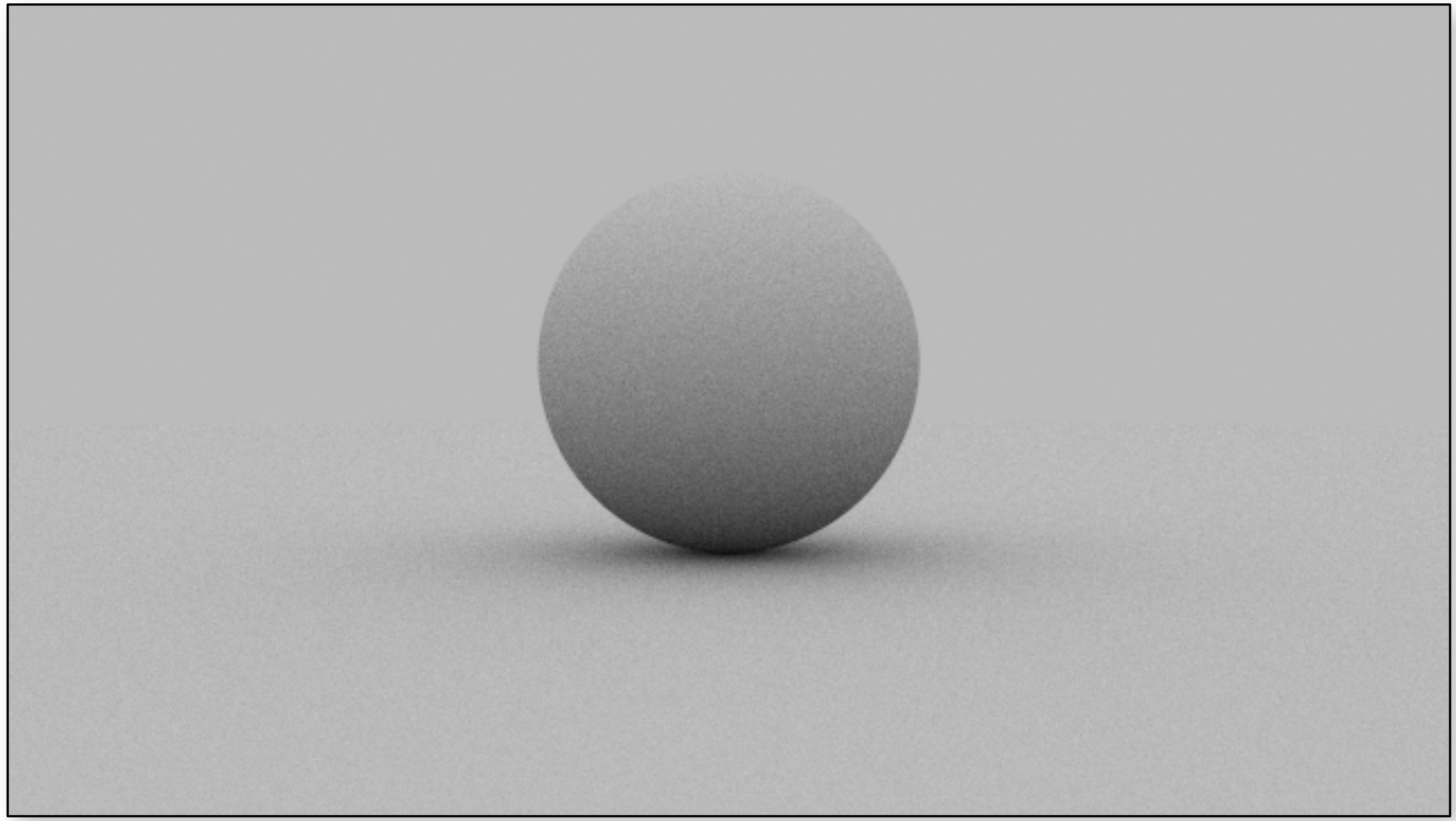


16 samples



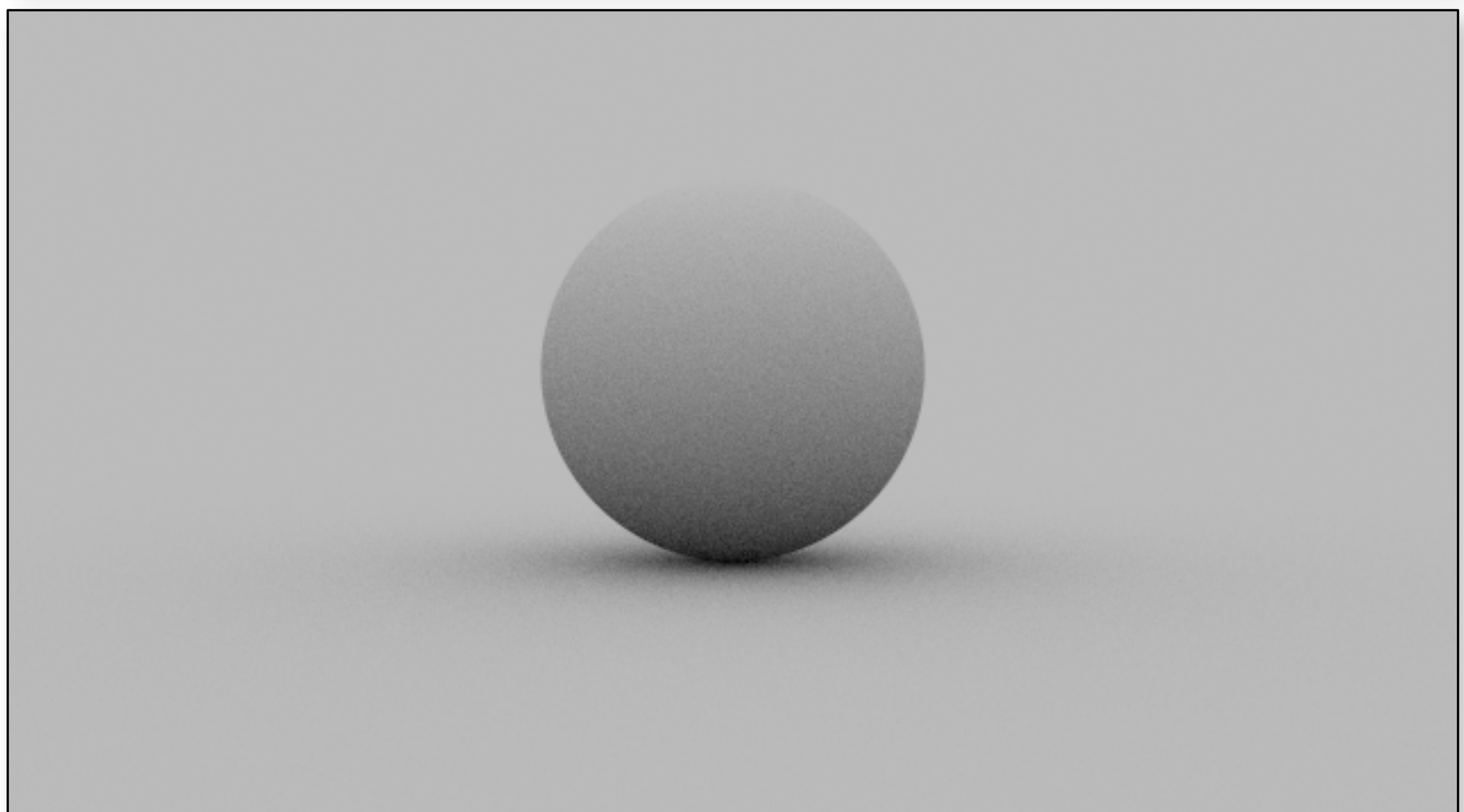
Cosine-weighted Hemispherical Sampling

Uniform Hemispherical Sampling

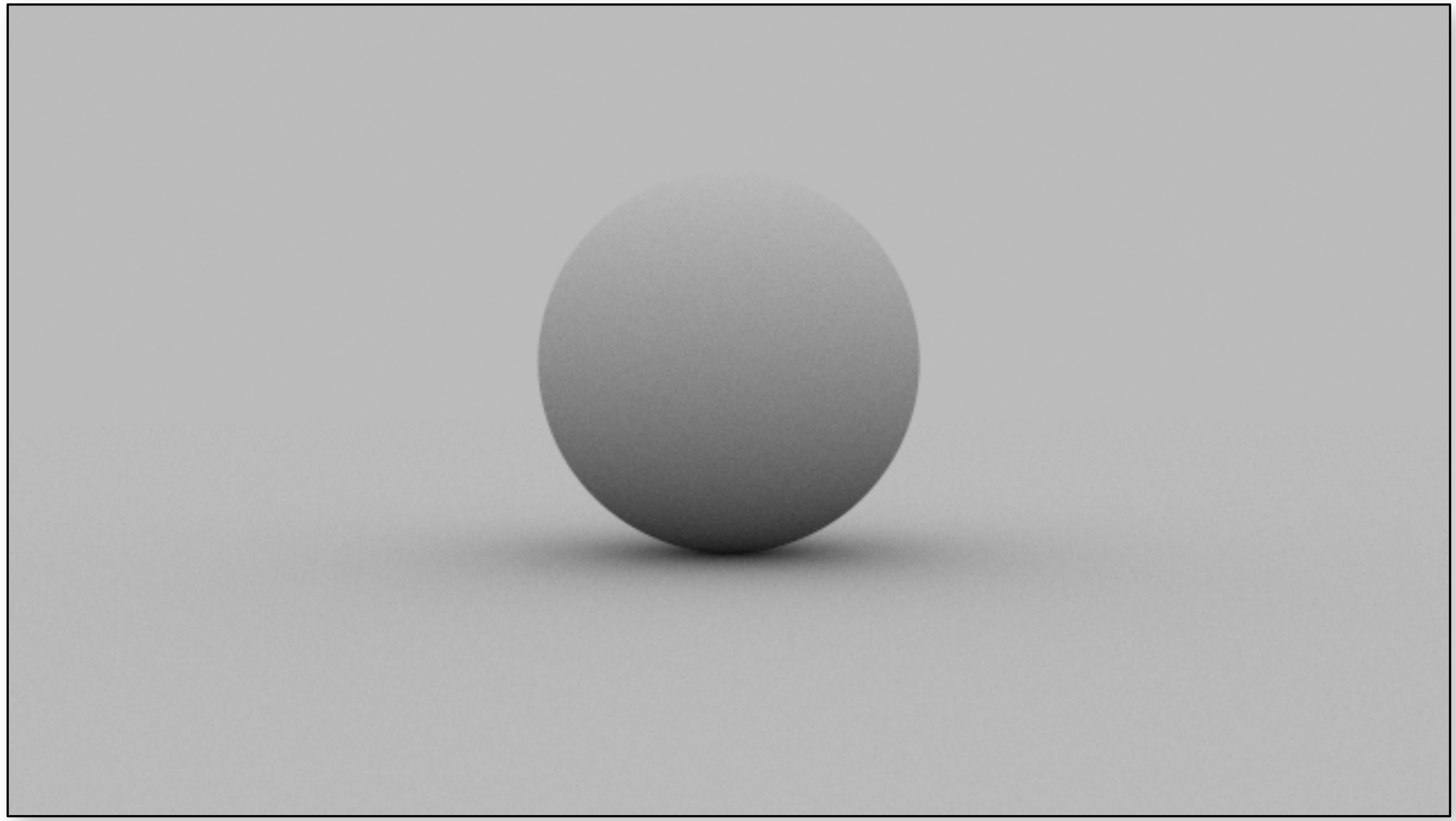


256 samples

Cosine-weighted Hemispherical Sampling

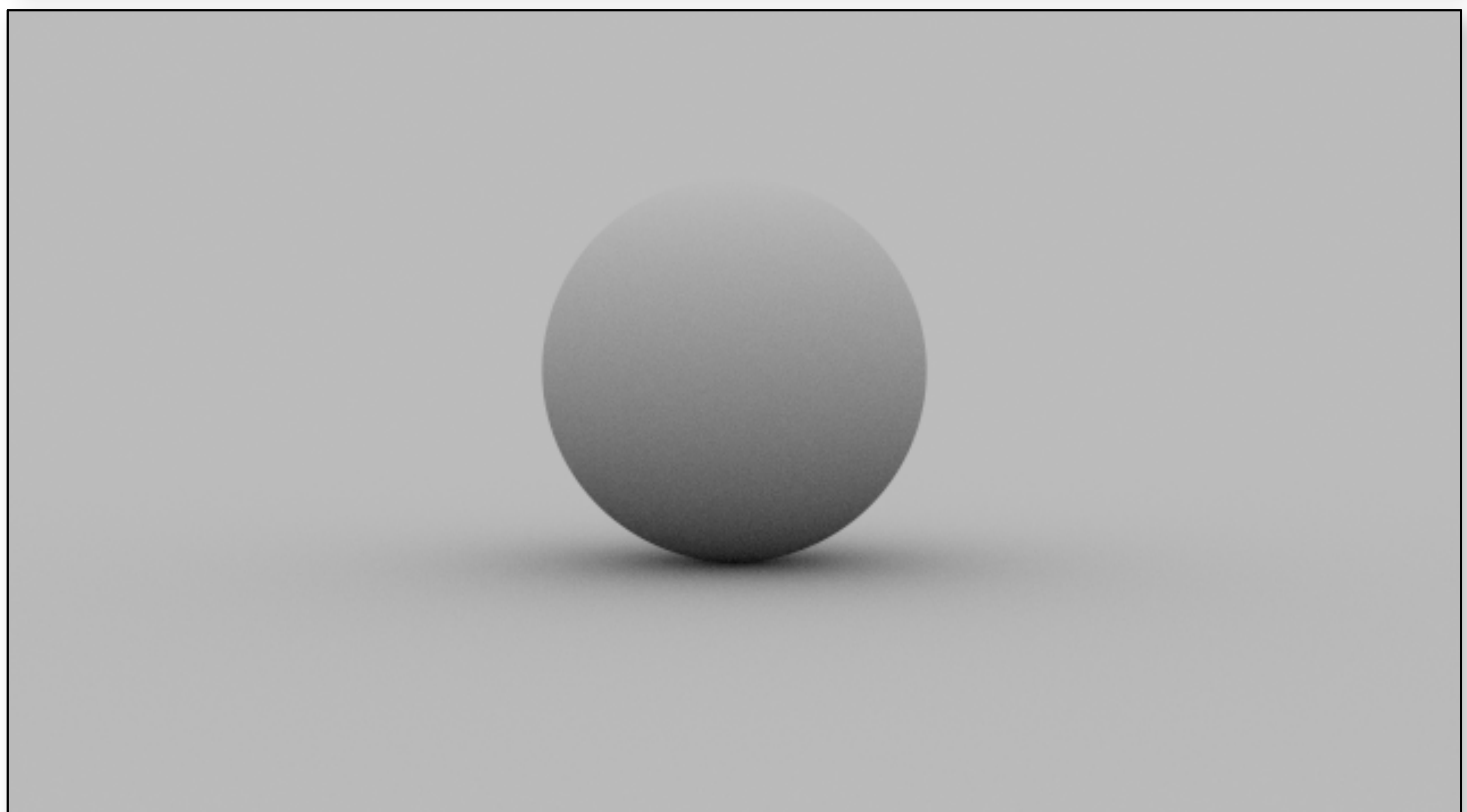


Uniform Hemispherical Sampling

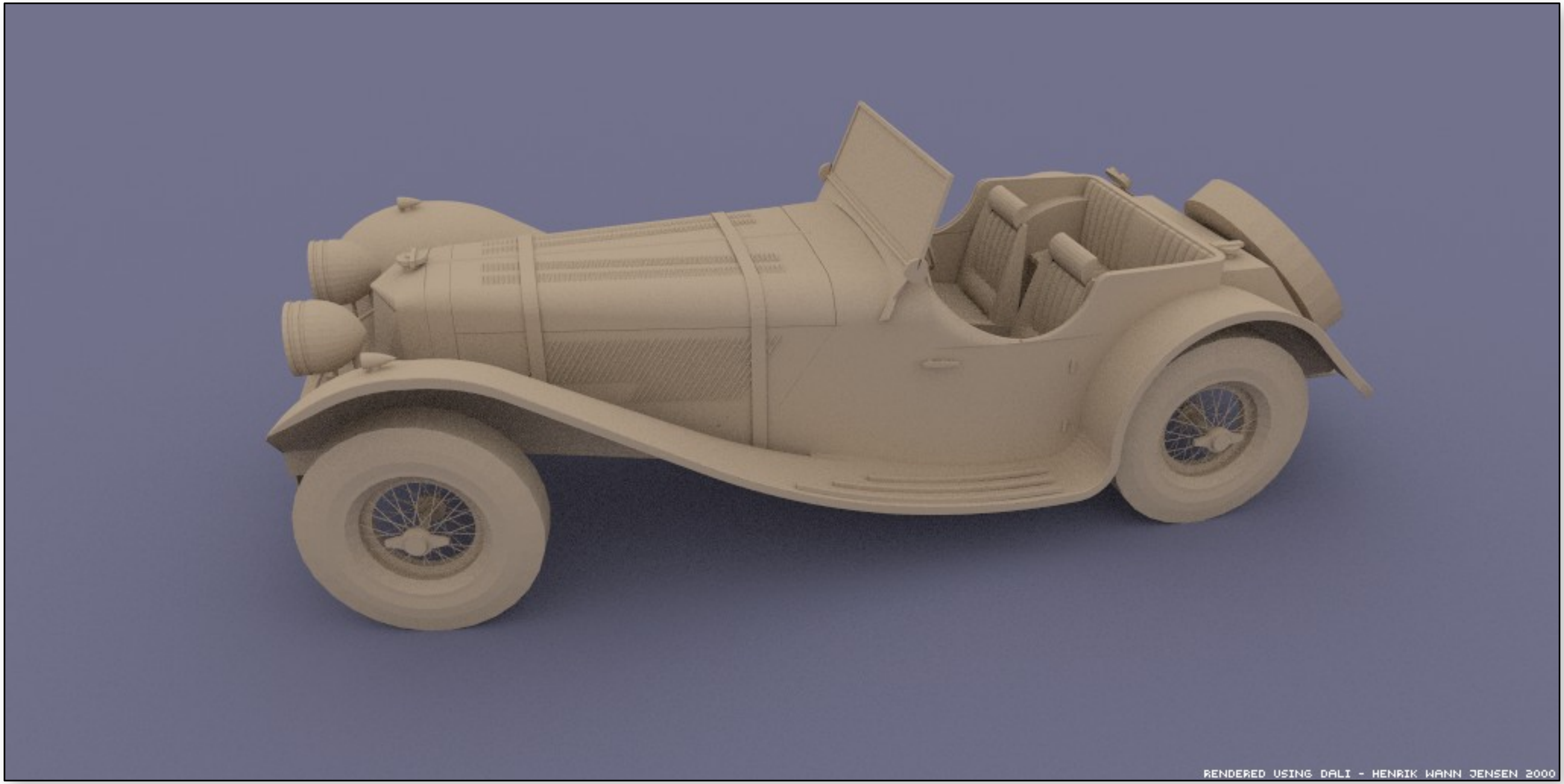


1024 samples

Cosine-weighted Hemispherical Sampling

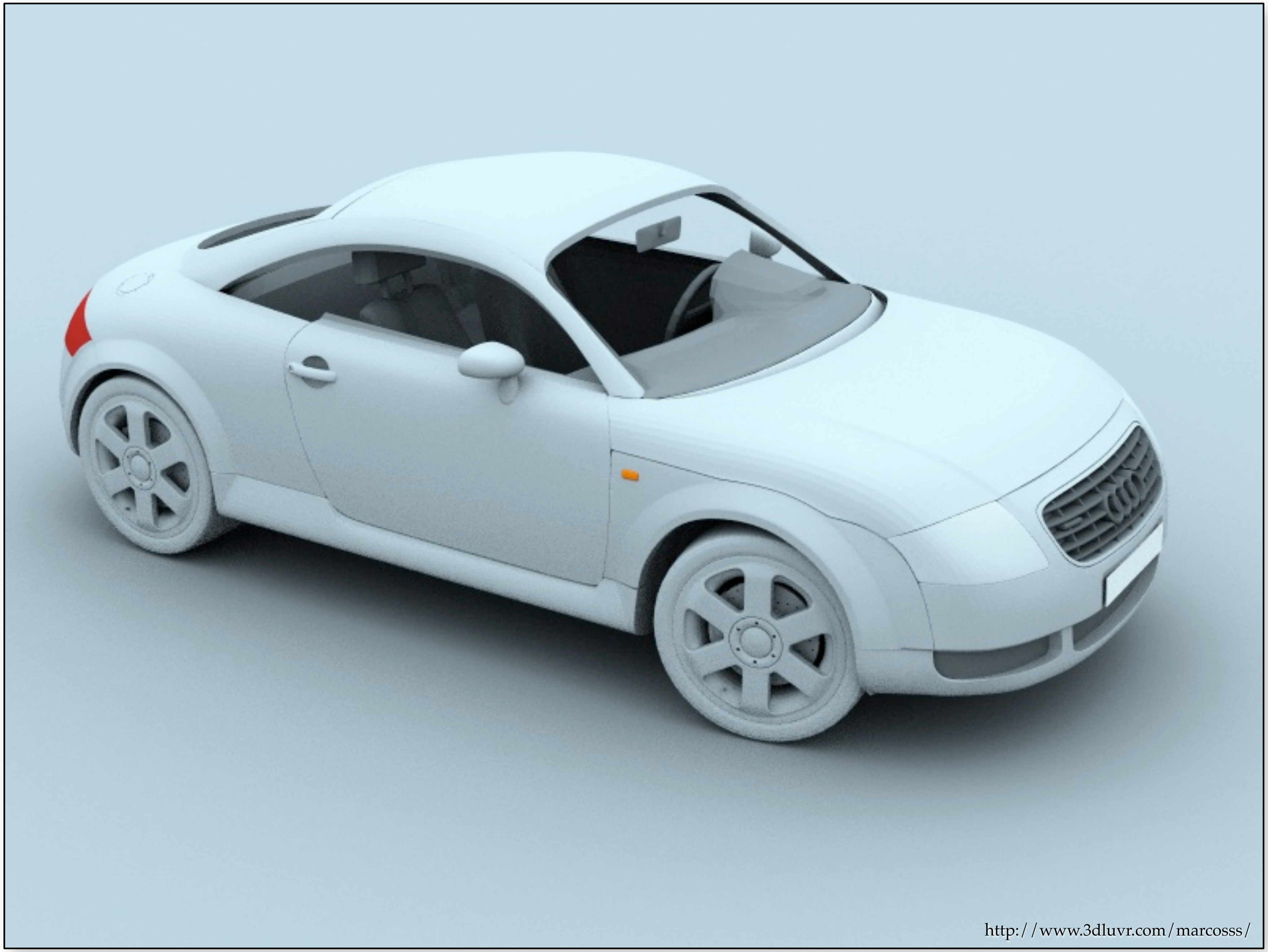


Ambient Occlusion

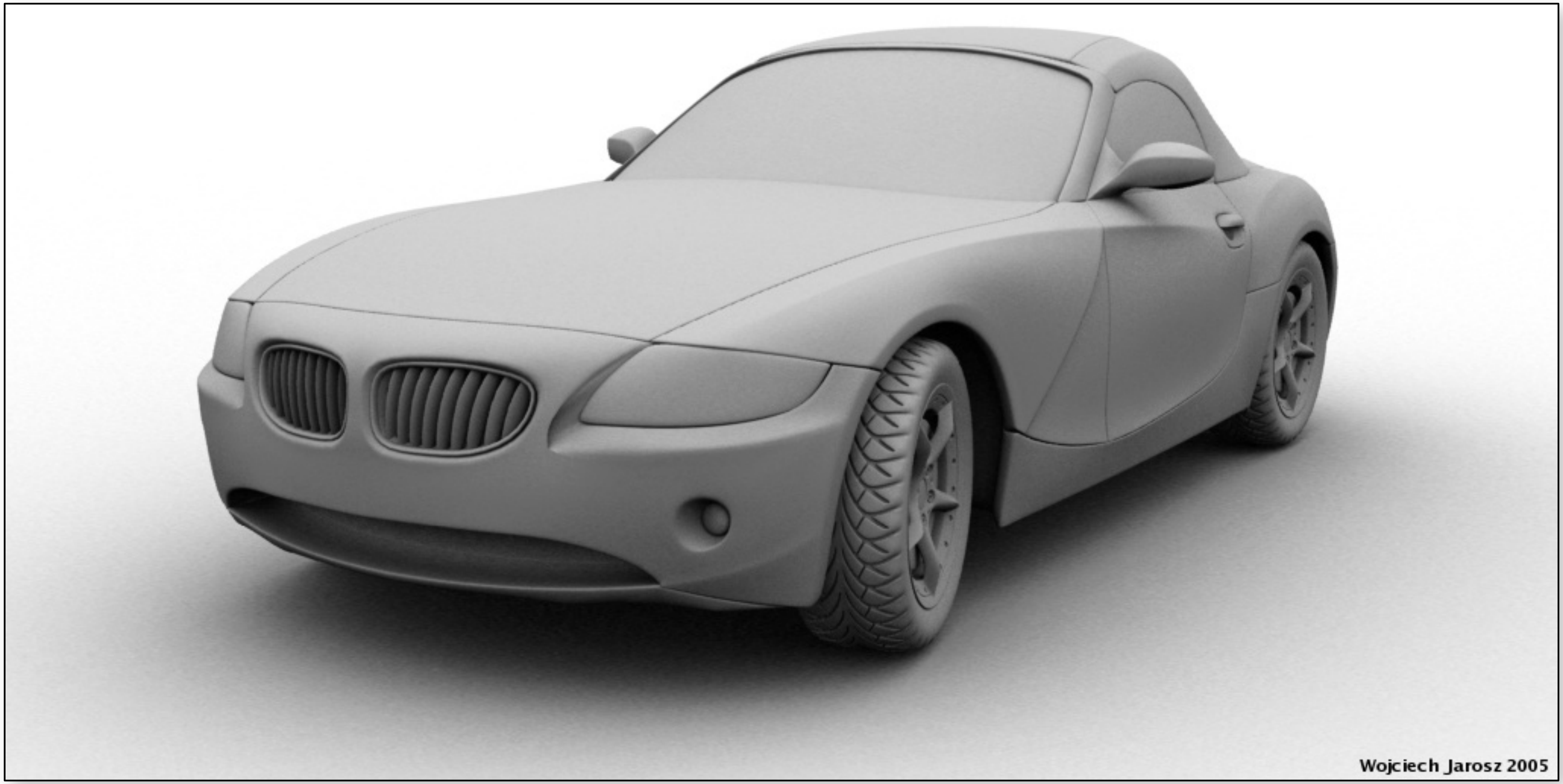


Ambient Occlusion

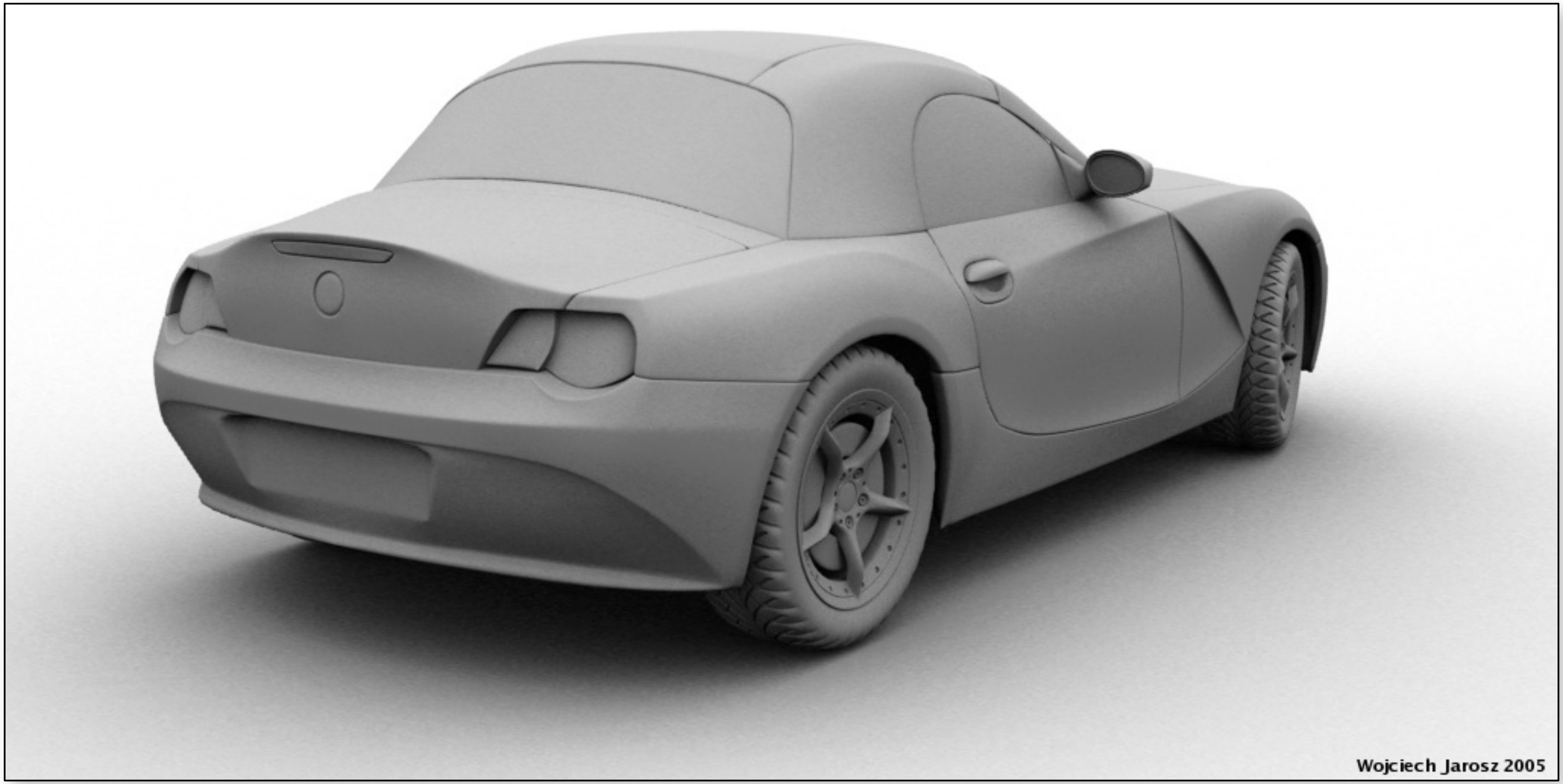
Ambient Occlusion



Overcast Sky Lighting



Overcast Sky Lighting



More Random Sampling

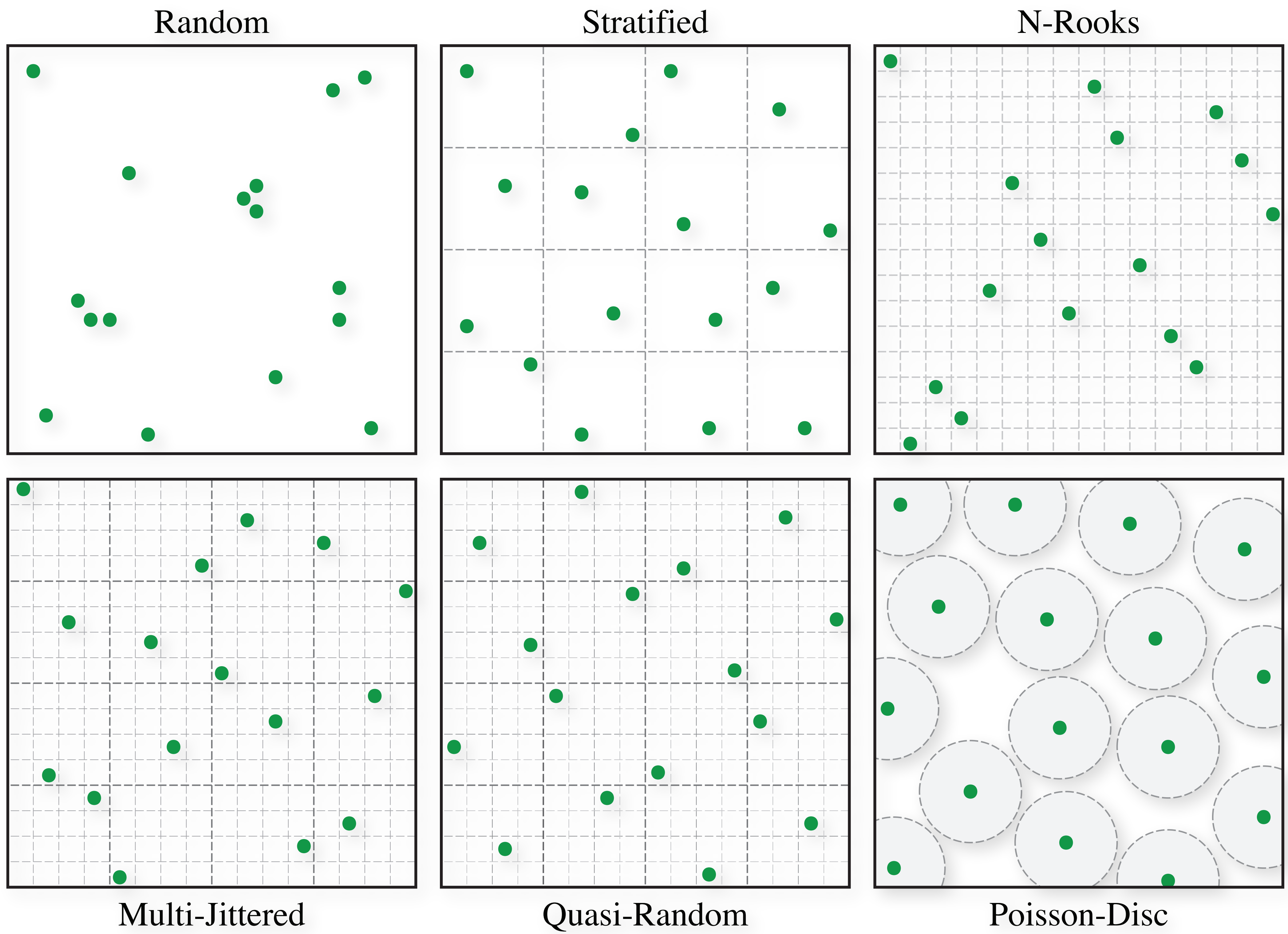
- Other useful sampling domains:
  - triangles
  - 1- or 2-D discrete PDFs (e.g. environment maps)
- Much more!

PBRe3: Chapters 13 & 14

More Integration Dimensions

- Anti-aliasing (image space)
- Light visibility (surface of area lights)
- Depth-of-field (camera aperture)
- Motion blur (time)
- Many lights
- Multiple bounces of light
- Participating media (volume)

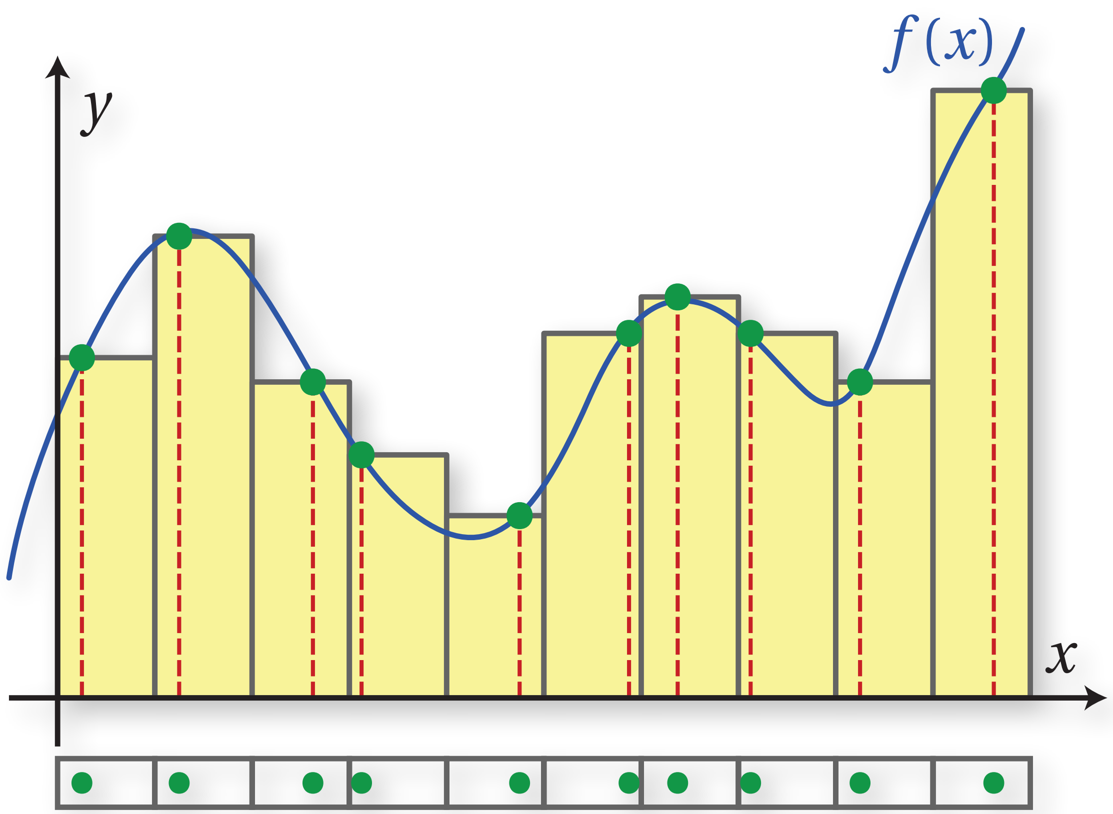
Careful Sample Placement



Variance Reduction: Stratification

- Subdivide integration domain into disjoint regions
- Place one random sample in each region
- Provably cannot increase variance
  - often reduces variance considerably

Stratified Monte Carlo integration



Riemann integration