Human Following Robot || Linear MPC

Full Derivation || Daksh Raval

February 19, 2026

Contents

1 Next State Model **1**

2 Nonlinear Continuous Model **1**

3 Linearization **2**

3.1 Jacobians 2
3.2 Plugging into Taylor Expansion 2
3.3 Simplifying 2
3.4 Putting Together 3
3.5 Finished Linearization 3

4 Discretization **3**

4.1 Original Full Form 3
4.2 $A_c^n = 0$ 3
4.3 Short Form 3
4.4 Short Form Further Derived 3
4.5 Expanded Short Form 4
4.6 Linearized State Space Model 4

5 Appendix **4**

1 Next State Model

Nutshell 1. *Linearize THEN discretize so you don't assume state constant through timestep [1]*

2 Nonlinear Continuous Model

Nutshell 2. *Accurately describes how state changes as function of different control and state inputs for bicycle kinematic model. [2][3]*

Remark 1. *All references (eg. [2]) in this file are clickable links*

$$\tilde{\mathbf{x}} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \theta \end{bmatrix} \Rightarrow \dot{\mathbf{x}} = f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} v \cdot cos(\theta) \\ v \cdot sin(\theta) \\ v \cdot tan(\delta)/L \end{bmatrix}$$

3 Linearization

Nutshell 3. *Use first order Taylor expansion and Jacobians*

Remark 2. *The **subscript O** means evaluated at **previous state/control at (t-1)**, ie. operating point*

$$\dot{\mathbf{x}} \approx f(x_o, u_o) + \frac{\partial f}{\partial \tilde{\mathbf{x}}}\bigg|_{u_o, x_o} \cdot (x - x_o) + \frac{\partial f}{\partial \tilde{\mathbf{u}}}\bigg|_{u_o, x_o} \cdot (u - u_o)$$

3.1 Jacobians

$$\frac{\partial f}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y & \partial f_1/\partial \theta \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & -v \cdot sin(\theta) \\ 0 & 0 & v \cdot cos(\theta) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial \tilde{\mathbf{u}}} = \begin{bmatrix} \partial f_1/\partial v & \partial f_2/\partial \delta \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} cos(\theta) & 0 \\ sin(\theta) & 0 \\ tan(\delta)/L & v \cdot sec^2(\delta)/L \end{bmatrix}$$

3.2 Plugging into Taylor Expansion

$$\dot{\mathbf{x}} \approx \begin{bmatrix} v_o \cdot cos(\theta_o) \\ v_o \cdot sin(\theta_o) \\ v_o \cdot tan(\delta_o)/L \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & -v_o \cdot sin(\theta_o) \\ 0 & 0 & v_o \cdot cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x - x_o \\ y - y_o \\ \theta - \theta_o \end{bmatrix}}_{\partial \tilde{\mathbf{x}}} + \underbrace{\begin{bmatrix} cos(\theta_o) & 0 \\ sin(\theta_o) & 0 \\ tan(\delta_o)/L & v_o \cdot sec^2(\delta_o)/L \end{bmatrix} \cdot \begin{bmatrix} v - v_o \\ \delta - \delta_o \end{bmatrix}}_{\partial \mathbf{u}}$$

$$\partial \tilde{\mathbf{x}} \Rightarrow (x - x_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (y - y_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (\theta - \theta_o) \cdot \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix}$$

$$\partial \tilde{\mathbf{u}} \Rightarrow (v - v_o) \cdot \begin{bmatrix} cos(\theta_o) \\ sin(\theta_o) \\ tan(\delta_o)/L \end{bmatrix} + (\delta - \delta_o) \cdot \begin{bmatrix} 0 \\ 0 \\ v_o \cdot sec^2(\delta_o)/L \end{bmatrix}$$

3.3 Simplifying

Variable Terms

$$\theta \cdot \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix} + \begin{bmatrix} cos(\theta_o) & 0 \\ sin(\theta_o) & 0 \\ tan(\delta_o)/L & v_o \cdot sec^2(\delta_o)/L \end{bmatrix} \cdot \begin{bmatrix} v \\ \delta \end{bmatrix}$$

Constant Terms

Nutshell 4. *Do I want steering third row or no? That would involve control vector, which wasn't in Hadis' derivation*

$$\underbrace{\begin{bmatrix} v_o \cdot cos(\theta_0) \\ v_o \cdot sin(\theta_o) \\ v_o \cdot tan(\delta_o)/L \end{bmatrix} - v_o \cdot \begin{bmatrix} cos(\theta_o) \\ sin(\theta_o) \\ tan(\delta_o)/L \end{bmatrix}}_{cancel} - \theta_o \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix} - \delta_o \cdot \begin{bmatrix} 0 \\ 0 \\ v_o \cdot sec^2(\delta_o)/L \end{bmatrix}$$

3.4 Putting Together

$$\underbrace{\begin{bmatrix} 0 & 0 & -v_o \cdot sin(\theta_o) \\ 0 & 0 & v_o \cdot cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix}}_{A_{continuous}} \cdot \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \underbrace{\begin{bmatrix} cos(\theta_o) & 0 \\ sin(\theta_o) & 0 \\ tan(\delta_o)/L & v_o \cdot sec^2(\delta_o)/L \end{bmatrix}}_{B_{continuous}} \cdot \begin{bmatrix} v \\ \delta \end{bmatrix} - \underbrace{\begin{bmatrix} -\theta_o \cdot v_o \cdot sin(\theta_o) \\ \theta_o \cdot v_o \cdot cos(\theta_o) \\ \delta_o \cdot v_o \cdot sec^2(\delta_o)/L \end{bmatrix}}_{\tilde{\mathbf{d}}_c}$$

3.5 Finished Linearization

$$\dot{\mathbf{x}} = A_c \cdot \tilde{\mathbf{x}} + B_c \cdot \tilde{\mathbf{u}} + (-\tilde{\mathbf{d}}_c)$$

4 Discretization

Nutshell 5. *Zero order hold*

4.1 Original Full Form

$$\tilde{\mathbf{x}}(k+1) = A_d(k) \cdot \tilde{\mathbf{x}}(k) + B_d(k) \cdot \tilde{\mathbf{u}}(k) + \tilde{\mathbf{d}}_d$$

$$A_d = e^{AT} = I + A_c \cdot (\Delta t) + \frac{1}{2!} A_c^2 \cdot (\Delta t^2) + \frac{1}{3!} A_c^3 \cdot (\Delta t^3)$$

$$B_d = B_c \cdot (\Delta t) + \frac{1}{2!} A_c B_c \cdot (\Delta t^2) + \frac{1}{3!} A_c^2 B_c \cdot (\Delta t^3) + \frac{1}{4!} A_c^3 B_c \cdot (\Delta t^4)$$

$$\tilde{\mathbf{d}}_d = \tilde{\mathbf{d}}_c \cdot (\Delta t) + \frac{1}{2!} A_c \cdot \tilde{\mathbf{d}}_c \cdot (\Delta t^2) + \frac{1}{3!} A_c^2 G_c \cdot (\Delta t^3) + \frac{1}{4!} A_c^3 G_c \cdot (\Delta t^4)$$

4.2 $A_c^n = 0$

Nutshell 6. *Zeros out because no term in third row of third column, so nothing multiplies rightmost basis vector*

$$\begin{bmatrix} 0 & 0 & -v_o \cdot sin(\theta_o) \\ 0 & 0 & v_o \cdot cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -v_o \cdot sin(\theta_o) \\ 0 & 0 & v_o \cdot cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix} = -v_o \cdot sin(\theta_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + v_o \cdot cos(\theta_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix} \boxed{= 0}$$

4.3 Short Form

$$A_d = I + A_c \cdot \Delta t \qquad B_d = B_c \cdot \Delta t + 0.5 \cdot \Delta t^2 \cdot A_c B_c \qquad \tilde{\mathbf{d}}_d = \tilde{\mathbf{d}}_c \cdot \Delta t + 0.5 \cdot \Delta t^2 \cdot A_c \cdot \tilde{\mathbf{d}}_c$$

4.4 Short Form Further Derived

$\mathbf{A_c} \cdot \mathbf{B_c}$:

$$\begin{bmatrix} \frac{tan(\delta_o)}{L} \cdot \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix} & \frac{v_o \cdot sec^2(\delta_o)}{L} \cdot \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix} \end{bmatrix} = \frac{v_o}{L} \begin{bmatrix} -sin(\theta_o) \cdot tan(\delta_o) & -v_o \cdot sin(\theta_o) \cdot sec^2(\delta_o) \\ cos(\theta_o) \cdot tan(\delta_o) & v_o \cdot cos(\theta_0) \cdot sec^2(\delta_o) \\ 0 & 0 \end{bmatrix}$$

$\mathbf{A_c} \cdot \mathbf{\tilde{d}_c}$

$$-\delta_o \cdot v_o \cdot sec^2(\delta_o)/L \cdot \begin{bmatrix} -v_o \cdot sin(\theta_o) \\ v_o \cdot cos(\theta_o) \\ 0 \end{bmatrix} = \delta_o \cdot v_o^2 \cdot sec^2(\delta_o)/L \cdot \begin{bmatrix} sin(\theta_o) \\ -cos(\theta_o) \\ 0 \end{bmatrix}$$

4.5 Expanded Short Form

$$\boxed{A_d = \begin{bmatrix} 1 & 0 & -v_o \cdot sin(\theta_o) \cdot \Delta t \\ 0 & 1 & v_o \cdot cos(\theta_o) \cdot \Delta t \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{B_d = \Delta t \cdot \begin{bmatrix} cos(\theta_o) & 0 \\ sin(\theta_o) & 0 \\ tan(\delta_o)/L & v_o \cdot sec^2(\delta_o)/L \end{bmatrix} + 0.5 \cdot \Delta t^2 \cdot \frac{v_o}{L} \begin{bmatrix} -sin(\theta_o) \cdot tan(\delta_o) & -v_o \cdot sin(\theta_o) \cdot sec^2(\delta_o) \\ cos(\theta_o) \cdot tan(\delta_o) & v_o \cdot cos(\theta_0) \cdot sec^2(\delta_o) \\ 0 & 0 \end{bmatrix}}$$

$$\boxed{\mathbf{\tilde{d}}_d = \Delta t \cdot \begin{bmatrix} \theta_o \cdot v_o \cdot sin(\theta_o) \\ -\theta_o \cdot v_o \cdot cos(\theta_o) \\ -\delta_o \cdot v_o \cdot sec^2(\delta_o)/L \end{bmatrix} + 0.5 \cdot \Delta t^2 \cdot \delta_o \cdot v_o^2 \cdot sec^2(\delta_o)/L \cdot \begin{bmatrix} sin(\theta_o) \\ -cos(\theta_o) \\ 0 \end{bmatrix}}$$

4.6 Linearized State Space Model

$$\boxed{\mathbf{\tilde{x}}(k+1) = A_d(k) \cdot \mathbf{\tilde{x}}(k) + B_d(k) \cdot \mathbf{\tilde{u}}(k) + \mathbf{\tilde{d}}_d}$$

5 Appendix

[1] ETH Paper on Bicycle Model

[2] MPC: Theory, Computation & Design

[3] Youtuber Control Bootcamp by Steve Brunton