LeastSquare

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1 Least Squares Approximations

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It often happens that $\mathbf{A}x = b$ has no solution. Several reason are :

- The usual reason is: too many equations.
- The matrix has more rows than columns.
- There are more equations than unknowns (m is greater than n). The n columns span a small part of m-dimensional space.
- Unless all measurements are perfect, b is outside that column space.

So elimination reaches an impossible equation and stops. But we can't stop just because measurements include noise.

When $\mathbf{A}x = b$ has no solution, multiply by $\mathbf{A}^\intercal$ and solve $\mathbf{A}^\intercal\mathbf{A}\hat{x} = \mathbf{A}^\intercal b$. So to get the approximations solution $\hat{x}$, we can get $\hat{x} = (\mathbf{A}^\intercal A)^{-1}\mathbf{A}^\intercal b$.

They are connected by the projection $p = A\hat{x}$.

For example we use : $Ax = b$

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

In this notebook we solve x using least square, get the projection, and the error

[1]:

```
import numpy as np
import math
A = np.array([[1. , 1.],
              [2., 1.],
              [3., 1.]])
b = np.array([1., 2., 2.])
```

[2]:

```
A
```

[2]:

```
array([[1., 1.],
       [2., 1.],
       [3., 1.]])
```

[3]:
```
b
```

[3]:
```
array([1., 2., 2.])
```

1.1 Least Squares Solution

$\hat{x} = (\mathbf{A}^\intercal A)^{-1}\mathbf{A}^\intercal b$

[4]:
```
x = np.linalg.inv(A.transpose().dot(A)).dot(A.transpose().dot(b))
x
```

[4]:
```
array([0.5       , 0.66666667])
```

1.2 The Projection matrix (p)

$p = A\hat{x}$

[5]:
```
p = A.dot(x)
p
```

[5]:
```
array([1.16666667, 1.66666667, 2.16666667])
```

1.3 The error

$e = b - A\hat{x}$ or $e = b - p$

[6]:
```
e = b - A.dot(x)
e
```

[6]:
```
array([-0.16666667,  0.33333333, -0.16666667])
```

[7]:
```
e = b - p
e
```

[7]:
```
array([-0.16666667,  0.33333333, -0.16666667])
```

[8]:
```
# MAE
mae = np.average(abs(e))
mae
```

[8]:
```
0.22222222222222263
```

[9]:
```
# MSE
power = np.power(e,2)
mse = np.average(power)
```

```
mse
```

[9]: 0.05555555555555556

[10]:
```
# RMSE
rmse = math.sqrt(mse)
rmse
```

[10]: 0.23570226039551584

When e is zero, $\hat{x}$ is an exact solution to $\mathbf{A}x = b$

2 References

- http://math.mit.edu/~gs/linearalgebra/ila0403.pdf
- https://medium.com/analytics-vidhya/forecast-kpi-rmse-mae-mape-bias-cdc5703d242d