1 Day 1

1.1 Syllabus Junk

- Pictures + Computer are ok so long as they're used for note taking.
- Expect for the tests to be at ends of the first third of the class, and the second third of the class.
- Theoretically this is a graduate course, and will be switched to 852, rather than remaining as 452.

1.2 The idea of algebraic topology

- Given topological spaces X and Y, how can we prove that X and Y are or aren't homeomorphic.
- To prove $X \cong Y$, we simply exhibit a homeomorphism.
  E.g. $(-1,1) \cong \mathbb{R}$, using $f(x) = \frac{x}{1-x^2}$
  E.g. $\square \cong \circ$
- To prove $X \ncong Y$, we'd find a topological invariant, (connected, compact, Hausdorff,...), that only one has.
  E.g. $(0,1) \ncong [0,1]$, here, the closed interval is compact, and the open interval is not.
  E.g. $(0,1) \ncong [0,1)$, because,

  $$\begin{aligned} [0,1) \setminus \{0\} = (0,1) &\text{ which is connected, but} \\ (0,1) \setminus \{\text{any point}\} &\text{ is disconnected} \end{aligned}$$

  Note, with the following exercise, If $X \cong Y$ via a homeomorphism, $\psi : X \to Y$, then $X \setminus \{p\} \cong Y \setminus \{\psi(p)\}$
- Show the following.

  $$\mathbb{R} \ncong \mathbb{R}^2$$

  Here, we note that $\mathbb{R} \setminus \{0\}$ is disconnected.
  Suppose towards contradiction that $\mathbb{R} \cong \mathbb{R}^2$, call the homeomorphism $\phi : \mathbb{R} \to \mathbb{R}^2$, because $\mathbb{R} \setminus \{0\}$, the excercise implies that $\mathbb{R} \setminus \{0\} \cong \mathbb{R}^2\{\phi(0)\}$, and therefore $\mathbb{R}^2 \setminus \{\phi(0)\}$ is disconnected, but that's just wrong, because $\mathbb{R}^2$ without a single point is still connected, rigorously showing this should be done through working with path connectedness. Therefore these are not homeomorphic.

  $$\mathbb{R}^2 \ncong \mathbb{R}^3$$

  This was a trick question, we don't actually have any topological properties that we can rely on. If we were to attempt to remove a line from $\mathbb{R}^2$, we don't have enough information about what the line is homeomorphic to in $\mathbb{R}^3$, which is the major stumbling block.
- <u>The Fundamental Group</u>

- The fundamental group is a waay to associate a topological space X to a group $\pi_1(X)$ so that $X \cong Y \Rightarrow \pi_1(X) \cong \pi_2(Y)$.
- We'll be able to use this to prove spaces aren't homeomorphic.
  Ex: In this course we'll learn the following.

$$\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{Z}$$
$$\pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) = \{1\}$$
$$\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \ncong \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\})$$
$$\mathbb{R}^2 \ncong \mathbb{R}^3$$

  Using this, we can show that these things are not homeomorphic, which is why we do algebraic topology. More powerful tools allow for more results.

- Note: It's not true that $\pi_1(X) \cong \pi_2(Y) \Rightarrow X \cong Y$
  More generally, algebraic topology is about associating the topological space X with the algebraic object $A(X)$, in such a way that $X \cong Y \Rightarrow A(X) \cong A(Y)$
  There's a spectrum though.
  1. Easy to compute and says nothing, $A(x)$ is the same for all of X
  2. Hard to compute, but says everything, $A(X) \cong A(Y) \iff X \cong Y$

2 Day 2

2.1 The Fundamental Group

- Idea: $\pi_1(X) = {\{\text{"loops" in X}\}}/{\sim}$, where $L_1 \equiv L_2$ if L_1 can be "deformed" inside X into L_2
- Ex: Last time it was claimed that $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{Z}$.
- Paths and Homotopies
- Let X be a topological space.

Definition 1. A path in X is a continuous map $f : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$ (with the subspace topology from the Euclidean topology on $\mathbb{R}$.)
If $f(0) = p$ and $f(1) = q$, we say f is a path from p to q.

- Ex:

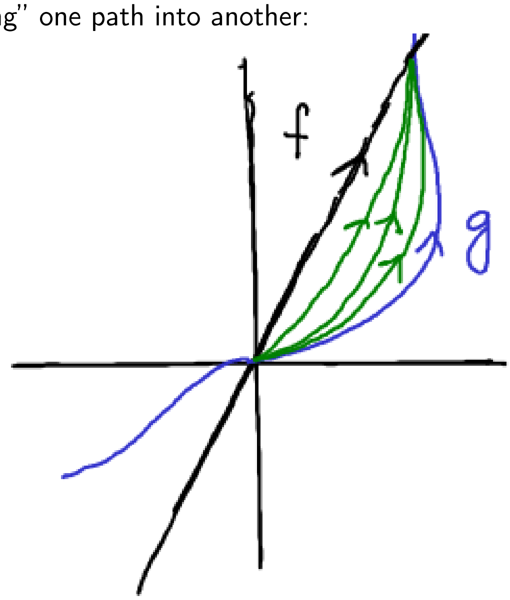
$$X = \mathbb{R}^2$$
$$f : I \to \mathbb{R}^2$$
$$f(t) = (1-2t, 0)$$

  f is a path in $\mathbb{R}^2$ from $(1,0)$ to $(-1,0)$.

- Another path in $\mathbb{R}^2$ from $(1,0)$ to $(-1,0)$ is,

$$g : I \to \mathbb{R}^2$$
$$g(t) = (\cos(\pi t), \sin(\pi t))$$

- To make precise, "Deforming" one path into another:



Definition 2. Let f and g be paths in X from p to q. A path homotopy from f to g is a continuous function,

$$H : I \times I \to X$$

(note that elements of $I \times I$ resemble, (s,t)) Such that,

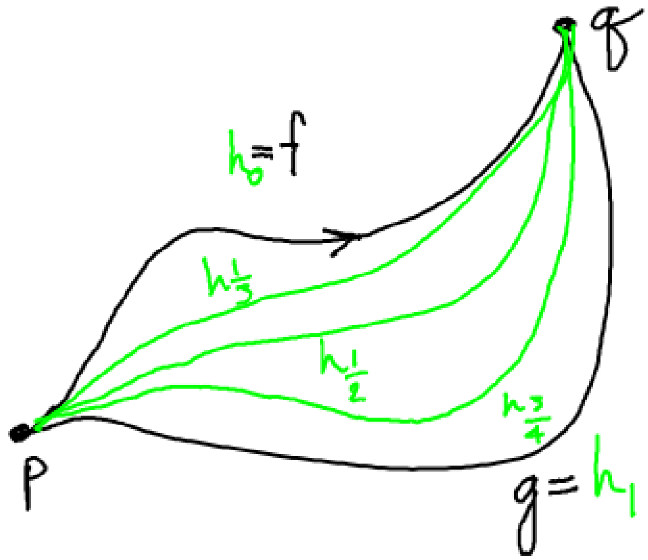
$$H(s,0) = f(s),\ \forall s$$
$$H(s,1) = g(s),\ \forall s$$
$$H(0,t) = p,\ \forall s$$
$$H(1,t) = q,\ \forall s$$

- To make sense of this, define, $\forall t$,

$$h_t : I \to X$$
$$h_t(s) = H(s,t)$$

Then, $\forall t$,

$$h_t = \text{ path in } X \text{ from } p \text{ to } q$$



This is continuous because H is continuous, and it goes from p to q, because $h_t(0) = H(0,t) = p$ and $h_t(1) = H(1,t) = q$. $h_0(s) = f$ because $h_0(s) = H(s,0) = f(s),\ \forall s$
and $h_1(s) = g$ because $h_1(s) = H(s,1) = g(s),\ \forall s$

Definition 3. If $\exists$ a path homotopy from f to g, we say f and g are path-homotopic, and $f \cong g$
Ex: $X = \mathbb{R}^2$, Let,

$$f(s) = (\cos(\pi s), \sin(\pi s))$$
$$f(s) = (\cos(\pi s), 2\sin(\pi s))$$

Both are paths in $\mathbb{R}^2$ from $(1,0)$ to $(-1,0)$.
Then,

$$H : I \times I \to \mathbb{R}^2$$
$$H(s,t) = (\cos(\pi s), (t+1) * \sin(\pi s))$$

H is a path homotopy from f to g, because,

$$H(s,0) = (\cos(\pi s), \sin(\pi s)) = f(s)$$
$$H(s,1) = (\cos(\pi s), 2\sin(\pi s)) = g(s)$$
$$H(0,t) = (\cos(0), (t+1)\sin(0)) = (1,0)\forall t$$
$$H(1,t) = (\cos(\pi), (t+1)\sin(\pi)) = (-1,0)\forall t$$

- Question: Find a path homotopy from $\mathbb{R}^2$ from $f(s) = (s,s)$, and $g(s) = (s, s^2)$
  Answer(June): $H(s,t) = (s, s^{t+1})$
  (see the notebook, there's a solution there. Keep in mind that you want to try to find p and q

first, before you do anything else)
Answer(Dr. Clader): General Trick In $\mathbb{R}^2$ let f and g be any two paths from p to q, then the straight line homotopy is as follows,

$$H : I \times I \to \mathbb{R}^2$$
$$H(s,t) = (1-t) * f(s) + t * g(s)$$

Note that this resembles the stuff you've seen in optimization and advanced linear algebra. This is a pretty powerful tool, remember and fear it.

- Ex: In the question above, $H(s,t) = (s, (1-t)s + ts^2)$

3 Day 3

3.1 Products of Paths

- Last time: If f and g are any two paths in $\mathbb{R}^2$ from p to q, then $f \cong_p q$.
- By contrast: In, $S' = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ if

$$f(s) = (\cos(\pi s), \sin(\pi s))$$
$$g(s) = (\cos(\pi s), -\sin(\pi s))$$

  Then $f \not\cong_p g$. (We'll prove this carefully later).

- Fact: (HW) $\cong_p$ is an equivalence relation on the set $\{$paths in X from x to $y\}$ Thus we can consider the set,

$$\{\text{paths in } X \text{ from } x \text{ to } y\} /_{\cong_p} = \{\text{path-homotopy classes of paths in } X \text{ from } x \text{ to } y\} \ni [f]$$

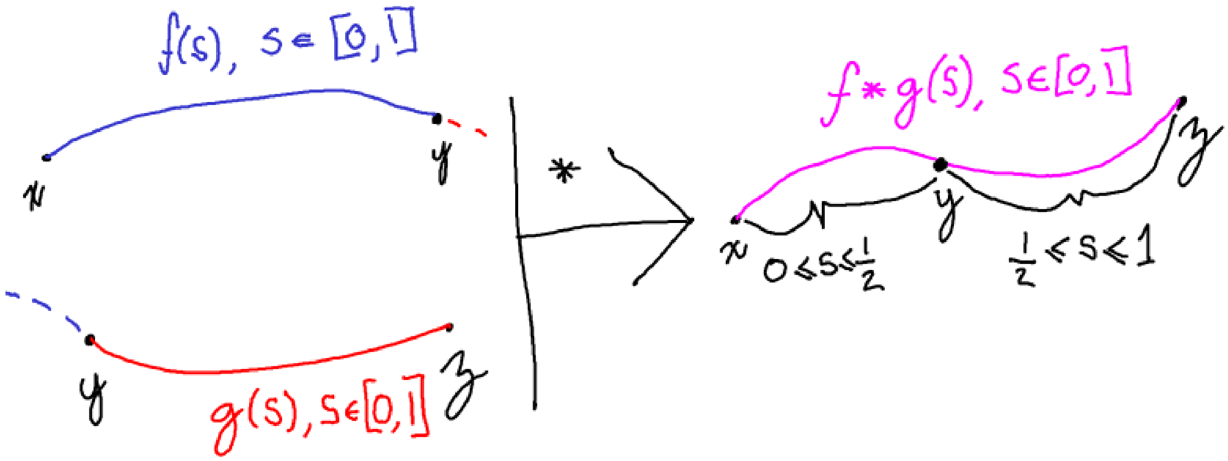
  E.g. in the S' example above, $[f] \neq [g]$

Definition 4. Let the following be so,

$$X = \text{ topological space}$$
$$f = \text{ path in } X \text{ from } x \text{ to } y$$
$$g = \text{ path in } X \text{ from } y \text{ to } z$$

Then the concatenation of f and g is the path $f * g$ from x to z given by,

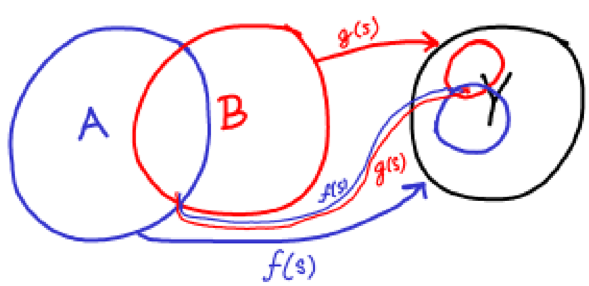
$$f * g : I \to X$$
$$(f*g)(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$



- Why is $f * g$ continuous?

Theorem 1. Gluing Lemma: Let the following be so,

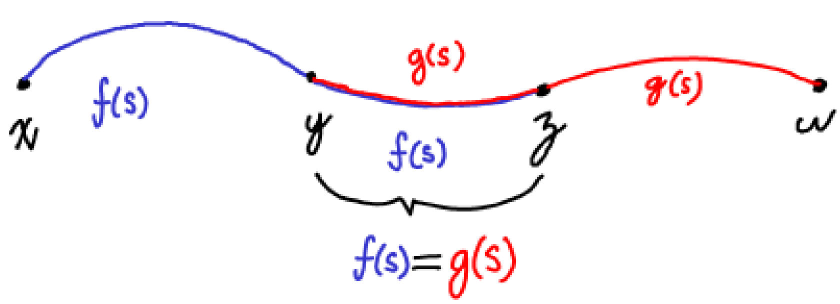
$$X = \text{topological space}$$
$$A, B \subseteq X, \text{ closed subsets such that } X = A \cup B$$
$$Y = \text{topological space}$$



Let the following continuous functions be defined,

$$f: \ A \to Y$$
$$g: \ B \to Y$$

such that $f(x) = g(x)\ \forall x \in A \cap B$.



Then the function,

$$h: \ X \to Y$$
$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous. The proof is left as an exercise to the reader. Thanks. (Homework Problem 1)

Note: Applying the gluing lemma to $I = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ shows that $f * g$ is continuous.

- Question: Let the following be so,

$$X = \mathbb{R}^2$$
$$f(s) = (s-1, s)$$
$$g(s) = (s, s+1)$$

What is $f * g$? Draw a picture.

- Answer:

$$f * g = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Which is a straight line from $(-1, 0)$ to $(1, 2)$.

- Proposition: $*$ is well defined on path-homotopy classes of paths
I.e., if,

$$f_0 \cong_p f_1$$
$$g_0 \cong_p g_1$$

then,

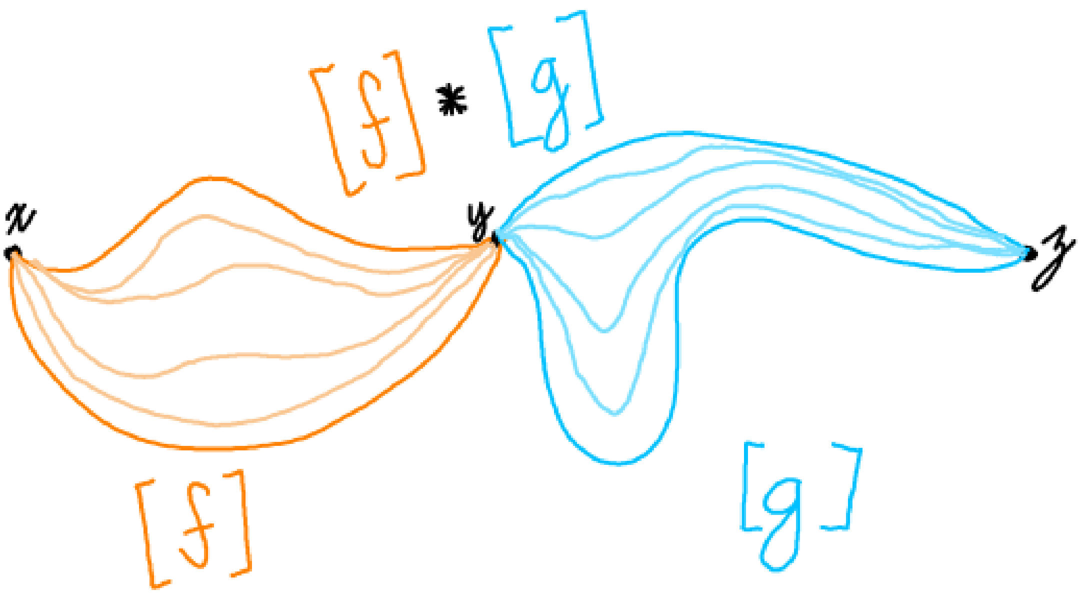
$$f_0 * g_0 \cong_p f_1 * g_1$$

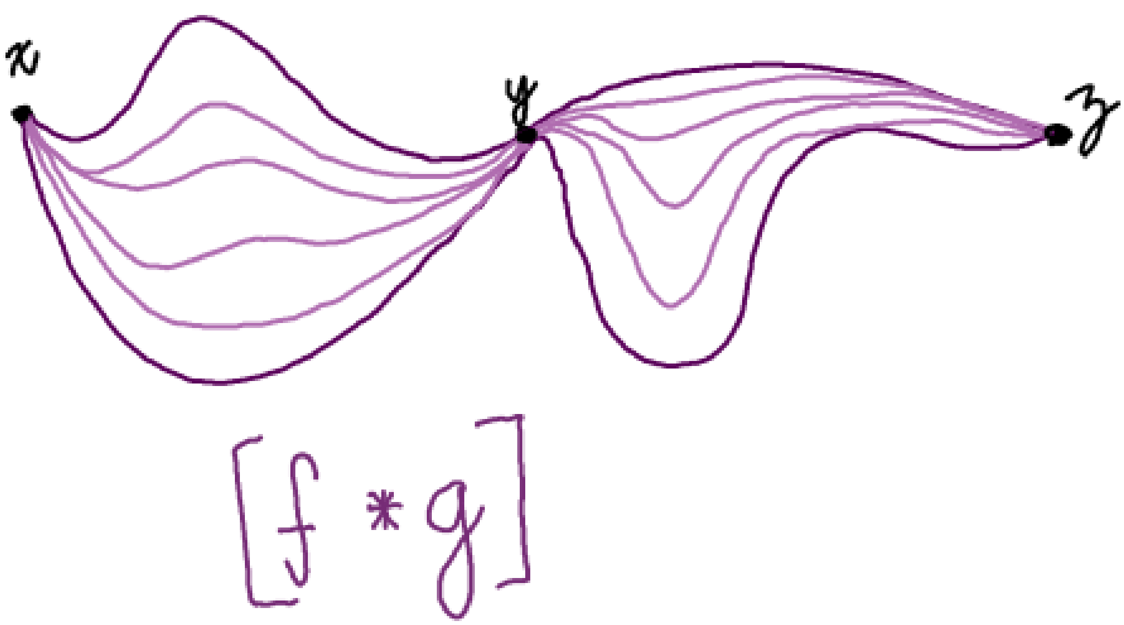
This means that if $[f] = \{$path-homotopy equivalence class of f$\}$ then we can define,

$$[f] * [g] := [f * g]$$

as long as the end point of f is the starting point of g.
So, now $*$ is an operation.

$$\{\text{ paths from } x \to y\}/_{\cong_p} * \{\text{paths } y \to z\}/_{\cong_p} \to \{\text{paths } x \to z\}/_{\cong_p}$$





- Idea of proof of proposition:
  Let,

$$F: \; I \times I \to X \text{ be a path homotopy from } f_0 \text{ to } f_1$$
$$G: \; I \times I \to X \text{ be a path homotopy from } g_0 \text{ to } g_1$$

Then we can define,

$$H: \ I \times I \to X$$
$$H(s,y) = \begin{cases} F(2s,t) & \text{if } 0 \le s \le \frac{1}{2} \\ G(2s-1,t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

Then,

$$h_0 = H(s,0) = (f_0 * g_0)(s)$$
$$h_1 = H(s,1) = (f_1 * g_1)(s)$$
$$h_t = H(s,t) = (f_t * g_t)(s) \text{ (some path between } x \text{ and } z \text{)}$$

So, H is a path homotopy from $(f_0 * g_0)$ to $(f_1 * g_1)$.

4 Day 4

4.0.1 Definition of Fundamental Group

- Recall: If,

$$f = \text{path in } X \text{ from x to y}$$
$$g = \text{path in } X \text{ from y to z}$$

Then,

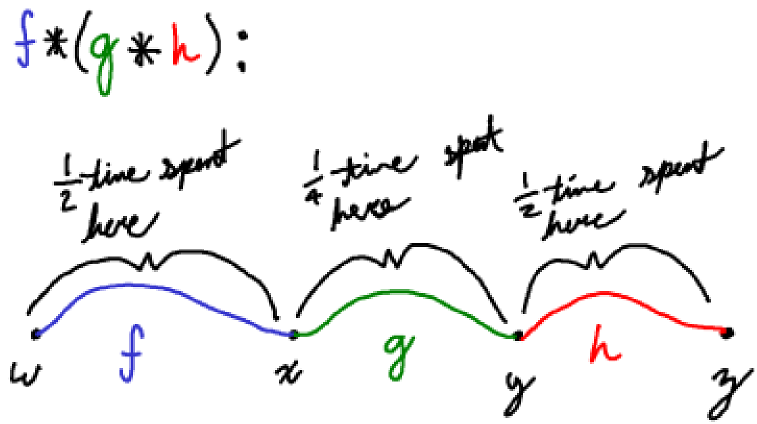
$$[f] * [g] := [\text{concatenation } f * g \text{ of } f \text{ and } g]$$

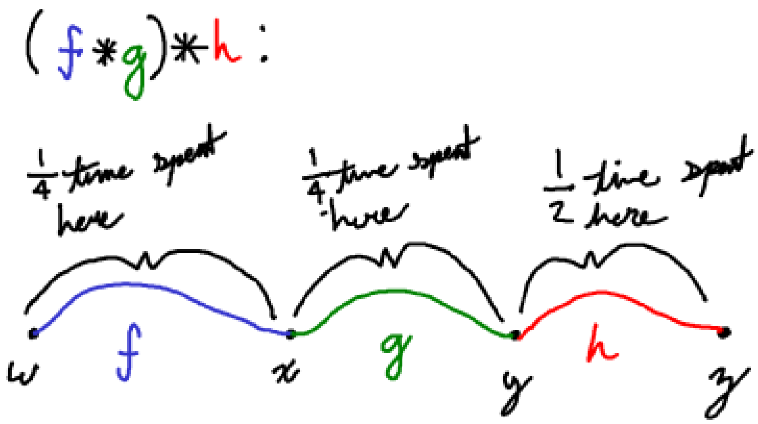
- Properties of $*$:

1. $*$ is associative, or

$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

The idea here is that we can adjust the time taken to travel on the path. These two paths are path-homotopic: interpolate between $f * (g * h)$ and $(f * g) * h$ by making f take less and less time and h take more and more time.

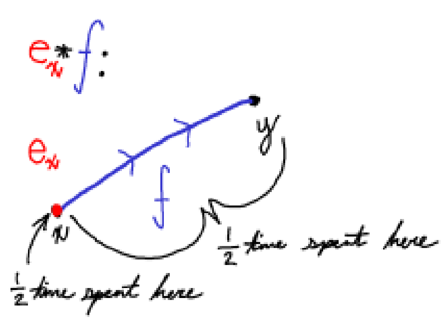




2. $*$ has left/right identities.
   Let

$$e_x : I \to X$$
$$e_x(s) = x, \ \forall \ s \in I, \text{“constant path at } x\text{”}$$

   Then, for all paths f from x to y, $[f] * [e_y] = [f]$, and $[e_x] * [f] = [f]$. The premise here is that e_x or e_y spend "half the time" sitting at either x or y.



   These are path-homotopic: interpolate between $f * e_y$ and f by making f take longer and longer.

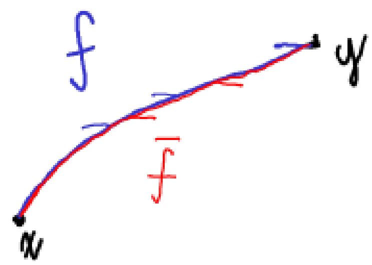
3. $*$ has inverses.

Let f be a path from x to y, and let $\overline{f}$ be the "reverse" path,

$$\overline{f}(s) = f(1-s)$$

Then,

$$[f] * [\overline{f}] = [e_x]$$
$$[\overline{f}] * [f] = [e_y]$$



Idea: The verbal gist of this is that the path takes half the time to travel to its destination, and is concatenated with a path that spends half the time to travel to the origin of the original function.

- These are path-homotopic: interpolate between $f * \overline{f}$ and e_x by doing less and less of f before turning around.
- Let,

$$X = \text{topological space}$$
$$x \in X$$

Definition 5. A loop in X based at $x \in X$ is a path,

$$f : I \to X$$
$$\text{such that} f(0) = f(1)$$

- Observation: If f and g are any two loops in X based at x, then $f * g$ is a loop.

Definition 6. The fundamental group of the X with basepoint x is:

$$\pi(X, x) = \{\text{path-homotopy classes of loops in } X \text{ based at } x\}$$

This is a group with the operation $*$

- e_x and e_y are loops.
- $f * \overline{f}$ and $\overline{f} * f$ are also loops.
- Good question Katy!
- Note: The fact that $\pi_1(X, x)$ satisfies the axioms of a group, and follows from the properties of $*$ we just checked.
  (E.g. the identity element is $[e_x]$)

- Question: What is $\pi_1(\mathbb{R}^2, (0, 0))$?
  Do you have a guess for $\pi_1(S', (1, 0))$?
  Answer 1: $\pi_1(\mathbb{R}^2, (0, 0)) \cong \{1\}$
  To prove this, it's enough to show that $\pi_1(\mathbb{R}^2, (0, 0))$ has just one element,
  i.e., any loop in $\mathbb{R}^2$ based at $(0, 0)$, is path-homotopic to any other. This is true via the straight line homotopy. Answer 2: $\pi_1(S', (1, 0)) \cong \mathbb{Z}$.

5 Day 5

5.0.1 To what extent does π_1 depend on x?

Theorem 2. Let X be a path-connected topological space, and let $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. This section builds off the worksheet provided in class.

1. Part 1: see drawing
2. Part 2: Let f and g be in $\pi_1(X, x_1)$

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha] \\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g])\end{aligned}$$

3. Part 3: Let $f \in \pi_1(X, x_1)$

$$\begin{aligned}\hat{\alpha}([f]) &= [\overline{\alpha}] * [f] * [\alpha] \\ \hat{\overline{\alpha}}([\overline{\alpha}] * [f] * [\alpha]) &= [\alpha] * [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] \\ &= [f]\end{aligned}$$

4. Therefore this mfer is an isomorphism.

5.0.2 For which topological spaces X can we actually compute $\pi_1(X, x)$?

Definition 7. A topological space X is simply-connected if

1. X is path connected
2. $\pi_1(X, x) = 1 \ \forall x \in X$
   (Because X is path connected, we only need to check this for one $x \in X$)

- Ex: $\mathbb{R}^2$ is simply connected
- Intuition: X is simply-connected if any loop in X if any loop in X can be "shrunk down" to a constant loop.
  (for all loops f in X saying f can be "shrunk down" means $f \cong_p c_x$ where c_x is a constant path)
- Next time: A convex subset of $\mathbb{R}^n$ is simply connected.

6 Day 6

- Goal: Prove that $\pi_1(S^1, x) \cong \mathbb{Z}$
- Idea: S^1 can be built by "wrapping $\mathbb{R}$ around itself".
  : Concretely, this is

$$p : \mathbb{R} \to S^1$$
$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

  We'll try to "unwrap" loops in S^1 to get paths in $\mathbb{R}$

- The above map p is an example of a "covering map". The ultimate goal of today is to understand what it means to be a covering map, before we get to the definition of it.

- Questions: Let the following be so,

$$u_1 = \{(x, y) \in S^1 | y > 0\}$$
$$u_2 = \{(x, y) \in S^1 | x > 0, \ y < 0\}$$

  Include the drawings from class, really get sick wit it.

- Observation: For any particular $n \in \mathbb{Z}$, the piece,

$$(n, n + \frac{1}{2}) \cong u_1$$

The homeomorphism in Dr. Clader's mind is,

$$\phi : (n, n+\frac{1}{2}) \to u_1$$
$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$
$$\text{i.e. } \phi = p_{|(n,n+\frac{1}{2})}$$

The inverse of ϕ is,

$$\phi^{-1} : u_1 \to (n, n+\frac{1}{2})$$
$$\phi^{-1} = \frac{\cos^{-1}(x)}{2\pi} + n$$
$$\text{(Recall: by definition } \cos^{-1}(x) \in [0, \pi])$$

Similarly, for u_2 for any particular $n \in \mathbb{Z}$, $(n - \frac{1}{4}, n) \cong u_2$.

Definition 8. Let $p : E \to B$ be a function between two topological spaces. We say p is a covering map if p is,

1. p is continuous and surjective
2. $\forall b \in B$ there exists a neighborhood u of b such that,

$$p^{-1}(u) = \cup_\alpha v_\alpha$$

where $v_\alpha \subseteq E$ are open, disjoint and,

$$p_{|v_\alpha} : v_\alpha \to u$$

is a homeomorphism for every α. Note that these open subsets with this property are called evenly covered
Note that b is one particular point or neighborhood, but there should be a neighborhood for every single point in B where all of this junk holds reasonably truish.

- Ex:

$$p : \mathbb{R} \to S^1$$
$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

p is a covering map. We just showed that u_1 is evenly covered:

$$p^{-1}(u_1) = \cup_{n\in\mathbb{Z}}(n, n+\frac{1}{2})$$

Note that in this case the $(n, n+\frac{1}{2})$ are the v_α from the definition of covering maps. u_2 is also evenly covered, but, $U = S^1$ is not evenly covered because, $p^{-1}(S^1) = \mathbb{R}$, and the only way to write $\mathbb{R}$ as a uniion of disjoint open sets v_α, is to take $v_\alpha = \mathbb{R}$, but $\mathbb{R} \ncong S^1$

- Ex:

$$B = \text{any space}$$
$$E = B \times \{1, 2, ..., n\} = \text{n discrete copies of } B$$

Where $\{1, 2, ..., n\}$ is equipped with the discrete topology.

7 Day 7

7.0.1 Guest lecturer: Mattias "your regular lecturer is more qualified for this" Beck

- Recalling the definition of an evenly covered set. New notation was introduced, but LATEXis behind the times. Let E and B be topological spaces

$$\phi : E \twoheadrightarrow B$$
$$\forall b \in B, \exists u \text{ a neighborhood of b} : p^{-1}(u) = \cup_\alpha v_\alpha$$
$$p_{|v_\alpha} : v_\alpha \to u$$

- Fun notation facts:
  $\twoheadrightarrow$ indicates a surjective function
  $\hookrightarrow$ indicates an injective function
  Combining the two gives you a bijective function, but that symbol doesn't exist in latex apparently.

- Example covering:

$$E = \mathbb{R}$$
$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$
$$B = S^1$$

Definition 9. Given a covering map from topological spaces E to B

$$p : E \to B$$

a path in our topological space B,

$$f : I \to B$$

A lift of f is a path, $\tilde{f} : I \to E$, such that $f = p \circ \tilde{f}$

- This is theoretically a theorem.
  Given covering map $p : E \to B$, $p(e) = b$, $f : I \to B$ path beginning at b, then there does not exist a left $\tilde{f}$, of f beginning at e Read Lemma 54.1 Munkres. (?!?!?)

Theorem 3. Let the following be so,

$$\begin{aligned} &E \text{ be a topological space} \\ &B \text{ be a topological space} \\ &p : E \to B \text{ a covering map} \\ &f : I \to B \text{ path beginning at } b \\ &e \in E,\ s.t.\ p(e) = b \end{aligned}$$

Then there exists a unique path, $\tilde{f}$ in E such that $p \circ \tilde{f} = f$, and $\tilde{f}(0) = e$

8 Day 8

8.0.1 Guest Lecturer: Mattias "you can have a hint, but you can't quote me on it" Beck

- ???????

9 Day 9

9.0.1 Guest Lecturers: Anastasia the Assassin, Deadly David, and Killa Katy

- Let p be a covering map.

  $$p : E \to B$$

  Let, $e \in E$, $b \in B$, such that $p(e) = b$.
  Summary of what we know about this situation,

  1. Any path f in B, beginning at b has a unique lift $\tilde{f}$ to a path in E beginning at e.
  2. If f and g are two paths in B, beginning at b, such that $f \cong_p g$, then $\tilde{f} \cong_p \tilde{g}$
  3. If f is a loop in B based at b, then $\tilde{f} \in p^{-1}(b)$

10 Day 10

10.0.1 $\pi_1(S^1)$, continued:

- Recap:

  $$p : \mathbb{R} \to S^1$$
  $$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

  Then there exists a function,

  $$\phi : \pi_1(S^1, b) \to p^{-1}(b)$$
  $$\phi([f]) = \tilde{f}(1)$$

Where $\tilde{f}$ is the lift of f to $\mathbb{R}$ starting at 0.
E.g.,(draw that spiraleboye)

$$\phi([\text{loop once counterclockwise}]) = 1$$
$$\phi([\text{loop twice counterclockwise}]) = 2$$
$$\phi([\text{loop once clockwise}]) = -1$$

The fact that there exists a unique lift, $\tilde{f}$ of any f is a feature of covering maps.
In fact,

$$p^{-1}(b) = \mathbb{Z}$$

and,

- Claim: $\phi : \pi_1(S^1, b) \to \mathbb{Z}$ is a bijection.

Proof. 1. Surjective: Given $c \in \mathbb{Z}$, choose a path, $\alpha : I \to \mathbb{R}$, from 0 to c in $\mathbb{R}$. Then let, $f : I \to S^1$ be $f = p \circ \alpha$
Then f is a loop in S^1 based at $b = (1,0)$ because

$$f(0) = p(\alpha(0)) = p(0) = (1,0)$$
$$f(1) = p(\alpha(1)) = p(c) = (1,0)$$

And, $\tilde{f} = \alpha$ because $p \circ \tilde{f} = p \circ \alpha = f$. Thus,

$$\phi([f]) = \tilde{f}(1) = \alpha(1) = c$$

2. Injective: Suppose,

$$\phi([f]) = \phi([g])$$
$$\implies \tilde{f}(1) = \tilde{g}(1)$$

Then, $\tilde{f}$ and $\tilde{g}$ are two paths in $\mathbb{R}$, that both start at 0 and both end at the same point.
$\Rightarrow$ (courtesy of homework 2) $\tilde{f} \cong_p \tilde{g}$ (because $\mathbb{R}$ is simply connected)
$\Rightarrow p \circ H$ is a path homotopy from $p \circ \tilde{f}$ to $p \circ \tilde{g}$.
$\Rightarrow f \cong_p g$
$\Rightarrow [f] = [g] \in \pi_1(S^1, b)$

$\square$

- Claim: ϕ is a group homomorphism (thus, an isomorphism).

Proof. Let $[f], [g] \in \pi\pi_1(S^1, b)$, we want to show that, $\phi([f] * [g]) = \phi([f]) + \phi([g])$
By definition,

$$\phi([f] * [g]) = \phi([f * g]) = f * \tilde{g}(1)$$

What is $f \tilde{*} g$? By definition $f \tilde{*} g$ is the lift of $f * g$ starting at 0 and,

$$\tilde{f} = \text{lift of } f \text{ starting at 0 ending at some } n$$
$$\tilde{g} = \text{lift of } g \text{ starting at 0 ending at some } m$$

So, $\tilde{f} * \tilde{g}$ doesn't make sense, but let:

$$\tilde{g}' = \text{``shift } \tilde{g} \text{ by n ''}$$
$$\text{i.e., } \tilde{g}^{'} = g(s) + n$$

Sow notice that $\tilde{f} * \tilde{g}'$ now makes sense, and $\tilde{g}^{'}$ is a lift of g, because:

$$\begin{aligned}(p \circ \tilde{g}^{'})(s) &= p(\tilde{g}(s)) \\ &= p(\tilde{g}(s) + n) \\ &= p(\tilde{g}(s))\end{aligned}$$

because $p(x + n) = p(x), \ \forall n \in \mathbb{Z}$

$$\begin{aligned}&= (p \circ \tilde{g})(s) \\ &= g(s)\end{aligned}$$

Thus, $\tilde{f} * \tilde{g}'$ is a lift of $f * g$ starting at 0

$$\begin{aligned}\implies \ \tilde{f} * \tilde{g} &= f \tilde{*} g \\ f \tilde{*} g(1) &= \tilde{f} * \tilde{g} \\ &= \text{endpoint of } \tilde{g}' \\ &= \tilde{g}(1) + n \\ &= m + n\end{aligned}$$

This shows that

$$\begin{aligned}\phi([f] * [g]) &= m + n \\ &= \tilde{f}(1) + \tilde{g}(1) \\ &= \phi([f]) + \phi([g])\end{aligned}$$

$\square$

- We want:

$$X \cong Y \implies \pi_1(X,x) \cong \pi_1(Y,y)$$
(X is homeomorphic to Y)

The big tool we'll use to do that is the tool from the second homework about maps between spaces being homomorphisms. That's for next time!

11 Day 11

11.0.1 Examining the group structure of $*$ functions

- Note that this Friday, office hours will be at 3-4pm.
- We want: If $X \cong Y$, then $\pi_1(X,x) \cong \pi_1(Y,y)$, or that, if two spaces are homeomorphic, then their fundamental groups are isomorphic. We will explore the tools used to show this in this lecture. From homework 2, we get the following definition

Definition 10. Let $\varphi : X \to Y$, be a continuous map, then the homomorphism induced by φ is:

$$\varphi_* : \pi_1(X,x) \to \pi_1(Y,y)$$
$$\varphi_*([f]) = [\varphi \circ f]$$

See the picture of the picture drawn on the board, make a drawyboye.

Lemma: (this is referred to lemma 1)If

$$X \to^{\varphi} Y \to^{\psi} Z$$

Where φ and ψ are both continuous, then,

$$(\psi \circ \varphi)_* = \psi \circ_* \varphi_*$$

Additionally, (This is referred to as lemma 2)

$$id_* = id$$

(or that given the $id : X \to Y$, the induced homomorphism, $\pi_1(X,x) \to \pi_1(Y,y)$ is the identity)

Proof. Firstly, Both sides are homomorphisms

$$\pi_1(X,x) \to \pi_1(Z,(\psi \circ \varphi)(x))$$

Given any $[f] \in \pi_1(X, x)$:

$$\begin{aligned}
(\psi \circ \varphi)_*([f]) &= [(\psi \circ \varphi) \circ f] \\
&= [\psi \circ (\varphi \circ f)] \\
&= \psi_*[\varphi \circ f] \\
&= \psi_*(\varphi_*(f)) \\
&= (\psi_* \circ \varphi_*)([f])
\end{aligned}$$

Given any $[f] \in \pi_1(X, x)$:

$$\begin{aligned}
id_*([f]) &= [id \circ f] \\
&= [f]
\end{aligned}$$

□

Theorem 4. if $\varphi : X \to Y$ is a homeomorphism, then $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.

Proof. We already know that φ_* is a homomorphism, to prove that it's a bijection, we'll find an inverse to φ_*. Claim that,

$$(\varphi)_* : \pi_1(Y, \varphi(x)) \to \pi_1(X, x)$$

is the inverse to φ_*.
(Note that this is doable, because φ is a homeomorphism, $\varphi^{-1} : Y \to X$ exists, and is continuous)
To check this:

$$\begin{aligned}
\varphi_* \circ (\varphi^{-1}) &= (\varphi \circ \varphi^{-1}), \text{ by lemma 1 shown today} \\
&= id_*, \text{ by definition of } \varphi^{-1} \text{ (identity on y)} \\
&= id, \text{ by lemma 2 shown today (identity on x)} \\
(\varphi^{-1})_* \circ \varphi_* &= (\varphi^{-1} \circ \varphi)_* = id_* = id
\end{aligned}$$

This by definition means φ_* and $(\varphi^{-1})_*$ are inverse functions. Additionally, this small red box has made it onto the board, for clarification.

$$\begin{aligned}
&id_x : X \to Y \\
&id_{\pi_1(X,x)} : \pi_1(X, x) \to \pi_1(X, x) \\
&\text{Lemma: } (id_x)_* id_{\pi_1(X,x)}
\end{aligned}$$

□

- This ends up proving that,

$$X \cong Y \implies \pi_1(X, x) \cong \pi_1(Y, \varphi(x))$$

But, non-homeomorphic spaces can have isomorphic π_1
Ex:

$$X = .$$
$$Y = \mathbb{R}^2$$

These are not homeomorphic, clearly X is compact and Y isn't, but their fundamental groups are isomorphic, since the fundamental group of X is just $\{1\}$, and clearly this is also true about $\mathbb{R}^2$

- So, given X and Y, how can we tell if $\pi_1(X) \cong \pi_1(Y)$?

11.0.2 Homotopy of Maps:

Definition 11. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Then a homotopy from f to g is a continuous function,

$$H : X \times I \to Y$$

such that,

$$H(x, 0) = f(x),\ \forall x \in X$$
$$H(x, 1) = f(x),\ \forall x \in X$$

Our goal is to make remark about the lower star versions of these maps, given their being homotopic.

12 Day 12

12.0.1 Homotopy of maps

Definition 12. Let $f : X \to Y$ be a continuous function. A homotopy from f to g is a continuous function,

$$H : X \times I \to Y$$

such that

$$H(x, 0) = f$$
$$H(x, 0) = g$$

We'll often write,

$$h_t : X \to Y$$
$$h_t(x) = H(x, t)$$

Then there's one h_t for each $t \in I$ and,

$$\begin{aligned} h_0 &= f \\ h_1 &= g \\ h_t &= \text{``A function interpolating between } f \text{ and } g\text{''} \end{aligned}$$

- Terminology/Notation: If there exists a homotopy from f to g, we'll say that f is homotopic to g and write $f \cong g$.

- Ex:

$$\begin{aligned} f : S^1 &\to \mathbb{R}^2 \\ g : S^1 &\to \mathbb{R}^2 \\ f(x, y) &= (x, y) \\ g(x, y) &= (0, 0) \end{aligned}$$

  Then $f \cong g$. A homotopy from f to g is,

$$\begin{aligned} H : S^1 \times I &\to \mathbb{R}^2 \\ H((x, y), t) &= ((1 - t)x, (1 - t)y) \end{aligned}$$

  Do the drawing from the board.

- Ex:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ g : \mathbb{R} &\to \mathbb{R} \\ f(x) &= x \\ g(x) &= x + 2 \end{aligned}$$

  Then $f \cong g$. A homotopy from f to g is:

$$\begin{aligned} H : \mathbb{R} \times I &\to \mathbb{R} \\ H(x, t) &= x + 2t \end{aligned}$$

  Refer again to the picture from the board.

- Questions:
  –
$$
\begin{aligned}
f &: \mathbb{R} \to \mathbb{R}^2 \\
g &: \mathbb{R} \to \mathbb{R}^2 \\
f(x) &= (x, 0) \\
g(x) &= (x, e^x)
\end{aligned}
$$
  –
$$
\begin{aligned}
f &: \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2 \setminus (0,0) \\
g &: \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2 \setminus (0,0) \\
f(x) &= (x, y) \\
g(x) &= \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
f &: \mathbb{R} \to \mathbb{R}^2 \\
g &: \mathbb{R} \to \mathbb{R}^2 \\
f(x) &= (x, 0) \\
g(x) &= (x, e^x)
\end{aligned}
$$

Just use the straight line homotopy it's not hard.

Maybe include the drawings?

Definition 13. Let $f : X \to Y$ and $g : X \to Y$ be continuous, and let $x_0 \in X$ be such that $f(x_0) = g(x_0) = y_0$. Then a homotopy from f to g relative to x_0 is a homotopy $H : X \times I \to Y$ from f to g such that $h_t(x_0) = y_0,\ \forall t$.
("x_0 doesn't move during the homotopy")

- Ex: in the second part of the questions from today, H was a homotopy relative to $(1, 0)$, or to any other point on the unit circle.
- Ex:
$$
X = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \le 1\}
$$

  (it's the 2 norm ball)
$$
\begin{aligned}
f &: X \to X \\
g &: X \to X
\end{aligned}
$$

Then,

$$H : X \times I \to X$$
$$H((x,y),t) = (1-t)x, (1-t)y)$$

is a homotopy relative to $(0,0)$.

Theorem 5. If $f : X \to Y$ and $g : X \to Y$ are homotopic relative to x_0, then:

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

are the same homomorphism.

13 Day 13

Theorem 6. Let

$$f : X \to Y$$
$$g : X \to Y$$

be a continuous function such that $f(x_0) = g(x_0) = y_0$. Suppose that f and g are homotopic relative to x_0. (there exists a homotopy H from f to g such that $H(x_0, t) = y_0, \ \forall t$).
Then,

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

are the same homomorphism.

Proof. Let $[\alpha] \in \pi_1(X, x_0)$. We want,

$$f * [\alpha] = g * [\alpha]$$
$$\iff [f \circ \alpha] = [g \circ \alpha]$$
$$\iff f \circ \alpha \cong_p g \circ \alpha$$

Define,

$$P : I \times I \to Y$$
$$P(s,t) = H(\alpha(s), t)$$

Equivalently,

$$p_t : I \to Y$$
$$p_t(s) = (h_t \circ \alpha)(s)$$

This is a path homotopy from $f \circ \alpha$ to $g \circ \alpha$. Firstly, because H is a homotopy relative to x_0.

$$P(0,t) = H(\alpha(0),t) = H(x_0,t) = y_0$$
$$P(1,t) = H(\alpha(1),t) = H(x_0,t) = y_0$$

Because H is a homotopy from f to g, the following is true.

$$P(s,0) = H(\alpha(s),0) = f(\alpha(s))$$
$$P(s,1) = H(\alpha(s),1) = g(\alpha(s))$$

□

- Application: Suppose $A \subseteq X$ and that there exists a homotopy H from

  $$id : X \to X$$

  to a continuous function

  $$r : X \to X$$

  such that,

  1. $r(x) \in A, \ \forall x \in X$
  2. $H(a,t) = a, \ \forall a \in A, \ \forall t \in I$
     ("every point of A stays fixed throughout the homotopy, or, H is a homotopy relative to every point in A)

  In this situation, we say that A is a deformation retract of X or that H is a deformation retraction of X onto A.

Theorem 7. If A is a deformation retract of X, then,

$$\pi_1(X,x_0) \cong \pi_1(A,x_0), \ \forall x_0 \in A$$

- Ex:

  $$X = \mathbb{R}^2$$
  $$A = S^1$$
  $$r : X \to X$$
  $$r(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$$

  On Friday, we saw that the straight line homotopy, $H : X \times I \to X$ is a homotopy from $id : X \to X$ to $r : X \to X$.

- Ex:

$$X = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$
$$A = \{(0,0)\}$$
$$r : X \to X$$
$$r(x,y) = (0,0)$$

On Friday, we saw that the straight line homotopy $H : X \times I \to X$ is a homotopy from $id : X \to X$ to $r : X \to X$, Thus,

$$\pi_1(X) \cong_p \pi_1(\{.\}) = \{1\}$$

- Question: Let,

$$X = \mathbb{R}^3 \setminus \{\text{z-axis}\}$$
$$A = \{(x,y,0) | x \neq 0,\ y \neq 0\}$$

Find a deformation retraction from X onto A. (Specify both r and H)
What does this tell us about $\pi_1(\mathbb{R}^3\{\text{z-axis}\})$

- Answer:

$$r(x,y,z) = (x,y,0)$$
$$H((x,y,z),t) = (x,y,(1-t)z)$$

Thus,

$$\pi_1(\mathbb{R}^3 \setminus \{\text{z-axis}\}) \cong \pi_1(A) \cong \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Proof. Let $x_0 \in A$. Let,

$$i : A \to X$$
$$i(a) = a$$
$$s : X \to A$$
$$s(x) = r(x)$$

Considering condition 2 in the definition of deformation retraction yields, $s \circ i = id_A$, because

$$s(i(a)) = s(a) = a$$

In the other direction,

$$i \circ s = r$$

The deformation retraction H is a homotopy relative to x_0 from r to id_X, so:

$$\begin{aligned} r_* &= (id_X)_* \\ \implies (i \circ s)_* &= (id_X)_* \\ \implies i_* \circ s_* &= id \end{aligned}$$

□

14 Day 14

- Quiz(Midterm) on Monday, whenever that is. Standard Dr. Clader Format. Last covered topic on that will be deformation retractions.
- Recall from last time:

Theorem 8.

$$\begin{aligned} &A \subseteq X \\ &x_0 \in A \\ &H = \text{ deformation retraction of } X \text{ onto } A \end{aligned}$$

Recall that H is a homotopy relative to x_0

$$\begin{aligned} id : X &\to X \\ r : X &\to X \\ s.t.\ r(x) \in A,\ \forall x &\in X \end{aligned}$$

Then,

$$\pi_1(X, x_0) \cong \pi_1(A, x_0)$$

(r is a retraction)

Proof. Consider, $X \leftrightharpoons A$, where $X \to A$ is s, the same function as r, and $A \to X$ is the inclusion map. Then,

$$\begin{aligned} s \circ i &= id : A \to A \\ \implies s_* \circ i_* &= id : \pi_1(A, x_0) \to \pi_1(A, x_0) \end{aligned}$$

In the other order:

$$i \circ s = r \cong id$$

Note that r is a homotopy relative to x_0, and that the next step follows from the theorem from the beginning of last class.

$$\implies i_* \circ s_* = id : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

So we have:

$$\begin{aligned} \pi_1(X, x_0) &\leftrightharpoons \pi_1(A, x_0) \\ s_* : \pi_1(X, x_0) &\to \pi_1(A, x_0) \\ i_* : \pi_1(A, x_0) &\to \pi_1(X, x_0) \end{aligned}$$

and we've shown s_* and i_* are inverses, giving

$$\pi_1(X, x_0) \cong \pi_1(A, x_0)$$

$\square$

- Fun Font Fabtacular Letter fundamental groups.

  1. **C family:** C,E,F,G,H,I,J,K,L,M,N,S,T,U,V,W,X,Y,Z
  2. **A family:** A,D,O,P,Q,R
  3. **B family:** B (fuckin loser.)

  The reason $\pi_1(E) \cong \pi_1(I)$ is that there is a deformation retraction.

  $$\begin{aligned} H : E \times I &\to E \\ H(x, 0) &= x \\ H(x, 1) &\in I, \text{ the letter ``I''} \end{aligned}$$

  The rest of this was erased, before I could write it down. Ahh damn.
  Let's talk about $\pi_1(B)$ though. What is that?

  1. It's the same as the fundamental group of a figure 8, because $B \cong \infty$
  2. It's also the same as:

     $$\pi_1(\mathbb{R}^2 \setminus \{p, q\})$$

     where p and q are unequal points in $\mathbb{R}^2$.
  3. Also the same as $\pi_1(\theta)$ (theta is just the letter theta)

14.1 Day 15

14.1.1 Continuation and finish of Day 14

- A space with the same π_1 as "B"
- Ex:

$$X = \mathbb{R}^2 \setminus \{p, q\}$$
$$p = (-1, 0)$$
$$q = (1, 0)$$

To see that $\pi_1(X) \cong \pi(\infty)$ (where infinity isn't actually infinity, but two circles joined together to look like a butt.), we can construct a deformation retraction of X onto,

$$A = \{(x+1)^2 + y^2 = 1\} \cup \{(x-1)^2 + y^2 = 1\}$$

Pictorially, refer to the picture taken in class.

1. Deformation retract X onto a closed disk of radius 2, centered at (0,0)
2. Then Deformation retract onto a union of two closed disks vertically, again, refer to the goddamn picture.

- Ex:

$$\theta = \{x^2 + y^2 = 1\} \cup \{(x, 0) | -1 \leq x \leq 1\}$$

Oh yeah, another goddamn picture. To see that $\pi_1(\theta) \cong \pi_1(\infty)$, (again using infinity in lieu of the double circle butt) we can construct a deformtaion retraction of $\mathbb{R}^2 \setminus \{p', q'\}$ onto θ where $p' = (0, \frac{1}{2})$ and $q' = (0, -\frac{1}{2})$

- Observation: This shows that,

$$\pi_1(\infty) = \pi_1(\theta)$$

because both of them are isomorphic to $\pi_1(\mathbb{R} \setminus \{\text{two points}\})$ But neither ∞ nor θ are deformation retracts of eachother.

- They're related by a more general relationship, that of homotopy equivalence.

14.1.2 Homotopy Equivalence

Definition 14. A continuous map,

$$f : X \to Y$$

is called a homotopy equivalence if there exists a $g : X \to Y$ such that $f \circ g \cong id_Y$, and $g \circ f \cong id_X$, with our equivalency being homotopic to.

- Goals: A homotopy equivalence induces an $\cong$ on π_1.
- Any deformation retraction "yields" a homotopy equivalence, but homotopy equivalence is an EQUIVALENCE relation. Sick.

- Ex:

$$X = \mathbb{R}^2$$
$$A = \{0, 0\}$$

Then A is a deformation retract of X.

$$H((x, y), t) = ((1 - t)x, (1 - t)y)$$

But, A is not homeomorphic to X

- Ex: Wow, yet another goddamn picture. Wonderful. Refer to the appropriate photograph. Closed disks in R^2, then A is a deformation retraction of X, and also A is homeomorphic to X. Look at the picture ya doink.
  X is homeomorphic to Y implies that $\pi_1(X) \cong \pi_1(Y)$, but the converse is not true, e.g.: X is a deformation retraction of Y implies that $\pi_1(X) \cong \pi_1(Y)$ But not converseley, e.g: X is homotopy equivalent to Y implies $\pi_1(X) \cong \pi_1(Y)$.
  None of these are conversely true. Wonderful! That was confusing.

14.2 Day 16

14.2.1 QUIZ DAY (it's a midterm)

14.3 Day 17

14.3.1 Homotopy Equivalence

- Goal: A homotopy equivalence induces an isomorphism on π_1
- This follows from:

Theorem 9. If $f : X \to Y$ and $g : X \to Y$ are continuous, $f \cong g$ (homotopic),

$$f(x_0) = y_0, \; g(x_0) = y_1$$

Then there exists a path, α from y_0 to y_1 such that $g_* = \hat{\alpha} \circ f_*$.
Schematically:

$$\underbrace{\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{\hat{\alpha}} \pi_1 Y, y_1}_{g_*}$$

Proof. Let

$$H : X \times I \to Y$$

be a homotopy from f to g (I.e, $h_t : X \to Y, \; \forall t \in I$). Let,

$$\alpha : I \to Y$$
$$\alpha(t) = h_t(x_0)$$

Note that we want,

$$\begin{aligned}
&\forall[\gamma] \in \pi_1(X, x_0) : \\
&g_*([\gamma]) = \hat{o} f_*([\gamma]) \\
\iff &[g \circ \gamma] = [\overline{\alpha} * (f \circ \gamma) * \alpha] \\
\iff &g \circ \gamma \cong \overline{\alpha} * (f \circ \gamma) * \alpha
\end{aligned}$$

We'll prove these are path homotopic by interpolating between them by the following loops: (There's some drawing that goes here) Explicitly, let,

$$\beta_t : I \to Y$$
$$\beta_t(s) = \overline{\alpha}((1-t)s)$$

Then,

$$\begin{aligned}
\beta_0 &= \overline{\alpha} \\
\beta_1 &= e_{y_1} \\
\beta_t &= \text{path from } y_1 \text{ fo } \alpha(t)
\end{aligned}$$

Now, define the followin loop at y_1:

$$\beta_t * (h_t \circ \gamma) * \overline{\beta_t}$$

This is:

1. When $t = 0$:
$$\beta_0 * (h_0 \circ \gamma) * \overline{\beta_0} = \overline{\alpha} * (f \circ \gamma) * \alpha$$
(this is the green loop from the hard to see picture)
2. When $t = 1$:
$$\beta_1 * (h_1 \circ \gamma) * \overline{\beta_1} = e_{y_1} * (f \circ \gamma) * e_{y_1}$$

Thus, $\beta_t * (h_t \circ \gamma) * \overline{\beta_t}$ give a path homotopy,

$$\overline{\alpha} * (f \circ \gamma) * \alpha \cong_p e_{y_1} * (f \circ \gamma) * e_{y_1} \cong_p g \circ \gamma$$

□

Corollary 9.1. If $f : X \to Y$ is a homotopy equivalence (recall that this means there exists a $g : Y \to X$ such that $f \circ g \cong id_Y$ and $g \circ f \cong id_X$). Then,

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is an isomorphism.

Proof. We know,

$$
\begin{aligned}
g \circ f &\cong id_X \\
\xRightarrow[\text{theorem}]{} (g \circ f)_* &= \hat{\alpha} \circ (id_X)_*, \text{ for some path } \alpha \\
\Rightarrow g_* \circ f_* &= \hat{\alpha}
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
f \circ g &\cong id_Y \\
(f \circ g) &= \hat{\beta} \circ (id_Y)_* \text{ for some path in } \beta \\
\rightarrow f_* \circ g_* &= \hat{\beta}
\end{aligned}
$$

Thus, if $f(x_0) = y_0$, $g(y_0) = x_1$, $f(x) = y_1$:

$$
\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\hat{\alpha}"] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \arrow[dl, "g_*"'] \\
\pi_1(X, x_1) \arrow[r, "f_*"] & \pi_1(Y, y_1)
\end{tikzcd}
$$

(Some fuckin arrow diagram Ah fuck) Therefore,

$$
\begin{aligned}
g_* \circ f_* = \hat{\alpha}, &\text{ an isomorphism!} \\
\Rightarrow g_* &\text{ is surjective}
\end{aligned}
$$

And similarly, because

$$
\begin{aligned}
f_* \circ g_* = \hat{\beta} &\text{ (an isomorphism!)} \\
\rightarrow g_* &\text{ is injective}
\end{aligned}
$$

□

14.4 Day 18

14.4.1 Homotopy equivalences, concluded

Recall,

Definition 15. A continuous function $f : X \to Y$ is a homotopy equivalence if there exists a continuous function $g : Y \to X$ such that $g \circ f \simeq id_x$ and $f \circ g \simeq id_Y$

Notation/terminology: We call g a homotopy inverse of f if there exists a homotopy equivalence, $f : X \to Y$, we say X and Y are homotopy equivalent and we write $X \simeq Y$

1. Ex:

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$$
$$f(x,y) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$$

is a homotopy equivalence with homotopy inverse,

$$i : S^1 \to R^2 \setminus \{(0,0)\}$$
$$i(x,y) = (x,y)$$

To check these are homotopy inverses:

$$f \circ i = id_{S^1}$$
$$(i \circ f)(x,y) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$$

So a homotopy between $f \circ i$ and id_{S^1} is,

$$H : S^1 \times I \to S^1$$
$$H((x,y),t) = (x,y),\ \forall t \in I$$

A homotopy between $i \circ f$ and $id_{\mathbb{R}^2 \setminus \{(0,0)\}}$ is the straight line homotopy. This is the deformation retraction.

2. Ex: Let,

$$D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

Then,

$$f : D \to \{(0,0)\}$$
$$f(x,y) = (0,0)$$

is a homotopy equivalence with homotopy inverse,

$$i : \{(0,0)\} \to D$$
$$i(0,0) = (0,0),\ \text{(inclusion)}$$

To check these are homotopy inverses:

$$f \circ i = id_{\{(0,0)\}}$$
$$(i \circ f)(x,y) = (0,0)$$

So, a homotopy betwen $i \circ f : D \to D$ and $id : D \to D$ is

$$H : D \times I \to D$$
$$H((x,y),t) = ((1-t)x, (1-t)y)$$

This is the straight-line homotopy and it is the deformation retraction of D onto $\{(0,0)\}$

3. Note:(HW) Any deformation retraction of X onto A gives rise to a homotopy equivalence $X \simeq A$ (Note that this side box was created at some point. $H : X \times I \to X$ a homotopy from $id_X : X \to X$ to $r : X \to X$ such that $r(x) \in A, \ \forall x \in X$)

Definition 16. If a topological space X is homotopy equivalent to a single point, we say that X is contractible.

1. Ex: The unit disk, D is retractible.
2. Note: By the theorem from last class,

   contractible $\Rightarrow$ simply-connected

14.5 Why care about homotopy equivalences?

Why do we care about homotopy equivalences instead of just using deformation retractions?

1. Deformation retraction is weirdly asymmetric. A is a deformation retraction of X but not vice versa, while homotopy equivalence is an equivalence relation (courtesy of HW). The fact that it's symmetric

   - Ex:

$$D \underbrace{\simeq}_{\text{last example}} \{.\} \underbrace{\simeq}_{\text{handout}} \mathbb{R}$$
$$\Rightarrow D \simeq \mathbb{R}$$
$$\Rightarrow \pi_1(D) \cong \pi_1(\mathbb{R})$$

   - Ex: Looking at the character θ,

$$\theta \underbrace{\simeq}_{\text{deformation retraction}} \mathbb{R}^2 \setminus x, y \underbrace{\simeq}_{\text{deformation retraction}} \infty$$
$$\Rightarrow \theta \cong \infty$$
$$\Rightarrow \pi_1(\theta) \cong \pi_1(\infty)$$

2. We don't have to worry about base points staying fixed throughout the homotopy.

15 Day 19

15.1 Calculating π_1 piecewise

- Goal: To calculate $\pi_1(X)$ with π_1("pieces of X")
- The ultimate theorem we'll prove for this is the Van Kampen Theorem, which will say, Let,

  $$X = U \cup V$$

  Where U and V are path connected open subsets of X. Furthermore their intersection is *also* path connected. Then,

  $$\pi_1(X) = \pi_1(U)?\pi_1(V)$$

  Note that the ? takes the place of some as yet undefined operation, and it will depend not only on $\pi_1(U)$ or $\pi_1(V)$, but their interaction.

15.2 Free product of groups

Definition 17. A word in G and H is a string of symbols,

$$a_1, a_2, a_3, \ldots, a_n$$

where a_i is either an element of G or an element of H

- $$\underbrace{G = \{a^0, a^1, a^2, a^3\}}_{\text{group under multiplication if we declare } a^4 = a^0}$$
  $$\underbrace{H = \{b^0, b^1, b^2\}}_{\text{group under multiplication if we declare } b^3 = b^0}$$

  An example of a word in G and H is

  $$a^1a^1b^2a^1b^0b^1$$

  Thus far, this is a different word from

  $$a^2b^2a^1b^1$$

  But we'd like for them to be equivalent. To achieve this end, we define two reducing operations on $\{$words in G and $H\}$

  1. If a word contains a copy of 1_g, the identity of G or 1_h remove that symbol from the word.
  2. If a word contains two consecutive terms from the same group, replace them with their product.

  This allows for these two things to be set equal by a sequence of the reducing operations.

  $$a^1a^1b^2a^1b^0b^1$$
  $$a^2b^2a^1b^0b^1$$
  $$a^2b^2a^1b^1$$

Definition 18. A word is called reduced if no further reducing operations can be applied to it.

- Ex: $a^1b^2a^1$ is the reduced form of $a^1a^{-1}a^1b^1b^1a^1$.
- Observation: Any word can be converted via a sequence of reducing operations to a unique reduced word.

Definition 19. Let H and G be any groups. Their free product

$$G * H = \{\text{reduced words in } G \text{ and } H\}$$

This is a group under the operation of concatenation followed by reduction. Note that $G * H$ is an infinite non-abelian group.

16 Day 20

16.1 Free products continued

- Question: Let,

$$G = \{\dots, a^{-2}, a^{-1}a^0, a^1a^2, \dots\}$$
$$H = \{\dots, b^{-2}, b^{-1}b^0, b^1b^2, \dots\}$$

  These are both groups under multiplication and they're isomorphic.

  - To what "familiar" group are G and H isomorphic.
  - What do elements of $G * H$ look like?

- Answer:

  - $$G \cong H \cong Z$$

    (via the isomorphism, $a^i \to i$ or $b^i \to i$)
  - Elements of $G * H$ look like,
    $$a^{i_1}b^{j_1}a^{i_2}b^{j_2}\dots a^{i_n}b^{j_n}$$
    (Where $i_k, j_k \in \mathbb{Z}$)
    This might start with b^{j_1} or it might end with a^{j_n}

- Observation:

16.1.1 Free Products with Amalgamation

- Let G_1, G_2 and H be groups, and let,

$$\varphi_1 : H \to G_1$$
$$\varphi_2 : H \to G_2$$

  be homomorphisms

$$\begin{tikzcd} G_1 & & G_2 \\ & H \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

- Observation: φ_1 and φ_2 induce homomorphisms,

$$\tilde{\varphi_1} : H \to G_1 * G_2$$
$$\tilde{\varphi_1} = \text{ the word that just contains } \varphi_1(h)$$

  and,

$$\tilde{\varphi_2} : H \to G_1 * G_2$$
$$\tilde{\varphi_2} = \text{ the word that just contains } \varphi_2(h)$$

Definition 20. The free product of G_1 and G_2 amalgamated over H is,

$$G_1 *_H := \frac{G_1 * G_2}{N}$$

where N is the smallest normal subgroup of $G_1 * G_2$ containing $\tilde{\varphi_1}(h)\tilde{\varphi_2}(h)^{-1}, \ \forall h \in H$

- Think: We're setting,

$$\tilde{\varphi_1}(h)\tilde{\varphi_2}(h)^{-1} = 1$$
$$\iff \tilde{\varphi_1}(h) = \tilde{\varphi_2}(h)$$

  (the sorts of things in N are $a\tilde{\varphi_1}(h)\tilde{\varphi_2}(h)^{-1}a^{-1}$, e.g.)

- Note: The notation $G_1 * G_2$ doesn't mention ϖ_1 and φ_2 but it depends on them!

- Ex:

$$G_1 = \langle a \rangle$$
$$G_2 = \{1\}$$
$$H = \langle b \rangle$$

  Let's form

$$G_1 *_H G_2$$

  where the amalgamation happens over the homomorphisms.

$$\varphi_1 : \langle b \rangle \to \langle a \rangle$$
$$\varphi_1(b^i) = a^{2i}$$

and

$$\begin{aligned}\varphi_2 : \langle b \rangle &\to \{1\} \\ \varphi_2(b^i) &= 1\end{aligned}$$

Then,

$$\begin{aligned}G_1 *_H G_2 &= \frac{G_1 * G_2}{\text{smallest nomal subgroup containing } \varphi_1(h)\varphi_2^{-1}(h) \forall h \in H} \\ &= \frac{\langle a \rangle}{\ldots \text{containing } a^2 1 \forall i \in \mathbb{Z}} = \frac{\langle a \rangle}{\langle a^2 \rangle}\end{aligned}$$

17 Day 21

17.1 Van Kampen Theorem

Theorem 10. Let X be a topological space such that,

$$X = U \cup V$$

where U, V and $U \cap V$ are all open, path-connected, subsets of X. Let $x_0 \in U \cap V$. Then,

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$$

Where amalgamation happens over the homomorphisms

$$i_{U*} : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$$

induced by the inclusion $i_U : U \cap V \to U$, and the homomorphism

$$i_{V*} : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$$

induced by the inclusion $i_V : U \cap V \to V$.

$$\begin{tikzcd}
 & \pi_1(U, x_0) \arrow[dr, "j_1"] \arrow[drr, bend left, "k_1"] & & \\
\pi_1(U \cap V, x_0) \arrow[ur, "i_{U*}"] \arrow[dr, "i_{V*}"'] & & \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0) \arrow[r, "\phi"] & \pi_1(X, x_0) \\
 & \pi_1(V, x_0) \arrow[ur, "j_2"] \arrow[urr, bend right, "k_2"'] & &
\end{tikzcd}$$

- <u>Ex:</u>(Algebra) $\mathbb{Z} *_{\{1\}} \mathbb{Z}$, where the amalgamation happens over the homomorphisms,

$$\begin{aligned}\varphi_1 &: \{1\} \to \mathbb{Z}, \text{ (trivial homomorphism)} \\ \varphi_2 &: \{1\} \to \mathbb{Z}, \text{ (trivial homomorphism)}\end{aligned}$$

(The trivial homomorphism just sends everything to the identity) By definition,

$$\begin{aligned}\mathbb{Z} *_{\{1\}} \mathbb{Z} &= \frac{\mathbb{Z} * \mathbb{Z}}{\underbrace{\text{smallest normal subgroup containing } \tilde{\varphi}_1(h)\tilde{\varphi}_2^{-1}(h) \forall h \in \{1\}}_{\tilde{\varphi}_1(1)\tilde{\varphi}_2^{-1}(1)=a^0(b^0)^{-1}=1}} \\ &= \frac{\langle a, b\rangle}{\{1\}} \\ &= \underbrace{\langle a, b\rangle}_{\text{Free group on two generators}}\end{aligned}$$

In general, if $\varphi_1 : H \to G_1$, then, $\tilde{\varphi}_1 : H \to G_1 * G_2$, and

$$\tilde{\varphi}(h) = \text{the word } \varphi_1(h) \text{ viewed as a word of length 1}$$

- Ex: (Topology), Note that there's a picture that should be here. One of these days I'll go through my notes and add all the missing drawings. Probably not gonna do that on the 19th of march though.

$$\begin{aligned}X = \text{Drawing of figure 8} &= \{(x, y) \in \mathbb{R} | (x-1)^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R} | (x+1)^2 + y^2 = 1\} \\ U &= \{(x, y) \in X | X \leq 1\} \\ V &= \{(x, y) \in X | X \geq -1\}\end{aligned}$$

Let $x_0 = (0, 0)$. Then, drawings, but mostly the following junk.

$$\begin{aligned}U \simeq S^1 &\implies \pi_1(U, x_0) \cong \mathbb{Z} \\ V \simeq S^1 &\implies \pi_1(V, x_0) \cong \mathbb{Z} \\ U \cap V \simeq \{(0,0)\} &\implies \pi_1(V, x_0) = \{1\}\end{aligned}$$

So, the Van Kampen Theorem says,

$$\pi_1(X, x_0) = \mathbb{Z} *_1 \mathbb{Z} = \langle a, p\rangle$$

Pictorially, an element of $\pi_1(X, x_0)$ may look like some as yet undrawn picture or,

$$a^3 b^5 a^{-2}$$

Where a^1 corresponds to a clockwise loop around one portion of the figure 8, a a^{-1} corresponds to a counterclockwise loop around that same portion, and switching to b^1 or b^{-1} switches which portion of the figure 8 the loop will be around.