

# 1 Day 1

## 1.1 Syllabus Junk

- Pictures + Computer are ok so long as they're used for note taking.
- Expect for the tests to be at ends of the first third of the class, and the second third of the class.
- Theoretically this is a graduate course, and will be switched to 852, rather than remaining as 452.

## 1.2 The idea of algebraic topology

- Given topological spaces  $X$  and  $Y$ , how can we prove that  $X$  and  $Y$  are or aren't homeomorphic.
- To prove  $X \cong Y$ , we simply exhibit a homeomorphism.  
E.g.  $(-1, 1) \cong \mathbb{R}$ , using  $f(x) = \frac{x}{1-x^2}$   
E.g.  $\square \cong \circ$
- To prove  $X \not\cong Y$ , we'd find a topological invariant, (connected, compact, Hausdorff, ...), that only one has.  
E.g.  $(0, 1) \not\cong [0, 1]$ , here, the closed interval is compact, and the open interval is not.  
E.g.  $(0, 1) \not\cong [0, 1)$ , because,

$$\begin{aligned} [0, 1) \setminus \{0\} &= (0, 1) \text{ which is connected, but} \\ (0, 1) \setminus \{\text{any point}\} &\text{ is disconnected} \end{aligned}$$

Note, with the following exercise, If  $X \cong Y$  via a homeomorphism,  $\psi : X \rightarrow Y$ , then  $X \setminus \{p\} \cong Y \setminus \{\psi(p)\}$

- Show the following.

$$\mathbb{R} \not\cong \mathbb{R}^2$$

Here, we note that  $\mathbb{R} \setminus \{0\}$  is disconnected.

Suppose towards contradiction that  $\mathbb{R} \cong \mathbb{R}^2$ , call the homeomorphism  $\phi : \mathbb{R} \rightarrow \mathbb{R}^2$ , because  $\mathbb{R} \setminus \{0\}$ , the exercise implies that  $\mathbb{R} \setminus \{0\} \cong \mathbb{R}^2 \setminus \{\phi(0)\}$ , and therefore  $\mathbb{R}^2 \setminus \{\phi(0)\}$  is disconnected, but that's just wrong, because  $\mathbb{R}^2$  without a single point is still connected, rigorously showing this should be done through working with path connectedness. Therefore these are not homeomorphic.

$$\mathbb{R}^2 \not\cong \mathbb{R}^3$$

This was a trick question, we don't actually have any topological properties that we can rely on. If we were to attempt to remove a line from  $\mathbb{R}^2$ , we don't have enough information about what the line is homeomorphic to in  $\mathbb{R}^3$ , which is the major stumbling block.

- The Fundamental Group

- The fundamental group is a way to associate a topological space  $X$  to a group  $\pi_1(X)$  so that  $X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y)$ .
- We'll be able to use this to prove spaces aren't homeomorphic.  
Ex: In this course we'll learn the following.

$$\begin{aligned}\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) &= \mathbb{Z} \\ \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) &= \{1\} \\ \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) &\not\cong \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) \\ \mathbb{R}^2 &\not\cong \mathbb{R}^3\end{aligned}$$

Using this, we can show that these things are not homeomorphic, which is why we do algebraic topology. More powerful tools allow for more results.

- Note: It's not true that  $\pi_1(X) \cong \pi_1(Y) \Rightarrow X \cong Y$   
 More generally, algebraic topology is about associating the topological space  $X$  with the algebraic object  $A(X)$ , in such a way that  $X \cong Y \Rightarrow A(X) \cong A(Y)$   
 There's a spectrum though.
  1. Easy to compute and says nothing,  $A(x)$  is the same for all of  $X$
  2. Hard to compute, but says everything,  $A(X) \cong A(Y) \iff X \cong Y$

## 2 Day 2

### 2.1 The Fundamental Group

- Idea:  $\pi_1(X) = \{\text{"loops" in } X\} / \sim$ , where  $L_1 \equiv L_2$  if  $L_1$  can be "deformed" inside  $X$  into  $L_2$
- Ex: Last time it was claimed that  $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{Z}$ .
- Paths and Homotopies
- Let  $X$  be a topological space.

**Definition 1.** A path in  $X$  is a continuous map  $f : I \rightarrow X$ , where  $I = [0, 1] \subseteq \mathbb{R}$  (with the subspace topology from the Euclidean topology on  $\mathbb{R}$ ).

If  $f(0) = p$  and  $f(1) = q$ , we say  $f$  is a path from  $p$  to  $q$ .

- Ex:

$$\begin{aligned}X &= \mathbb{R}^2 \\ f : I &\rightarrow \mathbb{R}^2 \\ f(t) &= (1 - 2t, 0)\end{aligned}$$

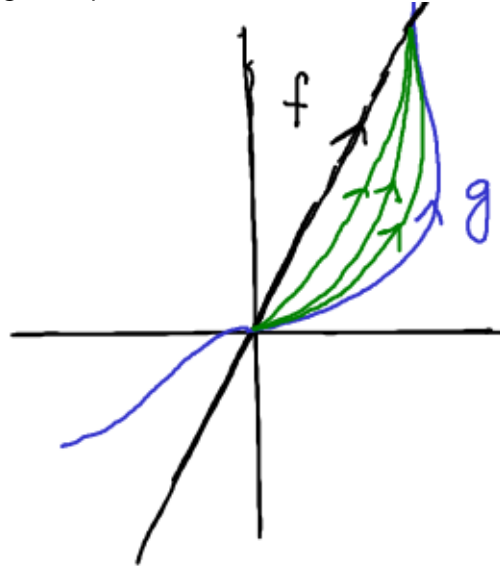
$f$  is a path in  $\mathbb{R}^2$  from  $(1, 0)$  to  $(-1, 0)$ .

- Another path in  $\mathbb{R}^2$  from  $(1, 0)$  to  $(-1, 0)$  is,

$$g : I \rightarrow \mathbb{R}^2$$

$$g(t) = (\cos(\pi t), \sin(\pi t))$$

- To make precise, “Deforming” one path into another:



**Definition 2.** Let  $f$  and  $g$  be paths in  $X$  from  $p$  to  $q$ . A path homotopy from  $f$  to  $g$  is a continuous function,

$$H : I \times I \rightarrow X$$

(note that elements of  $I \times I$  resemble,  $(s, t)$ ) Such that,

$$H(s, 0) = f(s), \forall s$$

$$H(s, 1) = g(s), \forall s$$

$$H(0, t) = p, \forall t$$

$$H(1, t) = q, \forall t$$

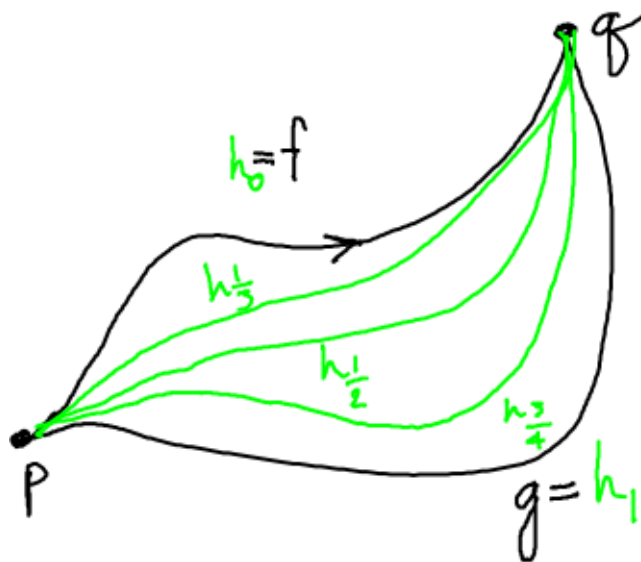
To make sense of this, define,  $\forall t$ ,

$$h_t : I \rightarrow X$$

$$h_t(s) = H(s, t)$$

Then,  $\forall t$ ,

$$h_t = \text{path in } X \text{ from } p \text{ to } q$$



This is continuous because  $H$  is continuous, and it goes from  $p$  to  $q$ , because  $h_t(0) = H(0, t) = p$  and  $h_t(1) = H(1, t) = q$ .  $h_0(s) = f$  because  $h_0(s) = H(s, 0) = f(s)$ ,  $\forall s$  and  $h_1(s) = g$  because  $h_1(s) = H(s, 1) = g(s)$ ,  $\forall s$

**Definition 3.** If  $\exists$  a path homotopy from  $f$  to  $g$ , we say  $f$  and  $g$  are path-homotopic, and  $f \cong g$   
Ex:  $X = \mathbb{R}^2$ , Let,

$$f(s) = (\cos(\pi s), \sin(\pi s))$$

$$g(s) = (\cos(\pi s), 2 \sin(\pi s))$$

Both are paths in  $\mathbb{R}^2$  from  $(1, 0)$  to  $(-1, 0)$ .  
 Then,

$$H : I \times I \rightarrow \mathbb{R}^2$$

$$H(s, t) = (\cos(\pi s), (t + 1) \sin(\pi s))$$

$H$  is a path homotopy from  $f$  to  $g$ , because,

$$H(s, 0) = (\cos(\pi s), \sin(\pi s)) = f(s)$$

$$H(s, 1) = (\cos(\pi s), 2 \sin(\pi s)) = g(s)$$

$$H(0, t) = (\cos(0), (t + 1) \sin(0)) = (1, 0) \forall t$$

$$H(1, t) = (\cos(\pi), (t + 1) \sin(\pi)) = (-1, 0) \forall t$$

- Question: Find a path homotopy from  $\mathbb{R}^2$  from  $f(s) = (s, s)$ , and  $g(s) = (s, s^2)$   
Answer(June):  $H(s, t) = (s, s^{t+1})$   
 (see the notebook, there's a solution there. Keep in mind that you want to try to find  $p$  and  $q$

first, before you do anything else)

Answer(Dr. Clader): General Trick In  $\mathbb{R}^2$  let  $f$  and  $g$  be any two paths from  $p$  to  $q$ , then the straight line homotopy is as follows,

$$H : I \times I \rightarrow \mathbb{R}^2$$

$$H(s, t) = (1 - t) * f(s) + t * g(s)$$

Note that this resembles the stuff you've seen in optimization and advanced linear algebra. This is a pretty powerful tool, remember and fear it.

- Ex: In the question above,  $H(s, t) = (s, (1 - t)s + ts^2)$

### 3 Day 3

#### 3.1 Products of Paths

- Last time: If  $f$  and  $g$  are any two paths in  $\mathbb{R}^2$  from  $p$  to  $q$ , then  $f \cong_p q$ .
- By contrast: In,  $S' = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$  if

$$f(s) = (\cos(\pi s), \sin(\pi s))$$

$$g(s) = (\cos(\pi s), -\sin(\pi s))$$

Then  $f \not\cong_p g$ . (We'll prove this carefully later).

- Fact: (HW)  $\cong_p$  is an equivalence relation on the set  $\{\text{paths in } X \text{ from } x \text{ to } y\}$  Thus we can consider the set,

$$\{\text{paths in } X \text{ from } x \text{ to } y\} / \cong_p = \{\text{path-homotopy classes of paths in } X \text{ from } x \text{ to } y\} \ni [f]$$

E.g. in the  $S'$  example above,  $[f] \neq [g]$

**Definition 4.** Let the following be so,

$$X = \text{topological space}$$

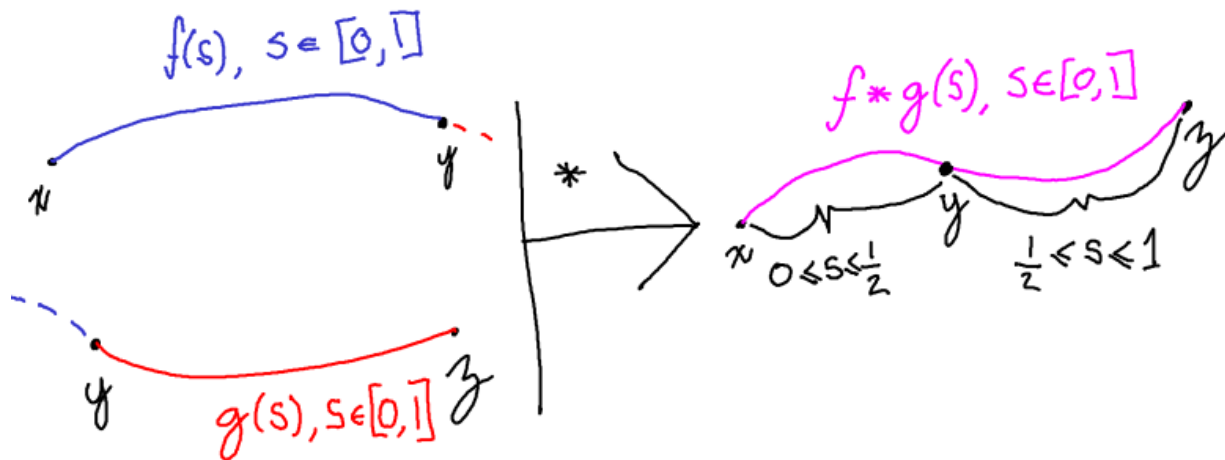
$$f = \text{path in } X \text{ from } x \text{ to } y$$

$$g = \text{path in } X \text{ from } y \text{ to } z$$

Then the concatenation of  $f$  and  $g$  is the path  $f * g$  from  $x$  to  $z$  given by,

$$f * g : I \rightarrow X$$

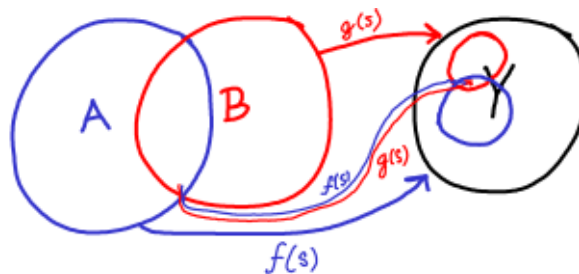
$$(f * g)(s) = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$



- Why is  $f * g$  continuous?

**Theorem 1. Gluing Lemma:** Let the following be so,

$X$  = topological space  
 $A, B \subseteq X$ , closed subsets such that  $X = A \cup B$   
 $Y$  = topological space

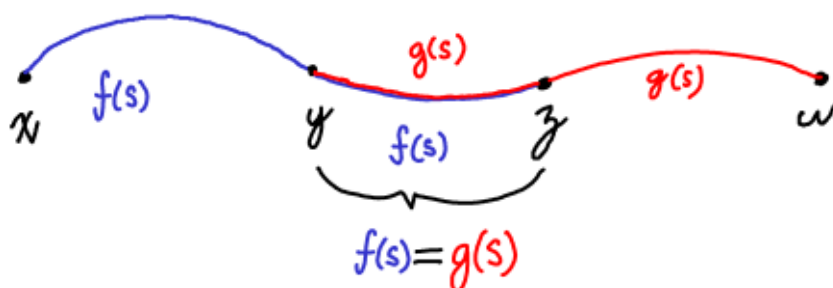


Let the following continuous functions be defined,

$$f : A \rightarrow Y$$

$$g : B \rightarrow Y$$

such that  $f(x) = g(x) \forall x \in A \cap B$ .



Then the function,

$$h : X \rightarrow Y$$

$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous. The proof is left as an exercise to the reader. Thanks. (Homework Problem 1)

Note: Applying the gluing lemma to  $I = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$  shows that  $f * g$  is continuous.

- Question: Let the following be so,

$$X = \mathbb{R}^2$$

$$f(s) = (s - 1, s)$$

$$g(s) = (s, s + 1)$$

What is  $f * g$ ? Draw a picture.

- Answer:

$$f * g = \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s - 1) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Which is a straight line from  $(-1, 0)$  to  $(1, 2)$ .

- Proposition:  $*$  is well defined on path-homotopy classes of paths  
i.e., if,

$$f_0 \cong_p f_1$$

$$g_0 \cong_p g_1$$

then,

$$f_0 * g_0 \cong_p f_1 * g_1$$

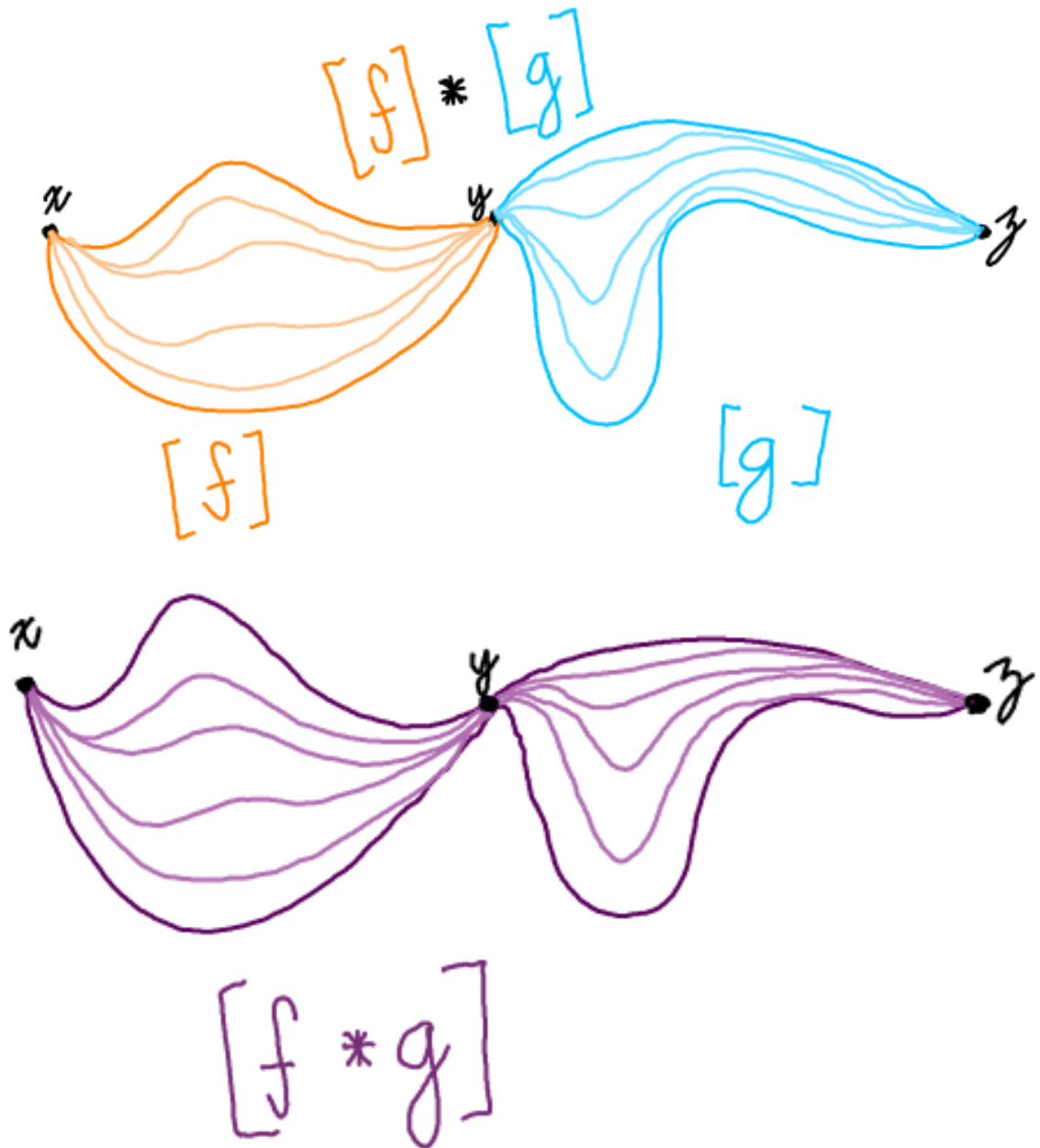
This means that if  $[f] = \{\text{path-homotopy equivalence class of } f\}$  then we can define,

$$[f] * [g] := [f * g]$$

as long as the end point of  $f$  is the starting point of  $g$ .

So, now  $*$  is an operation.

$$\{\text{paths from } x \rightarrow y\} / \cong_p * \{\text{paths } y \rightarrow z\} / \cong_p \rightarrow \{\text{paths } x \rightarrow z\} / \cong_p$$



- Idea of proof of proposition:

Let,

$F : I \times I \rightarrow X$  be a path homotopy from  $f_0$  to  $f_1$

$G : I \times I \rightarrow X$  be a path homotopy from  $g_0$  to  $g_1$



Then we can define,

$$H : I \times I \rightarrow X$$

$$H(s, t) = \begin{cases} F(2s, t) & \text{if } 0 \leq s \leq \frac{1}{2} \\ G(2s - 1, t) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

Then,

$$h_0 = H(s, 0) = (f_0 * g_0)(s)$$

$$h_1 = H(s, 1) = (f_1 * g_1)(s)$$

$$h_t = H(s, t) = (f_t * g_t)(s) \text{ (some path between } x \text{ and } z \text{)}$$

So,  $H$  is a path homotopy from  $(f_0 * g_0)$  to  $(f_1 * g_1)$ .

## 4 Day 4

### 4.0.1 Definition of Fundamental Group

- Recall: If,

$$f = \text{path in } X \text{ from } x \text{ to } y$$

$$g = \text{path in } X \text{ from } y \text{ to } z$$

Then,

$$[f] * [g] := [\text{concatenation } f * g \text{ of } f \text{ and } g]$$

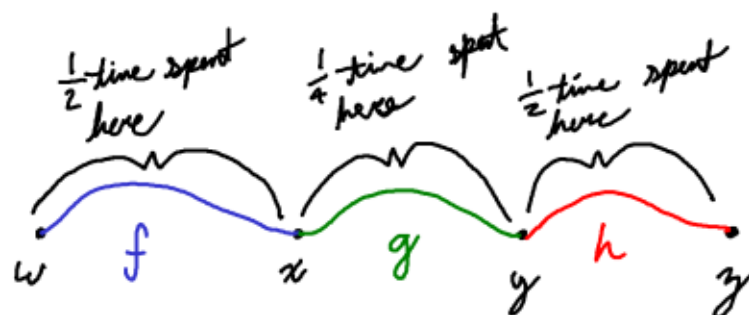
- Properties of  $*$ :

1.  $*$  is associative, or

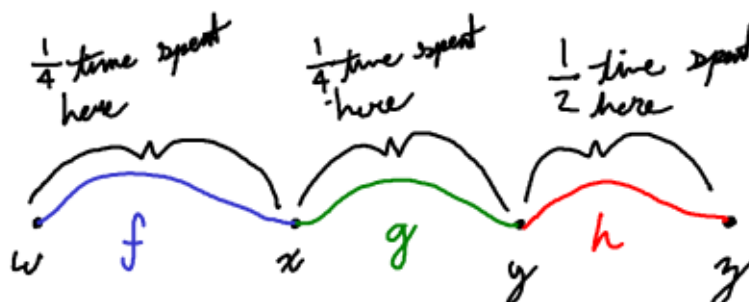
$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

The idea here is that we can adjust the time taken to travel on the path. These two paths are path-homotopic: interpolate between  $f * (g * h)$  and  $(f * g) * h$  by making  $f$  take less and less time and  $h$  take more and more time.

$$f * (g * h):$$



$$(f * g) * h:$$



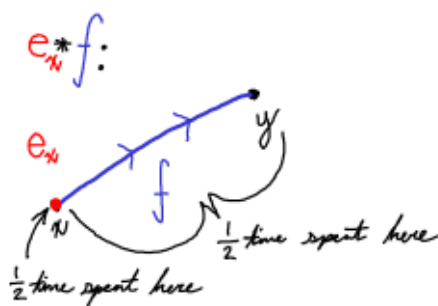
2.  $*$  has left/right identities.

Let

$$e_x : I \rightarrow X$$

$$e_x(s) = x, \forall s \in I, \text{ "constant path at } x \text{ "}$$

Then, for all paths  $f$  from  $x$  to  $y$ ,  $[f] * [e_y] = [f]$ , and  $[e_x] * [f] = [f]$ . The premise here is that  $e_x$  or  $e_y$  spend "half the time" sitting at either  $x$  or  $y$ .



These are path-homotopic: interpolate between  $f * e_y$  and  $f$  by making  $f$  take longer and longer.

3.  $*$  has inverses.

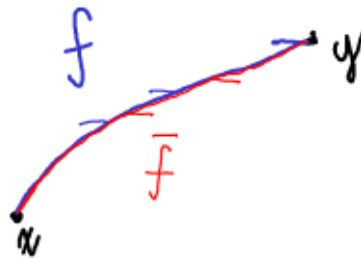
Let  $f$  be a path from  $x$  to  $y$ , and let  $\bar{f}$  be the “reverse” path,

$$\bar{f}(s) = f(1 - s)$$

Then,

$$[f] * [\bar{f}] = [e_x]$$

$$[\bar{f}] * [f] = [e_y]$$



Idea: The verbal gist of this is that the path takes half the time to travel to its destination, and is concatenated with a path that spends half the time to travel to the origin of the original function.

These are path-homotopic: interpolate between  $f * \bar{f}$  and  $e_x$  by doing less and less of  $f$  before turning around. Let,

$X$  = topological space

$x \in X$

A loop in  $X$  based at  $x \in X$  is a path,

$$f : I \rightarrow X$$

such that  $f(0) = f(1)$



– Observation: If  $f$  and  $g$  are any two loops in  $X$  based at  $x$ , then  $f * g$  is a loop.

**Definition 6.** The fundamental group of the  $X$  with basepoint  $x$  is:

$$\pi(X, x) = \{\text{path-homotopy classes of loops in } X \text{ based at } x\}$$

This is a group with the operation  $*$

- $e_x$  and  $e_y$  are loops.
- $f * \bar{f}$  and  $\bar{f} * f$  are also loops.
- Good question Katy!
- Note: The fact that  $\pi_1(X, x)$  satisfies the axioms of a group, and follows from the properties of  $*$  we just checked.  
(E.g. the identity element is  $[e_x]$ )
- Question: What is  $\pi_1(\mathbb{R}^2, (0, 0))$ ?  
Do you have a guess for  $\pi_1(S', (1, 0))$ ?  
Answer 1:  $\pi_1(\mathbb{R}^2, (0, 0)) \cong \{1\}$   
To prove this, it's enough to show that  $\pi_1(\mathbb{R}^2, (0, 0))$  has just one element, i.e., any loop in  $\mathbb{R}^2$  based at  $(0, 0)$ , is path-homotopic to any other. This is true via the straight line homotopy. Answer 2:  $\pi_1(S', (1, 0)) \cong \mathbb{Z}$ .

## 5 Day 5

### 5.0.1 To what extent does $\pi_1$ depend on $x$ ?

**Theorem 2.** Let  $X$  be a path-connected topological space, and let  $x_0, x_1 \in X$ , then  $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ . This section builds off the worksheet provided in class.

1. Part 1: see drawing
2. Part 2: Let  $f$  and  $g$  be in  $\pi_1(X, x_1)$

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= [\bar{\alpha}] * [f] * [g] * [\alpha] \\ &= [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g]) \end{aligned}$$

3. Part 3: Let  $f \in \pi_1(X, x_1)$

$$\begin{aligned} \hat{\alpha}([f]) &= [\bar{\alpha}] * [f] * [\alpha] \\ \hat{\bar{\alpha}}([\bar{\alpha}] * [f] * [\alpha]) &= [\alpha] * [\bar{\alpha}] * [f] * [\alpha] * [\bar{\alpha}] \\ &= [f] \end{aligned}$$

4. Therefore this mfer is an isomorphism.

### 5.0.2 For which topological spaces $X$ can we actually compute $\pi_1(X, x)$ ?

**Definition 7.** A topological space  $X$  is simply-connected if

1.  $X$  is path connected
  2.  $\pi_1(X, x) = 1 \ \forall x \in X$   
(Because  $X$  is path connected, we only need to check this for one  $x \in X$ )
- Ex:  $\mathbb{R}^2$  is simply connected
  - Intuition:  $X$  is simply-connected if any loop in  $X$  if any loop in  $X$  can be “shrunk down” to a constant loop.  
(for all loops  $f$  in  $X$  saying  $f$  can be “shrunk down” means  $f \cong_p c_x$  where  $c_x$  is a constant path)
  - Next time: A convex subset of  $\mathbb{R}^n$  is simply connected.

## 6 Day 6

- Goal: Prove that  $\pi_1(S^1, x) \cong \mathbb{Z}$
- Idea:  $S^1$  can be built by “wrapping  $\mathbb{R}$  around itself”.
- : Concretely, this is

$$p : \mathbb{R} \rightarrow S^1$$

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

We’ll try to “unwrap” loops in  $S^1$  to get paths in  $\mathbb{R}$

- The above map  $p$  is an example of a “covering map”. The ultimate goal of today is to understand what it means to be a covering map, before we get to the definition of it.
- Questions: Let the following be so,

$$u_1 = \{(x, y) \in S^1 | y > 0\}$$

$$u_2 = \{(x, y) \in S^1 | x > 0, y < 0\}$$

Include the drawings from class, really get sick wit it.

- Observation: For any particular  $n \in \mathbb{Z}$ , the piece,

$$(n, n + \frac{1}{2}) \cong u_1$$

The homeomorphism in Dr. Clader’s mind is,

$$\phi : (n, n + \frac{1}{2}) \rightarrow u_1$$

$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$

i.e.  $\phi = p|_{(n, n + \frac{1}{2})}$

The inverse of  $\phi$  is,

$$\begin{aligned}\phi^{-1} : u_1 &\rightarrow (n, n + \frac{1}{2}) \\ \phi^{-1} &= \frac{\cos^{-1}(x)}{2\pi} + n \\ (\text{Recall: by definition } \cos^{-1}(x) &\in [0, \pi])\end{aligned}$$

Similarly, for  $u_2$  for any particular  $n \in \mathbb{Z}$ ,  $(n - \frac{1}{4}, n) \cong u_2$ .

**Definition 8.** Let  $p : E \rightarrow B$  be a function between two topological spaces. We say  $p$  is a covering map if  $p$  is,

1.  $p$  is continuous and surjective
2.  $\forall b \in B$  there exists a neighborhood  $u$  of  $b$  such that,

$$p^{-1}(u) = \cup_{\alpha} v_{\alpha}$$

where  $v_{\alpha} \subseteq E$  are open, disjoint and,

$$p|_{v_{\alpha}} : v_{\alpha} \rightarrow u$$

is a homeomorphism for every  $\alpha$ . Note that these open subsets with this property are called evenly covered

Note that  $b$  is one particular point or neighborhood, but there should be a neighborhood for every single point in  $B$  where all of this junk holds reasonably truish.

– Ex:

$$\begin{aligned}p : \mathbb{R} &\rightarrow S^1 \\ p(x) &= (\cos(2\pi x), \sin(2\pi x))\end{aligned}$$

$p$  is a covering map. We just showed that  $u_1$  is evenly covered:

$$p^{-1}(u_1) = \cup_{n \in \mathbb{Z}} (n, n + \frac{1}{2})$$

Note that in this case the  $(n, n + \frac{1}{2})$  are the  $v_{\alpha}$  from the definition of covering maps.  $u_2$  is also evenly covered, but,  $U = S^1$  is not evenly covered because,  $p^{-1}(S^1) = \mathbb{R}$ , and the only way to write  $\mathbb{R}$  as a union of disjoint open sets  $v_{\alpha}$ , is to take  $v_{\alpha} = \mathbb{R}$ , but  $\mathbb{R} \not\cong S^1$

– Ex:

$$\begin{aligned}B &= \text{any space} \\ E &= B \times \{1, 2, \dots, n\} = n \text{ discrete copies of } B\end{aligned}$$

Where  $\{1, 2, \dots, n\}$  is equipped with the discrete topology.

## 7 Day 7

### 7.0.1 Guest lecturer: Mattias “your regular lecturer is more qualified for this” Beck

- Recalling the definition of an evenly covered set. New notation was introduced, but  $\text{\LaTeX}$  is behind the times. Let  $E$  and  $B$  be topological spaces

$$\begin{aligned}\phi : E &\twoheadrightarrow B \\ \forall b \in B, \exists u \text{ a neighborhood of } b : p^{-1}(u) &= \cup_{\alpha} v_{\alpha} \\ p|_{v_{\alpha}} : v_{\alpha} &\rightarrow u\end{aligned}$$

- Fun notation facts:

$\twoheadrightarrow$  indicates a surjective function

$\hookrightarrow$  indicates an injective function

Combining the two gives you a bijective function, but that symbol doesn't exist in latex apparently.

- Example covering:

$$\begin{aligned}E &= \mathbb{R} \\ \phi(x) &= (\cos(2\pi x), \sin(2\pi x)) \\ B &= S^1\end{aligned}$$

**Definition 9.** Given a covering map from topological spaces  $E$  to  $B$

$$p : E \rightarrow B$$

a path in our topological space  $B$ ,

$$f : I \rightarrow B$$

A lift of  $f$  is a path,  $\tilde{f} : I \rightarrow E$ , such that  $f = p \circ \tilde{f}$

- This is theoretically a theorem.

Given covering map  $p : E \rightarrow B$ ,  $p(e) = b$ ,  $f : I \rightarrow B$  path beginning at  $b$ , then there does not exist a left  $\tilde{f}$ , of  $f$  beginning at  $e$  Read Lemma 54.1 Munkres. (?!?!?)

**Theorem 3.** Let the following be so,

$$\begin{aligned}E &\text{ be a topological space} \\ B &\text{ be a topological space} \\ p : E &\rightarrow B \text{ a covering map} \\ f : I &\rightarrow B \text{ path beginning at } b \\ e \in E, &\text{ s.t. } p(e) = b\end{aligned}$$

Then there exists a unique path,  $\tilde{f}$  in  $E$  such that  $p \circ \tilde{f} = f$ , and  $\tilde{f}(0) = e$

## 8 Day 8

### 8.0.1 Guest Lecturer: Mattias “you can have a hint, but you can’t quote me on it” Beck

– ???????

## 9 Day 9

### 9.0.1 Guest Lecturers: Anastasia the Assassin, Deadly David, and Killa Katy

– Let  $p$  be a covering map.

$$p : E \rightarrow B$$

Let,  $e \in E$ ,  $b \in B$ , such that  $p(e) = b$ .

Summary of what we know about this situation,

1. Any path  $f$  in  $B$ , beginning at  $b$  has a unique lift  $\tilde{f}$  to a path in  $E$  beginning at  $e$ .
2. If  $f$  and  $g$  are two paths in  $B$ , beginning at  $b$ , such that  $f \cong_p g$ , then  $\tilde{f} \cong_p \tilde{g}$
3. If  $f$  is a loop in  $B$  based at  $b$ , then  $\tilde{f} \in p^{-1}(b)$

## 10 Day 10

### 10.0.1 $\pi_1(S^1)$ , continued:

– Recap:

$$p : \mathbb{R} \rightarrow S^1$$

$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

Then there exists a function,

$$\phi : \pi_1(S^1, b) \rightarrow p^{-1}(b)$$

$$\phi([f]) = \tilde{f}(1)$$

Where  $\tilde{f}$  is the lift of  $f$  to  $\mathbb{R}$  starting at 0.

E.g., (draw that spiraleboye)

$$\phi([\text{loop once counterclockwise}]) = 1$$

$$\phi([\text{loop twice counterclockwise}]) = 2$$

$$\phi([\text{loop once clockwise}]) = -1$$



The fact that there exists a unique lift,  $\tilde{f}$  of any  $f$  is a feature of covering maps.  
In fact,

$$p^{-1}(b) = \mathbb{Z}$$

and,

– Claim:  $\phi : \pi_1(S^1, b) \rightarrow \mathbb{Z}$  is a bijection.

*Proof.* 1. Surjective: Given  $c \in \mathbb{Z}$ , choose a path,  $\alpha : I \rightarrow \mathbb{R}$ , from 0 to  $c$  in  $\mathbb{R}$ . Then let,  
 $f : I \rightarrow S^1$  be  $f = p \circ \alpha$   
Then  $f$  is a loop in  $S^1$  based at  $b = (1, 0)$  because

$$\begin{aligned} f(0) &= p(\alpha(0)) = p(0) = (1, 0) \\ f(1) &= p(\alpha(1)) = p(c) = (1, 0) \end{aligned}$$

And,  $\tilde{f} = \alpha$  because  $p \circ \tilde{f} = p \circ \alpha = f$ . Thus,

$$\phi([f]) = \tilde{f}(1) = \alpha(1) = c$$

2. Injective: Suppose,

$$\begin{aligned} \phi([f]) &= \phi([g]) \\ \implies \tilde{f}(1) &= \tilde{g}(1) \end{aligned}$$

Then,  $\tilde{f}$  and  $\tilde{g}$  are two paths in  $\mathbb{R}$ , that both start at 0 and both end at the same point.  
 $\implies$  (courtesy of homework 2)  $\tilde{f} \cong_p \tilde{g}$  (because  $\mathbb{R}$  is simply connected)  
 $\implies p \circ H$  is a path homotopy from  $p \circ \tilde{f}$  to  $p \circ \tilde{g}$ .  
 $\implies f \cong_p g$   
 $\implies [f] = [g] \in \pi_1(S^1, b)$

□

– Claim:  $\phi$  is a group homomorphism ( thus, an isomorphism ).

*Proof.* Let  $[f], [g] \in \pi_1(S^1, b)$ , we want to show that,  $\phi([f] * [g]) = \phi([f]) + \phi([g])$   
By definition,

$$\phi([f] * [g]) = \phi([f * g]) = f * \tilde{g}(1)$$

What is  $f * \tilde{g}$ ? By definition  $f * \tilde{g}$  is the lift of  $f * g$  starting at 0 and,

$$\begin{aligned} \tilde{f} &= \text{lift of } f \text{ starting at } 0 \text{ ending at some } n \\ \tilde{g} &= \text{lift of } g \text{ starting at } 0 \text{ ending at some } m \end{aligned}$$

So,  $\tilde{f} * \tilde{g}$  doesn't make sense, but let:

$$\begin{aligned}\tilde{g}' &= \text{"shift } \tilde{g} \text{ by } n \text{"} \\ \text{i.e., } \tilde{g}' &= g(s) + n\end{aligned}$$

Now notice that  $\tilde{f} * \tilde{g}'$  now makes sense, and  $\tilde{g}'$  is a lift of  $g$ , because:

$$\begin{aligned}(p \circ \tilde{g}')(s) &= p(\tilde{g}(s)) \\ &= p(\tilde{g}(s) + n) \\ &= p(\tilde{g}(s))\end{aligned}$$

because  $p(x + n) = p(x) \forall n \in \mathbb{Z}$

$$\begin{aligned}&= (p \circ \tilde{g})(s) \\ &= g(s)\end{aligned}$$

Thus,  $\tilde{f} * \tilde{g}'$  is a lift of  $f * g$  starting at 0

$$\begin{aligned}\implies \tilde{f} * \tilde{g} &= f * g \\ f * g(1) &= \tilde{f} * \tilde{g} \\ &= \text{endpoint of } \tilde{g}' \\ &= \tilde{g}(1) + n \\ &= m + n\end{aligned}$$

This shows that

$$\begin{aligned}\phi([f] * [g]) &= m + n \\ &= \tilde{f}(1) + \tilde{g}(1) \\ &= \phi([f]) + \phi([g])\end{aligned}$$

□

– We want:

$$\begin{aligned}X \cong Y &\implies \pi_1(X, x) \cong \pi_1(Y, y) \\ &(\text{X is homeomorphic to Y})\end{aligned}$$

The big tool we'll use to do that is the tool from the second homework about maps between spaces being homomorphisms. That's for next time!

## 11 Day 11

### 11.0.1 Examining the group structure of $*$ functions

- Note that this Friday, office hours will be at 3-4pm.
- We want: If  $X \cong Y$ , then  $\pi_1(X, x) \cong \pi_1(Y, y)$ , or that, if two spaces are homeomorphic, then their fundamental groups are isomorphic. We will explore the tools used to show this in this lecture. From homework 2, we get the following definition

**Definition 10.** Let  $\varphi : X \rightarrow Y$ , be a continuous map, then the homomorphism induced by  $\varphi$  is:

$$\begin{aligned}\varphi_* : \pi_1(X, x) &\rightarrow \pi_1(Y, y) \\ \varphi_*([f]) &= [\varphi \circ f]\end{aligned}$$

See the picture of the picture drawn on the board, make a drawyboye.

Lemma: (this is referred to lemma 1) If

$$X \xrightarrow{\varphi} Y \xrightarrow{\psi} Z$$

Where  $\varphi$  and  $\psi$  are both continuous, then,

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$$

Additionally, (This is referred to as lemma 2)

$$id_* = id$$

(or that given the  $id : X \rightarrow Y$ , the induced homomorphism,  $\pi_1(X, x) \rightarrow \pi_1(Y, y)$  is the identity)

*Proof.* Firstly, Both sides are homomorphisms

$$\pi_1(X, x) \rightarrow \pi_1(Z, (\psi \circ \varphi)(x))$$

Given any  $[f] \in \pi_1(X, x)$ :

$$\begin{aligned}(\psi \circ \varphi)_*([f]) &= [(\psi \circ \varphi) \circ f] \\ &= [\psi \circ (\varphi \circ f)] \\ &= \psi_*[\varphi \circ f] \\ &= \psi_*(\varphi_*([f])) \\ &= (\psi_* \circ \varphi_*)([f])\end{aligned}$$

Given any  $[f] \in \pi_1(X, x)$ :

$$\begin{aligned} id_*([f]) &= [id \circ f] \\ &= [f] \end{aligned}$$

□

**Theorem 4.** if  $\varphi : X \rightarrow Y$  is a homeomorphism, then  $\varphi_* : \pi_1(X, x) \rightarrow \pi_1(Y, y)$  is an isomorphism.

*Proof.* We already know that  $\varphi_*$  is a homomorphism, to prove that it's a bijection, we'll find an inverse to  $\varphi_*$ . Claim that,

$$(\varphi)_* : \pi_1(Y, \varphi(x)) \rightarrow \pi_1(X, x)$$

is the inverse to  $\varphi_*$ .

(Note that this is doable, because  $\varphi$  is a homeomorphism,  $\varphi^{-1} : Y \rightarrow X$  exists, and is continuous)

To check this:

$$\begin{aligned} \varphi_* \circ (\varphi^{-1})_* &= (\varphi \circ \varphi^{-1})_*, \text{ by lemma 1 shown today} \\ &= id_*, \text{ by definition of } \varphi^{-1} \text{ (identity on } y) \\ &= id, \text{ by lemma 2 shown today (identity on } x) \\ (\varphi^{-1})_* \circ \varphi_* &= (\varphi^{-1} \circ \varphi)_* = id_* = id \end{aligned}$$

This by definition means  $\varphi_*$  and  $(\varphi^{-1})_*$  are inverse functions. Additionally, this small red box has made it onto the board, for clarification.

$$\begin{aligned} id_x : X &\rightarrow Y \\ id_{\pi_1(X, x)} : \pi_1(X, x) &\rightarrow \pi_1(X, x) \\ \text{Lemma: } (id_x)_* id_{\pi_1(X, x)} & \end{aligned}$$

□

– This ends up proving that,

$$X \cong Y \implies \pi_1(X, x) \cong \pi_1(Y, \varphi(x))$$

But, non-homeomorphic spaces can have isomorphic  $\pi_1$

Ex:

$$\begin{aligned} X &= . \\ Y &= \mathbb{R}^2 \end{aligned}$$

These are not homeomorphic, clearly  $X$  is compact and  $Y$  isn't, but their fundamental groups are isomorphic, since the fundamental group of  $X$  is just  $\{1\}$ , and clearly this is also true about  $\mathbb{R}^2$

– So, given  $X$  and  $Y$ , how can we tell if  $\pi_1(X) \cong \pi_1(Y)$ ?

### 11.0.2 Homotopy of Maps:

**Definition 11.** Let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  be continuous functions. Then a homotopy from  $f$  to  $g$  is a continuous function,

$$H : X \times I \rightarrow Y$$

such that,

$$\begin{aligned} H(x, 0) &= f(x), \forall x \in X \\ H(x, 1) &= g(x), \forall x \in X \end{aligned}$$

Our goal is to make remark about the lower star versions of these maps, given their being homotopic.

## 12 Day 12

### 12.0.1 Homotopy of maps

**Definition 12.** Let  $f : X \rightarrow Y$  be a continuous function. A homotopy from  $f$  to  $g$  is a continuous function,

$$H : X \times I \rightarrow Y$$

such that

$$\begin{aligned} H(x, 0) &= f \\ H(x, 1) &= g \end{aligned}$$

We'll often write,

$$\begin{aligned} h_t &: X \rightarrow Y \\ h_t(x) &= H(x, t) \end{aligned}$$

Then there's one  $h_t$  for each  $t \in I$  and,

$$\begin{aligned} h_0 &= f \\ h_1 &= g \\ h_t &= \text{"A function interpolating between } f \text{ and } g" \end{aligned}$$

- Terminology/Notation: If there exists a homotopy from  $f$  to  $g$ , we'll say that  $f$  is homotopic to  $g$  and write  $f \cong g$ .

– Ex:

$$\begin{aligned} f : S^1 &\rightarrow \mathbb{R}^2 \\ g : S^1 &\rightarrow \mathbb{R}^2 \\ f(x, y) &= (x, y) \\ g(x, y) &= (0, 0) \end{aligned}$$

Then  $f \cong g$ . A homotopy from  $f$  to  $g$  is,

$$\begin{aligned} H : S^1 \times I &\rightarrow \mathbb{R}^2 \\ H((x, y), t) &= ((1 - t)x, (1 - t)y) \end{aligned}$$

Do the drawing from the board.

– Ex:

$$\begin{aligned} f : \mathbb{R} &\rightarrow \mathbb{R} \\ g : \mathbb{R} &\rightarrow \mathbb{R} \\ f(x) &= x \\ g(x) &= x + 2 \end{aligned}$$

Then  $f \cong g$ . A homotopy from  $f$  to  $g$  is:

$$\begin{aligned} H : \mathbb{R} \times I &\rightarrow \mathbb{R} \\ H(x, t) &= x + 2t \end{aligned}$$

Refer again to the picture from the board.

– Questions:

\*

$$f : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$g : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$f(x) = (x, 0)$$

$$g(x) = (x, e^x)$$

\*

$$f : \mathbb{R}^2 \setminus (0, 0) \rightarrow \mathbb{R}^2 \setminus (0, 0)$$

$$g : \mathbb{R}^2 \setminus (0, 0) \rightarrow \mathbb{R}^2 \setminus (0, 0)$$

$$f(x) = (x, y)$$

$$g(x) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

$$f : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$g : \mathbb{R} \rightarrow \mathbb{R}^2$$

$$f(x) = (x, 0)$$

$$g(x) = (x, e^x)$$

Just use the straight line homotopy it's not hard.

Maybe include the drawings?

**Definition 13.** Let  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  be continuous, and let  $x_0 \in X$  be such that  $f(x_0) = g(x_0) = y_0$ . Then a homotopy from  $f$  to  $g$  relative to  $x_0$  is a homotopy  $H : X \times I \rightarrow Y$  from  $f$  to  $g$  such that  $h_t(x_0) = y_0, \forall t$ .  
 (“ $x_0$  doesn't move during the homotopy”)

- Ex: in the second part of the questions from today,  $H$  was a homotopy relative to  $(1, 0)$ , or to any other point on the unit circle.
- Ex:

$$X = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

(it's the 2 norm ball)

$$f : X \rightarrow X$$

$$g : X \rightarrow X$$

Then,

$$H : X \times I \rightarrow X$$

$$H((x, y), t) = (1 - t)x, (1 - t)y)$$

is a homotopy relative to  $(0, 0)$ .

**Theorem 5.** If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are homotopic relative to  $x_0$ , then:

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

$$g_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

are the same homomorphism.

## 13 Day 13

**Theorem 6.** Let

$$f : X \rightarrow Y$$

$$g : X \rightarrow Y$$

be a continuous function such that  $f(x_0) = g(x_0) = y_0$ . Suppose that  $f$  and  $g$  are homotopic relative to  $x_0$ . (there exists a homotopy  $H$  from  $f$  to  $g$  such that  $H(x_0, t) = y_0, \forall t$ ).

Then,

$$f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

$$g_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$$

are the same homomorphism.

*Proof.* Let  $[\alpha] \in \pi_1(X, x_0)$ . We want,

$$f_*[\alpha] = g_*[\alpha]$$

$$\iff [f \circ \alpha] = [g \circ \alpha]$$

$$\iff f \circ \alpha \cong_p g \circ \alpha$$

Define,

$$P : I \times I \rightarrow Y$$

$$P(s, t) = H(\alpha(s), t)$$



Equivalently,

$$p_t : I \rightarrow Y$$

$$p_t(s) = (h_t \circ \alpha)(s)$$

This is a path homotopy from  $f \circ \alpha$  to  $g \circ \alpha$ . Firstly, because  $H$  is a homotopy relative to  $x_0$ .

$$P(0, t) = H(\alpha(0), t) = H(x_0, t) = y_0$$

$$P(1, t) = H(\alpha(1), t) = H(x_0, t) = y_0$$

Because  $H$  is a homotopy from  $f$  to  $g$ , the following is true.

$$P(s, 0) = H(\alpha(s), 0) = f(\alpha(s))$$

$$P(s, 1) = H(\alpha(s), 1) = g(\alpha(s))$$

□

– Application: Suppose  $A \subseteq X$  and that there exists a homotopy  $H$  from

$$id : X \rightarrow X$$

to a continuous function

$$r : X \rightarrow X$$

such that,

1.  $r(x) \in A, \forall x \in X$
2.  $H(a, t) = a, \forall a \in A, \forall t \in I$   
 (“every point of  $A$  stays fixed throughout the homotopy, or,  $H$  is a homotopy relative to every point in  $A$ )

In this situation, we say that  $A$  is a deformation retract of  $X$  or that  $H$  is a deformation retraction of  $X$  onto  $A$ .

**Theorem 7.** If  $A$  is a deformation retract of  $X$ , then,

$$\pi_1(X, x_0) \cong \pi_1(A, x_0), \forall x_0 \in A$$

– Ex:

$$X = \mathbb{R}^2$$

$$A = S^1$$

$$r : X \rightarrow X$$

$$r(x, y) = \left( \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

On Friday, we saw that the straight line homotopy,  $H : X \times I \rightarrow X$  is a homotopy from  $id : X \rightarrow X$  to  $r : X \rightarrow X$ .

– Ex:

$$\begin{aligned} X &= \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \\ A &= \{(0, 0)\} \\ r : X &\rightarrow X \\ r(x, y) &= (0, 0) \end{aligned}$$

On Friday, we saw that the straight line homotopy  $H : X \times I \rightarrow X$  is a homotopy from  $id : X \rightarrow X$  to  $r : X \rightarrow X$ , Thus,

$$\pi_1(X) \cong_p \pi_1(\{.\}) = \{1\}$$

– Question: Let,

$$\begin{aligned} X &= \mathbb{R}^3 \setminus \{\text{z-axis}\} \\ A &= \{(x, y, 0) \mid x \neq 0, y \neq 0\} \end{aligned}$$

Find a deformation retraction from  $X$  onto  $A$ . (Specify both  $r$  and  $H$ )  
What does this tell us about  $\pi_1(\mathbb{R}^3 \setminus \{\text{z-axis}\})$

– Answer:

$$\begin{aligned} r(x, y, z) &= (x, y, 0) \\ H((x, y, z), t) &= (x, y, (1 - t)z) \end{aligned}$$

Thus,

$$\pi_1(\mathbb{R}^3 \setminus \{\text{z-axis}\}) \cong \pi_1(A) \cong \pi_1(\mathbb{R}^2 \setminus \{(0, 0)\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

*Proof.* Let  $x_0 \in A$ . Let,

$$\begin{aligned} i : A &\rightarrow X \\ i(a) &= a \\ s : X &\rightarrow A \\ s(x) &= r(x) \end{aligned}$$

Considering condition 2 in the definition of deformation retraction yields,  $s \circ i = id_A$ , because

$$s(i(a)) = s(a) = a$$

In the other direction,

$$i \circ s = r$$

The deformation retraction  $H$  is a homotopy relative to  $x_0$  from  $r$  to  $id_X$ , so:

$$\begin{aligned} r_* &= (id_X)_* \\ \implies (i \circ s)_* &= (id_X)_* \\ \implies i_* \circ s_* &= id \end{aligned}$$

□

## 14 Day 14

- Quiz(Midterm) on Monday, whenever that is. Standard Dr. Clader Format. Last covered topic on that will be deformation retractions.
- Recall from last time:

**Theorem 8.**

$$\begin{aligned} A &\subseteq X \\ x_0 &\in A \\ H &= \text{deformation retraction of } X \text{ onto } A \end{aligned}$$

Recall that  $H$  is a homotopy relative to  $x_0$

$$\begin{aligned} id &: X \rightarrow X \\ r &: X \rightarrow X \\ \text{s.t. } r(x) &\in A, \forall x \in X \end{aligned}$$

Then,

$$\pi_1(X, x_0) \cong \pi_1(A, x_0)$$

( $r$  is a retraction)

*Proof.* Consider,  $X \rightleftharpoons A$ , where  $X \rightarrow A$  is  $s$ , the same function as  $r$ , and  $A \rightarrow X$  is the inclusion map. Then,

$$\begin{aligned} s \circ i &= id : A \rightarrow A \\ \implies s_* \circ i_* &= id : \pi_1(A, x_0) \rightarrow \pi_1(A, x_0) \end{aligned}$$

In the other order:

$$i \circ s = r \cong id$$

Note that  $r$  is a homotopy relative to  $x_0$ , and that the next step follows from the theorem from the beginning of last class.

$$\implies i_* \circ s_* = id : \pi_1(X, x_0) \rightarrow \pi_1(X, x_0)$$

So we have:

$$\begin{aligned} \pi_1(X, x_0) &\Leftarrow \pi_1(A, x_0) \\ s_* : \pi_1(X, x_0) &\rightarrow \pi_1(A, x_0) \\ i_* : \pi_1(A, x_0) &\rightarrow \pi_1(X, x_0) \end{aligned}$$

and we've shown  $s_*$  and  $i_*$  are inverses, giving

$$\pi_1(X, x_0) \cong \pi_1(A, x_0)$$

□

- Fun Font Fabtacular Letter fundamental groups.

1. **C family:** C, E, F, G, H, I, J, K, L, M, N, S, T, U, V, W, X, Y, Z
2. **A family:** A, D, O, P, Q, R
3. **B family:** B (fuckin loser.)

The reason  $\pi_1(E) \cong \pi_1(I)$  is that there is a deformation retraction.

$$\begin{aligned} H : E \times I &\rightarrow E \\ H(x, 0) &= x \\ H(x, 1) &\in I, \text{ the letter "I"} \end{aligned}$$

The rest of this was erased, before I could write it down. Ahh damn. Let's talk about  $\pi_1(B)$  though. What is that?

1. It's the same as the fundamental group of a figure 8, because  $B \cong \infty$
2. It's also the same as:

$$\pi_1(\mathbb{R}^2 \setminus \{p, q\})$$

where  $p$  and  $q$  are unequal points in  $\mathbb{R}^2$ .

3. Also the same as  $\pi_1(\theta)$  (theta is just the letter theta)

## 14.1 Day 15

### 14.1.1 Continuation and finish of Day 14

- A space with the same  $\pi_1$  as “B”
- Ex:

$$\begin{aligned} X &= \mathbb{R}^2 \setminus \{p, q\} \\ p &= (-1, 0) \\ q &= (1, 0) \end{aligned}$$

To see that  $\pi_1(X) \cong \pi_1(\infty)$  (where infinity isn't actually infinity, but two circles joined together to look like a butt.), we can construct a deformation retraction of  $X$  onto,

$$A = \{(x+1)^2 + y^2 = 1\} \cup \{(x-1)^2 + y^2 = 1\}$$

Pictorially, refer to the picture taken in class.

1. Deformation retract  $X$  onto a closed disk of radius 2, centered at  $(0,0)$
2. Then Deformation retract onto a union of two closed disks vertically, again, refer to the goddamn picture.

- Ex:

$$\theta = \{x^2 + y^2 = 1\} \cup \{(x, 0) \mid -1 \leq x \leq 1\}$$

Oh yeah, another goddamn picture. To see that  $\pi_1(\theta) \cong \pi_1(\infty)$ , (again using infinity in lieu of the double circle butt) we can construct a deformation retraction of  $\mathbb{R}^2 \setminus \{p', q'\}$  onto  $\theta$  where  $p' = (0, \frac{1}{2})$  and  $q' = (0, -\frac{1}{2})$

- Observation: This shows that,

$$\pi_1(\infty) = \pi_1(\theta)$$

because both of them are isomorphic to  $\pi_1(\mathbb{R} \setminus \{\text{two points}\})$  But neither  $\infty$  nor  $\theta$  are deformation retracts of each other.

- They're related by a more general relationship, that of homotopy equivalence.

### 14.1.2 Homotopy Equivalence

**Definition 14.** A continuous map,

$$f : X \rightarrow Y$$

is called a homotopy equivalence if there exists a  $g : X \rightarrow Y$  such that  $f \circ g \cong id_Y$ , and  $g \circ f \cong id_X$ , with our equivalency being homotopic to.

- Goals: A homotopy equivalence induces an  $\cong$  on  $\pi_1$ .
- Any deformation retraction “yields” a homotopy equivalence, but homotopy equivalence is an EQUIVALENCE relation. Sick.

- Ex:

$$X = \mathbb{R}^2$$

$$A = \{0, 0\}$$

Then  $A$  is a deformation retract of  $X$ .

$$H((x, y), t) = ((1 - t)x, (1 - t)y)$$

But,  $A$  is not homeomorphic to  $X$

- Ex: Wow, yet another goddamn picture. Wonderful. Refer to the appropriate photograph. Closed disks in  $\mathbb{R}^2$ , then  $A$  is a deformation retraction of  $X$ , and also  $A$  is homeomorphic to  $X$ . Look at the picture ya doink.  
 $X$  is homeomorphic to  $Y$  implies that  $\pi_1(X) \cong \pi_1(Y)$ , but the converse is not true, e.g.:  $X$  is a deformation retraction of  $Y$  implies that  $\pi_1(X) \cong \pi_1(Y)$  But not converseley, e.g:  $X$  is homotopy equivalent to  $Y$  implies  $\pi_1(X) \cong \pi_1(Y)$ .  
 None of these are conversely true. Wonderful! That was confusing.

## 14.2 Day 16

### 14.2.1 QUIZ DAY (it's a midterm)

## 14.3 Day 17

### 14.3.1 Homotopy Equivalence

- Goal: A homotopy equivalence induces an isomorphism on  $\pi_1$
- This follows from:

**Theorem 9.** If  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$  are continuous,  $f \cong g$  (homotopic),

$$f(x_0) = y_0, g(x_0) = y_1$$

Then there exists a path,  $\alpha$  from  $y_0$  to  $y_1$  such that  $g_* = \hat{\alpha} \circ f_*$ .

Schematically:

$$\underbrace{\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{\hat{\alpha}} \pi_1(Y, y_1)}_{g_*}$$

*Proof.* Let

$$H : X \times I \rightarrow Y$$

be a homotopy from  $f$  to  $g$  (i.e,  $h_t : X \rightarrow Y, \forall t \in I$ ). Let,

$$\alpha : I \rightarrow Y$$

$$\alpha(t) = h_t(x_0)$$

Note that we want,

$$\begin{aligned}
 & \forall [\gamma] \in \pi_1(X, x_0) : \\
 & g_*([\gamma]) = \hat{\alpha} f_*([\gamma]) \\
 \iff & [g \circ \gamma] = [\bar{\alpha} * (f \circ \gamma) * \alpha] \\
 \iff & g \circ \gamma \cong \bar{\alpha} * (f \circ \gamma) * \alpha
 \end{aligned}$$

We'll prove these are path homotopic by interpolating between them by the following loops: (There's some drawing that goes here) Explicitly, let,

$$\begin{aligned}
 \beta_t &: I \rightarrow Y \\
 \beta_t(s) &= \bar{\alpha}((1-t)s)
 \end{aligned}$$

Then,

$$\begin{aligned}
 \beta_0 &= \bar{\alpha} \\
 \beta_1 &= e_{y_1} \\
 \beta_t &= \text{path from } y_1 \text{ to } \alpha(t)
 \end{aligned}$$

Now, define the following loop at  $y_1$ :

$$\beta_t * (h_t \circ \gamma) * \bar{\beta}_t$$

This is:

1. When  $t = 0$ :

$$\beta_0 * (h_0 \circ \gamma) * \bar{\beta}_0 = \bar{\alpha} * (f \circ \gamma) * \alpha$$

(this is the green loop from the hard to see picture)

2. When  $t = 1$ :

$$\beta_1 * (h_1 \circ \gamma) * \bar{\beta}_1 = e_{y_1} * (f \circ \gamma) * e_{y_1}$$

Thus,  $\beta_t * (h_t \circ \gamma) * \bar{\beta}_t$  give a path homotopy,

$$\bar{\alpha} * (f \circ \gamma) * \alpha \cong_p e_{y_1} * (f \circ \gamma) * e_{y_1} \cong_p g \circ \gamma$$

□

**Corollary 9.1.** If  $f : X \rightarrow Y$  is a homotopy equivalence (recall that this means there exists a  $g : Y \rightarrow X$  such that  $f \circ g_Y$  and  $g \circ f \cong id_X$ ). Then,  $f_* : \pi_1(X, x_0) \rightarrow \pi_1(Y, y_0)$  is an isomorphism.

*Proof.* We know,

$$\begin{aligned}
 & g \circ f \cong id_X \\
 \xRightarrow{\text{theorem}} & (g \circ f)_* = \hat{\alpha} \circ (id_X)_*, \text{ for some path } \alpha \\
 & \Rightarrow g_* \circ f_* = \hat{\alpha}
 \end{aligned}$$

and similarly,

$$\begin{aligned}
 f \circ g &\cong id_Y \\
 (f \circ g) &= \hat{\beta} \circ (id_Y)_* \text{ for some path in } \beta \\
 &\rightarrow f_* \circ g_* = \hat{\beta}
 \end{aligned}$$

Thus, if  $f(x_0) = y_0$ ,  $g(y_0) = x_1$ ,  $f(x) = y_1$ :

$$\begin{array}{ccc}
 \pi_1(X, x_0) & \xrightarrow{f_*} & \pi_1(Y, y_0) \\
 \downarrow \hat{\alpha} & \swarrow g_* & \downarrow \hat{\beta} \\
 \pi_1(X, x_1) & \xrightarrow{f_*} & \pi_1(Y, y_1)
 \end{array}$$

(Some fuckin arrow diagram Ah fuck) Therefore,

$$\begin{aligned}
 g_* \circ f_* &= \hat{\alpha}, \text{ an isomorphism!} \\
 &\Rightarrow g_* \text{ is surjective}
 \end{aligned}$$

And similarly, because

$$\begin{aligned}
 f_* \circ g_* &= \hat{\beta} \text{ (an isomorphism!)} \\
 &\rightarrow g_* \text{ is injective}
 \end{aligned}$$