EC 702
Spring, 2018
Homework 1

Due Wednesday, Feb. 7, 2018

This homework reviews the concept of covariance propagation, and has two initial estimation projects.

1. (Covariance Computation) Consider the linear discrete-time state space model

$$\underline{x}(t+1) = A\underline{x}(t) + G\underline{w}(t)$$
$$y(t) = C\underline{x}(t) + \underline{v}(t)$$

Assume that $\underline{x}(0), \underline{w}(s), \underline{x}(t)$ are jointly Gaussian for all s, t. Assume in addition that $\underline{w}, \underline{v}$ are white and zero mean, with covariance

$$E[\underline{w}(s)\underline{w}^T(t)] = Q\delta(t-s), \qquad E[\underline{v}(s)\underline{v}^T(t)] = R\delta(t-s)$$

and that the initial density of $\underline{x}(0)$ is Gaussian $N(\underline{x}_0, \Sigma_0)$, independent of $\underline{w}, \underline{v}$. Assume in addition that $\underline{w}$ and $\underline{v}$ are correlated, as

$$E[\underline{w}(s)\underline{v}^T(t)] = S\delta(t-s)$$

Obtain a recursive formula for computing the covariances of $\underline{x}(t)$, denoted as $\Sigma_x(t)$ and $\underline{y}(t)$, denoted as $\Sigma_y(t)$.

2. Consider the following *matrix* recursive equation

$$P(t+1) = AP(t)B + Q$$

where $A, B, P(t)$ and Q are square $n \times n$ matrices, and A, B, Q are constant in time. Assume the initial matrix is $P(0) = D$. Prove by induction that

$$P(t) = A^t D B^t + \sum_{n=0}^{t-1} A^{t-1-n} Q B^{t-1-n}$$

3. (Nonlinear Least Squares) The problem of regression can often be posed as a parameter estimation problem. On the web site, there is a file of data

`nls_data.txt`

that contains 100 pairs of data points (x, y), where the x values are in the interval [0,10]. You would like to find a model for $y = f(x)$, assuming some parametric description of the function f. In particular, if $f(x)$ is a polynomial, the parameters would be the coefficients of the polynomial.

Your goal is to estimate the parameters. Unfortunately, you don't know the function parameterization, so you will have to test a few possible parameterizations to test this.

(a) Find the best *linear* fit to the data, assuming a form $z = ax + b$. This is a least-squares problem of the form unknowns a, b: Minimize

$$\sum_{i=1}^{100} (z_i - ax_i - b)^2 = (\underline{z} - a\underline{x} - b\underline{1})^T(\underline{z} - a\underline{x} - b\underline{1}) = (\underline{z} - A\begin{pmatrix} a \\ b \end{pmatrix})^T(\underline{z} - A\begin{pmatrix} a \\ b \end{pmatrix})$$

where $A = \begin{pmatrix} \underline{x} & \underline{1} \end{pmatrix}$, $\underline{1}$ is the vector of all 1s. Setting the gradient equal to 0 gives:

$$A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T \underline{z}$$

In MATLAB, one solves linear equations $Ax = b$ as $x = A\backslash b$. Solve for the optimal values of a, b.

(b) Find the error $\underline{e} = \underline{z} - A\begin{pmatrix} a \\ b \end{pmatrix}$, and compute the error in the fit $e^T e$.

(c) Repeat the above two steps for a quadratic fit $z = a + bx + cx^2$.

(d) Repeat the above two steps for a cubic fit $z = a + bx + cx^2 + dx^3$.

(e) Repeat the above two steps for a quartic fit $z = a + bx + cx^2 + dx^3 + ex^4$.

(f) The next step is to find the best fit for a nonlinear function, of the form

$$z = ce^{ax}$$

for some constants c, d. Initially, this may not appear to be a linear least squares problem. However, consider the logarithmic transformation

$$\log z = \log c + ax = b + ax$$

for $b = \log c$. This is now least-squares problem, with unknowns a, b: Minimize

$$\sum_{i=1}^{20} (\log z_i - ax_i - b)^2 = (\log \underline{z} - a\underline{x} - b\underline{1})^T (\log \underline{z} - a\underline{x} - b\underline{1}) = (\log \underline{z} - A\begin{pmatrix} a \\ b \end{pmatrix})^T (\log \underline{z} - A\begin{pmatrix} a \\ b \end{pmatrix})$$

where $A = \begin{pmatrix} \underline{x} & \underline{1} \end{pmatrix}$, $\underline{1}$ is the vector of all 1s. Setting the gradient equal to 0 gives:

$$A^T A \begin{pmatrix} a \\ b \end{pmatrix} = A^T \log \underline{z}$$

Solve for the optimal values of a, b.

(g) Find the error $\underline{e} = \underline{z} - e^{A\begin{pmatrix} a & b \end{pmatrix}^T}$, and compute the error in the fit $e^T e$.

(h) Which one is the best model?

4. In this problem, you are to implement a simple Kalman filter to estimate the two-dimensional position of a moving particle. The underlying state of the particle is defined in terms of its two-dimensional position and velocity, as

$$\underline{x} = \begin{bmatrix} x \\ v_x \\ y \\ v_y \end{bmatrix}$$

Assume that the initial position is random, and Gaussian distributed, with the identity covariance and mean $\underline{0}$. The dynamics of this state, in sampled time, are described as:

$$\underline{x}(t+1) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

where the process noises $w_1(t), w_2(t)$ are zero-mean, white Gaussian processes independent of the initial condition $\underline{x}(0)$, with mean 0 and variance 0.09.

The observation process is given by:

$$\underline{y}(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \underline{x} + \begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix}$$

where v_1, v_2 are white, zero mean Gaussian processes with variance 0.16 each, independent of $\underline{x}_0, w_1, w_2$.

On the web site, you will find a file **trackdata.txt** that contains the observations $\underline{y}$ for times from 0 to 19 (so 20 pairs of measurements.) Implement a Kalman filter to estimate the updated trajectories of $\underline{x}(t)$. As an output, plot the points $(\hat{x}_1(t|t), \hat{x}_3(t|t))$ for $t = 0, \ldots, 19$. Note that these are the first and third entries in the vector $\underline{x}$.