1 Day 1

1.1 Syllabus Junk

- Pictures + Computer are ok so long as they're used for note taking.
- Expect for the tests to be at ends of the first third of the class, and the second third of the class.
- Theoretically this is a graduate course, and will be switched to 852, rather than remaining as 452.

1.2 The idea of algebraic topology

- Given topological spaces X and Y, how can we prove that X and Y are or aren't homeomorphic.
- To prove $X \cong Y$, we simply exhibit a homeomorphism.
  E.g. $(-1, 1) \cong \mathbb{R}$, using $f(x) = \frac{x}{1-x^2}$
  E.g. $\square \cong \circ$
- To prove $X \ncong Y$, we'd find a topological invariant, (connected, compact, Hausdorff,...), that only one has.
  E.g. $(0, 1) \ncong [0, 1]$, here, the closed interval is compact, and the open interval is not.
  E.g. $(0, 1) \ncong [0, 1)$, because,

  $$[0,1) \setminus \{0\} = (0,1) \text{ which is connected, but}$$
  $$(0,1) \setminus \{\text{any point}\} \text{ is disconnected}$$

  Note, with the following exercise, If $X \cong Y$ via a homeomorphism, $\psi : X \to Y$, then $X \setminus \{p\} \cong Y \setminus \{\psi(p)\}$
- Show the following.

  $$\mathbb{R} \ncong \mathbb{R}^2$$

  Here, we note that $\mathbb{R} \setminus \{0\}$ is disconnected.
  Suppose towards contradiction that $\mathbb{R} \cong \mathbb{R}^2$, call the homeomorphism $\phi : \mathbb{R} \to \mathbb{R}^2$, because $\mathbb{R} \setminus \{0\}$, the excercise implies that $\mathbb{R} \setminus \{0\} \cong \mathbb{R}^2\{\phi(0)\}$, and therefore $\mathbb{R}^2 \setminus \{\phi(0)\}$ is disconnected, but that's just wrong, because $\mathbb{R}^2$ without a single point is still connected, rigorously showing this should be done through working with path connectedness. Therefore these are not homeomorphic.

  $$\mathbb{R}^2 \ncong \mathbb{R}^3$$

  This was a trick question, we don't actually have any topological properties that we can rely on. If we were to attempt to remove a line from $\mathbb{R}^2$, we don't have enough information about what the line is homeomorphic to in $\mathbb{R}^3$, which is the major stumbling block.
- <u>The Fundamental Group</u>

- The fundamental group is a waay to associate a topological space X to a group $\pi_1(X)$ so that $X \cong Y \Rightarrow \pi_1(X) \cong \pi_2(Y)$.
- We'll be able to use this to prove spaces aren't homeomorphic.
  Ex: In this course we'll learn the following.

$$\begin{aligned} \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) &= \mathbb{Z} \\ \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) &= \{1\} \\ \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) &\not\cong \pi_2(\mathbb{R}^3 \setminus \{\text{any point}\}) \\ \mathbb{R}^2 &\not\cong \mathbb{R}^3 \end{aligned}$$

  Using this, we can show that these things are not homeomorphic, which is why we do algebraic topology. More powerful tools allow for more results.
- Note: It's not true that $\pi_1(X) \cong \pi_2(Y) \Rightarrow X \cong Y$
  More generally, algebraic topology is about associating the topological space X with the algebraic object $A(X)$, in such a way that $X \cong Y \Rightarrow A(X) \cong A(Y)$
  There's a spectrum though.
  1. Easy to compute and says nothing, $A(x)$ is the same for all of X
  2. Hard to compute, but says everything, $A(X) \cong A(Y) \iff X \cong Y$

2 Day 2

2.1 The Fundamental Group

- Idea: $\pi_1(X) = {\{\text{“loops” in X}\}}/{\sim}$, where $L_1 \equiv L_2$ if L_1 can be "deformed" inside X into L_2
- Ex: Last time it was claimed that $\pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) = \mathbb{Z}$.
- Paths and Homotopies
- Let X be a topological space.

Definition 1. A path in X is a continuous map $f : I \to X$, where $I = [0,1] \subseteq \mathbb{R}$ (with the subspace topology from the Euclidean topology on $\mathbb{R}$.)
If $f(0) = p$ and $f(1) = q$, we say f is a path from p to q.

- Ex:

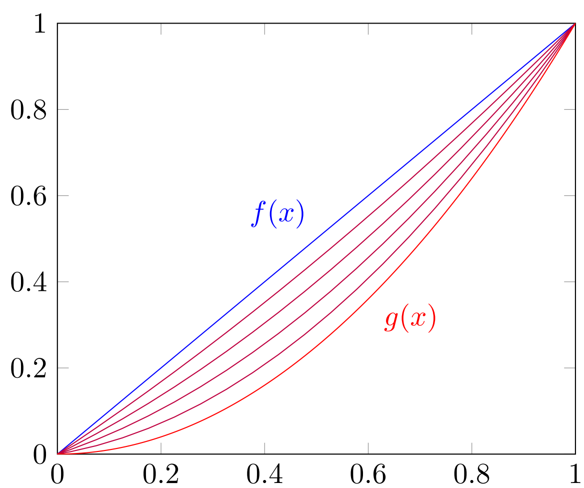
$$\begin{aligned} X &= \mathbb{R}^2 \\ f &: I \to \mathbb{R}^2 \\ f(t) &= (1-2t, 0) \end{aligned}$$

  f is a path in $\mathbb{R}^2$ from $(1,0)$ to $(-1,0)$.

- Another path in $\mathbb{R}^2$ from $(1,0)$ to $(-1,0)$ is,

$$g : I \to \mathbb{R}^2$$
$$g(t) = (\cos(\pi t), \sin(\pi t))$$

- To make precise, "Deforming" one path into another:



Definition 2. Let f and g be paths in X from p to q. A path homotopy from f to g is a continuous function,

$$H : I \times I \to X$$

(note that elements of $I \times I$ resemble, (s,t)) Such that,

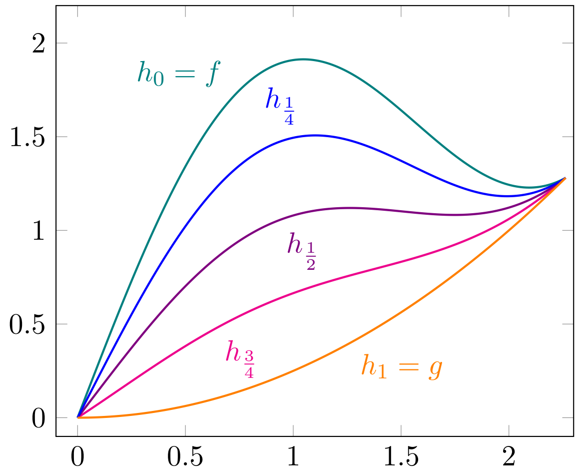
$$H(s,0) = f(s),\ \forall s$$
$$H(s,1) = g(s),\ \forall s$$
$$H(0,t) = p,\ \forall s$$
$$H(1,t) = q,\ \forall s$$

- To make sense of this, define, $\forall t$,

$$h_t : I \to X$$
$$h_t(s) = H(s,t)$$

Then, $\forall t$,

$$h_t = \text{ path in } X \text{ from } p \text{ to } q$$



This is continuous because H is continuous, and it goes from p to q, because $h_t(0) = H(0,t) = p$ and $h_t(1) = H(1,t) = q$. $h_0(s) = f$ because $h_0(s) = H(s,0) = f(s),\ \forall s$ and $h_1(s) = g$ because $h_1(s) = H(s,1) = g(s),\ \forall s$

Definition 3. If $\exists$ a path homotopy from f to g, we say f and g are path-homotopic, and $f \cong g$
Ex: $X = \mathbb{R}^2$, Let,

$$f(s) = (\cos(\pi s), \sin(\pi s))$$
$$f(s) = (\cos(\pi s), 2\sin(\pi s))$$

Both are paths in $\mathbb{R}^2$ from $(1,0)$ to $(-1,0)$.
Then,

$$H : I \times I \to \mathbb{R}^2$$
$$H(s,t) = (\cos(\pi s), (t+1) * \sin(\pi s))$$

H is a path homotopy from f to g, because,

$$H(s,0) = (\cos(\pi s), \sin(\pi s)) = f(s)$$
$$H(s,1) = (\cos(\pi s), 2\sin(\pi s)) = g(s)$$
$$H(0,t) = (\cos(0), (t+1)\sin(0)) = (1,0)\forall t$$
$$H(1,t) = (\cos(\pi), (t+1)\sin(\pi)) = (-1,0)\forall t$$

- Question: Find a path homotopy from $\mathbb{R}^2$ from $f(s) = (s,s)$, and $g(s) = (s, s^2)$
  Answer(June): $H(s,t) = (s, s^{t+1})$
  (see the notebook, there's a solution there. Keep in mind that you want to try to find p and q first, before you do anything else)
  Answer(Dr. Clader): General Trick In $\mathbb{R}^2$ let f and g be any two paths from p to q, then the

straight line homotopy is as follows,

$$\begin{aligned} H :& I \times I \to \mathbb{R}^2 \\ H(s,t) &= (1-t) * f(s) + t * g(s) \end{aligned}$$

Note that this resembles the stuff you've seen in optimization and advanced linear algebra. This is a pretty powerful tool, remember and fear it.

- Ex: In the question above, $H(s,t) = (s, (1-t)s + ts^2)$

3 Day 3

3.1 Products of Paths

- Last time: If f and g are any two paths in $\mathbb{R}^2$ from p to q, then $f \cong_p q$.
- By contrast: In, $S' = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ if

$$\begin{aligned} f(s) &= (\cos(\pi s), \sin(\pi s)) \\ g(s) &= (\cos(\pi s), -\sin(\pi s)) \end{aligned}$$

  Then $f \ncong_p g$. (We'll prove this carefully later).

- Fact: (HW) $\cong_p$ is an equivalence relation on the set $\{$paths in X from x to $y\}$ Thus we can consider the set,

$$\{\text{paths in } X \text{ from } x \text{ to } y\} \big/ \cong_p = \{\text{path-homotopy classes of paths in } X \text{ from } x \text{ to } y\} \ni [f]$$

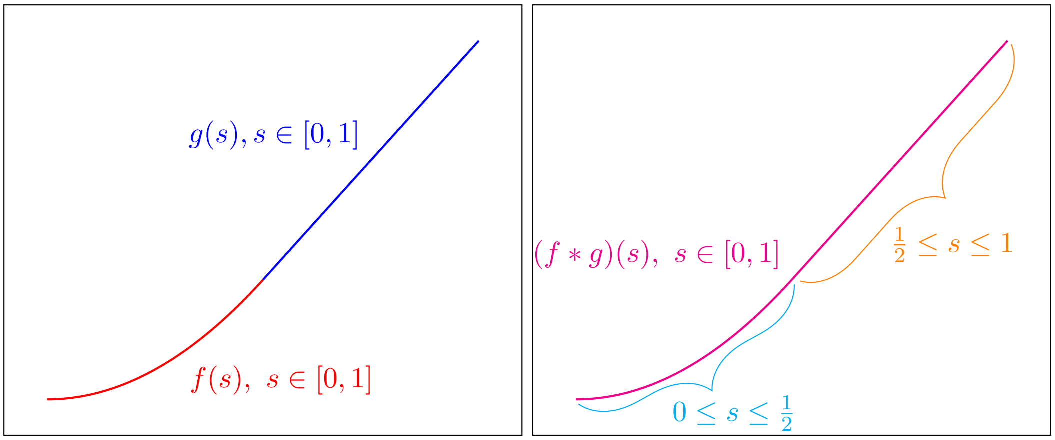
  E.g. in the S' example above, $[f] \neq [g]$

Definition 4. Let the following be so,

$$\begin{aligned} X =& \text{ topological space} \\ f =& \text{ path in } X \text{ from } x \text{ to } y \\ g =& \text{ path in } X \text{ from } y \text{ to } z \end{aligned}$$

Then the concatenation of f and g is the path $f * g$ from x to z given by,

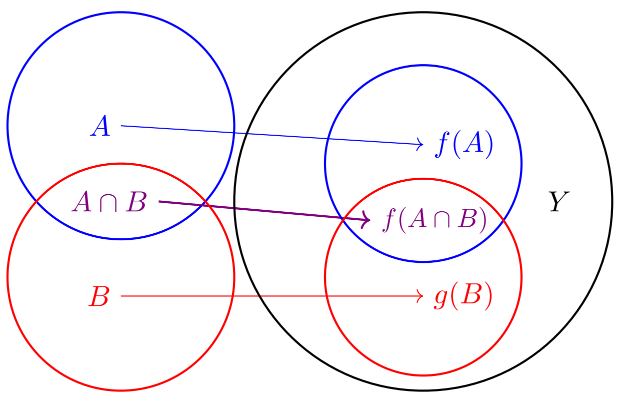
$$\begin{aligned} f * g &: I \to X \\ (f * g)(s) &= \begin{cases} f(2s) & \text{if } 0 \leq s \leq \frac{1}{2} \\ g(2s) & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases} \end{aligned}$$



- Why is $f * g$ continuous?

Theorem 1. Gluing Lemma: Let the following be so,

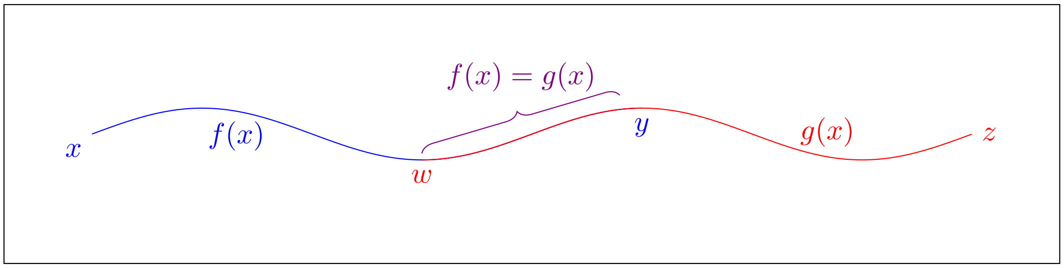
$$X = \text{topological space}$$
$$A, B \subseteq X, \text{ closed subsets such that } X = A \cup B$$
$$Y = \text{topological space}$$



Let the following continuous functions be defined,

$$f: \ A \to Y$$
$$g: \ B \to Y$$

such that $f(x) = g(x)\ \forall x \in A \cap B$.



Then the function,

$$h: X \to Y$$
$$h(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in B \end{cases}$$

is continuous. The proof is left as an exercise to the reader. Thanks. (Homework Problem 1)

Note: Applying the gluing lemma to $I = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$ shows that $f * g$ is continuous.

- Question: Let the following be so,

$$X = \mathbb{R}^2$$
$$f(s) = (s-1, s)$$
$$g(s) = (s, s+1)$$

What is $f * g$? Draw a picture.

- Answer:

$$f * g = \begin{cases} f(2s) & \text{if } 0 \le s \le \frac{1}{2} \\ g(2s-1) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

Which is a straight line from $(-1, 0)$ to $(1, 2)$.

- Proposition: $*$ is well defined on path-homotopy classes of paths
I.e., if,

$$f_0 \cong_p f_1$$
$$g_0 \cong_p g_1$$

then,

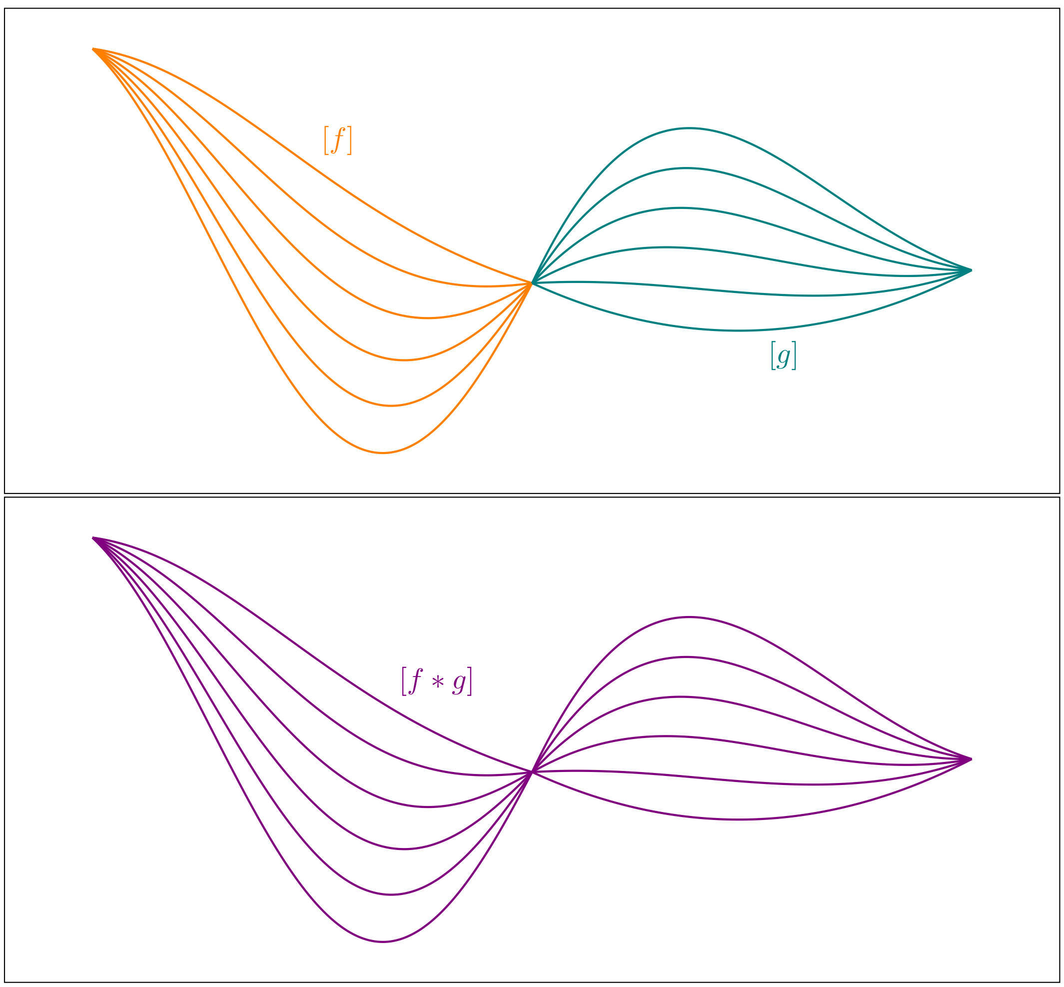
$$f_0 * g_0 \cong_p f_1 * g_1$$

This means that if $[f] = \{\text{path-homotopy equivalence class of f}\}$ then we can define,

$$[f] * [g] := [f * g]$$

as long as the end point of f is the starting point of g.
So, now $*$ is an operation.

$$\{\text{ paths from } x \to y\}\big/_{\cong_p} * \{\text{paths } y \to z\}\big/_{\cong_p} \to \{\text{paths } x \to z\}\big/_{\cong_p}$$



- <u>Idea of proof of proposition:</u>
  Let,

$$F: \ I \times I \to X \text{ be a path homotopy from } f_0 \text{ to } f_1$$
$$G: \ I \times I \to X \text{ be a path homotopy from } g_0 \text{ to } g_1$$

Then we can define,

$$H:\ I \times I \to X$$
$$H(s,y) = \begin{cases} F(2s,t) & \text{if } 0 \le s \le \frac{1}{2} \\ G(2s-1,t) & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

Then,

$$h_0 = H(s,0) = (f_0 * g_0)(s)$$
$$h_1 = H(s,1) = (f_1 * g_1)(s)$$
$$h_t = H(s,t) = (f_t * g_t)(s) \text{ (some path between } x \text{ and } z \text{)}$$

So, H is a path homotopy from $(f_0 * g_0)$ to $(f_1 * g_1)$.

4 Day 4

4.0.1 Definition of Fundamental Group

- Recall: If,

$$f = \text{path in } X \text{ from x to y}$$
$$g = \text{path in } X \text{ from y to z}$$

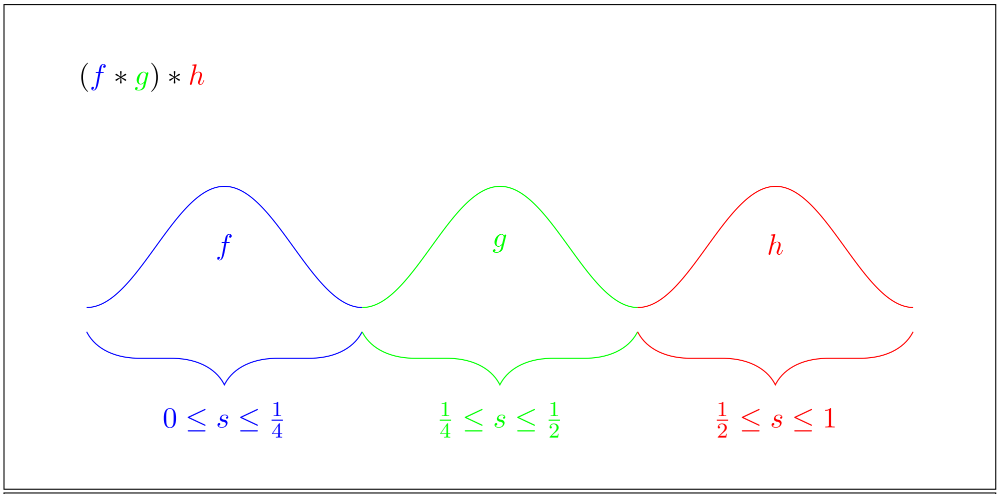
Then,

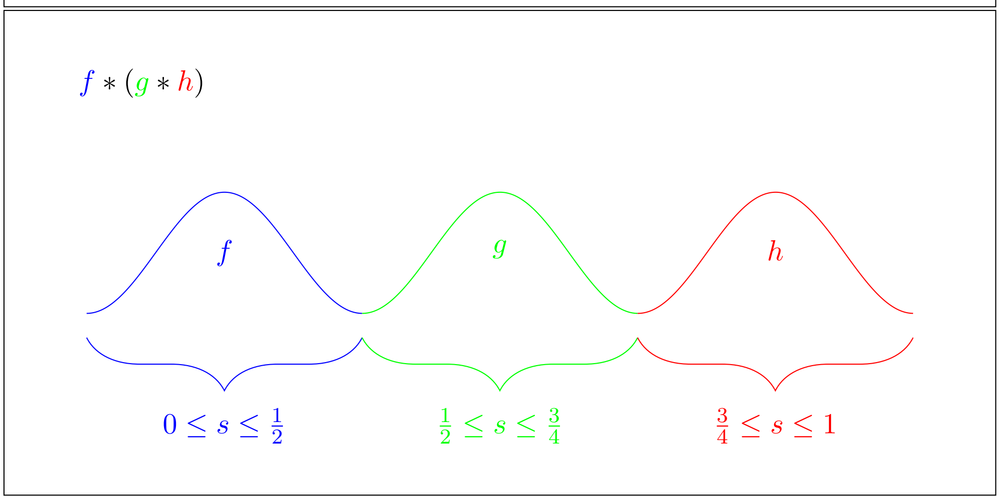
$$[f] * [g] := [\text{concatenation } f * g \text{ of } f \text{ and } g]$$

- Properties of $*$:
  1. $*$ is associative, or

$$[f] * ([g] * [h]) = ([f] * [g]) * [h]$$

The idea here is that we can adjust the time taken to travel on the path. These two paths are path-homotopic: interpolate between $f * (g * h)$ and $(f * g) * h$ by making f take less and less time and h take more and more time.

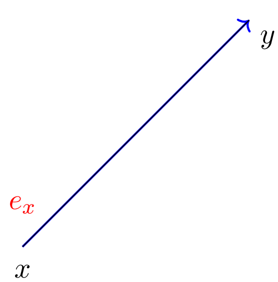


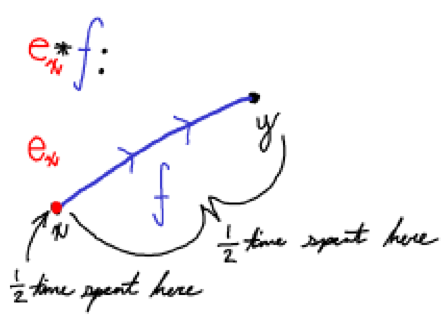


2. $*$ has left/right identities.
 Let

$$e_x : I \to X$$
$$e_x(s) = x, \ \forall \ s \in I, \text{“constant path at } x\text{”}$$

Then, for all paths f from x to y, $[f] * [e_y] = [f]$, and $[e_x] * [f] = [f]$. The premise here is that e_x or e_y spend "half the time" sitting at either x or y.





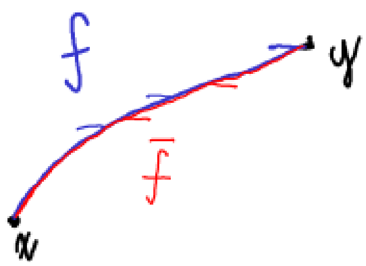
These are path-homotopic: interpolate between $f * e_y$ and f by making f take longer and longer.

3. $*$ has inverses.
   Let f be a path from x to y, and let $\overline{f}$ be the "reverse" path,

$$\overline{f}(s) = f(1-s)$$

   Then,

$$[f] * [\overline{f}] = [e_x]$$
$$[\overline{f}] * [f] = [e_y]$$



   Idea: The verbal gist of this is that the path takes half the time to travel to its destination, and is concatenated with a path that spends half the time to travel to the origin of the original function.

- These are path-homotopic: interpolate between $f * \overline{f}$ and e_x by doing less and less of f before turning around.
- Let,

$$X = \text{topological space}$$
$$x \in X$$

Definition 5. A loop in X based at $x \in X$ is a path,

$$f : I \to X$$
$$\text{such that} f(0) = f(1)$$

- Observation: If f and g are any two loops in X based at x, then $f * g$ is a loop.

Definition 6. The fundamental group of the X with basepoint x is:

$$\pi(X, x) = \{\text{path-homotopy classes of loops in } X \text{ based at } x\}$$

This is a group with the operation $*$

- e_x and e_y are loops.
- $f * \overline{f}$ and $\overline{f} * f$ are also loops.
- Note: The fact that $\pi_1(X, x)$ satisfies the axioms of a group, and follows from the properties of $*$ we just checked.
  (E.g. the identity element is $[e_x]$)

- Question: What is $\pi_1(\mathbb{R}^2, (0, 0))$?
  Do you have a guess for $\pi_1(S', (1, 0))$?
  Answer 1: $\pi_1(\mathbb{R}^2, (0, 0)) \cong \{1\}$
  To prove this, it's enough to show that $\pi_1(\mathbb{R}^2, (0, 0))$ has just one element,
  i.e., any loop in $\mathbb{R}^2$ based at $(0, 0)$, is path-homotopic to any other. This is true via the straight line homotopy. Answer 2: $\pi_1(S', (1, 0)) \cong \mathbb{Z}$.

5 Day 5

5.0.1 To what extent does π_1 depend on x?

Theorem 2. Let X be a path-connected topological space, and let $x_0, x_1 \in X$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. This section builds off the worksheet provided in class.

1. Part 1: see drawing
2. Part 2: Let f and g be in $\pi_1(X, x_1)$

$$\begin{aligned}\hat{\alpha}([f] * [g]) &= [\overline{\alpha}] * [f] * [g] * [\alpha] \\ &= [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] * [g] * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g])\end{aligned}$$

3. Part 3: Let $f \in \pi_1(X, x_1)$

$$\begin{aligned}\hat{\alpha}([f]) &= [\overline{\alpha}] * [f] * [\alpha] \\ \hat{\overline{\alpha}}([\overline{\alpha}] * [f] * [\alpha]) &= [\alpha] * [\overline{\alpha}] * [f] * [\alpha] * [\overline{\alpha}] \\ &= [f]\end{aligned}$$

4. Therefore this mfer is an isomorphism.

5.0.2 For which topological spaces X can we actually compute $\pi_1(X,x)$?

Definition 7. A topological space X is simply-connected if

1. X is path connected
2. $\pi_1(X,x) = 1 \; \forall x \in X$
   (Because X is path connected, we only need to check this for one $x \in X$)

- Ex: $\mathbb{R}^2$ is simply connected
- Intuition: X is simply-connected if any loop in X if any loop in X can be "shrunk down" to a constant loop.
  (for all loops f in X saying f can be "shrunk down" means $f \cong_p c_x$ where c_x is a constant path)
- Next time: A convex subset of $\mathbb{R}^n$ is simply connected.

6 Day 6

- Goal: Prove that $\pi_1(S^1, x) \cong \mathbb{Z}$
- Idea: S^1 can be built by "wrapping $\mathbb{R}$ around itself".
  : Concretely, this is

$$
\begin{aligned}
p &: \mathbb{R} \to S^1 \\
p(x) &= (\cos(2\pi x), \sin(2\pi x))
\end{aligned}
$$

  We'll try to "unwrap" loops in S^1 to get paths in $\mathbb{R}$

- The above map p is an example of a "covering map". The ultimate goal of today is to understand what it means to be a covering map, before we get to the definition of it.

- Questions: Let the following be so,

$$
\begin{aligned}
u_1 &= \{(x,y) \in S^1 | y > 0\} \\
u_2 &= \{(x,y) \in S^1 | x > 0,\ y < 0\}
\end{aligned}
$$

  Include the drawings from class, really get sick wit it.

- Observation: For any particular $n \in \mathbb{Z}$, the piece,

$$(n, n + \frac{1}{2}) \cong u_1$$

The homeomorphism in Dr. Clader's mind is,

$$\phi : (n, n + \frac{1}{2}) \to u_1$$
$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$
$$\text{i.e. } \phi = p_{|(n,n+\frac{1}{2})}$$

The inverse of ϕ is,

$$\phi^{-1} : u_1 \to (n, n + \frac{1}{2})$$
$$\phi^{-1} = \frac{\cos^{-1}(x)}{2\pi} + n$$
$$\text{(Recall: by definition } \cos^{-1}(x) \in [0, \pi])$$

Similarly, for u_2 for any particular $n \in \mathbb{Z}$, $(n - \frac{1}{4}, n) \cong u_2$.

Definition 8. Let $p : E \to B$ be a function between two topological spaces. We say p is a covering map if p is,

1. p is continuous and surjective
2. $\forall b \in B$ there exists a neighborhood u of b such that,

$$p^{-1}(u) = \cup_\alpha v_\alpha$$

where $v_\alpha \subseteq E$ are open, disjoint and,

$$p_{|v_\alpha} : v_\alpha \to u$$

is a homeomorphism for every α. Note that these open subsets with this property are called evenly covered
Note that b is one particular point or neighborhood, but there should be a neighborhood for every single point in B where all of this junk holds reasonably truish.

- Ex:

$$p : \mathbb{R} \to S^1$$
$$p(x) = (\cos(2\pi x), \sin(2\pi x))$$

p is a covering map. We just showed that u_1 is evenly covered:

$$p^{-1}(u_1) = \cup_{n\in\mathbb{Z}}(n, n + \frac{1}{2})$$

Note that in this case the $(n, n + \frac{1}{2})$ are the v_α from the definition of covering maps. u_2 is also evenly covered, but, $U = S^1$ is not evenly covered because, $p^{-1}(S^1) = \mathbb{R}$, and the only way to write $\mathbb{R}$ as a uniion of disjoint open sets v_α, is to take $v_\alpha = \mathbb{R}$, but $\mathbb{R} \ncong S^1$

- Ex:

$$B = \text{any space}$$
$$E = B \times \{1, 2, ..., n\} = \text{n discrete copies of } B$$

Where $\{1, 2, ..., n\}$ is equipped with the discrete topology.

7 Day 7

7.0.1 Guest lecturer: Mattias "your regular lecturer is more qualified for this" Beck

- Recalling the definition of an evenly covered set. New notation was introduced, but LaTeXis behind the times. Let E and B be topological spaces

$$\phi : E \twoheadrightarrow B$$
$$\forall b \in B, \exists u \text{ a neighborhood of b} : p^{-1}(u) = \cup_\alpha v_\alpha$$
$$p_{|v_\alpha} : v_\alpha \to u$$

- Fun notation facts:
  $\twoheadrightarrow$ indicates a surjective function
  $\hookrightarrow$ indicates an injective function
  Combining the two gives you a bijective function, but that symbol doesn't exist in latex apparently.

- Example covering:

$$E = \mathbb{R}$$
$$\phi(x) = (\cos(2\pi x), \sin(2\pi x))$$
$$B = S^1$$

Definition 9. Given a covering map from topological spaces E to B

$$p : E \to B$$

a path in our topological space B,

$$f : I \to B$$

A lift of f is a path, $\tilde{f} : I \to E$, such that $f = p \circ \tilde{f}$

- This is theoretically a theorem.
  Given covering map $p : E \to B$, $p(e) = b$, $f : I \to B$ path beginning at b, then there does not exist a left $\tilde{f}$, of f beginning at e Read Lemma 54.1 Munkres. (?!?!?)

Theorem 3. Let the following be so,

$$
\begin{aligned}
&E \text{ be a topological space} \\
&B \text{ be a topological space} \\
&p : E \to B \text{ a covering map} \\
&f : I \to B \text{ path beginning at } b \\
&e \in E,\ s.t.\ p(e) = b
\end{aligned}
$$

Then there exists a unique path, $\tilde{f}$ in E such that $p \circ \tilde{f} = f$, and $\tilde{f}(0) = e$

8 Day 8

8.0.1 Guest Lecturer: Mattias "you can have a hint, but you can't quote me on it" Beck

- ???????

9 Day 9

9.0.1 Guest Lecturers: Anastasia the Assassin, Deadly David, and Killa Katy

- Let p be a covering map.

$$p : E \to B$$

Let, $e \in E$, $b \in B$, such that $p(e) = b$.
Summary of what we know about this situation,

1. Any path f in B, beginning at b has a unique lift $\tilde{f}$ to a path in E beginning at e.
2. If f and g are two paths in B, beginning at b, such that $f \cong_p g$, then $\tilde{f} \cong_p \tilde{g}$
3. If f is a loop in B based at b, then $\tilde{f} \in p^{-1}(b)$

10 Day 10

10.0.1 $\pi_1(S^1)$, continued:

- Recap:

$$
\begin{aligned}
p &: \mathbb{R} \to S^1 \\
p(x) &= (\cos(2\pi x), \sin(2\pi x))
\end{aligned}
$$

Then there exists a function,

$$
\begin{aligned}
\phi &: \pi_1(S^1, b) \to p^{-1}(b) \\
\phi([f]) &= \tilde{f}(1)
\end{aligned}
$$

Where $\tilde{f}$ is the lift of f to $\mathbb{R}$ starting at 0.
E.g.,(draw that spiraleboye)

$$\phi([\text{loop once counterclockwise}]) = 1$$
$$\phi([\text{loop twice counterclockwise}]) = 2$$
$$\phi([\text{loop once clockwise}]) = -1$$

The fact that there exists a unique lift, $\tilde{f}$ of any f is a feature of covering maps.
In fact,

$$p^{-1}(b) = \mathbb{Z}$$

and,

- Claim: $\phi : \pi_1(S^1, b) \to \mathbb{Z}$ is a bijection.

Proof. 1. Surjective: Given $c \in \mathbb{Z}$, choose a path, $\alpha : I \to \mathbb{R}$, from 0 to c in $\mathbb{R}$. Then let, $f : I \to S^1$ be $f = p \circ \alpha$
Then f is a loop in S^1 based at $b = (1, 0)$ because

$$f(0) = p(\alpha(0)) = p(0) = (1, 0)$$
$$f(1) = p(\alpha(1)) = p(c) = (1, 0)$$

And, $\tilde{f} = \alpha$ because $p \circ \tilde{f} = p \circ \alpha = f$. Thus,

$$\phi([f]) = \tilde{f}(1) = \alpha(1) = c$$

2. Injective: Suppose,

$$\phi([f]) = \phi([g])$$
$$\implies \tilde{f}(1) = \tilde{g}(1)$$

Then, $\tilde{f}$ and $\tilde{g}$ are two paths in $\mathbb{R}$, that both start at 0 and both end at the same point.
$\Rightarrow$ (courtesy of homework 2) $\tilde{f} \cong_p \tilde{g}$ (because $\mathbb{R}$ is simply connected)
$\Rightarrow p \circ H$ is a path homotopy from $p \circ \tilde{f}$ to $p \circ \tilde{g}$.
$\Rightarrow f \cong_p g$
$\Rightarrow [f] = [g] \in \pi_1(S^1, b)$

$\square$

- Claim: ϕ is a group homomorphism (thus, an isomorphism).

Proof. Let $[f], [g] \in \pi\pi_1(S^1, b)$, we want to show that, $\phi([f] * [g]) = \phi([f]) + \phi([g])$
By definition,

$$\phi([f] * [g]) = \phi([f * g]) = f \tilde{*} g(1)$$

What is $f \tilde{*} g$? By definition $f \tilde{*} g$ is the lift of $f * g$ starting at 0 and,

$$\begin{aligned} \tilde{f} &= \text{lift of } f \text{ starting at 0 ending at some } n \\ \tilde{g} &= \text{lift of } g \text{ starting at 0 ending at some } m \end{aligned}$$

So, $\tilde{f} * \tilde{g}$ doesn't make sense, but let:

$$\begin{aligned} \tilde{g}' &= \text{``shift } \tilde{g} \text{ by n ''} \\ \text{i.e., } \tilde{g}^{'} &= g(s) + n \end{aligned}$$

Sow notice that $\tilde{f} * \tilde{g}^{'}$ now makes sense, and $\tilde{g}^{'}$ is a lift of g, because:

$$\begin{aligned} (p \circ \tilde{g}^{'})(s) &= p(\tilde{g}(s)) \\ &= p(\tilde{g}(s) + n) \\ &= p(\tilde{g}(s)) \end{aligned}$$

because $p(x + n) = p(x), \ \forall n \in \mathbb{Z}$

$$\begin{aligned} &= (p \circ \tilde{g})(s) \\ &= g(s) \end{aligned}$$

Thus, $\tilde{f} * \tilde{g}^{'}$ is a lift of $f * g$ starting at 0

$$\begin{aligned} \implies \tilde{f} * \tilde{g} &= f \tilde{*} g \\ f \tilde{*} g(1) &= \tilde{f} * \tilde{g} \\ &= \text{endpoint of } \tilde{g}' \\ &= \tilde{g}(1) + n \\ &= m + n \end{aligned}$$

This shows that

$$\begin{aligned} \phi([f] * [g]) &= m + n \\ &= \tilde{f}(1) + \tilde{g}(1) \\ &= \phi([f]) + \phi([g]) \end{aligned}$$

$\square$

- We want:

$$X \cong Y \implies \pi_1(X, x) \cong \pi_1(Y, y)$$
(X is homeomorphic to Y)

The big tool we'll use to do that is the tool from the second homework about maps between spaces being homomorphisms. That's for next time!

11 Day 11

11.0.1 Examining the group structure of $*$ functions

- Note that this Friday, office hours will be at 3-4pm.
- We want: If $X \cong Y$, then $\pi_1(X, x) \cong \pi_1(Y, y)$, or that, if two spaces are homeomorphic, then their fundamental groups are isomorphic. We will explore the tools used to show this in this lecture. From homework 2, we get the following definition

Definition 10. Let $\varphi : X \to Y$, be a continuous map, then the homomorphism induced by φ is:

$$\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$$
$$\varphi_*([f]) = [\varphi \circ f]$$

See the picture of the picture drawn on the board, make a drawyboye.

Lemma: (this is referred to lemma 1)If

$$X \to^{\varphi} Y \to^{\psi} Z$$

Where φ and ψ are both continuous, then,

$$(\psi \circ \varphi)_* = \psi \circ_* \varphi_*$$

Additionally, (This is referred to as lemma 2)

$$id_* = id$$

(or that given the $id : X \to Y$, the induced homomorphism, $\pi_1(X, x) \to \pi_1(Y, y)$ is the identity)

Proof. Firstly, Both sides are homomorphisms

$$\pi_1(X, x) \to \pi_1(Z, (\psi \circ \varphi)(x))$$

Given any $[f] \in \pi_1(X, x)$:

$$\begin{aligned}(\psi \circ \varphi)_*([f]) &= [(\psi \circ \varphi) \circ f] \\ &= [\psi \circ (\varphi \circ f)] \\ &= \psi_*[\varphi \circ f] \\ &= \psi_*(\varphi_*(f)) \\ &= (\psi_* \circ \varphi_*)([f])\end{aligned}$$

Given any $[f] \in \pi_1(X, x)$:

$$\begin{aligned}id_*([f]) &= [id \circ f] \\ &= [f]\end{aligned}$$

□

Theorem 4. if $\varphi : X \to Y$ is a homeomorphism, then $\varphi_* : \pi_1(X, x) \to \pi_1(Y, y)$ is an isomorphism.

Proof. We already know that φ_* is a homomorphism, to prove that it's a bijection, we'll find an inverse to φ_*. Claim that,

$$(\varphi)_* : \pi_1(Y, \varphi(x)) \to \pi_1(X, x)$$

is the inverse to φ_*.
(Note that this is doable, because φ is a homeomorphism, $\varphi^{-1} : Y \to X$ exists, and is continuous)
To check this:

$$\begin{aligned}\varphi_* \circ (\varphi^{-1}) &= (\varphi \circ \varphi^{-1}), \text{ by lemma 1 shown today} \\ &= id_*, \text{ by definition of } \varphi^{-1} \text{ (identity on y)} \\ &= id, \text{ by lemma 2 shown today (identity on x)} \\ (\varphi^{-1})_* \circ \varphi_* &= (\varphi^{-1} \circ \varphi)_* = id_* = id\end{aligned}$$

This by definition means φ_* and $(\varphi^{-1})_*$ are inverse functions. Additionally, this small red box has made it onto the board, for clarification.

$$\begin{aligned}&id_x : X \to Y \\ &id_{\pi_1(X,x)} : \pi_1(X, x) \to \pi_1(X, x) \\ &\text{Lemma: } (id_x)_* id_{\pi_1(X,x)}\end{aligned}$$

□

- This ends up proving that,

$$X \cong Y \implies \pi_1(X, x) \cong \pi_1(Y, \varphi(x))$$

But, non-homeomorphic spaces can have isomorphic π_1
Ex:

$$X = .$$
$$Y = \mathbb{R}^2$$

These are not homeomorphic, clearly X is compact and Y isn't, but their fundamental groups are isomorphic, since the fundamental group of X is just $\{1\}$, and clearly this is also true about $\mathbb{R}^2$

- So, given X and Y, how can we tell if $\pi_1(X) \cong \pi_1(Y)$?

11.0.2 Homotopy of Maps:

Definition 11. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions. Then a homotopy from f to g is a continuous function,

$$H : X \times I \to Y$$

such that,

$$H(x, 0) = f(x),\ \forall x \in X$$
$$H(x, 1) = f(x),\ \forall x \in X$$

Our goal is to make remark about the lower star versions of these maps, given their being homotopic.

12 Day 12

12.0.1 Homotopy of maps

Definition 12. Let $f : X \to Y$ be a continuous function. A homotopy from f to g is a continuous function,

$$H : X \times I \to Y$$
$$\text{such that}$$
$$H(x, 0) = f$$
$$H(x, 0) = g$$

We'll often write,

$$h_t : X \to Y$$
$$h_t(x) = H(x, t)$$

Then there's one h_t for each $t \in I$ and,

$$\begin{aligned} h_0 &= f \\ h_1 &= g \\ h_t &= \text{“A function interpolating between } f \text{ and } g\text{”} \end{aligned}$$

- Terminology/Notation: If there exists a homotopy from f to g, we'll say that f is homotopic to g and write $f \cong g$.

- Ex:

$$\begin{aligned} f : S^1 &\to \mathbb{R}^2 \\ g : S^1 &\to \mathbb{R}^2 \\ f(x, y) &= (x, y) \\ g(x, y) &= (0, 0) \end{aligned}$$

  Then $f \cong g$. A homotopy from f to g is,

$$\begin{aligned} H : S^1 \times I &\to \mathbb{R}^2 \\ H((x, y), t) &= ((1-t)x, (1-t)y) \end{aligned}$$

  Do the drawing from the board.

- Ex:

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ g : \mathbb{R} &\to \mathbb{R} \\ f(x) &= x \\ g(x) &= x + 2 \end{aligned}$$

  Then $f \cong g$. A homotopy from f to g is:

$$\begin{aligned} H : \mathbb{R} \times I &\to \mathbb{R} \\ H(x, t) &= x + 2t \end{aligned}$$

  Refer again to the picture from the board.

- <u>Questions:</u>
  -
$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}^2 \\ g &: \mathbb{R} \to \mathbb{R}^2 \\ f(x) &= (x, 0) \\ g(x) &= (x, e^x) \end{aligned}$$
  -
$$\begin{aligned} f &: \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2 \setminus (0,0) \\ g &: \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2 \setminus (0,0) \\ f(x) &= (x, y) \\ g(x) &= (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}) \end{aligned}$$

$$\begin{aligned} f &: \mathbb{R} \to \mathbb{R}^2 \\ g &: \mathbb{R} \to \mathbb{R}^2 \\ f(x) &= (x, 0) \\ g(x) &= (x, e^x) \end{aligned}$$

    Just use the straight line homotopy it's not hard.

  Maybe include the drawings?

Definition 13. Let $f : X \to Y$ and $g : X \to Y$ be continuous, and let $x_0 \in X$ be such that $f(x_0) = g(x_0) = y_0$. Then a <u>homotopy from f to g relative to x_0</u> is a homotopy $H : X \times I \to Y$ from f to g such that $h_t(x_0) = y_0, \ \forall t$.
("x_0 doesn't move during the homotopy")

- <u>Ex:</u> in the second part of the questions from today, H was a homotopy relative to $(1, 0)$, or to any other point on the unit circle.
- <u>Ex:</u>
$$X = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

  (it's the 2 norm ball)
$$\begin{aligned} f &: X \to X \\ g &: X \to X \end{aligned}$$

Then,

$$H : X \times I \to X$$
$$H((x, y), t) = (1 - t)x, (1 - t)y)$$

is a homotopy relative to $(0, 0)$.

Theorem 5. If $f : X \to Y$ and $g : X \to Y$ are homotopic relative to x_0, then:

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$g_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

are the same homomorphism.

13 Day 13

Theorem 6. Let

$$f : X \to Y$$
$$g : X \to Y$$

be a continuous function such that $f(x_0) = g(x_0) = y_0$. Suppose that f and g are homotopic relative to x_0. (there exists a homotopy H from f to g such that $H(x_0, t) = y_0, \ \forall t$).
Then,

$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$
$$f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

are the same homomorphism.

Proof. Let $[\alpha] \in \pi_1(X, x_0)$. We want,

$$f * [\alpha] = g * [\alpha]$$
$$\iff [f \circ \alpha] = [g \circ \alpha]$$
$$\iff f \circ \alpha \cong_p g \circ \alpha$$

Define,

$$P : I \times I \to Y$$
$$P(s, t) = H(\alpha(s), t)$$

Equivalently,

$$p_t : I \to Y$$
$$p_t(s) = (h_t \circ \alpha)(s)$$

This is a path homotopy from $f \circ \alpha$ to $g \circ \alpha$. Firstly, because H is a homotopy relative to x_0.

$$P(0,t) = H(\alpha(0),t) = H(x_0,t) = y_0$$
$$P(1,t) = H(\alpha(1),t) = H(x_0,t) = y_0$$

Because H is a homotopy from f to g, the following is true.

$$P(s,0) = H(\alpha(s),0) = f(\alpha(s))$$
$$P(s,1) = H(\alpha(s),1) = g(\alpha(s))$$

□

- Application: Suppose $A \subseteq X$ and that there exists a homotopy H from

  $$id : X \to X$$

  to a continuous function

  $$r : X \to X$$

  such that,

  1. $r(x) \in A,\ \forall x \in X$
  2. $H(a,t) = a,\ \forall a \in A,\ \forall t \in I$
     ("every point of A stays fixed throughout the homotopy, or, H is a homotopy relative to every point in A)

  In this situation, we say that A is a deformation retract of X or that H is a deformation retraction of X onto A.

Theorem 7. If A is a deformation retract of X, then,

$$\pi_1(X,x_0) \cong \pi_1(A,x_0),\ \forall x_0 \in A$$

- Ex:

  $$X = \mathbb{R}^2$$
  $$A = S^1$$
  $$r : X \to X$$
  $$r(x,y) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$$

  On Friday, we saw that the straight line homotopy, $H : X \times I \to X$ is a homotopy from $id : X \to X$ to $r : X \to X$.

- Ex:

$$\begin{aligned} X &= \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\} \\ A &= \{(0,0)\} \\ r &: X \to X \\ r(x,y) &= (0,0) \end{aligned}$$

On Friday, we saw that the straight line homotopy $H : X \times I \to X$ is a homotopy from $id : X \to X$ to $r : X \to X$, Thus,

$$\pi_1(X) \cong_p \pi_1(\{.\}) = \{1\}$$

- Question: Let,

$$\begin{aligned} X &= \mathbb{R}^3 \setminus \{\text{z-axis}\} \\ A &= \{(x,y,0) | x \neq 0,\ y \neq 0\} \end{aligned}$$

Find a deformation retraction from X onto A. (Specify both r and H)
What does this tell us about $\pi_1(\mathbb{R}^3\{\text{z-axis}\})$

- Answer:

$$\begin{aligned} r(x,y,z) &= (x,y,0) \\ H((x,y,z),t) &= (x,y,(1-t)z) \end{aligned}$$

Thus,

$$\pi_1(\mathbb{R}^3 \setminus \{\text{z-axis}\}) \cong \pi_1(A) \cong \pi_1(\mathbb{R}^2 \setminus \{(0,0)\}) \cong \pi_1(S^1) \cong \mathbb{Z}$$

Proof. Let $x_0 \in A$. Let,

$$\begin{aligned} i &: A \to X \\ i(a) &= a \\ s &: X \to A \\ s(x) &= r(x) \end{aligned}$$

Considering condition 2 in the definition of deformation retraction yields, $s \circ i = id_A$, because

$$s(i(a)) = s(a) = a$$

In the other direction,

$$i \circ s = r$$

The deformation retraction H is a homotopy relative to x_0 from r to id_X, so:

$$\begin{aligned} r_* &= (id_X)_* \\ \implies (i \circ s)_* &= (id_X)_* \\ \implies i_* \circ s_* &= id \end{aligned}$$

□

14 Day 14

- Quiz(Midterm) on Monday, whenever that is. Standard Dr. Clader Format. Last covered topic on that will be deformation retractions.
- Recall from last time:

Theorem 8.

$$\begin{aligned} &A \subseteq X \\ &x_0 \in A \\ &H = \text{ deformation retraction of } X \text{ onto } A \end{aligned}$$

Recall that H is a homotopy relative to x_0

$$\begin{aligned} id &: X \to X \\ r &: X \to X \\ s.t.\ r(x) &\in A,\ \forall x \in X \end{aligned}$$

Then,

$$\pi_1(X, x_0) \cong \pi_1(A, x_0)$$

(r is a retraction)

Proof. Consider, $X \leftrightharpoons A$, where $X \to A$ is s, the same function as r, and $A \to X$ is the inclusion map. Then,

$$\begin{aligned} &s \circ i = id : A \to A \\ \implies &s_* \circ i_* = id : \pi_1(A, x_0) \to \pi_1(A, x_0) \end{aligned}$$

In the other order:

$$i \circ s = r \cong id$$

Note that r is a homotopy relative to x_0, and that the next step follows from the theorem from the beginning of last class.

$$\implies i_* \circ s_* = id : \pi_1(X, x_0) \to \pi_1(X, x_0)$$

So we have:

$$\begin{aligned} \pi_1(X, x_0) &\leftrightarrows \pi_1(A, x_0) \\ s_* : \pi_1(X, x_0) &\to \pi_1(A, x_0) \\ i_* : \pi_1(A, x_0) &\to \pi_1(X, x_0) \end{aligned}$$

and we've shown s_* and i_* are inverses, giving

$$\pi_1(X, x_0) \cong \pi_1(A, x_0)$$

□

- <u>Fun Font Fabtacular</u> Letter fundamental groups.
  1. **C family:** C,E,F,G,H,I,J,K,L,M,N,S,T,U,V,W,X,Y,Z
  2. **A family:** A,D,O,P,Q,R
  3. **B family:** B (fuckin loser.)

  The reason $\pi_1(E) \cong \pi_1(I)$ is that there is a deformation retraction.

  $$\begin{aligned} H &: E \times I \to E \\ H(x, 0) &= x \\ H(x, 1) &\in I, \text{ the letter "I"} \end{aligned}$$

  The rest of this was erased, before I could write it down. Ahh damn.
  Let's talk about $\pi_1(B)$ though. What is that?

  1. It's the same as the fundamental group of a figure 8, because $B \cong \infty$
  2. It's also the same as:

     $$\pi_1(\mathbb{R}^2 \setminus \{p, q\})$$

     where p and q are unequal points in $\mathbb{R}^2$.
  3. Also the same as $\pi_1(\theta)$ (theta is just the letter theta)

15 Day 15

15.1 Continuation and finish of Day 14

- A space with the same π_1 as "B"
- Ex:

$$X = \mathbb{R}^2 \setminus \{p, q\}$$
$$p = (-1, 0)$$
$$q = (1, 0)$$

To see that $\pi_1(X) \cong \pi(\infty)$ (where infinity isn't actually infinity, but two circles joined together to look like a figure eight.), we can construct a deformation retraction of X onto,

$$A = \{(x+1)^2 + y^2 = 1\} \cup \{(x-1)^2 + y^2 = 1\}$$

Pictorially, refer to the picture taken in class.

1. Deformation retract X onto a closed disk of radius 2, centered at (0,0)
2. Then Deformation retract onto a union of two closed disks vertically, again, refer to the picture.

- Ex:

$$\theta = \{x^2 + y^2 = 1\} \cup \{(x, 0) | -1 \leq x \leq 1\}$$

Oh yeah, another picture. To see that $\pi_1(\theta) \cong \pi_1(\infty)$, (again using infinity in lieu of the double circle figure eight) we can construct a deformtaion retraction of $\mathbb{R}^2 \setminus \{p', q'\}$ onto θ where $p' = (0, \frac{1}{2})$ and $q' = (0, -\frac{1}{2})$

- Observation: This shows that,

$$\pi_1(\infty) = \pi_1(\theta)$$

because both of them are isomorphic to $\pi_1(\mathbb{R} \setminus \{\text{two points}\})$ But neither ∞ nor θ are deformation retracts of eachother.

- They're related by a more general relationship, that of homotopy equivalence.

15.1.1 Homotopy Equivalence

Definition 14. A continuous map,

$$f : X \to Y$$

is called a homotopy equivalence if there exists a $g : X \to Y$ such that $f \circ g \cong id_Y$, and $g \circ f \cong id_X$, with our equivalency being homotopic to.

- Goals: A homotopy equivalence induces an $\cong$ on π_1.
- Any deformation retraction "yields" a homotopy equivalence, but homotopy equivalence is an EQUIVALENCE relation. Sick.

- Ex:

$$X = \mathbb{R}^2$$
$$A = \{0, 0\}$$

Then A is a deformation retract of X.

$$H((x, y), t) = ((1-t)x, (1-t)y)$$

But, A is not homeomorphic to X

- Ex: Wow, yet another picture. Wonderful. Refer to the appropriate photograph. Closed disks in R^2, then A is a deformation retraction of X, and also A is homeomorphic to X. Look at the picture ya doink.
  X is homeomorphic to Y implies that $\pi_1(X) \cong \pi_1(Y)$, but the converse is not true, e.g.: X is a deformation retraction of Y implies that $\pi_1(X) \cong \pi_1(Y)$ But not converseley, e.g: X is homotopy equivalent to Y implies $\pi_1(X) \cong \pi_1(Y)$.
  None of these are conversely true. Wonderful! That was confusing.

16 Day 16

16.1 QUIZ DAY (it's a midterm)

17 Day 17

17.1 Homotopy Equivalence

- Goal: A homotopy equivalence induces an isomorphism on π_1
- This follows from:

Theorem 9. If $f : X \to Y$ and $g : X \to Y$ are continuous, $f \cong g$ (homotopic),

$$f(x_0) = y_0,\ g(x_0) = y_1$$

Then there exists a path, α from y_0 to y_1 such that $g_* = \hat{\alpha} \circ f_*$.
Schematically:

$$\underbrace{\pi_1(X, x_0) \xrightarrow{f_*} \pi_1(Y, y_0) \xrightarrow{\hat{\alpha}} \pi_1 Y, y_1}_{g_*}$$

Proof. Let

$$H : X \times I \to Y$$

be a homotopy from f to g (I.e, $h_t : X \to Y,\ \forall t \in I$). Let,

$$\alpha : I \to Y$$
$$\alpha(t) = h_t(x_0)$$

Note that we want,

$$\begin{aligned}&\forall[\gamma]\in\pi_1(X,x_0):\\&g_*([\gamma])=\hat{\circ}f_*([\gamma])\\\iff&[g\circ\gamma]=[\overline{\alpha}*(f\circ\gamma)*\alpha]\\\iff&g\circ\gamma\cong\overline{\alpha}*(f\circ\gamma)*\alpha\end{aligned}$$

We'll prove these are path homotopic by interpolating between them by the following loops: (There's some drawing that goes here) Explicitly, let,

$$\beta_t:I\to Y$$
$$\beta_t(s)=\overline{\alpha}((1-t)s)$$

Then,

$$\begin{aligned}\beta_0&=\overline{\alpha}\\\beta_1&=e_{y_1}\\\beta_t&=\text{path from } y_1 \text{ fo } \alpha(t)\end{aligned}$$

Now, define the followin loop at y_1:

$$\beta_t*(h_t\circ\gamma)*\overline{\beta_t}$$

This is:

1. When $t=0$:
$$\beta_0*(h_0\circ\gamma)*\overline{\beta_0}=\overline{\alpha}*(f\circ\gamma)*\alpha$$
(this is the green loop from the hard to see picture)
2. When $t=1$:
$$\beta_1*(h_1\circ\gamma)*\overline{\beta_1}=e_{y_1}*(f\circ\gamma)*e_{y_1}$$

Thus, $\beta_t*(h_t\circ\gamma)*\overline{\beta_t}$ give a path homotopy,

$$\overline{\alpha}*(f\circ\gamma)*\alpha\cong_p e_{y_1}*(f\circ\gamma)*e_{y_1}\cong_p g\circ\gamma$$

□

Corollary 9.1. If $f:X\to Y$ is a homotopy equivalence (recall that this means there exists a $g:Y\to X$ such that $f\circ g\cong id_Y$ and $g\circ f\cong id_X$). Then,

$$f_*:\pi_1(X,x_0)\to\pi_1(Y,y_0)$$

is an isomorphism.

Proof. We know,

$$\begin{aligned} & g \circ f \cong id_X \\ \xRightarrow[\text{theorem}]{} & (g \circ f)_* = \hat{\alpha} \circ (id_X)_*, \text{ for some path } \alpha \\ & \Rightarrow g_* \circ f_* = \hat{\alpha} \end{aligned}$$

and similarly,

$$\begin{aligned} & f \circ g \cong id_Y \\ (f \circ g) = \hat{\beta} \circ (id_Y)_* & \text{ for some path in } \beta \\ & \to f_* \circ g_* = \hat{\beta} \end{aligned}$$

Thus, if $f(x_0) = y_0$, $g(y_0) = x_1$, $f(x) = y_1$:

$$\begin{tikzcd} \pi_1(X, x_0) \arrow[r, "f_*"] \arrow[d, "\hat{\alpha}"] & \pi_1(Y, y_0) \arrow[d, "\hat{\beta}"] \arrow[dl, "g_*"'] \\ \pi_1(X, x_1) \arrow[r, "f_*"] & \pi_1(Y, y_1) \end{tikzcd}$$

(Some fuckin arrow diagram Ah fuck) Therefore,

$$\begin{aligned} g_* \circ f_* = \hat{\alpha}, & \text{ an isomorphism!} \\ & \Rightarrow g_* \text{ is surjective} \end{aligned}$$

And similarly, because

$$\begin{aligned} f_* \circ g_* = \hat{\beta} & \text{ (an isomorphism!)} \\ & \to g_* \text{ is injective} \end{aligned}$$

□

18 Day 18

18.1 Homotopy equivalences, concluded

Recall,

Definition 15. A continuous function $f : X \to Y$ is a homotopy equivalence if there exists a continuous function $g : Y \to X$ such that $g \circ f \simeq id_x$ and $f \circ g \simeq id_Y$

Notation/terminology: We call g a homotopy inverse of f if there exists a homotopy equivalence, $f : X \to Y$, we say X and Y are homotopy equivalent and we write $X \simeq Y$

1. Ex:

$$f : \mathbb{R}^2 \setminus \{(0,0)\} \to S^1$$
$$f(x,y) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$$

is a homotopy equivalence with homotopy inverse,

$$i : S^1 \to R^2 \setminus \{(0,0)\}$$
$$i(x,y) = (x,y)$$

To check these are homotopy inverses:

$$f \circ i = id_{S^1}$$
$$(i \circ f)(x,y) = (\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}})$$

So a homotopy between $f \circ i$ and id_{S^1} is,

$$H : S^1 \times I \to S^1$$
$$H((x,y),t) = (x,y), \ \forall t \in I$$

A homotopy between $i \circ f$ and $id_{\mathbb{R}^2 \setminus \{(0,0)\}}$ is the straight line homotopy. This is the deformation retraction.

2. Ex: Let,

$$D = \{(x,y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$$

Then,

$$f : D \to \{(0,0)\}$$
$$f(x,y) = (0,0)$$

is a homotopy equivalence with homotopy inverse,

$$i : \{(0,0)\} \to D$$
$$i(0,0) = (0,0), \text{ (inclusion)}$$

To check these are homotopy inverses:

$$f \circ i = id_{\{(0,0)\}}$$
$$(i \circ f)(x,y) = (0,0)$$

So, a homotopy betwen $i \circ f : D \to D$ and $id : D \to D$ is

$$H : D \times I \to D$$
$$H((x, y), t) = ((1-t)x, (1-t)y)$$

This is the straight-line homotopy and it is the deformation retraction of D onto $\{(0,0)\}$

3. Note:(HW) Any deformation retraction of X onto A gives rise to a homotopy equivalence $X \simeq A$ (Note that this side box was created at some point. $H : X \times I \to X$ a homotopy from $id_X : X \to X$ to $r : X \to X$ such that $r(x) \in A,\ \forall x \in X$)

Definition 16. If a topological space X is homotopy equivalent to a single point, we say that X is contractible.

1. Ex: The unit disk, D is retractible.
2. Note: By the theorem from last class,

contractible $\Rightarrow$ simply-connected

18.2 Why care about homotopy equivalences?

Why do we care about homotopy equivalences instead of just using deformation retractions?

1. Deformation retraction is weirdly asymmetric. A is a deformation retraction of X but not vice versa, while homotopy equivalence is an equivalence relation (courtesy of HW). The fact that it's symmetric
   - Ex:

$$D \underbrace{\simeq}_{\text{last example}} \{.\} \underbrace{\simeq}_{\text{handout}} \mathbb{R}$$
$$\Rightarrow D \simeq \mathbb{R}$$
$$\Rightarrow \pi_1(D) \cong \pi_1(\mathbb{R})$$

   - Ex: Looking at the character θ,

$$\theta \underbrace{\simeq}_{\text{deformation retraction}} \mathbb{R}^2 \setminus x, y \underbrace{\simeq}_{\text{deformation retraction}} \infty$$
$$\Rightarrow \theta \cong \infty$$
$$\Rightarrow \pi_1(\theta) \cong \pi_1(\infty)$$

2. We don't have to worry about base points staying fixed throughout the homotopy.

19 Day 19

19.1 Calculating π_1 piecewise

- Goal: To calculate $\pi_1(X)$ with π_1("pieces of X")
- The ultimate theorem we'll prove for this is the Van Kampen Theorem, which will say, Let,

  $$X = U \cup V$$

  Where U and V are path connected open subsets of X. Furthermore their intersection is *also* path connected. Then,

  $$\pi_1(X) = \pi_1(U)?\pi_1(V)$$

  Note that the ? takes the place of some as yet undefined operation, and it will depend not only on $\pi_1(U)$ or $\pi_1(V)$, but their interaction.

19.2 Free product of groups

Definition 17. A word in G and H is a string of symbols,

$$a_1, a_2, a_3, \dots, a_n$$

where a_i is either an element of G or an element of H

- $$\underbrace{G = \{a^0, a^1, a^2, a^3\}}_{\text{group under multiplication if we declare } a^4 = a^0}$$
  $$\underbrace{H = \{b^0, b^1, b^2\}}_{\text{group under multiplication if we declare } b^3 = b^0}$$

  An example of a word in G and H is

  $$a^1a^1b^2a^1b^0b^1$$

  Thus far, this is a different word from

  $$a^2b^2a^1b^1$$

  But we'd like for them to be equivalent. To achieve this end, we define two reducing operations on $\{$words in G and $H\}$

  1. If a word contains a copy of 1_g, the identity of G or 1_h remove that symbol from the word.
  2. If a word contains two consecutive terms from the same group, replace them with their product.

  This allows for these two things to be set equal by a sequence of the reducing operations.

  $$a^1a^1b^2a^1b^0b^1$$
  $$a^2b^2a^1b^0b^1$$
  $$a^2b^2a^1b^1$$

Definition 18. A word is called reduced if no further reducing operations can be applied to it.

- Ex: $a^1b^2a^1$ is the reduced form of $a^1a^{-1}a^1b^1b^1a^1$.
- Observation: Any word can be converted via a sequence of reducing operations to a unique reduced word.

Definition 19. Let H and G be any groups. Their free product

$$G * H = \{\text{reduced words in } G \text{ and } H\}$$

This is a group under the operation of concatenation followed by reduction. Note that $G * H$ is an infinite non-abelian group.

20 Day 20

20.1 Free products continued

- Recall:

$$G * H = \{\text{reduced words in } G \text{ and } H\}$$

  which is a group under concatenation followed by reduction.

- Question: Let,

$$G = \{\dots, a^{-2}, a^{-1}a^0, a^1a^2, \dots\}$$
$$H = \{\dots, b^{-2}, b^{-1}b^0, b^1b^2, \dots\}$$

  These are both groups under multiplication and they're isomorphic.

  - To what "familiar" group are G and H isomorphic.
  - What do elements of $G * H$ look like?

- Answer:
  -
$$G \cong H \cong Z$$
    (via the isomorphism, $a^i \to i$ or $b^i \to i$)
  - Elements of $G * H$ look like,
$$a^{i_1}b^{j_1}a^{i_2}b^{j_2}\dots a^{i_n}b^{j_n}$$
    (Where $i_k, j_k \in \mathbb{Z}$)
    This might start with b^{j_1} or it might end with a^{j_n}

- Observation: Though H and G are abelian in this example, $G*H$ is not abelian. Both $1_H \in G*H$ and $1_G \in G * H$ reduce to 0

$$G * H \cong \mathbb{Z} * \mathbb{Z}$$

  This example is called the "free group with two generators", and is denoted,

$$\langle g, h\rangle$$

20.1.1 Free Products with Amalgamation

- Let G_1, G_2 and H be groups, and let,

$$\varphi_1 : H \to G_1$$
$$\varphi_2 : H \to G_2$$

  be homomorphisms

$$\begin{tikzcd} G_1 & & G_2 \\ & H \arrow[ul, "\varphi_1"] \arrow[ur, "\varphi_2"'] & \end{tikzcd}$$

- <u>Observation:</u> φ_1 and φ_2 induce homomorphisms,

$$\tilde{\varphi_1} : H \to G_1 * G_2$$
$$\tilde{\varphi_1} = \text{ the word that just contains } \varphi_1(h)$$

  and,

$$\tilde{\varphi_2} : H \to G_1 * G_2$$
$$\tilde{\varphi_2} = \text{ the word that just contains } \varphi_2(h)$$

Definition 20. The <u>free product of G_1 and G_2 amalgamated over H</u> is,

$$G_1 *_H := \frac{G_1 * G_2}{N}$$

where N is the smallest normal subgroup of $G_1 * G_2$ containing $\tilde{\varphi_1}(h)\tilde{\varphi_2}(h)^{-1}, \ \forall h \in H$

- <u>Think:</u> We're setting,

$$\tilde{\varphi_1}(h)\tilde{\varphi_2}(h)^{-1} = 1$$
$$\iff \tilde{\varphi_1}(h) = \tilde{\varphi_2}(h)$$

  (the sorts of things in N are $a\tilde{\varphi_1}(h)\tilde{\varphi_2}(h)^{-1}a^{-1}$, e.g.)

- <u>Note:</u> The notation $G_1 * G_2$ doesn't mention ϖ_1 and φ_2 but it depends on them!

- Ex:

$$G_1 = \langle a \rangle$$
$$G_2 = \{1\}$$
$$H = \langle b \rangle$$

Let's form

$$G_1 *_H G_2$$

where the amalgamation happens over the homomorphisms.

$$\varphi_1 : \langle b \rangle \to \langle a \rangle$$
$$\varphi_1(b^i) = a^{2i}$$

and

$$\varphi_2 : \langle b \rangle \to \{1\}$$
$$\varphi_2(b^i) = 1$$

Then,

$$G_1 *_H G_2 = \frac{G_1 * G_2}{\text{smallest nomal subgroup containing } \varphi_1(h)\varphi_2^{-1}(h) \forall h \in H}$$
$$= \frac{\langle a \rangle}{\ldots\text{containing } a^2 1 \forall i \in \mathbb{Z}} = \frac{\langle a \rangle}{\langle a^2 \rangle}$$

21 Day 21

21.1 Van Kampen Theorem

Theorem 10. Let X be a topological space such that,

$$X = U \cup V$$

where U, V and $U \cap V$ are all open, path-connected, subsets of X. Let $x_0 \in U \cap V$. Then,

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) *_{\pi_1(U \cap V, x_0)} \pi_1(V, x_0)$$

Where amalgamation happens over the homomorphisms

$$i_{U*} : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$$

induced by the inclusion $i_U : U \cap V \to U$, and the homomorphism

$$i_{V*} : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$$

induced by the inclusion $i_V : U \cap V \to V$.

$$\begin{tikzcd} & \pi_1(U,x_0) \arrow[dr, "j_1"] \arrow[drr, bend left, "k_1"] & & \\ \pi_1(U\cap V,x_0) \arrow[ur, "i_{U*}"] \arrow[dr, "i_{V*}"'] & & \pi_1(U,x_0) *_{\pi_1(U\cap V,x_0)} \pi_1(V,x_0) \arrow[r, "\phi"] & \pi_1(X,x_0) \\ & \pi_1(V,x_0) \arrow[ur, "j_2"] \arrow[urr, bend right, "k_2"'] & & \end{tikzcd}$$

- Ex:(Algebra) $\mathbb{Z} *_{\{1\}} \mathbb{Z}$, where the amalgamation happens over the homomorphisms,

$$\begin{aligned} \varphi_1 &: \{1\} \to \mathbb{Z}, \text{ (trivial homomorphism)} \\ \varphi_2 &: \{1\} \to \mathbb{Z}, \text{ (trivial homomorphism)} \end{aligned}$$

(The trivial homomorphism just sends everything to the identity) By definition,

$$\begin{aligned} \mathbb{Z} *_{\{1\}} \mathbb{Z} &= \frac{\mathbb{Z} * \mathbb{Z}}{\underbrace{\text{smallest normal subgroup containing } \tilde{\varphi}_1(h)\tilde{\varphi}_2^{-1}(h)\forall h \in \{1\}}_{\tilde{\varphi}_1(1)\tilde{\varphi}_2^{-1}(1)=a^0(b^0)^{-1}=1}} \\ &= \frac{\langle a,b\rangle}{\{1\}} \\ &= \underbrace{\langle a,b\rangle}_{\text{Free group on two generators}} \end{aligned}$$

In general, if $\varphi_1 : H \to G_1$, then, $\tilde{\varphi}_1 : H \to G_1 * G_2$, and

$$\tilde{\varphi}(h) = \text{the word } \varphi_1(h) \text{ viewed as a word of length 1}$$

- Ex: (Topology), Note that there's a picture that should be here. One of these days I'll go through my notes and add all the missing drawings. Probably not gonna do that on the 19th of march though.

$$\begin{aligned} X = \text{Drawing of figure 8} &= \{(x,y) \in \mathbb{R} | (x-1)^2 + y^2 = 1\} \cup \{(x,y) \in \mathbb{R} | (x+1)^2 + y^2 = 1\} \\ U &= \{(x,y) \in X | X \le 1\} \\ V &= \{(x,y) \in X | X \ge -1\} \end{aligned}$$

Let $x_0 = (0,0)$. Then, drawings, but mostly the following junk.

$$\begin{aligned} U \simeq S^1 &\implies \pi_1(U,x_0) \cong \mathbb{Z} \\ V \simeq S^1 &\implies \pi_1(V,x_0) \cong \mathbb{Z} \\ U \cap V \simeq \{(0,0)\} &\implies \pi_1(V,x_0) = \{1\} \end{aligned}$$

So, the Van Kampen Theorem says,

$$\pi_1(X, x_0) = \mathbb{Z} *_1 \mathbb{Z} = \langle a, p \rangle$$

Pictorially, an element of $\pi_1(X, x_0)$ may look like some as yet undrawn picture or,

$$a^3 b^5 a^{-2}$$

Where a^1 corresponds to a clockwise loop around one portion of the figure 8, a a^{-1} corresponds to a counterclockwise loop around that same portion, and switching to b^1 or b^{-1} switches which portion of the figure 8 the loop will be around.

- Ex:(Algebra)

$$\{1\} *_H \{1\}$$

Where the amalgamation happens over the homomorphisms,

$$\varphi_1 : H \to \{1\}$$
$$\varphi_1(h) = 1, \ \forall h \in H$$
$$\varphi_2 : H \to \{1\}$$
$$\varphi_2(h) = 1, \ \forall h \in H$$

Then by definition,

$$\{1\} *_H \{1\} = \frac{\{1\} * \{1\}}{\text{smallest normal subgroup stuff}} = \frac{\{1\}}{\dots} = \{1\}$$

- Ex: (Topology)

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

Again, a drawing is referenced. I should learn how to draw quickky. We select our two subsets of X as such,

$$U = \{(x, y, z) \in X | Z \geq \frac{-1}{2}\}$$
$$V = \{(x, y, z) \in X | Z \leq \frac{1}{2}\}$$

We let $x_0 = (0, 0, 0)$, from the drawing we kinda see,

$$U \simeq \{.\} \implies \pi_1(U, x_0) \cong \{1\}$$
$$V \simeq \{.\} \implies \pi_1(V, x_0) \cong \{1\}$$
$$U \cap V \simeq S^1 \ \text{(by deformation retract)} \implies \pi_1(U \cap V, x_0) \cong \mathbb{Z}$$

So, the Van Kampen theorem says,

$$\pi_1(S^2, x_0) \cong \{1\} *_{\mathbb{Z}} \{1\} = \{1\}$$

Which implies that any 1-loop on the sphere can be contracted to a single point.

22 Day 22

Recall the Van Kampen theorem: (on page 38 of these notes)

- Ex:

$$X = \text{torus} = \text{Complicated scribble} = \frac{[0,1]\times[0,1]}{\sim}$$
$$(0,y) \sim (1,y)$$
$$(x,1) \sim (x,1)$$

$$U = \{[p] \in \frac{[0,1]\times[0,1]}{\sim} \mid p \in B_{\frac{1}{3}}(\frac{1}{2},\frac{1}{2})\}$$
$$V = \{[p] \in \frac{[0,1]\times[0,1]}{\sim} \mid p \notin B_{\frac{1}{4}}(\frac{1}{2},\frac{1}{2})\}$$

Then, with complicated doodles afoot,

$$U \simeq \{.\} \Rightarrow \pi_1(U,x_0) = \{1\}$$
$$V \simeq \{\text{a drawing}\} \Rightarrow \pi_1(V,x_0) = \langle a,b\rangle$$
$$U\cap V \simeq S^1 \Rightarrow \pi_1(U\cap V,x_0) = \mathbb{Z}$$

The homomorphisms induced by i_U and i_V are,

$$i_{U*} : \underbrace{\pi_1(U\cap V,x_0)}_{=\mathbb{Z}} \to \underbrace{\pi_1(U,x_0)}_{=\{1\}} i_{U*}(g) = 1,\ \forall g$$

With the other as,

$$i_{V*} : \underbrace{\pi_1(U\cap V,x_0)}_{=\mathbb{Z}=\langle c\rangle} \to \underbrace{\pi_1(V,x_0)}_{\langle a,b\rangle}$$

There is some inscrutable drawing here that encapsulates the logic, maybe I'll add it. Who knows? I sure don't!

$$i_{V*}(c) = aba^{-1}b^{-1}$$

So, the Van Kampen Theorem says,

$$\pi_1(X) \cong \{1\} *_{\langle c\rangle} \langle a,b\rangle$$
$$= \frac{\{1\} * \langle a,b\rangle}{\text{smallest yadda yadda yadda}}$$

With the smallest yadda yadda yadda containing,

$$i_{U*}(h)i_{V*}(h),\ \forall h \in \langle c \rangle$$

In this quotient,

$$aba^{-1}b^{-1} = 1$$
$$\iff ab = ba$$

Thus, in this quotient, any word is equivalent to a word of the form,

$$a^k b^l$$

Where $k, l \in \mathbb{Z}$, which is to say that,

$$\pi_1(X, x_0) \cong \mathbb{Z} \oplus \mathbb{Z}$$

- Question: What is the $\pi_1(K)$ where K is a klein bottle?
(unanswered at the moment)

23 Day 23

23.1 Conclusion of Van Kampen

- Recall: Proving the following,

$$\pi_1(X) \cong \frac{\pi_1(U) * \pi_1(V)}{N}$$

Where,

$$N = \text{Smallest normal subgroup containing } i_{U*}(h)i_{V*}^{-1}(h)\forall h \in \pi_1(U \cap V)$$

Using

$$j : \pi_1(U) * \pi_1(V) \to \pi_1(X)$$

induced by the homomorphisms

$$j_{U*} : \pi_1(U) \to \pi_1(X) j_{V*} : \pi_1(V) \to \pi_1(X)$$

So far, we proved

1. j is surjective
2. $N \subseteq ker(j)$

For step 3, we need to show that $ker(j) \subseteq N$, then we're finished by the first isomorphism theorem.

- Idea of proof Let,

$$w = [w_1][w_2]...[w_k] \in ker(j)$$

This means if we define

$$f_i = \begin{cases} j_U * w_i & \text{if} [w_i] \in \pi_1(U) \\ j_V * w_i & \text{if} [w_i] \in \pi_1(V) \end{cases}$$

Then,

$$\underbrace{[f_1 * f_2 * \cdots * f_k]}_{\text{this is } j(w)} = [e_{x_0}]$$
$$\iff f_1 * \cdots * f_k \cong_p e_{x_0}$$

We want, $w \in N$ First, we define a "move" on words $w \in \pi_1(U) * \pi_1(V)$:

– If $[w_i] \in \pi_1(U)$, but w_i actually lives in $U \cap V$, then view $[w_i] \in \pi_1(V)$ instead. (or vice versa)

This amounts to multiplying w by an element of N! Why though? "$[w_i] \in \pi_1(U)$ but w_i actually lies in $U \cap V$" means,

$$[w_i] = i_{U*}([\tilde{w}_i]) \text{ for some } [\tilde{w}_i] \in \pi_1(U \cap V)$$

"view $[w_i] \in \pi_1(V)$ instead", means to replace $[w_i]$ by $[w' = i_{V*}([\tilde{w}_i])]$, Thus, a move means,

$$w = [w_1][w_2]\ldots \underbrace{[w_i]}_{=i_{U*}([\tilde{w}_i])} \ldots [w_k] \rightsquigarrow [w_1'][w_2']\ldots \underbrace{[w_i']}_{=i_{V*}([\tilde{w}_i])} \ldots [w_k']$$

To get from LHS to RHS, multiply w by

$$([w_{i+1}]\ldots[w_k])i_{U*}^{-1}([\tilde{w}_i])i_{V*}([\tilde{w}_i])([w_{i+1}]\ldots[w_k])$$

This is an element of N. From here, we use the path homotopy,

$$f_1 * \ldots f_k \cong_p e_{x_0}$$

to cook up a sequence of moves taking w to $[e_{x_0}]$. This means

$$w(\text{something in } N)$$
$$\Rightarrow w \in N$$

- Wrapup of π_1 Where do we go from here?
  – Knot theory: A "knot" in $\mathbb{R}^3$ is $K : I \to \mathbb{R}^3$ such that, $K(0) = K(1)$ but $K_{|_{(0,1)}}$ is injective. That is to say that two knots, K_0 and K_1 in $\mathbb{R}^3$ are equivalent if there exists a family of homeomorphisms
  $$h_t : \mathbb{R}^3 \to \mathbb{R}^3$$
  such that $h_0 = id$ and $h_0 \circ K_0 = K_1$
  Turns out that you can use $\pi_1(\mathbb{R}^3 \setminus im(K))$ in a sneaky way to detect if two knots are equivalent.
  – The Fundamental Theorem of Algebra: This can be proven with the fundamental group. Has a proof via π_1 that involves viewing a polynomial with no roots in $\mathbb{C}$ as a map
  $$p : \mathbb{C} \to \mathbb{C}\{0\}$$
  and arguing,
  $$p_*([\text{some loop}]) \neq 1$$
  Which yields a contradiction because $\pi_1(\mathbb{C}) = \{1\}$
  – Siilar ideas prove Borsuk-Ulam Theorem and Hairy Ball Theorem, and yadda yadda yadda.

24 Day 24

24.1 Introduction to Simplicial Homology

- We've learned that $\pi_1(X)$ is one way to measure the "number of holes" in X.
- Advantages:
  1. Not too hard to visualize
  2. Some computational tools
- Disdvantages:
  1. Nonabelian group means that it is algebraically weird.
  2. Can't detect "higher dimensional holes", (e.g.$\pi_1(S^2) = \{1\}$).
- Homology is an attempt to remediate these two issues, and provides a different methodology for measuring the number of holes:
  - $\oplus$ is an abelian group
  - $\oplus$ can detect holes of different dimensions
  - $\oplus$ has good computational holes
  - $\ominus$ has a what Dr. Clader thinks is a "super non-intuitive definition"
- There are several variants of homology, and we're gonna start with the simplicial one. It's simplices, the simplest thing there is. This is called the simplicial homology.

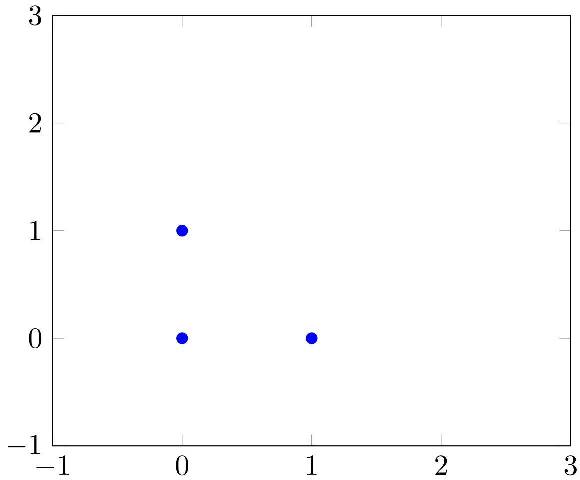
24.1.1 Simplices, the simplest things that exist

Definition 21. A set of points, $\{p_0, p_1, p_2, \ldots, p_n\} \subseteq \mathbb{R}^N$ is affinely independent or geometrically independent if

$$\{p_1 - p_0, \ldots, p_n - p_0\} \subseteq \mathbb{R}^N$$

is a linearly independent set of vectors.

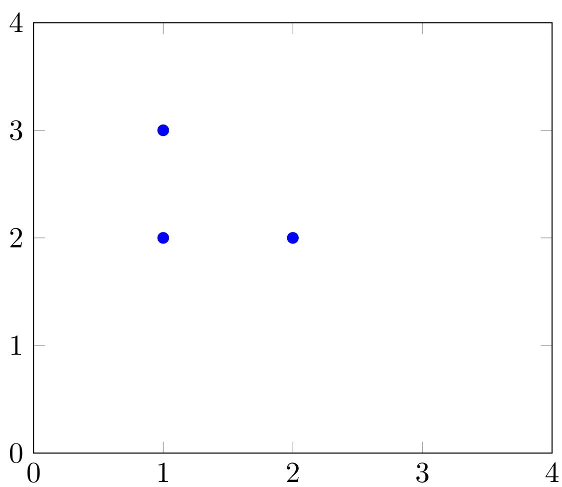
- <u>Ex:</u>



$$\{(0,0),(1,0),(0,1)\} \subseteq \mathbb{R}^2$$

is affinely independent.

- <u>Ex:</u>

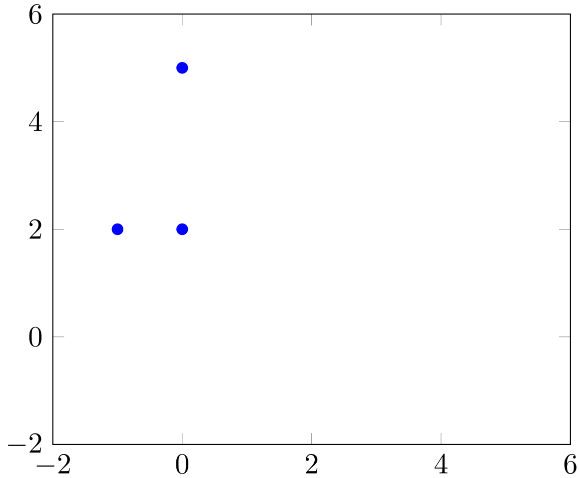


$$\{(1,2),(2,2),(1,3)\} \subseteq \mathbb{R}^2$$

That's affine independent set of vectors.
(it's affinely independent)

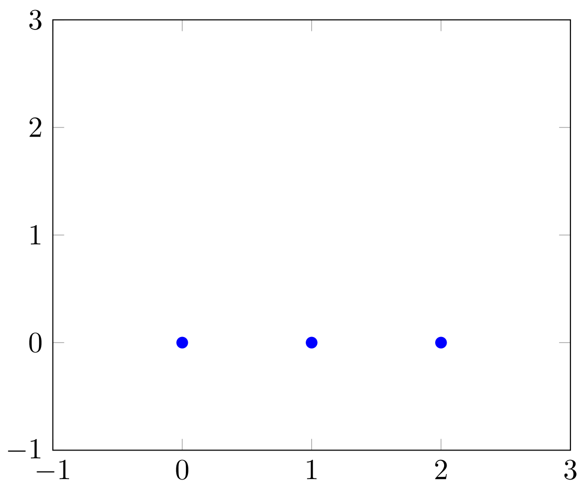
- More generally, if we apply any invertible linear transformation, for example translation, rotation, to $\{(0,0),(1,0),(0,1)\}$, the result is affinely independent.

- <u>Ex:</u>



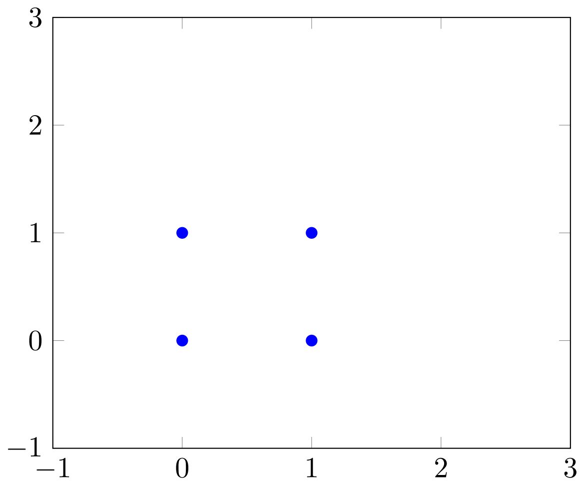
$$\{(0,2),(-1,2),(0,5)\} \subseteq \mathbb{R}^2$$

- <u>Non-Ex:</u>



$$\{(0,0),(1,0),(2,0)\} \subseteq \mathbb{R}^2$$

- Non-Ex:



$$\{(0,0),(1,0),(0,1),(1,1)\} \subseteq \mathbb{R}^2$$

Definition 22. Let $\{p_0, p_1, \ldots, p_n\} \subseteq \mathbb{R}^N$ be an affinely independent set. Then the simplex spanned by $\{p_0, p_1, \ldots, p_n\}$ is:

$$\sigma = \{a_0p_0 + a_1p_1 + \cdots + a_np_n | a_i \in R^{\geq 0} \forall i, \ \sum_0^n a_i = 1\}$$

Additionally, we could consider this to be the convex hull of $p_0, \ldots, p_n$. We say that σ is a simplex of dimension n or an n-simplex

- Ex: $(n = 0)$
  $\{p_0\}$ is affinely indpendent for all $p_0 \in \mathbb{R}^N$, then the 0-simplex spanned by $\{p_0\}$ is,

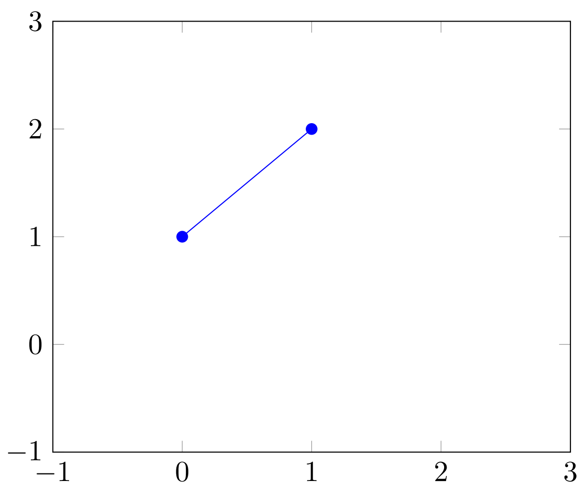
  $$\sigma = \{a_0p_0 | a_0 \in R^{\geq 0}, \ a_0 = 1\} = \{p_0\}$$

  It's just the point p_0

- Ex: $(n = 1)$
  $\{p_0, p_1\}$ is affinely independent as long as $p_0 \neq p_1$. The 1-simplex spanned by $\{p_0, p_1\}$ is:

$$\sigma = \{a_0 p_0 + a_1 p_1 | a_0, a_1 \in R^{\geq 0},\ a_0 + a_1 = 1\}$$

  With $p_0 = (0, 1)$, $p_1 = (1, 2)$, we have

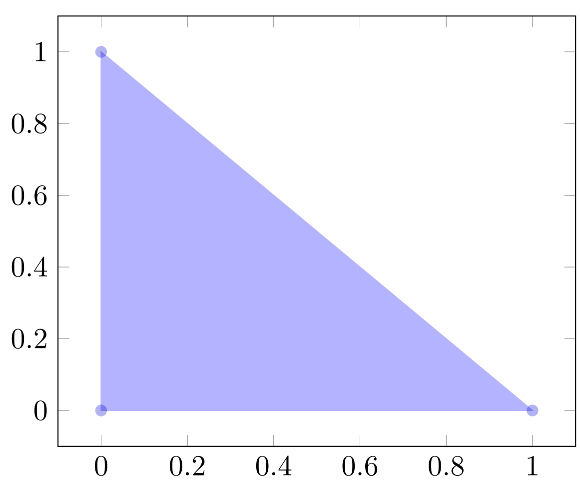


- Ex: $(n = 2)$
  E.g.

$$\sigma = \{a_0(0,0) + a_1(1,0) + a_2(0,1), (1,1) | a_0, a_1, a_2 \in R^{\geq 0},\ a_0 + a_1 + a_2 = 1\}$$

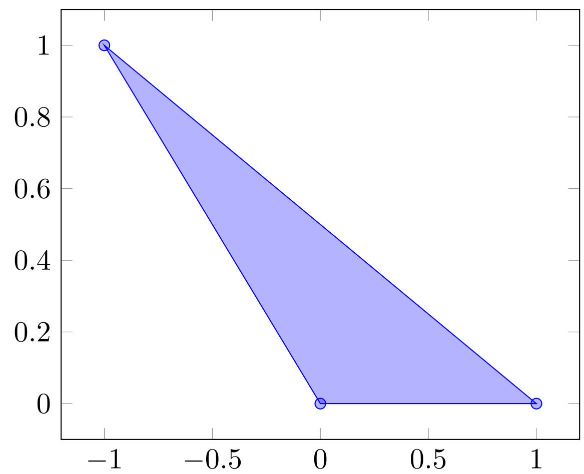
  However, since $p_0 = (0, 0)$, this is jut,

$$\sigma = \{a_1(1,0) + a_2(0,1), (1,1) | a_1, a_2 \in R^{\geq 0},\ a_1 + a_2 \leq 1\}$$



- Fact: More generally:
  - The 2-simplex spanned by $\{p_0, p_1, p_2\} \subseteq \mathbb{R}^N$ is a triangle with p_0, p_1, p_2 as vertices.
  - The 3-simplex spanned by $\{p_0, p_1, p_2, p_3\} \subseteq \mathbb{R}^N$ is a tetrahedron with p_0, p_1, p_2, p_3 as vertices.

- Ex:



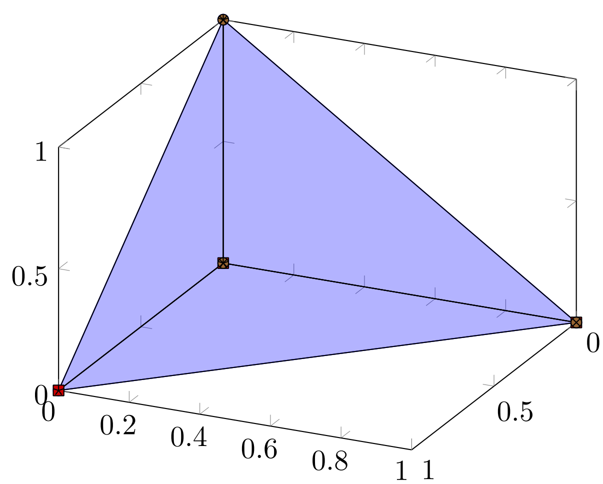
$$\{(0,0),(1,0),(-1,1)\}$$

is affinely independent, and also a simplex.

Definition 23. Let σ be a simplex spanned by $\{p_0,\ldots,p_n\}$. Then a face of σ is any simplex spanned by a nonempty subset of σ

- Ex:

$$\sigma = \text{3-simplex spanned by } \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$$



Faces:

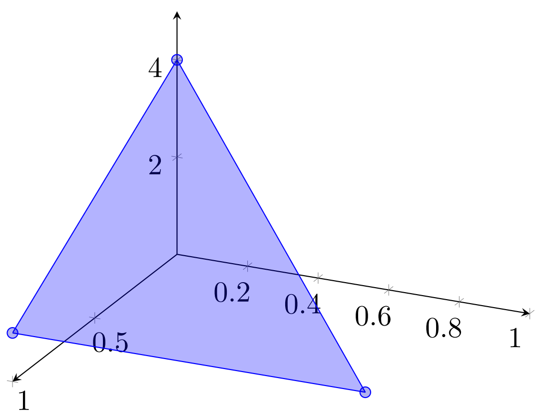
- Each vertex (spanned by a 1-element subset)
- Each edge, (spanned by 2-element subsets)
- Each surface, (spanned by 3-element subsets)
- The whole thing, (spanned by 4-element subsets)

25 Day 25

- Let

$$\sigma = \text{ simplex spanned by } \{(0,1,1),(0,0,4),(1,1,1)\}$$

- Convince yourself that this is affinely independent
  - Draw a picture of σ



  - Label the faces of σ

- Note that this is affinely independent because

$$\{(0,-1,3),(1,0,0)\}$$

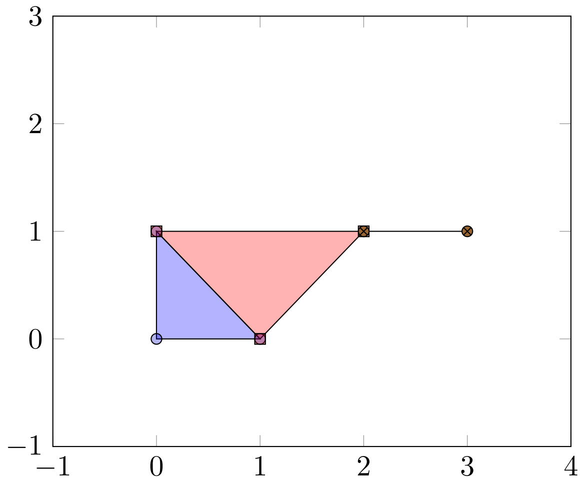
is linearly independent

- The faces of σ are,
  - The three vertices
  - The three edges
  - All of σ

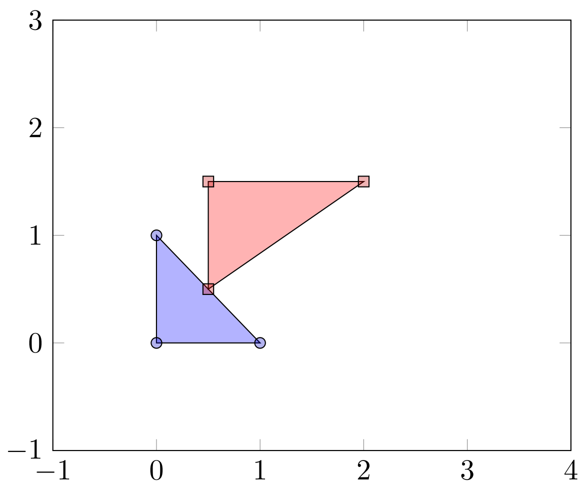
Definition 24. A simplicial complex K is a finite set of simplices $\sigma \in \mathbb{R}^N$ such that,

- If $\sigma \in K$, then every face of σ is also in K
- If $\sigma, \tau \in K$ then $\sigma \cap \tau$ is a face of both σ and τ.

- <u>Ex:</u>

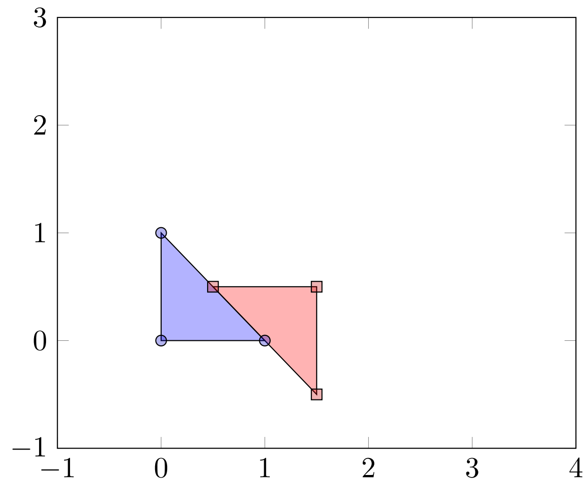


- <u>Non-Ex:</u>



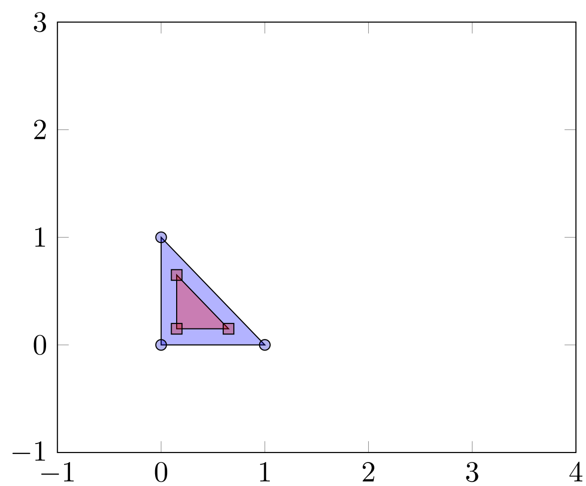
This is not a simplicial complex, because the intersection of the orange triangle and the blue triangle is not a face of both.

- Non-ex:



Again, this is not a simplicial complex, because the intersection of the orange triangle and the blue triangle is not a face of both.

- Non-ex:



Yet again, this is not a simplicial complex, because the intersection of the orange triangle and the blue triangle is not a face of both.

- Observation: If K is a simplicial complex, there's an associated topological space,

$$|K| = \bigcup_{\sigma \in K} \subseteq \mathbb{R}^N$$

(topological space with the subspace topology from the euclidean topology)
We call $|K|$ the underlying topological space of K.

25.0.1 Towards simplicial homology

Definition 25. Let,

$$K = \text{Simplicial complex}$$
$$K_n = \{\text{n-simplices in } K \subseteq K$$

The group of n-chains in K is:

$$C_n(K) = \{a_1\sigma_1 + a_2\sigma_2 + \cdots + a_r\sigma_r | \sigma_i \in K,\ a_i \in \mathbb{Z}\}$$

(Formal linear combinations, in other words, $C_n(K) \cong \mathbb{Z}^{\# K_n}$)

- Ex:

$$5\sigma - 7\tau \in C_2(K)$$
$$-v_1 + 2v_2 - 3v_4 \in C_0(K)$$

- This is a group under addition, e.g.:

$$(5\sigma - 7\tau) + (3\sigma + 2\tau) = 8\sigma - 5\tau$$

- Want to define "boundary maps":

$$C_n(K) \to C_{n-1}(K)$$

- Notation: We denote the simplex spanned by $v_0, \ldots, v_n \in \mathbb{R}^n$ by $[v_0, \ldots, v_n]$

Definition 26. Let $n \geq 1.$ Define the boundary homomorphism

$$\partial_n : C_n(K) \to C_{n-1}(K)$$

on each

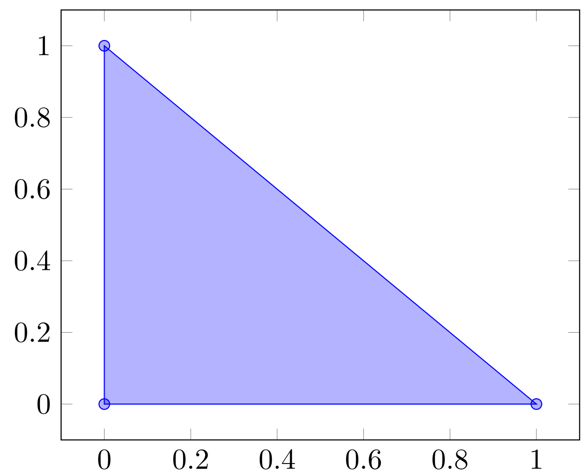
$$\sigma = [v_0, v_1, \ldots, v_n]$$

by:

$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j [v_0, \ldots, \hat{v}_j, \ldots, v_n]$$

With $\hat{v}_j$ indicating that we remove this from the simplex

- Ex:



$$\partial_2 : C_2(K) \to C_1(K)$$
$$\partial([(0,1),(0,0),(1,0)]) = [(0,0),(1,0)] - [(0,1),(1,0)] + [(0,1),(0,0)]$$

26 Day 26

26.1 Simplicial Homology

- Let the following be so,

$$K = \text{Simplicial complex}$$
$$K_n = \{ \text{ n-simplices of } K \}$$
$$C_n(K) = \{ \text{ n-chains } \}$$
$$= \{\mathbb{Z}\text{-linear combinations of n-simplices of } K$$

- Choose an ordering of the vertices, (0-simplices) of K,

$$K_0 = \{v_0, v_1, \ldots, v_k\}$$

- Then,

$$\partial_n : C_n(K) \to C_{n-1}(K)$$
$$\partial_n(\underbrace{[v_{i_0}, v_{i_1}, \ldots, v_{i_n}])}_{\text{n-simplex spanned by } v_{i_0}, v_{i_1}, \ldots, v_{i_n}} = \sum_{j=0}^{n} (-1)^j [v_{i_0}, v_{i_1}, \ldots, \hat{v}_{i_j} \ldots, v_{i_n}])$$

Where, $i_0 \leq_1 \cdots \leq i_n$.

Definition 27. The simplicial chain complex of K is,

$$\{0\} \to \cdots \to^{\partial_3} C_2(k) \to^{\partial_2} C_1(k) \to^{\partial_1} C_0(k) \to \{0\}$$

By convention,

$$C_n(K) = \{0\}$$

Whenever K has no n-simplices.

Definition 28. The n^{th} simplicial homology group of K is,

$$H_n(K) = \frac{ker(\partial_n)}{im(\partial_{n+1})}$$

- Note: This quotient makes sense because,

  $$im(\partial_{n+1}) \subseteq ker(\partial_n)$$

  or equivalently,

  $$\partial_n \circ \partial_{n+1} = 0$$

26.1.1 Worksheet stuff

1. Simplicial chain complex for worksheet

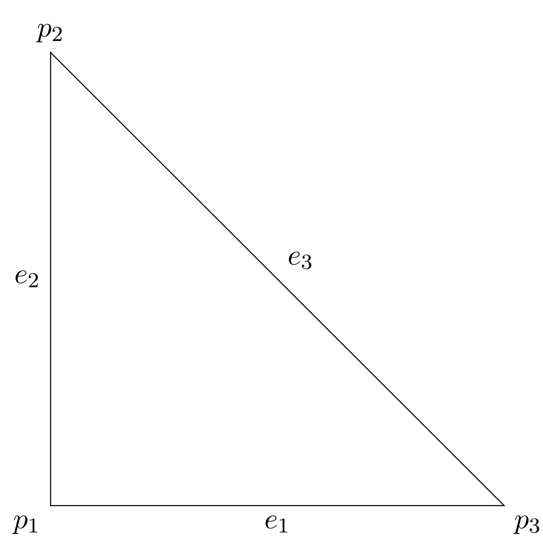
$$\{0\} \to C_2(k) \to^{\partial_2} C_1(k) \to^{\partial_1} C_0(k) \to \{0\}$$

2.

$$\begin{aligned}
\partial_1(e_1) = \partial_1([v_1, v_2]) = v_2 - v_1 \\
\partial_1(e_2) = \partial_1([v_1, v_3]) = v_3 - v_1 \\
\partial_1(e_3) = \partial_1([v_2, v_3]) = v_3 - v_2
\end{aligned}$$

3.

$$\begin{aligned}
\partial_2([v_1, v_2, v_3]) = [v_2, v_3] - [v_1, v_3] + [v_1, v_2] \\
\partial_2([v_1, v_2, v_3]) = e_3 - e_1 + e_2
\end{aligned}$$



4.

$$ker(\partial_2) = \{af_1 | ker(af_1) = 0\}$$

Since $\partial_2(f_1) \neq 0$, the only multiple of f_1 sent to 0 is $0f_1$

5.

$$ker(\partial_1) = \{ae_1 - ae_2 + ae_3 |\ a \in \mathbb{Z}\}$$

27 Day 27

Recall from yesterday,

1. Simplicial chain complex from worksheet

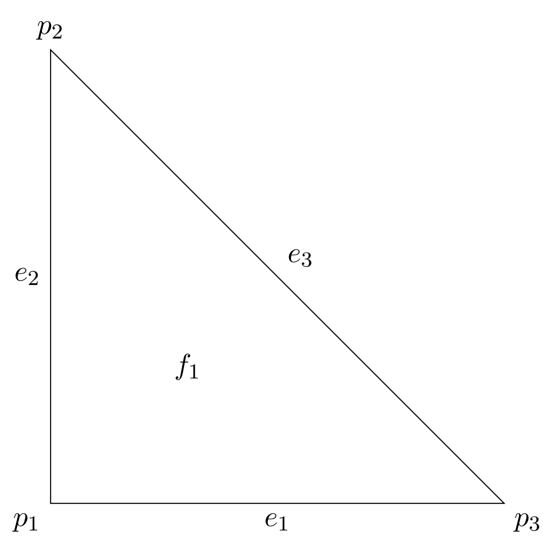
$$\{0\} \to C_2(k) \to^{\partial_2} C_1(k) \to^{\partial_1} C_0(k) \to \{0\}$$

2.

$$\begin{aligned} \partial_1(e_1) = \partial_1([v_1, v_2]) &= v_2 - v_1 \\ \partial_1(e_2) = \partial_1([v_1, v_3]) &= v_3 - v_1 \\ \partial_1(e_3) = \partial_1([v_2, v_3]) &= v_3 - v_2 \\ \partial_2(f_1) &= e_3 - e_2 + e_1 \end{aligned}$$

3.

$$\begin{aligned} \partial_2([v_1, v_2, v_3]) &= [v_2, v_3] - [v_1, v_3] + [v_1, v_2] \\ \partial_2([v_1, v_2, v_3]) &= e_3 - e_1 + e_2 \end{aligned}$$



4.
$$ker(\partial_2) = \{af_1 | ker(af_1) = 0\}$$

Since $\partial_2(f_1) \neq 0$, the only multiple of f_1 sent to 0 is $0f_1$

5. Kernels,
$$\begin{aligned} ker(\partial_1) &= \{ae_1 - ae_2 + ae_3 |\ a \in \mathbb{Z}\} \\ &= \langle e_1 - e_2 + e_3 \rangle \\ ker(\partial_2) &= \{0\} \end{aligned}$$

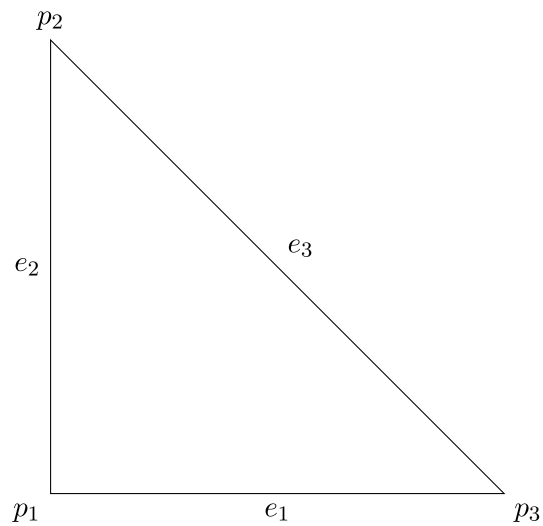
Images,
$$\begin{aligned} im(\partial_2) &= \langle e_3 - e_2 + e_1 \rangle \\ im(\partial_1) &= \langle v_2 - v_1, v_3 - v_1, v_3 - v_2 \rangle \end{aligned}$$

Homologies,
$$\begin{aligned} H_2(K) &= \frac{ker(\partial_2)}{im(\partial_3)} = \frac{\{0\}}{\{0\}} \\ H_1(K) &= \frac{ker(\partial_1)}{im(\partial_2)} = \frac{\langle e_1 - e_2 + e_3 \rangle}{\langle e_3 - e_2 + e_1 \rangle} \\ H_0(K) &= \frac{ker(\partial_0)}{im(\partial_1)} = \frac{C_0(K)}{\langle v_2 - v_1, v_3 - v_1, v_3 - v_2 \rangle} \cong \mathbb{Z} \end{aligned}$$

Where,
$$a_1v_1 + a_2v_2 + a_3v_3 \in C_0(K)$$

Note that this is like setting,
$$\begin{aligned} v_2 - v_1 &= 0 \\ v_3 - v_1 &= 0 \\ v_3 - v_2 &= 0 \end{aligned}$$

6. <u>Ex:</u> Given the following simplicial complex,



$$\{0\} \to^{\partial_2} C_1(K) \to^{\partial_1} C_0(K) \to \{0\}$$

With,

$$C_1(K) \cong \mathbb{Z}^3$$
$$C_0(K) \cong \mathbb{Z}^3$$

$ker(\partial_1)$ and $im(\partial_2)$ are as before, but now,

$$ker(\partial_2) = \{0\}$$
$$im(\partial_2) = \{0\}$$

Meaning that our homology ends up as,

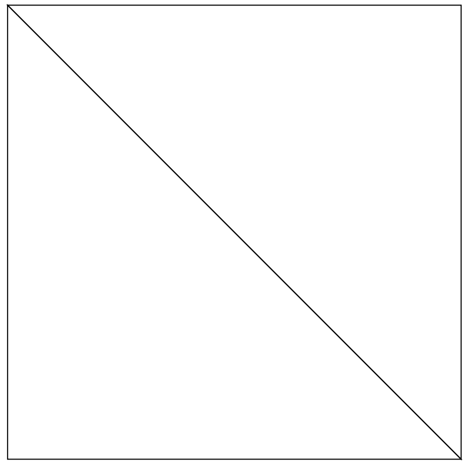
$$H_2(K) = \frac{ker(\partial_2)}{im(\partial_3)} = \frac{\{0\}}{\{0\}}$$
$$H_1(K) = \frac{ker(\partial_1)}{im(\partial_2)} = \frac{\langle e_1 - e_2 + e_3 \rangle}{\{0\}} \cong \mathbb{Z}$$
$$H_0(K) = \frac{ker(\partial_0)}{im(\partial_1)} = \frac{C_0(K)}{\langle v_2 - v_1, v_3 - v_1, v_3 - v_2 \rangle} \cong \mathbb{Z}$$

7. Note: Just like π_1 H_1 detected the "holes" in the 2nd example. In general, nonzero elements of $H_n(X)$:

   $ker(\partial_n) \ni n$-chains that could be the boundary of an $(n+1)$-simplex

   But aren't $\notin im(\partial_{n+1})$

8. Ex:



$H_1(K) \cong \mathbb{Z}^2$

- Option 1: Given X look for a simplicial complex K such that $|K| \cong X$ and define,

$$H_n(X) = H_n(K)$$

- Option 2: "triangulate" X

Definition 29. The standard n-simplex

$$\Delta^n = \text{ simplex spanned by, } \underbrace{(0,0,\ldots,0)}_{v_0}\underbrace{(1,0,\ldots,0)}_{v_1},\ldots,\underbrace{(0,0,\ldots,1)}_{v_n} \text{ in } \mathbb{R}^n$$

- Ex: OH GREAT TIME TO DRAW SOME SIMPLICES REALLY GREAT JUST WONDERFUL

Definition 30. The interior of Δ^n is,

$$int(\Delta^n) = \Delta^n \setminus \bigcup \{ \text{ proper faces } \}$$

Definition 31. Let x be any topological space. A Δ-complex structure on X is a collection of sets,

$$S_n = \{\sigma_1^n, \ldots, \sigma_{k_n}^n\}$$

(one of these sets for each $n \in \mathbb{Z}^{\geq 0}$) where,

$$\sigma_i^n : \Delta^n \to X$$

are continuous maps $\forall i$

- $$\sigma^n_{i\ |int(\Delta^n)}$$

  is injective $\forall i, n$ and each $x \in X$ is in the image of exactly one $\sigma^n_{i\ |int(\Delta^n)}$

28 Day 28

28.1 Δ-complexes

Definition 32. Let X be any topological space. A Δ-complex structure on X is a collection of sets:

$$\begin{aligned} S_0 &= \{\sigma_1^0, \sigma_2^0, \dots, \sigma_{k_0}^0\} \\ S_1 &= \{\sigma_1^1, \sigma_2^1, \dots, \sigma_{k_1}^1\} \\ S_2 &= \{\sigma_1^2, \sigma_2^2, \dots, \sigma_{k_2}^2\} \\ &\vdots \\ S_n &= \{\sigma_1^n, \sigma_2^n, \dots, \sigma_{k_n}^n\} \end{aligned}$$

Where,

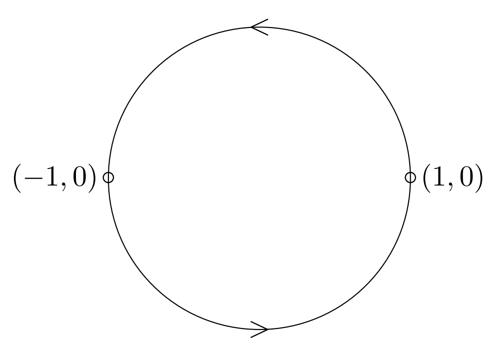
$$\underbrace{\sigma_i^n : \Delta^n \to X}_{\text{“}n\text{-simplex in } X\text{”}}$$

(note that Δ^n is the standard n-simplex) and that the maps are continuous such that,

1. “simplices are glued together only along their boundaries and they glue to form all of X”
2. “a face of a simplex in X is another simplex”
3. “the topology on X is compatible with the topologies on the simplices”

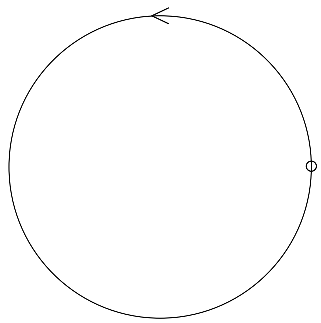
- E.g. In condition 3, we're excluding the possibility of forming a set by gluing the simplices then giving that set the trivial topology. CHECK IT OUT THER'S A DOODLING.
- Ex: $X = S^1$

$$\begin{aligned} S_0 &= \{\sigma_1^0, \sigma_2^0\} \\ \sigma_1^0 &: \Delta^0 \to X \text{ constant map at } (1,0) \\ \sigma_2^0 &: \Delta^0 \to X \text{ constant map at } (-1,0) \\ S_1 &= \{\sigma_1^1, \sigma_2^1\} \\ \sigma_1^1 &: \Delta^1 \to X \\ \sigma_2^1 &: \Delta^1 \to X \\ \sigma_1^1(t) &= (\cos(\pi t), \sin(\pi t)) \\ \sigma_1^1(t) &= (\cos(\pi t), -\sin(\pi t)) \end{aligned}$$



- Ex: $X = S^1$

$$\begin{aligned} S_0 &= \{\sigma_1^0\} \text{ (constant map at } (1,0) \text{)} \\ S_1 &= \{\sigma_1^1\} \\ \sigma_1^1 &: \Delta^1 \to X \\ \sigma_1^1(t) &= (\cos(2\pi t), \sin(2\pi t)) \end{aligned}$$



- Ex: $X =$ torus

$$\begin{aligned} S_0 &= \{\sigma_1^0\} \\ &\underbrace{S_1 = \{\sigma_1^1, \sigma_2^1, \sigma_3^1\}}_{\text{map to } e_1, e_2, e_3 \text{ respectively in the direction shown}} \\ S_2 &= \{\sigma_1^2, \sigma_2^2\} \end{aligned}$$

Fuckin lost me brah

29 Day 29

- So far: Defined $H_n^\Delta(X)$, the simplicial homology of a space X with a Δ-complex structure
- Issues:
  - Does it depend on the Δ-complex structure?
  - $X \cong_{homeo} Y \implies H_n^\Delta(X) \cong H_n^\Delta(Y)$?
  - Does $f : X \to Y$ induce $f_* : H_n^\Delta(X) \to H_n^\Delta(Y)$?

  To resolve these issues, we'll replace simplicial homology with a (seemingly much larger), version of homology.

29.1 Singular Homology

Let the following be so,

$$\begin{aligned} X &= \text{ any topological space} \\ \Delta^n &= \text{ standard } n\text{-simplex } \subseteq \mathbb{R}^{n+1} \end{aligned}$$

Definition 33.

$$S_n(X) = \{\text{continuous maps from } \sigma : \Delta^n \to X\}$$

Note that, $S_n(X)$ is absurdly huge and that our continuous maps can be singularly horrible.

$$C_n(X) = \{a_1\sigma_1 + \cdots + a_r\sigma_r |\ a_1, \ldots, a_r \in \mathbb{Z},\ \sigma_1, \ldots, \sigma_r \in S_n(X)\}$$

Note that this is equivalent to a product of uncountably many copies of $\mathbb{Z}$ We call $C_n(X)$ the group of singular n-chains

Definition 34. The (singular) boundary homomorphisms are:

$$\partial_n : C_n(X) \to C_{n-1}(X)$$
$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j (\sigma \circ f_j)$$

Definition 35. The singular homology of X is,

$$\frac{ker(\partial_n)}{im(\partial_{n+1})}$$

- Note: The "same" proof for simplicial homology shows,

  $$\partial_n \circ \partial_{n+1} = 0$$

  This $\implies$

  $$im(\partial_n + 1) \subseteq ker(\partial_n)$$

  So, the definition of $H_n(x)$ makes sense.

- So, what's craaaazy about this definition,

  $$H_n(X) = \frac{\mathbb{Z}^{\text{uncountable}}}{\mathbb{Z}^{\text{uncountable}}}$$

  but crazily:

Theorem 11.

$$H_n(X) \cong H_n^{\Delta}(X)$$

for any Δ-complex structure on X

- We won't prove this theorem, but we'll see some examples.
- Note: This theorem implies that $H_n^{\Delta}(X)$ is independent of the Δ-complex structure you choose. Look into relative homology to show some of this I guess? Read chapter 2 of Hatcher "The equivalence of simplicial and singular homology"
- Note: This theorem implies that $H_n^{\Delta}(X)$ is independent of the Δ-complex structure you choose.

29.2 Induced Homomorphisms

- Let $f : X \to Y$ be continuous
- Then f induces a homomorphism,

$$f_\# : C_n(X) \to C_n(Y)$$
$$a_1\sigma_1 + \dots a_r\sigma_r \in C_n(X), \text{ etc.}$$
$$f_\#(a_1\sigma_1 + \dots + a_r\sigma_r) = a_1(f \circ \sigma_1) + \dots + a_r(f \circ \sigma_r)$$

With σ_i and $f \circ \sigma_i$ are maps,

$$\sigma_i : \Delta^n \to X$$
$$f \circ \sigma_i : \Delta^n \to Y$$

To check that this induces a homomorphism,

$$f_* : H_n(X) \to H_n(Y)$$

With,

$$H_n(X) = \frac{ker(\partial_n^X)}{im(\partial_{n+1}^X)}$$
$$H_n(Y) = \frac{ker(\partial_n^Y)}{im(\partial_{n+1}^Y)}$$

we'll need to check the fact that $f_\#$ takes $ker(\partial_n^X)$ to ∂_n^Y and $im(\partial_{n+1}^X)$ to $im(\partial_{n+1}^Y)$

- These both follow from:

Let,

$$\partial_n^X : C_n(X) \to C_{n-1}(X)$$
$$\partial_n^Y : C_n(Y) \to C_{n-1}(Y)$$

be the boundary homomorphisms, then,

$$\partial_n^Y \circ f_\# = f_\# \circ \partial_n^X$$

Proof. Suffices to prove on n-simplices σ:

$$\begin{aligned}(\partial_n^Y \circ f_\#)(\sigma) &= \partial_n^Y(f \circ \sigma) \\ &= \sum_{j=0}^{n} (-i)^j (f \circ \sigma) \circ f_j \\ &= \sum_{j=0}^{n} (-i)^j f \circ (\sigma \circ f_j) \\ &= f_\#(\sum_{j=0}^{n} (-i)^j (\sigma \circ f_j)) \\ &= f_\#(\partial_n^X(\sigma))\end{aligned}$$

□

- We'll show next time that,

$$f_* : H_n(X) \to H_n(Y)$$
$$f_*([a_1\sigma_1 + \cdots + a_r\sigma_r]) = [f_\#(a_1\sigma_1 + \cdots + a_r\sigma_r)]$$

30 Day 30

30.1 Singular Homology, continued

- Recall: A continuous map $f : X \to Y$ induces
  - $$f_\# : C_n(X) \to C_n(Y)$$
    such that
    $$\partial_n^Y \circ f_\# = f_\# \circ \partial_n^X$$
  - $$f_* : H_n(X) \to H_n(Y)$$
    $$f_*([\underbrace{a_1\sigma_1, \ldots, a_r\sigma_r}_{\alpha}]) = [f_\#([a_1\sigma_1, \ldots, a_r\sigma_r])]$$

  With sigma as the "simplex in X"
  $$\sigma_n : \Delta^n \to X$$
- Questions/excercises:
  1. Prove that:
     $$\alpha \in ker(\partial_n^X) \implies f_\#(\alpha) \in \ker(\partial_n^Y)$$
  2. Prove that:
     $$\alpha \in im(\partial_{n+1}^X) \implies f_\#(\alpha) \in (\partial_{n+1}^Y)$$

3. Convince yourself that points 1 and 2 imply that f_* is well defined

- Answers:

1.

$$\begin{aligned} \alpha \in ker(\partial_n^X) &\implies \partial_n^X = 0 \\ &\implies \partial_n^Y(f_\#)(\alpha)) = f_\#(\partial_n^X(\alpha)) = f_\#(0) = 0 \\ &\implies f_\#(\alpha) \in ker(\partial_n^Y) \end{aligned}$$

2.

$$\begin{aligned} \alpha \in im(\partial_{n+1}^X) &\implies \alpha = \partial_{n+1}^X, \text{ for some } \beta \\ &\implies f_\#(\alpha) = f_\#(\partial_{n+1}(\beta)) \\ &\implies f_\#(\alpha) \in (\partial_{n+1}^Y) \end{aligned}$$

3.

$$f_* : \frac{ker(\partial_n^X)}{im(\partial_{n+1}^X)} \to \frac{ker(\partial_n^X)}{im(\partial_{n+1}^X)}$$

Answer one implies that the numerator maps to the numerator.
Answer two implies that the image of 0 is 0.
Both of these together imply that f_* is well defined.

- Facts:

1. $(id_X) = id_{H_n(X)}$
2. $(f \circ g) = f_* \circ g_*$

With these, it follows that we can use the same proof that we used for π_1

Theorem 12. If X is homeomorphic to Y, then $H_n(X) \cong H_n(Y),\ \forall n$

- How can we possibly ever compute $H_n(X)$?
- Lets start small: $H_0(X)$ for any X and $H_n(\cdot)$ for any n
- Proposition: If X is path-connected, then $H_0(X) \cong \mathbb{Z}$

Proof. Define

$$\phi : H_0(X) \to \mathbb{Z}$$
$$H_0(X) = \frac{ker(\partial_0)}{im(\partial_1)} = \frac{C_0(X))}{im(\partial_1)}$$
$$\phi([a_1\sigma_1 + \cdots + a_r\sigma_r]) = a_1 + \cdots + a_r$$

This is a homomorphism and is

- Well-Defined: (i.e.if $a_1\sigma_1 + \dots + a_r\sigma_r \in im(\partial_1)$, then $a_1 + \dots + a_r = 0$)
  For any 1-simplex, $\tau : \Delta^1 \to X$,

$$\begin{aligned}\partial_1(\tau) &= \sum_{j=0}^{1}(-1)^j(\tau \circ f_j)\\ &= \underbrace{1(\tau \circ f_0)}_{\text{a 0-simplex in } X} + \underbrace{(-1)(\tau \circ f_0)}_{\text{a 0-simplex in } X}\\ &\implies \text{sum of coefficients in } \partial_1(\tau) \text{ is } 1 + (-1) = 0\end{aligned}$$

  This implies that the sum of the coefficients in ∂_1(anything) $= 0$. This is one of the things we wanted.

- Injective: suppose,

$$\phi([a_1\sigma_1 + \dots + a_r\sigma_r]) = 0 \implies a_1 + \dots + a_r = 0$$

  implies that by allowing repeats I can assume that every a_i is either 1 or -1, and since the sum is 0,

$$\#\{a_i = 1\} = \#\{a_i = -1\}$$

  This implies that,

$$[a_1\sigma_1 + \dots + r\sigma_r] = \underbrace{[\sigma_1 - \sigma_2 + \dots + \sigma_{r-1} - \sigma_r]}_{\text{(may require some relabelling)}}$$

  But each $\sigma_i : \Delta^0 \to X$ is a constant map to some point $x_i \in X$. Choose paths

$$\begin{aligned}x_1 &\to x_2\\ x_3 &\to x_4\end{aligned}$$

  These are 1-simplices whose boundaries are $\sigma_2 - \sigma_1, \sigma_4 - \sigma_3, \dots$ which implies,

$$[\sigma_2 - \sigma_1 + \sigma_4 - \sigma_3 + \dots] = [0]$$

□

31 Day 31

31.1 Last time

- Proving,

$$\begin{aligned}\phi : H_0(X) &\to \mathbb{Z}\\ \phi([\alpha]) &= \text{ sum of coefficients in } \alpha\end{aligned}$$

  is an isomorphism when X is path-connected

- Injectivity: Suppose

$$\begin{aligned} \phi([\alpha]) &= 0 \\ \Rightarrow \text{Sum of coefficients in } \alpha &= 0 \\ \Rightarrow [\alpha] &= [\sigma_1 - \sigma_2, \sigma_3 - \sigma_4, \dots, \sigma_{r-1} - \sigma_r] \end{aligned}$$

E.g. if

$$\begin{aligned} \alpha &= 3\sigma_1 - 2\sigma_2 - \sigma_3 \\ &= \sigma_1 + \sigma_1 + \sigma_1 - \sigma_2 - \sigma_2 - \sigma_3 \\ &= \sigma_1 - \sigma_2 + \sigma_1 - \sigma_2 + \sigma_1 - \sigma_3 \end{aligned}$$

Each $\sigma_i : \Delta^0 \to X$ is a constant map from Δ^0 to a single point, $x_i \in X$. Choose a path $\delta_i : I \to X$ from x_{i+1} to x_i. This is a 1-simplex in X, namely,

$$\begin{aligned} \tau_i &: \Delta^1 \to X \\ \tau_i(1-t, t) &= \delta_i(t) \end{aligned}$$

Then,

$$\begin{aligned} \partial_1(\tau_i) &= \tau_i \circ f_0 - \tau_i \circ f_1 \\ = \quad \underbrace{\tau_{i|t=1}}_{\text{constant map at } x_i} \quad &- \quad \underbrace{\tau_{i|t=1}}_{\text{constant map at } x_{i+1}} \\ &= \sigma_i - \sigma_{i+1} \end{aligned}$$

Thus,

$$\begin{aligned} [\alpha] &= [\underbrace{\sigma_1 - \sigma_2}_{\partial_1(\tau_1} + \underbrace{\sigma_3 - \sigma_4}_{\partial_1(\tau_2} + \cdots + \underbrace{\sigma_{r-1} - \sigma_r}_{\partial_1(\tau_{t-1}})] \\ &= [\partial_1(\tau_1 + \cdots + \tau_{t-1})] \end{aligned}$$

- Surjectivity: For any $a \in \mathbb{Z}$,

$$\phi([a \cdot \text{any } \sigma\text{-simplex}]) = a$$

- $\underbrace{Pictorially}$:

SOME GODDAMN PICTURE I CAN'T MAKE YET AHH FUCK

$$\begin{aligned} H_0(X) &= \frac{C_0(X)}{im(\partial_1)} \\ \partial_1(\tau) &= w - v \\ &\implies w - v \in im(\partial_1) \\ &\implies [w] = [v] \in H_0(X) \\ &\implies \text{every generator of } C_0 \text{ equals every other} \end{aligned}$$

Theorem 13. Suppose,

$$X = X_1 \sqcup X_2 \sqcup \cdots \sqcup X_k$$

Where $X_1, \ldots, X_k$ are path-connected ("path components of X"). Then,

$$H_n(X) \cong H_n(X_1) \oplus H_n(X_2) \oplus \cdots \oplus H_n(X_k), \ \forall n \geq 0$$

Corollary 13.1.

$$H_0(X) \cong \mathbb{Z}^{\# \text{ number of path components}}$$

Proof. For simplicity, let,

$$X = X_1 \sqcup X_2$$

First, claim

$$C_n(X) \cong C_n(X_1) \oplus C_n(X_2)$$

To see this, let:

$$\begin{aligned} \phi : C_n(X) \cong C_n(X_1) \oplus C_n(X_2) \to C_n(X) \\ \phi(\alpha_1, \alpha_2) = (i_1)_\# \alpha_1 + (i_2)_\# \alpha_2 \end{aligned}$$

Where $i_1 : X_1 \to X$ and $i_2 : X_2 \to X$ are the inclusions. Note that α_1 and α_2 are linear combinations of simplices in their respective components.
This is

- Injective, because the images of $(i_1)_\#(\alpha_1)$ and $(i_2)_\#(\alpha_2)$ are disjoint, so no cancellation can occur.
- Surjective, because if,

  $$\alpha = a_1\sigma_1 + \cdots + k\alpha_r \in C_n(X)$$

  $\implies$ each σ_i must land entirely in either X_1 or X_2, (Because $\sigma_i : \Delta^n \to X$ is continuous and Δ^n is path-connected)

$$\begin{aligned} &\implies \alpha = (i_1)_\#(\text{stuff in } X_1) + (i_2)_\#(\text{stuff in } X_2) \\ &\implies \alpha \in im(\phi) \end{aligned}$$

Thus, $C_n(X) \cong C_n(X_1) \oplus C_n(X_2)$. And ∂_n respects this decomposition:

$$C_n(X) \xrightarrow{\partial_n} C_{n-1}(X)$$

$$C_n(X_1) \xrightarrow{\partial_n^1} C_{n-1}(X_1)$$

$$C_n(X_2) \xrightarrow{\partial_n^2} C_{n-1}(X_2)$$

$$C_n(X) \cong C_n(X_1) \oplus C_n(X_2)$$
$$C_{n-1}(X) \cong C_{n-1}(X_1) \oplus C_{n-1}(X_2)$$

Thus,

$$H_n(X) = \frac{ker(\partial_n)}{im(\partial_{n+1})} = \frac{ker(\partial_n^1) \oplus ker(\partial_n^2)}{im(\partial_{n+1}^1) \oplus im(\partial_{n+1}^2)}$$

□

- The last homology we can compute by force is:

Theorem 14. Let $X = \{\cdot\}$. Then,

$$H_n(X) = \{0\}$$

for all $n0$

Proof. For any n there's only one possible n-simplec,

$$\sigma^n : \Delta^n \to X$$
$$\sigma^n = \text{ constant map}$$

Thus,

$$C_n(X) = \{a \cdot \sigma^n | \ a \in \mathbb{Z}\} \cong \mathbb{Z}$$

The boundary maps are,

$$\partial_n : C_n(X) \to C_{n-1}$$
$$\partial_n(\sigma^n) = \sum_{j=0}^{n} (-1)^j)(\underbrace{\sigma^n \circ f_j}_{(n-1)\text{-simplices, so must } = \sigma^{n-1}})$$
$$= \sigma^{n-1} - \sigma^{n-1} + \sigma^{n-1} - \sigma^{n-1} \cdots + (-1)^n \sigma^{n-1}$$

□

32 Day 32

32.1 Singular Homology Goals

- (Eventually) version of the Van Kampen theorem,
- Prove that if $X \simeq Y$(homotopy equivalent), then $H_n(X) \cong H_n(Y), \ \forall n$

32.2 Homological Algebra

Definition 36. A chain complex is a collection of abelian groups.

$$\ldots, C_2, C_1, C_0, C_{-1}, C_{-2}, \ldots$$

and group homomorphisms

$$\partial_n : C_n \to C_{n-1}$$

Such that

$$\partial_{n-1} \circ \partial_n = 0$$

$$C. = (\dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} C_{-2} \xrightarrow{\partial_{-2}} \dots)$$

- Main Example:

$$\begin{aligned} C_n &= C_n(X) \\ &= \{\mathbb{Z}\text{-linear combinations of n-simplices in } X\} \\ &\text{or} \\ C_n &= C_n^{\Delta}(X) \\ &= \{\underbrace{\mathbb{Z}\text{-linear combinations of the } n\text{-simplices in } S_n}_{\text{(particular } \Delta\text{-complex str)}}\} \\ &\text{or} \\ C_n &= C_n(K), \text{ Where } K \text{ is a simplicial complex} \end{aligned}$$

Definition 37. If $C.$ is any chain complex, it's n^{th}-homology group is

$$H_n(C.) := \frac{ker(\partial_n)}{im(\partial_{n+1})}$$

- Note: This makes sense because $im(\partial_{n+1}) \subseteq ker(\partial_n)$, which follos from $\partial_n \circ \partial_{n+1} = 0$.

Definition 38. If $C.$ and $D.$ are chain compexes, then a chain map from $C.$ to $C.$ to $D.$ is a collection of homomorphisms

$$f_n : C_n \to D_n$$

Such that,

$$\partial_n^D \circ f_n = f_{n-1} \circ \partial_n^C, \ \forall n \in \mathbb{Z}$$

Graphically,

$$
\begin{array}{ccccccccccccc}
\dots & \xrightarrow{\partial_3} & D_2 & \xrightarrow{\partial_2} & D_1 & \xrightarrow{\partial_1} & D_0 & \xrightarrow{\partial_0} & D_{-1} & \xrightarrow{\partial_{-1}} & D_{-2} & \xrightarrow{\partial_{-2}} & \dots \\
 & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_{-1}} & & \downarrow{\scriptstyle f_{-2}} & & \\
\dots & \xrightarrow{\partial_3} & C_2 & \xrightarrow{\partial_2} & C_1 & \xrightarrow{\partial_1} & C_0 & \xrightarrow{\partial_0} & C_{-1} & \xrightarrow{\partial_{-1}} & C_{-2} & \xrightarrow{\partial_{-2}} & \dots
\end{array}
$$

Such that this commutes.

- If X and Y are topological spaces and $g : X \to Y$ is a continuous function, then

$$
\begin{aligned}
g_{\#} &: C_n(X) \to C_n(Y) \\
g_{\#}(\alpha_1\sigma_1 + \cdots + \alpha_r\sigma_r) &= a_1(g \circ \sigma_1) + \cdots a_r(g \circ \sigma_r)
\end{aligned}
$$

  is a chain map because we checked $\partial_n^Y \circ g_{\#} = g_{\#} \circ \partial_n^X$

- Exercise: If $f. : C. \to D.$ is any chain map, then

$$
\begin{aligned}
f_* &: H_n(C.) \to H_n(D.) \\
H_n(C.) &= \frac{ker(\partial_n^C)}{im(\partial_{n+1}^C)} \\
H_n(D.) &= \frac{ker(\partial_n^D)}{im(\partial_{n+1}^D)} \\
f_*([\alpha]) &= [f_n(\alpha)]
\end{aligned}
$$

  Where,

$$
\begin{aligned}
\alpha \in ker(\partial_n^C) &\subseteq C_n \\
f_n(\alpha) &\in D_n
\end{aligned}
$$

  is well defined. (Proof is exactly what we did to show that $g_* : H_n(X) \to H_n(Y)$ is well defined

- Question: When do two different chain maps,

$$
\begin{aligned}
f. &: C. \to D. \\
g. &: C. \to D.
\end{aligned}
$$

  induce the same homomorphism

$$
f_* = g_* : H_n(C.) \to H_n(D.)
$$

  (Structurally: if X and Y are homotopy equivalent, then there exists

$$
\begin{tikzcd}
X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"]
\end{tikzcd}
$$

such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. For π_1 we "showed" $(f \circ g)_* = (id_Y)_*$ and $(g \circ f)_* = (id_X)_*$, thus f_* and g_* were inverse isomorphisms.

$$\begin{tikzcd} \pi_1(X) \arrow[r, bend left, "f_*"] & \pi_1(Y) \arrow[l, bend left, "g_*"] \end{tikzcd}$$

To do this argument for H_n instead of π_1 we'll need to know that homotopic maps between spaces induce the <u>same</u> homomorphism on homology.)

Definition 39. A <u>chain homotopy</u> $f. : C. \to D.$ to $g. : C. \to D.$ is a collectopn of homomorphisms

$$h_n : C_n \to D_{n+1},\ \forall n \in \mathbb{Z}$$

such that

$$\partial^D_{n+1} \circ h_n + h_n \circ \partial^C_{n+1} = f_n - g_n$$

$$\begin{tikzcd} \dots \arrow[r, "\partial^D_3"] & D_2 \arrow[r, "\partial^D_2"] & D_1 \arrow[r, "\partial^D_1"] & D_0 \arrow[r, "\partial^D_0"] & \dots \\ \dots \arrow[r, "\partial^C_3"] & C_2 \arrow[r, "\partial^C_2"] \arrow[u, dashed, shift right, "f_2"'] \arrow[u, dashed, shift left, "g_2"] & C_1 \arrow[r, "\partial^C_1"] \arrow[u, dashed, shift right, "f_1"'] \arrow[u, dashed, shift left, "g_1"] \arrow[ul, dashed, "h_1"] & C_0 \arrow[r, "\partial^C_0"] \arrow[u, dashed, shift right, "f_0"'] \arrow[u, dashed, shift left, "g_0"] \arrow[ul, dashed, "h_0"] & \dots \end{tikzcd}$$

If there exists a chain homotopy from $f.$ and $g.$ then $f_* = g_*$

Proof. Both f_* and g_* are homomorphisms,

$$H_n(C.) \to H_n(D.)$$
$$H_n(C.) = \frac{ker(\partial^C_n)}{im(\partial^C_{n+1})}$$

Let $[\alpha] \in H_n(C)$

$$\begin{aligned} f_*([\alpha]) - g_*([\alpha]) &= [f_n(\alpha)] - [g_n(\alpha)] \\ &= [f_n(\alpha) - g_n(\alpha)] \\ &= [\partial^D_{n+1} \circ h_n(\alpha)] + [h_{n-1} \circ \partial^C_n(\alpha)] \\ &= 0 \end{aligned}$$

□