EC 702
Spring, 2018
Homework 2

Due Tuesday, Feb. 20, 2018

1. (Data Fusion) Consider the following estimation problem:

$$\underline{x}(t+1) = A\underline{x}(t)$$
$$\underline{y}_1(t) = C_1\underline{x}(t) + \underline{v}_1(t); \qquad \underline{y}_2(t) = C_2\underline{x}(t) + \underline{v}_2(t)$$

Assume that A is invertible, and no information is provided for the initial condition $\underline{x}(0)$ (i.e. infinite initial uncertainty). Assume that the noises $\underline{v}_1, \underline{v}_2$ are zero-mean, white, Gaussian and independent of each other, with covariances R_1, R_2 respectively. Note that this system has no process noise, so that the evolution of the state $\underline{x}$ is deterministic.

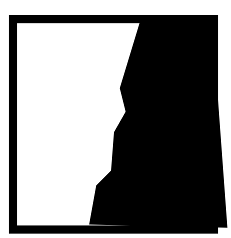
Since we have no information, an appropriate form for this estimation problem is the *information form* of the Kalman Filter, where one propagates the information estimates $P^{-1}(t|t)\hat{x}(t|t)$ and the inverse covariances $P^{-1}(t|t)$. Note that having no prior information corresponds to $P^{-1}(0|-1) = 0$. The above problem is a simpler version of the information form Kalman filter in that no process noise is present.

   (a) Write the information form Kalman filter for the estimate $\hat{\underline{z}}_{12}(t|t) = (P_{12}^{-1}(t|t)\hat{\underline{x}}_{12}(t|t))$ and the evolution of the inverse error covariance $P_{12}^{-1}(t|t)$ when both observations $\underline{y}_1, \underline{y}_2$ are provided from 0 to t.

   (b) Write the information form Kalman filter for the estimate $\hat{\underline{z}}_1(t|t)- = (P_1^{-1}(t|t)\hat{\underline{x}}_1(t|t))$ and the evolution of the inverse error covariance $P_1^{-1}(t|t)$ when only the observations $\underline{y}_1$ are provided.

   (c) Assume you process the observations $\underline{y}_2(0), \ldots, \underline{y}_2(t)$ through another information filter to obtain $\hat{\underline{z}}_2(t|t) = (P_2^{-1}(t|t)\underline{x}_2(t|t))$ and the evolution of the inverse error covariance $P_2^{-1}(t)$. By using the superposition property of linear systems, show how $P_{12}^{-1}(t|t)$ can be directly obtained from $P_1^{-1}(t|t)$ and $P_2^{-1}(t|t)$. Similarly, show how $P_{12}^{-1}(t|t)\hat{\underline{x}}_{12}(t|t)$ can be obtained by combining $P_1^{-1}(t|t)\hat{\underline{x}}_1(t|t)$ and $P_2^{-1}(t|t)\hat{\underline{x}}_2(t|t)$.

2. An important problem in image processing is edge location. Consider the image below. The edge to be located is the quasilinear boundary between the bright and dark portions of the image. A series of N horizontal lines are scanned across this image; these lines are equally spaced vertically, with $t = 0$ corresponding to the bottom of the image and $t = N-1$ corresponding to the top of the image.



   (a) Let $f(t)$ denote the horizontal coordinate of the edge on the t-th line. Assume that a model for $f(t)$ is given by:

$$f(t) = at + b + \sum_{\tau=0}^{t-1} w(\tau), \qquad 0 \le t < N$$

where a, b are statistically independent, zero-mean Gaussian random variables with positive variances σ_a^2, σ_b^2 , and $w(t)$ is a zero-mean, Gaussian white noise sequence of variance q that is statistically independent of a, b. That is, the image consists of a straight line with slope $1/a$, starting at $(b, 0)$, plus some noise, which grows as t increases. Define a state variable

$$\underline{x}(t) = \begin{pmatrix} f(t) \\ a \\ b \end{pmatrix}$$

Write a state equation driven by Gaussian white noise for the $\underline{x}$ process.

(b) Assume that the output electronics on the t-th line is noisy, and provides a horizontal coordinate y corresponding to a noisy version of the true horizontal coordinate.

$$y(t) = f(t) + v(t), \quad 0 \leq t < N$$

where $\{v(t)\}$ is a zero-mean, white-noise Gaussian sequence with variance r, statistically uncorrelated with $a, b, w(s)$. The problem now fits the assumptions of a Kalman filter. Write the relevant equations for the corresponding Kalman filter estimate and variance.

(c) Assume that the initial variances are $\sigma_b^2 = 10000, \sigma_a^2 = 16$, and the process noise $w(t)$ has variance $q = 0.04$. Assume apriori that the mean of a is zero, and the mean of b is 256. Assume that the measurement noise $v(t)$ has variance $r = 100$. Write a MATLAB program (function) to implement a Kalman filter to process a sequence of measurements $y(t)$ stored as a vector Y, and generate the predicted and updated estimates of $\underline{x}$ as a function of time. The input file "linemeas.txt" in the assignment directory contains a vector of 512 mesurements. Process all 512 measurements through the Kalman filter to get the best estimate of the coefficients a, b from the image data. Report the estimates.

(d) Use the outputs of the Kalman filter to estimate the variances of the estimates of a, b at the final time generated after processing all 512 measurements. Report the individual variances of a, b and their cross-covariance. Note that the final estimate of a, b correspond to a fixed-point smoother estimate of these parameters using all the data.

3. Consider the discrete time dynamical system

$$\underline{x}(t+1) = \underline{x}(t) + G\underline{w}(t)$$
$$\underline{y}(t) = C\underline{x}(t) + \underline{v}(t)$$

where $\underline{w}(), \underline{v}()$ are independent, zero-mean Gaussian white-noise sequences with variances Q, R respectively, and $\underline{x}(0)$ is an independent Gaussian random vector with mean 0 and covariance P_0. Let $P(t)$ denote the unconditional covariance of $\underline{x}(t)$.

(a) What is the unconditional expected value of $\underline{x}(t)$, given no measurements?

(b) Compute the expected value of $\underline{w}(t)$ given observation of the exact value of the state $\underline{x}(t+1)$, in terms of the matrices G and $P(t+1)$.

(c) Let $\underline{w}_a(t) = \underline{w}(t) - E\underline{w}(t)|\underline{x}(t+1)$. Show that $\underline{w}_a(t)$ is a white noise sequence! (This is the hardest part ....)

More remarkably, this sequence is independent of $\underline{x}(T)$ (you don't have to show this, but it is interesting...) so that you have constructed a Markov model for designing a Kalman filter which evolves backwards in time!

4. This problem is an example of a model used for tracking objects moving using pulsed radar systems. Radar systems measure time-of-flight of the signal using correlation between the transmitted and received signals, and translate this to an estimate of radial distance from the radar. They also measure angle from the radar because they know the direction of the radar antenna when the pulse was transmitted. Thus, the position and velocity of the objects are often defined in polar coordinates with the radar located at the origin, in

terms of range and range rate, as well as azimuth angle and azimuth angle rate. A dynamical model for the system motion, with sampling time interval T, would be given as:

$$\underline{x}(t+1) = \begin{pmatrix} 1 & T & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & T \\ 0 & 0 & 0 & 1 \end{pmatrix} \underline{x}(t) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

where the state vector is

$$\underline{x}(t) = \begin{pmatrix} r \\ v_r \\ \theta \\ v_\theta \end{pmatrix}$$

The corresponding measurement equation is

$$\underline{y}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \underline{x}(t) + \underline{v}(t)$$

This corresponds to a model with accelerations u_1 denoting the random radial acceleration, and u_2 the random angular acceleration. Usually, these accelerations are not modeled as white noise, but have some correlation, which is described by the dynamical system model:

$$u_1(t+1) = \rho u_1(t) + w_1(t); \quad u_2(t+1) = \rho u_2(t) + w_2(t)$$

where $w_1(t), w_2(t)$ are independent white noise processes. The parameter ρ is inversely related to the average maneuver duration time.

To keep the problem well-grounded, the units of r will be in meters, and θ will be in radians.

(a) Augment the original state variables to obtain a new state space model, with states defined as

$$\underline{x}(t) = \begin{pmatrix} r \\ v_r \\ u_1 \\ \theta \\ v_\theta \\ u_2 \end{pmatrix}$$

(b) Implement the equations of a Kalman filter for the resulting model in MATLAB. Use the following parameter choices: $T = 5$ seconds, $\rho = 0.5$, variance of $w_1(t) = 35^2 = 1225$, variance of $w_2(t) = 0.0001$. The covariance of the measurement noise $\underline{v}(t)$ is given by

$$R = \begin{pmatrix} 10^4 & 0 \\ 0 & 0.0025 \end{pmatrix}$$

which corresponds to a range accuracy of 100 meters standard deviation and angle accuracy of 0.05 radians standard deviation in angle (not our finest radar...) Assume the initial state covariance is given by $\Sigma_0 = \text{diagonal}[10^8, 10^8, 10^4, 100, 100, 100]0^8$.

(c) In the homework folder, there is a file called **radar-measurements.txt**, wich corresponds to 50 measurements of an object flying through this two-dimensional air space, sampled at 5 second intervals. The measurements are stored as a 50 x 2 matrix, with the first row corresponding to the noisy measurement (r, θ) at time 0. Run your Kalman filter with these measurements, and obtain the updated estimate of (r, θ) after 25 measurements are processed ($\hat{\underline{x}}(24|24)$). Record these and the values of the standard deviation of the error in estimating r and θ, obtained from the respective diagonal entries of the updated covariance $P(t|t)$.

(d) Implement a non-causal smoothing algorithm for this problem, and process all 50 measurements to generate the smoothed estimate of (r, θ) corresponding to $\hat{\underline{x}}(24|49)$. Report this value and the standard deviations for estimating r, θ obtained from the respective diagonal entries of the smoothed covariance $P(24|49)$.

(e) Show a single plot comparing the filtered estimate of r versus discrete time t and the smoothed estimate of r versus discrete time t. Show another plot for the filtered estimate of θ versus discrete time t and the smoothed estimate of $theta$.