

## 1. SYLLABUS REVIEW

- (1) Pictures + Computer are ok so long as they're used for note taking.
- (2) Expect for the tests to be at ends of the first third of the class, and the second third of the class.
- (3) Theoretically this is a graduate course, and will be switched to 852, rather than remaining as 452.

### 1.1. Day 1.

- (1) The idea of algebraic topology
- (2) Given topological spaces  $X$  and  $Y$ , how can we prove that  $X$  and  $Y$  are or aren't homeomorphic.
- (3) To prove  $X \cong Y$ , we simply exhibit a homeomorphism.  
E.g.  $(-1, 1) \cong \mathbb{R}$ , using  $f(x) = \frac{x}{1-x^2}$   
E.g.  $\square \cong \circ$
- (4) To prove  $X \not\cong Y$ , we'd find a topological invariant, (connected, compact, Hausdorff,...), that only one has.  
E.g.  $(0, 1) \not\cong [0, 1]$ , here, the closed interval is compact, and the open interval is not.  
E.g.  $(0, 1) \not\cong [0, 1)$ , because,

$$\begin{aligned}[0, 1) \setminus \{0\} &= (0, 1) \text{ which is connected, but} \\ (0, 1) \setminus \{\text{any point}\} &\text{ is disconnected}\end{aligned}$$

Note, with the following exercise, If  $X \cong Y$  via a homeomorphism,  $\psi : X \rightarrow Y$ , then  $X \setminus \{p\} \cong Y \setminus \{\psi(p)\}$

- (5) Show the following.

$$\mathbb{R} \not\cong \mathbb{R}^2$$

Here, we note that  $\mathbb{R} \setminus \{0\}$  is disconnected.

Suppose towards contradiction that  $\mathbb{R} \cong \mathbb{R}^2$ , call the homeomorphism  $\phi : \mathbb{R} \rightarrow \mathbb{R}^2$ , because  $\mathbb{R} \setminus \{0\}$ , the exercise implies that  $\mathbb{R} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{\phi(0)\}$ , and therefore  $\mathbb{R}^2 \setminus \{\phi(0)\}$  is disconnected, but that's just wrong, because  $\mathbb{R}^2$  without a single point is still connected, rigorously showing this should be done through working with path connectedness. Therefore these are not homeomorphic.

$$\mathbb{R}^2 \not\cong \mathbb{R}^3$$

This was a trick question, we don't actually have any topological properties that we can rely on. If we were to attempt to remove a line from  $\mathbb{R}^2$ , we don't have enough information about what the line is homeomorphic to in  $\mathbb{R}^3$ , which is the major stumbling block.

- (6) The Fundamental Group

- (7) The fundamental group is a way to associate a topological space  $X$  to a group  $\pi_1(X)$  so that  $X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$ .
- (8) We'll be able to use this to prove spaces aren't homeomorphic.  
Ex: In this course we'll learn the following.

$$\begin{aligned}\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) &= \mathbb{Z} \\ \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) &= \{1\} \\ \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) &\not\cong \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) \\ \mathbb{R}^2 &\not\cong \mathbb{R}^3\end{aligned}$$

Using this, we can show that these things are not homeomorphic, which is why we do algebraic topology. More powerful tools allow for more results.

- (9) Note: It's not true that  $\pi_1(X) \cong \pi_1(Y) \Rightarrow X \cong Y$   
 More generally, algebraic topology is about associating the topological space  $X$  with the algebraic object  $A(X)$ , in such a way that  $X \cong Y \Rightarrow A(X) \cong A(Y)$   
 There's a spectrum though.
- (a) Easy to compute and says nothing,  $A(x)$  is the same for all of  $X$
  - (b) Hard to compute, but says everything,  $A(X) \cong A(Y) \iff X \cong Y$

## 1.2. Day 2.

- (1) The Fundamental Group
- (2) Idea:  $\pi_1(X) = \{\text{"loops" in } X\} / \sim$ , where  $L_1 \equiv L_2$  if  $L_1$  can be "deformed" inside  $X$  into  $L_2$
- (3) Ex: Last time it was claimed that  $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{Z}$ .
- (4) Paths and Homotopies
- (5) Let  $X$  be a topological space.
- (6) Def: A path in  $X$  is a continuous map  $f : I \rightarrow X$ , where  $I = [0, 1] \subseteq \mathbb{R}$  (with the subspace topology from the Euclidean topology on  $\mathbb{R}$ ).  
 If  $f(0) = p$  and  $f(1) = q$ , we say  $f$  is a path from  $p$  to  $q$ .
- (7) Ex:

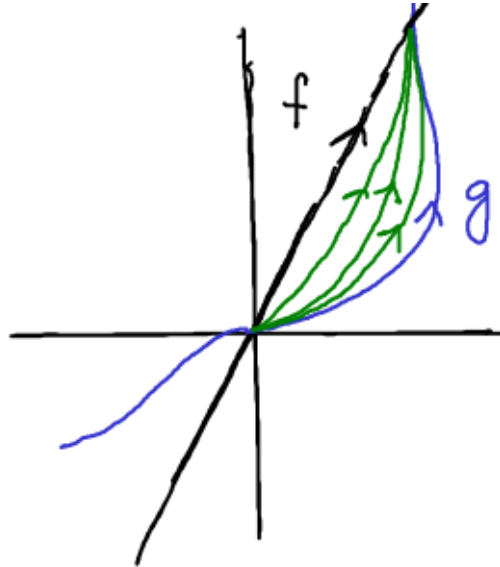
$$\begin{aligned}X &= \mathbb{R}^2 \\ f : I &\rightarrow \mathbb{R}^2 \\ f(t) &= (1 - 2t, 0)\end{aligned}$$

$f$  is a path in  $\mathbb{R}^2$  from  $(1,0)$  to  $(-1,0)$ .

- (8) Another path in  $\mathbb{R}^2$  from  $(1,0)$  to  $(-1,0)$  is,

$$\begin{aligned}g : I &\rightarrow \mathbb{R}^2 \\ g(t) &= (\cos(\pi t), \sin(\pi t))\end{aligned}$$

(9) To make precise, “Deforming” one path into another:



(10) Def: Let  $f$  and  $g$  be paths in  $X$  from  $p$  to  $q$ . A path homotopy from  $f$  to  $g$  is a continuous function,

$$H : I \times I \rightarrow X$$

(note that elements of  $I \times I$  resemble,  $(s, t)$ ) Such that,

$$H(s, 0) = f(s), \forall s$$

$$H(s, 1) = g(s), \forall s$$

$$H(0, t) = p, \forall t$$

$$H(1, t) = q, \forall t$$

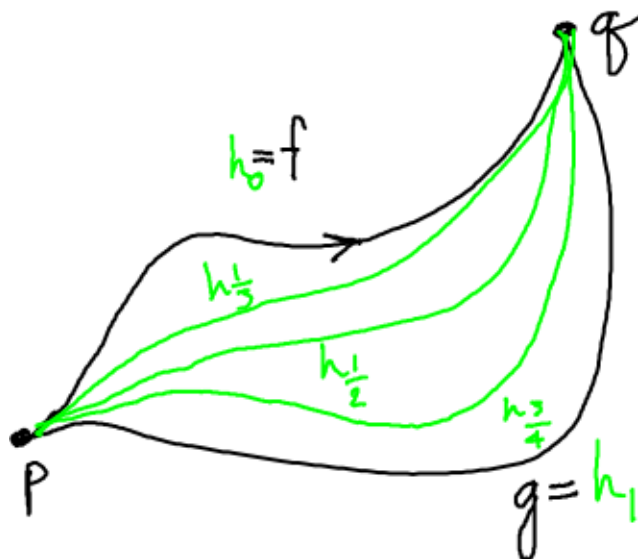
To make sense of this, define,  $\forall t$ ,

$$h_t : I \rightarrow X$$

$$h_t(s) = H(s, t)$$

Then,  $\forall t$ ,

$$h_t = \text{path in } X \text{ from } p \text{ to } q$$



This is continuous because  $H$  is continuous, and it goes from  $p$  to  $q$ , because  $h_t(0) = H(0, t) = p$  and  $h_t(1) = H(1, t) = q$ .  $h_0(s) = f$  because  $h_0(s) = H(s, 0) = f(s)$ ,  $\forall s$  and  $h_1(s) = g$  because  $h_1(s) = H(s, 1) = g(s)$ ,  $\forall s$

- (11) Def: If  $\exists$  a path homotopy from  $f$  to  $g$ , we say  $f$  and  $g$  are path-homotopic, and  $f \cong g$   
Ex:  $X = \mathbb{R}^2$ , Let,

$$f(s) = (\cos(\pi s), \sin(\pi s))$$

$$g(s) = (\cos(\pi s), 2 \sin(\pi s))$$

Both are paths in  $\mathbb{R}^2$  from  $(1, 0)$  to  $(-1, 0)$ .  
 Then,

$$H : I \times I \rightarrow \mathbb{R}^2$$

$$H(s, t) = (\cos(\pi s), (t + 1) \sin(\pi s))$$

$H$  is a path homotopy from  $f$  to  $g$ , because,

$$H(s, 0) = (\cos(\pi s), \sin(\pi s)) = f(s)$$

$$H(s, 1) = (\cos(\pi s), 2 \sin(\pi s)) = g(s)$$

$$H(0, t) = (\cos(0), (t + 1) \sin(0)) = (1, 0) \forall t$$

$$H(1, t) = (\cos(\pi), (t + 1) \sin(\pi)) = (-1, 0) \forall t$$

- (12) Question: Find a path homotopy from  $\mathbb{R}^2$  from  $f(s) = (s, s)$ , and  $g(s) = (s, s^2)$

Answer(June):  $H(s, t) = (s, s^{t+1})$

(see the notebook, there's a solution there. Keep in mind that you want to try to find  $p$  and  $q$  first, before you do anything else)

Answer(Dr. Clader): General Trick In  $\mathbb{R}^2$  let  $f$  and  $g$  be any two paths from  $p$  to  $q$ , then

the straight line homotopy is as follows,

$$H : I \times I \rightarrow \mathbb{R}^2$$

$$H(s, t) = (1 - t) * f(s) + t * g(s)$$

Note that this resembles the stuff you've seen in optimization and advanced linear algebra. This is a pretty powerful tool, remember and fear it.

- (13) Ex: In the question above,  $H(s, t) = (s, (1 - t)s + ts^2)$

### 1.3. Day 3.

- (1) Products of Paths

- (2) Last time: If  $f$  and  $g$  are any two paths in  $\mathbb{R}^2$  from  $p$  to  $q$ , then  $f \cong_p q$ .

- (3) By contrast: In,  $S' = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$  if

$$f(s) = (\cos(\pi s), \sin(\pi s))$$

$$g(s) = (\cos(\pi s), -\sin(\pi s))$$

Then  $f \not\cong_p g$ . (We'll prove this carefully later).

- (4) Fact: (HW)  $\cong_p$  is an equivalence relation on the set {paths in  $X$  from  $x$  to  $y$ } Thus we can consider the set,

$$\{\text{paths in } X \text{ from } x \text{ to } y\} / \cong_p$$

$$= \{\text{path-homotopy classes of paths in } X \text{ from } x \text{ to } y\} \ni [f]$$

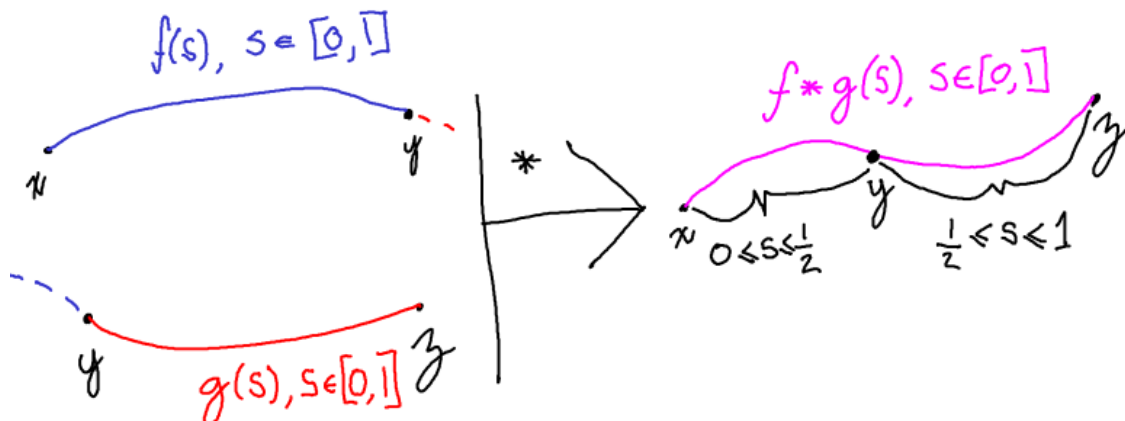
E.g. in the  $S'$  example above,  $[f] \neq [g]$

- (5) Def: Let the following be so,

$X =$  topological space

$f =$  path in  $X$  from  $x$  to  $y$

$g =$  path in  $X$  from  $y$  to  $z$



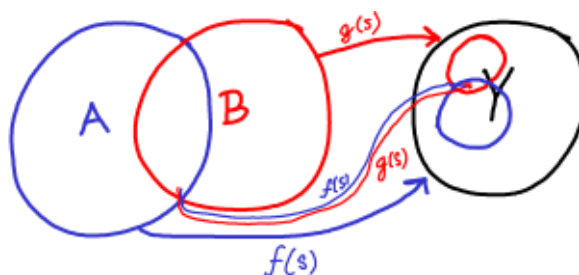
Then the concatenation of  $f$  and  $g$  is the path  $f * g$  from  $x$  to  $z$  given by,

$$f * g : I \rightarrow X$$
$$(f * g)(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

(6) Why is  $f * g$  continuous?

(7) Gluing Lemma: Let the following be so,

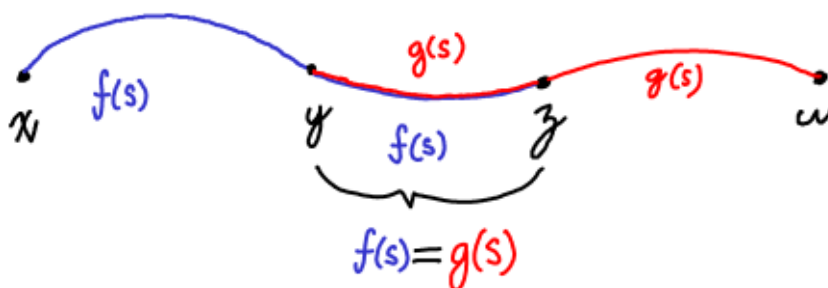
$$\begin{aligned} X &= \text{topological space} \\ A, B &\subseteq X, \text{ closed subsets such that } X = A \cup B \\ Y &= \text{topological space} \end{aligned}$$



Let the following continuous functions be defined,

$$\begin{aligned} f &: A \rightarrow Y \\ g &: B \rightarrow Y \end{aligned}$$

such that  $f(x) = g(x) \forall x \in A \cap B$ .



Then the function,

$$\begin{aligned} h &: X \rightarrow Y \\ h(x) &= \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases} \end{aligned}$$

is continuous. The proof is left as an exercise to the reader. Thanks. (Homework Problem 1)

Note: Applying the gluing lemma to  $I = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$  shows that  $f * g$  is continuous.

(8) Question: Let the following be so,

$$\begin{aligned} X &= \mathbb{R}^2 \\ f(s) &= (s - 1, s) \\ g(s) &= (s, s + 1) \end{aligned}$$

What is  $f * g$ ? Draw a picture.

(9) Answer:

$$f * g = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Which is a straight line from  $(-1, 0)$  to  $(1, 2)$ .

(10) Proposition:  $*$  is well defined on path-homotopy classes of paths  
I.e., if,

$$\begin{aligned} f_0 &\cong_p f_1 \\ g_0 &\cong_p g_1 \end{aligned}$$

then,

$$f_0 * g_0 \cong_p f_1 * g_1$$

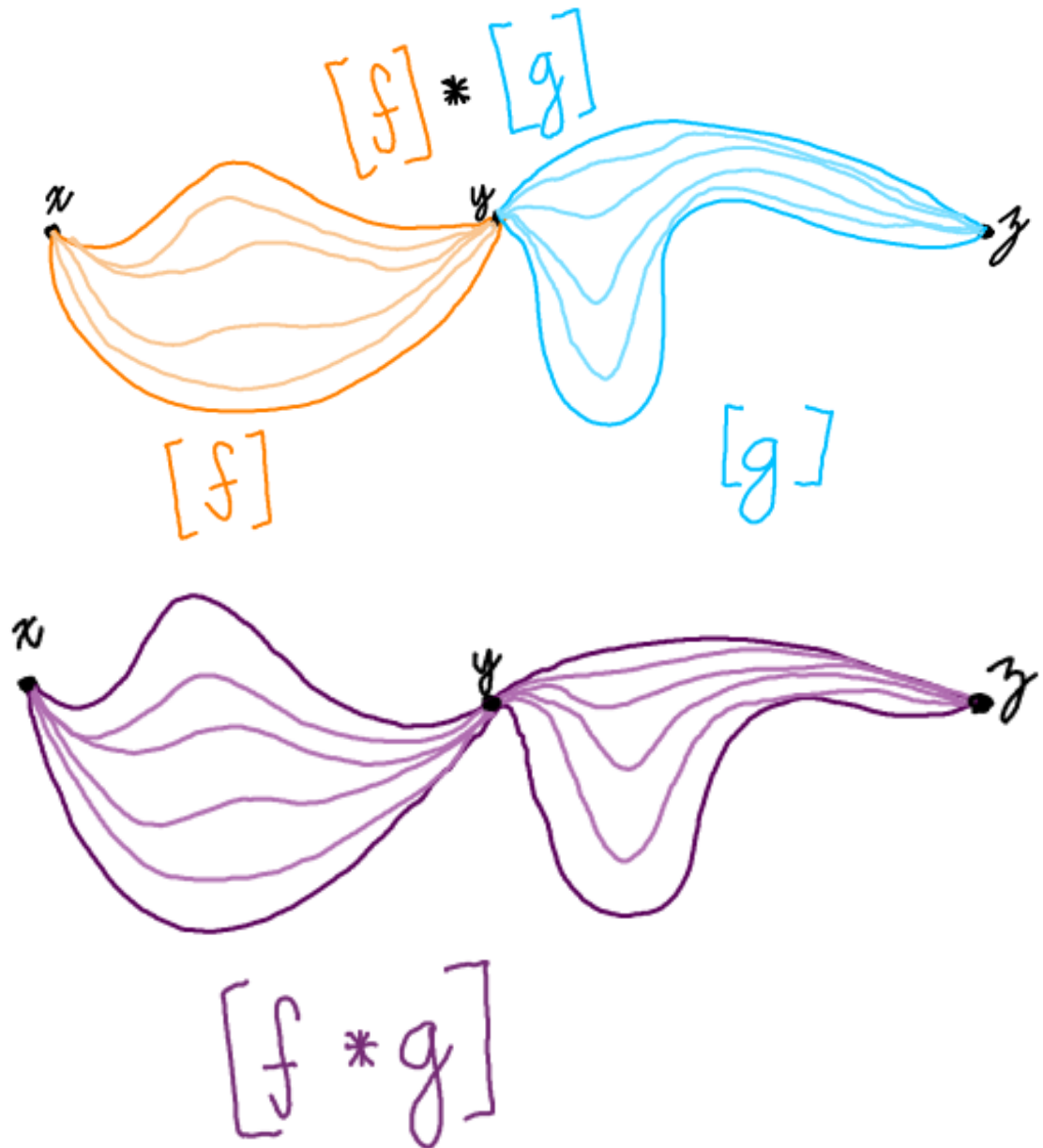
This means that if  $[f] = \{\text{path-homotopy equivalence class of } f\}$  then we can define,

$$[f] * [g] := [f * g]$$

as long as the end point of  $f$  is the starting point of  $g$ .  
So, now  $*$  is an operation.

$$\{\text{paths from } x \rightarrow y\}_{/\cong_p} * \{\text{paths } y \rightarrow z\}_{/\cong_p} \rightarrow \{\text{paths } x \rightarrow z\}_{/\cong_p}$$





(11) Idea of proof of proposition:  
 Let,

$F : I \times I \rightarrow X$  be a path homotopy from  $f_0$  to  $f_1$   
 $G : I \times I \rightarrow X$  be a path homotopy from  $g_0$  to  $g_1$

Then we can define,

$$H : I \times I \rightarrow X$$

$$H(s, t) = \begin{cases} F(2s, t) & \text{if } 0 \leq s \leq \frac{1}{2} \\ G(2s - 1, t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Then,

$$\begin{aligned} h_0 &= H(s, 0) = (f_0 * g_0)(s) \\ h_1 &= H(s, 1) = (f_1 * g_1)(s) \\ h_t &= H(s, t) = (f_t * g_t)(s) \text{ (some path between } x \text{ and } z \text{ )} \end{aligned}$$

So,  $H$  is a path homotopy from  $(f_0 * g_0)$  to  $(f_1 * g_1)$ .

#### 1.4. Day 4.

- (1) Definition of Fundamental Group
- (2) Recall: If,

$$\begin{aligned} f &= \text{path in } X \text{ from } x \text{ to } y \\ g &= \text{path in } X \text{ from } y \text{ to } z \end{aligned}$$

Then,

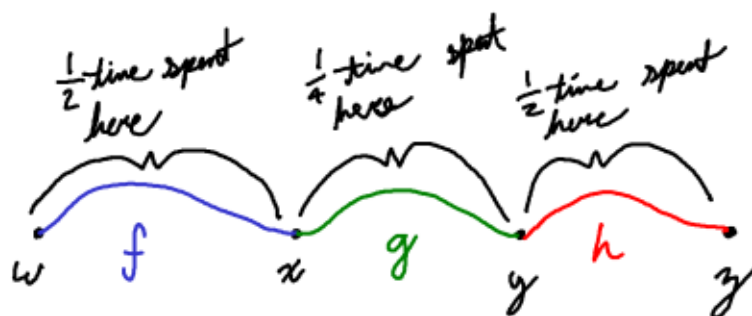
$$[f] * [g] := [\text{concatenation } f * g \text{ of } f \text{ and } g]$$

- (3) Properties of  $*$ :
  - (a)  $*$  is associative, or

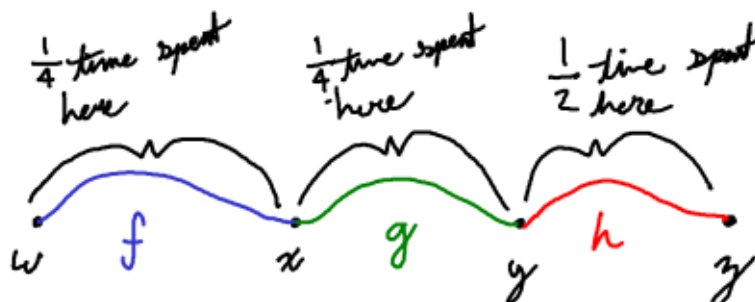
$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

The idea here is that we can adjust the time taken to travel on the path. These two paths are path-homotopic: interpolate between  $f * (g * h)$  and  $(f * g) * h$  by making  $f$  take less and less time and  $h$  take more and more time.

$$f * (g * h):$$



$$(f * g) * h:$$

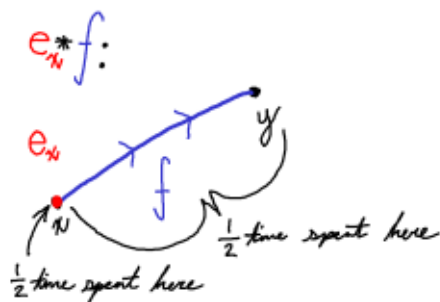


- (b)  $*$  has left/right identities.  
Let

$$e_x : I \rightarrow X$$

$$e_x(s) = x, \forall s \in I, \text{ "constant path at } x \text{"}$$

Then, for all paths  $f$  from  $x$  to  $y$ ,  $[f] * [e_y] = [f]$ , and  $[e_x] * [f] = [f]$ . The premise here is that  $e_x$  or  $e_y$  spend "half the time" sitting at either  $x$  or  $y$ .



These are path-homotopic: interpolate between  $f * e_y$  and  $f$  by making  $f$  take longer and longer.

(c)  $*$  has inverses.

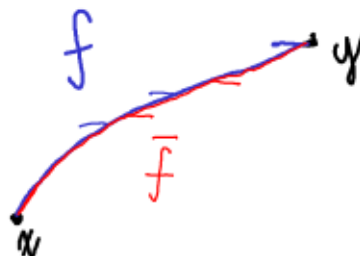
Let  $f$  be a path from  $x$  to  $y$ , and let  $\bar{f}$  be the “reverse” path,

$$\bar{f}(s) = f(1 - s)$$

Then,

$$[f] * [\bar{f}] = [e_x]$$

$$[\bar{f}] * [f] = [e_y]$$



Idea: The verbal gist of this is that the path takes half the time to travel to its destination, and is concatenated with a path that spends half the time to travel to the origin of the original function.

These are path-homotopic: interpolate between  $f * \bar{f}$  and  $e_x$  by doing less and less of  $f$  before turning around.

(d) Let,

$X$  = topological space

$x \in X$

Definition: A loop in  $X$  based at  $x \in X$  is a path,

$$f : I \rightarrow X$$

such that  $f(0) = f(1)$



(e) Observation: If  $f$  and  $g$  are any two loops in  $X$  based at  $x$ , then  $f * g$  is a loop.

(f) Definition: The fundamental group of the  $X$  with basepoint  $x$  is:

$$\pi(X, x) = \{\text{path-homotopy classes of loops in } X \text{ based at } x\}$$

This is a group with the operation  $*$

- (i)  $e_x$  and  $e_y$  are loops.
- (ii)  $f * \bar{f}$  and  $\bar{f} * f$  are also loops.
- (iii) Good question Katy!
- (g) Note: The fact that  $\pi_1(X, x)$  satisfies the axioms of a group, and follows from the properties of  $*$  we just checked.  
(E.g. the identity element is  $[e_x]$ )
- (h) Question: What is  $\pi_1(\mathbb{R}^2, (0, 0))$ ?  
Do you have a guess for  $\pi_1(S', (1, 0))$ ?  
Answer 1:  $\pi_1(\mathbb{R}^2, (0, 0)) \cong \{1\}$   
To prove this, it's enough to show that  $\pi_1(\mathbb{R}^2, (0, 0))$  has just one element, i.e., any loop in  $\mathbb{R}^2$  based at  $(0, 0)$ , is path-homotopic to any other. This is true via the straight line homotopy. Answer 2:  $\pi_1(S', (1, 0)) \cong \mathbb{Z}$ .

### 1.5. Day 5.

- (1)  $\pi_1$  continued: To what extent does  $\pi_1$  depend on  $x$ ?
- (2) Theorem: Let  $X$  be a path-connected topological space, and let  $x_0, x_1 \in X$ , then  $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ . This section builds off the worksheet provided in class.
  - (a) Part 1: see drawing
  - (b) Part 2: Let  $f$  and  $g$  be in  $\pi_1(X, x_1)$ 

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g])\end{aligned}$$
  - (c) Part 3: Let  $f \in \pi_1(X, x_1)$ 

$$\begin{aligned}\hat{\alpha}([f]) &= [\bar{\alpha}] * [f] * [\alpha] \\ \hat{\alpha}([\bar{\alpha}] * [f] * [\alpha]) &= [\alpha] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] \\ &= [f]\end{aligned}$$
  - (d) Therefore this mfer is an isomorphism.
- (3) For which topological spaces  $X$  can we actually compute  $\pi_1(X, x)$ ?
- (4) Definition: A topological space  $X$  is simply-connected if
  - (a)  $X$  is path connected
  - (b)  $\pi_1(X, x) = 1 \ \forall x \in X$   
(Because  $X$  is path connected, we only need to check this for one  $x \in X$ )
- (5) Ex:  $\mathbb{R}^2$  is simply connected
- (6) Intuition:  $X$  is simply-connected if any loop in  $X$  if any loop in  $X$  can be “shrunk down” to a constant loop.  
(for all loops  $f$  in  $X$  saying  $f$  can be “shrunk down” means  $f \cong_p c_x$  where  $c_x$  is a constant path)
- (7) Next time: A convex subset of  $\mathbb{R}^n$  is simply connected.

### 1.6. Day 6.

- (1) Goal: Prove that  $\pi_1(S^1, x) \cong \mathbb{Z}$

- (2) Idea:  $S^1$  can be built by “wrapping  $\mathbb{R}$  around itself”.  
: Concretely, this is

$$p : \mathbb{R} \rightarrow S^1$$

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

We’ll try to “unwrap” loops in  $S^1$  to get paths in  $\mathbb{R}$

- (3) The above map  $p$  is an example of a “covering map”. The ultimate goal of today is to understand what it means to be a covering map, before we get to the definition of it.

- (4) Questions: Let the following be so,

$$u_1 = \{(x, y) \in S^1 | y > 0\}$$

$$u_2 = \{(x, y) \in S^1 | x > 0, y < 0\}$$

Include the drawings from class, really get sick wit it.

- (5) Observation: For any particular  $n \in \mathbb{Z}$ , the piece,

$$(n, n + \frac{1}{2}) \cong u_1$$

The homeomorphism in Dr. Clader’s mind is,

$$\phi : (n, n + \frac{1}{2}) \rightarrow u_1$$

$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$

i.e.  $\phi = p|_{(n, n + \frac{1}{2})}$

The inverse of  $\phi$  is,

$$\phi^{-1} : u_1 \rightarrow (n, n + \frac{1}{2})$$

$$\phi^{-1} = \frac{\cos^{-1}(x)}{2\pi} + n$$

(Recall: by definition  $\cos^{-1}(x) \in [0, \pi]$ )

Similarly, for  $u_2$  for any particular  $n \in \mathbb{Z}$ ,  $(n - \frac{1}{4}, n) \cong u_2$ .

- (6) Definition: Let  $p : E \rightarrow B$  be a function between two topological spaces. We say  $p$  is a covering map if  $p$  is,

- (a)  $p$  is continuous and surjective  
(b)  $\forall b \in B$  there exists a neighborhood  $u$  of  $b$  such that,

$$p^{-1}(u) = \cup_{\alpha} v_{\alpha}$$

where  $v_{\alpha} \subseteq E$  are open, disjoint and,

$$p|_{v_{\alpha}} : v_{\alpha} \rightarrow u$$

is a homeomorphism for every  $\alpha$ . Note that these open subsets with this property are called evenly covered

Note that  $b$  is one particular point or neighborhood, but there should be a neighborhood for every single point in  $B$  where all of this junk holds reasonably truish.

(7) Ex:

$$p : \mathbb{R} \rightarrow S^1$$

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

$p$  is a covering map. We just showed that  $u_1$  is evenly covered:

$$p^{-1}(u_1) = \cup_{n \in \mathbb{Z}} (n, n + \frac{1}{2})$$

Note that in this case the  $(n, n + \frac{1}{2})$  are the  $v_\alpha$  from the definition of covering maps.  $u_2$  is also evenly covered, but,  $U = S^1$  is not evenly covered because,  $p^{-1}(S^1) = \mathbb{R}$ , and the only way to write  $\mathbb{R}$  as a union of disjoint open sets  $v_\alpha$ , is to take  $v_\alpha = \mathbb{R}$ , but  $\mathbb{R} \not\cong S^1$

(8) Ex:

$$B = \text{any space}$$

$$E = B \times \{1, 2, \dots, n\} = n \text{ discrete copies of } B$$

Where  $\{1, 2, \dots, n\}$  is equipped with the discrete topology.

## 1.7. Day 7.

- (1) Guest lecturer: Mattias “i think your regular lecturer is more qualified for this” Beck
- (2) Recalling the definition of an evenly covered set. New notation was introduced, but L<sup>A</sup>T<sub>E</sub>X is behind the times. Let  $E$  and  $B$  be topological spaces

$$\phi : E \rightarrow B$$

$$\forall b \in B, \exists u \text{ a neighborhood of } b : p^{-1}(u) = \cup_\alpha v_\alpha$$

$$p|_{v_\alpha} : v_\alpha \rightarrow u$$

(3) Fun notation facts:

$\rightarrow$  indicates a surjective function

$\hookrightarrow$  indicates an injective function

Combining the two gives you a bijective function, but that symbol doesn't exist in latex apparently.

(4) Example covering:

$$E = \mathbb{R}$$

$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$

$$B = S^1$$

- (5) Definition: Given a covering map from topological spaces  $E$  to  $B$

$$p : E \rightarrow B$$

a path in our topological space  $B$ ,

$$f : I \rightarrow B$$

A lift of  $f$  is a path,  $\tilde{f} : I \rightarrow E$ , such that  $f = p \circ \tilde{f}$

- (6) Theorem: Given covering map  $p : E \rightarrow B$ ,  $p(e) = b$ ,  $f : I \rightarrow B$  path beginning at  $b$ , then there does not exist a left  $\tilde{f}$ , of  $f$  beginning at  $e$  Read Lemma 54.1 Munkres. (?!?!?)
- (7) The same theorem but reworded:  
Let the following be so,

$E$  be a topological space

$B$  be a topological space

$p : E \rightarrow B$  a covering map

$f : I \rightarrow B$  path beginning at  $b$

$e \in E$ , s.t.  $p(e) = b$

Then there exists a unique path,  $\tilde{f}$  in  $E$  such that  $p \circ \tilde{f} = f$ , and  $\tilde{f}(0) = e$

### 1.8. Day 8.

- (1) Guest Lecturer: Matthias “you can have a hint, but you can’t quote me on it” Beck
- (2) ???????

### 1.9. Day 9.

- (1) Guest Lecturer: Anastasia the Assassin, Deadly David, and Killa Katy
- (2) Let  $p$  be a covering map.

$$p : E \rightarrow B$$

Let,  $e \in E$ ,  $b \in B$ , such that  $p(e) = b$ .

Summary of what we know about this situation,

- (a) Any path  $f$  in  $B$ , beginning at  $b$  has a unique lift  $\tilde{f}$  to a path in  $E$  beginning at  $e$ .
- (b) If  $f$  and  $g$  are two paths in  $B$ , beginning at  $b$ , such that  $f \cong_p g$ , then  $\tilde{f} \cong_p \tilde{g}$
- (c) If  $f$  is a loop in  $B$  based at  $b$ , then  $\tilde{f} \in p^{-1}(b)$

### 1.10. Day 10.

- (1)  $p_1(S^1)$ , continued:
- (2) Recap:

$$p : \mathbb{R} \rightarrow S^1$$

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$



Then there exists a function,

$$\begin{aligned}\phi : \pi_1(S^1, b) &\rightarrow p^{-1}(b) \\ \phi([f]) &= \tilde{f}(1)\end{aligned}$$

Where  $\tilde{f}$  is the lift of  $f$  to  $\mathbb{R}$  starting at 0.  
E.g., (draw that spiraleboye)

$$\begin{aligned}\phi([\text{loop once counterclockwise}]) &= 1 \\ \phi([\text{loop twice counterclockwise}]) &= 2 \\ \phi([\text{loop once clockwise}]) &= -1\end{aligned}$$

The fact that there exists a unique lift,  $\tilde{f}$  of any  $f$  is a feature of covering maps.  
In fact,

$$p^{-1}(b) = \mathbb{Z}$$

and,

(3) Claim:  $\phi : \pi_1(S^1, b) \rightarrow \mathbb{Z}$  is a bijection.

*Proof.* (a) Surjective: Given  $c \in \mathbb{Z}$ , choose a path,  $\alpha : I \rightarrow \mathbb{R}$ , from 0 to  $c$  in  $\mathbb{R}$ . Then let,  
 $f : I \rightarrow S^1$  be  $f = p \circ \alpha$   
Then  $f$  is a loop in  $S^1$  based at  $b = (1, 0)$  because

$$\begin{aligned}f(0) &= p(\alpha(0)) = p(0) = (1, 0) \\ f(1) &= p(\alpha(1)) = p(c) = (1, 0)\end{aligned}$$

And,  $\tilde{f} = \alpha$  because  $p \circ \tilde{f} = p \circ \alpha = f$ . Thus,

$$\phi([f]) = \tilde{f}(1) = \alpha(1) = c$$

(b) Injective: Suppose,

$$\begin{aligned}\phi([f]) &= \phi([g]) \\ \implies \tilde{f}(1) &= \tilde{g}(1)\end{aligned}$$

Then,  $\tilde{f}$  and  $\tilde{g}$  are two paths in  $\mathbb{R}$ , that both start at 0 and both end at the same point.  
 $\Rightarrow$  (courtesy of homework 2)  $\tilde{f} \cong_p \tilde{g}$  (because  $\mathbb{R}$  is simply connected)  
 $\Rightarrow p \circ \tilde{f}$  is a path homotopy from  $p \circ \tilde{f}$  to  $p \circ \tilde{g}$ .  
 $\Rightarrow f \cong_p g$   
 $\Rightarrow [f] = [g] \in \pi_1(S^1, b)$

□

(4) Claim:  $\phi$  is a group homomorphism ( thus, an isomorphism ).

*Proof.* Let  $[f], [g] \in \pi\pi_1(S^1, b)$ , we want to show that,  $\phi([f] * [g]) = \phi([f]) + \phi([g])$   
 By definition,

$$\phi([f] * [g]) = \phi([f * g]) = f * \tilde{g}(1)$$

What is  $f * \tilde{g}$ ? By definition  $f * \tilde{g}$  is the lift of  $f * g$  starting at 0 and,

$$\begin{aligned}\tilde{f} &= \text{lift of } f \text{ starting at 0 ending at some } n \\ \tilde{g} &= \text{lift of } g \text{ starting at 0 ending at some } m\end{aligned}$$

So,  $\tilde{f} * \tilde{g}$  doesn't make sense, but let:

$$\begin{aligned}\tilde{g}' &= \text{"shift } \tilde{g} \text{ by } n \text{"} \\ \text{i.e., } \tilde{g}' &= g(s) + n\end{aligned}$$

Now notice that  $\tilde{f} * \tilde{g}'$  now makes sense, and  $\tilde{g}'$  is a lift of  $g$ , because:

$$\begin{aligned}(p \circ \tilde{g}')(s) &= p(\tilde{g}(s)) \\ &= p(\tilde{g}(s) + n) \\ &= p(\tilde{g}(s)) \\ \text{because } p(x + n) &= p(x), \forall n \in \mathbb{Z} \\ &= (p \circ \tilde{g})(s) \\ &= g(s)\end{aligned}$$

Thus,  $\tilde{f} * \tilde{g}'$  is a lift of  $f * g$  starting at 0

$$\begin{aligned}\implies \tilde{f} * \tilde{g}' &= f * \tilde{g} \\ f * \tilde{g}'(1) &= \tilde{f} * \tilde{g} \\ &= \text{endpoint of } \tilde{g}' \\ &= \tilde{g}'(1) + n \\ &= m + n\end{aligned}$$

This shows that

$$\begin{aligned}\phi([f] * [g]) &= m + n \\ &= \tilde{f}(1) + \tilde{g}(1) \\ &= \phi([f]) + \phi([g])\end{aligned}$$

□

(5) We want:

$$X \cong Y \implies \pi_1(X, x) \cong \pi_1(Y, y)$$

(X is homeomorphic to Y)

The big tool we'll use to do that is the tool from the second homework about maps between spaces being homomorphisms. That's for next time!

### 1.11. Day 11.

- (1) Note that this Friday, office hours will be at 3-4pm.
- (2) We want: If  $X \cong Y$ , then  $\pi_1(X, x) \cong \pi_1(Y, y)$ , or that, if two spaces are homeomorphic, then their fundamental groups are isomorphic. We will explore the tools used to show this in this lecture
- (3) Definition(HW2): Let  $\varphi : X \rightarrow Y$ , be a continuous map, then the homomorphism induced by  $\varphi$  is:

$$\begin{aligned}\varphi_* : \pi_1(X, x) &\rightarrow \pi_1(Y, y) \\ \varphi_*([f]) &= [\varphi \circ f]\end{aligned}$$

See the picture of the picture drawn on the board, make a drawyboye.

- (4) Lemma: (this is referred to lemma 1) If

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

Where  $\varphi$  and  $\psi$  are both continuous, then,

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$$

Additionally, (This is referred to as lemma 2)

$$id_* = id$$

(or that given the  $id : X \rightarrow Y$ , the induced homomorphism,  $\pi_1(X, x) \rightarrow \pi_1(Y, y)$  is the identity)

- (5) *Proof.* (a) Both sides are homomorphisms,

- (1)  $\pi_1(X, x) \rightarrow \pi_1(Z, (\psi \circ \varphi)(x))$

- (2)

Given any  $[f] \in \pi_1(X, x)$ :

$$\begin{aligned}(\psi \circ \varphi)_*([f]) &= [(\psi \circ \varphi) \circ f] \\ &= [\psi \circ (\varphi \circ f)] \\ &= \psi_*[\varphi \circ f] \\ &= \psi_*(\varphi_*([f])) \\ &= (\psi_* \circ \varphi_*)([f])\end{aligned}$$

(b) Given any  $[f] \in \pi_1(X, x)$ :

$$\begin{aligned} id_*([f]) &= [id \circ f] \\ &= [f] \end{aligned}$$

□

(6) Theorem: if  $\varphi : X \rightarrow Y$  is a homeomorphism, then  $\varphi_* : \pi_1(X, x) \rightarrow \pi_1(Y, y)$  is an isomorphism.

*Proof.* We already know that  $\varphi_*$  is a homomorphism, to prove that it's a bijection, we'll find an inverse to  $\varphi_*$ . Claim that,

$$(\varphi)_* : \pi_1(Y, \varphi(x)) \rightarrow \pi_1(X, x)$$

is the inverse to  $\varphi_*$ .

(Note that this is doable, because  $\varphi$  is a homeomorphism,  $\varphi^{-1} : Y \rightarrow X$  exists, and is continuous)

To check this:

$$\begin{aligned} &\varphi_* \circ (\varphi^{-1})_* \\ &= (\varphi \circ \varphi^{-1})_*, \text{ by lemma 1 shown today} \\ &= id_*, \text{ by definition of } \varphi^{-1} \text{ (identity on } y) \\ &= id, \text{ by lemma 2 shown today (identity on } x) \\ &(\varphi^{-1})_* \circ \varphi_* = (\varphi^{-1} \circ \varphi)_* = id_* = id \end{aligned}$$

This by definition means  $\varphi_*$  and  $(\varphi^{-1})_*$  are inverse functions. Additionally, this small red box has made it onto the board, for clarification.

$$\begin{aligned} id_x : X &\rightarrow Y \\ id_{\pi_1(X, x)} : \pi_1(X, x) &\rightarrow \pi_1(X, x) \\ \text{Lemma: } (id_x)_* id_{\pi_1(X, x)} & \end{aligned}$$

□

(7) This ends up proving that,

$$X \cong Y \implies \pi_1(X, x) \cong \pi_1(Y, \varphi(x))$$

But, non-homeomorphic spaces can have isomorphic  $\pi_1$

Ex:

$$\begin{aligned} X &= . \\ Y &= \mathbb{R}^2 \end{aligned}$$

These are not homeomorphic, clearly  $X$  is compact and  $Y$  isn't, but their fundamental groups are isomorphic, since the fundamental group of  $X$  is just  $\{1\}$ , and clearly this is also true about  $\mathbb{R}^2$

(8) So, given  $X$  and  $Y$ , how can we tell if  $\pi_1(X) \cong \pi_1(Y)$ ?

(9) Homotopy of Maps:

Definition: Let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  be continuous functions. Then a homotopy from  $f$  to  $g$  is a continuous function,

$$H : X \times I \rightarrow Y$$

such that,

$$H(x, 0) = f(x), \forall x \in X$$

$$H(x, 1) = g(x), \forall x \in X$$

Our goal is to make remark about the lower star versions of these maps, given their being homotopic.

1.12. **Day 12.**

(1) Homotopy of maps: Definition: Let  $f : X \rightarrow Y$  be a continuous function. A homotopy from  $f$  to  $g$  is a continuous function,

$$H : X \times I \rightarrow Y$$

such that

$$H(x, 0) = f$$

$$H(x, 1) = g$$

We'll often write,

$$h_t : X \rightarrow Y$$

$$h_t(x) = H(x, t)$$

Then there's one  $h_t$  for each  $t \in I$  and,

$$h_0 = f$$

$$h_1 = g$$

$h_t$  = "A function interpolating between  $f$  and  $g$ "

(2) Terminology/Notation: If there exists a homotopy from  $f$  to  $g$ , we'll say that  $f$  is homotopic to  $g$  and write  $f \cong g$ .

(3) Ex:

$$f : S^1 \rightarrow \mathbb{R}^2$$

$$g : S^1 \rightarrow \mathbb{R}^2$$

$$f(x, y) = (x, y)$$

$$g(x, y) = (0, 0)$$

Then  $f \cong g$ . A homotopy from  $f$  to  $g$  is,

$$H : S^1 \times I \rightarrow \mathbb{R}^2$$

$$H((x, y), t) = ((1 - t)x, (1 - t)y)$$

Do the drawing from the board.

(4) Ex:

$$f : \mathbb{R} \rightarrow \mathbb{R}$$

$$g : \mathbb{R} \rightarrow \mathbb{R}$$

$$f(x) = x$$

$$g(x) = x + 2$$

Then  $f \cong g$ . A homotopy from  $f$  to  $g$  is:

$$H : \mathbb{R} \times I \rightarrow \mathbb{R}$$

$$H(x, t) = x + 2t$$

Refer again to the picture from the board.

(5) Questions:

(a)

$$f : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$g : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$f(x) = (x, 0)$$

$$g(x) = (x, e^x)$$

(b)

$$f : \mathbb{R}^2 \setminus (0, 0) \rightarrow \mathbb{R}^2 \setminus (0, 0)$$

$$g : \mathbb{R}^2 \setminus (0, 0) \rightarrow \mathbb{R}^2 \setminus (0, 0)$$

$$f(x) = (x, y)$$

$$g(x) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$g : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$f(x) = (x, 0)$$

$$g(x) = (x, e^x)$$

Just use the straight line homotopy it's not hard.  
Maybe include the drawings?

- (6) Definition: Let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  be continuous, and let  $x_0 \in X$  be such that  $f(x_0) = g(x_0) = y_0$ . Then a homotopy from  $f$  to  $g$  relative to  $x_0$  is a homotopy  $H : X \times I \rightarrow Y$  from  $f$  to  $g$  such that  $h_t(x_0) = y_0, \forall t$ .  
 (“ $x_0$  doesn’t move during the homotopy”)
- (7) Ex: in the second part of the questions from today,  $H$  was a homotopy relative to  $(1, 0)$ , or to any other point on the unit circle.
- (8) Ex:

$$X = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

(it’s the 2 norm ball)

$$f : X \rightarrow X$$

$$g : X \rightarrow X$$

Then,

$$H : X \times I \rightarrow X$$

$$H((x, y), t) = (1 - t)x, (1 - t)y)$$

is a homotopy relative to  $(0, 0)$ .

- (9) Theorem: If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are homotopic relative to  $x_0$ , then:

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

$$g_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

are the same homomorphism.