

MPC Linearization & Discretization Derivation

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Nutshell 1. *Derivation questions*

1 Next State Model

Nutshell 2. *Linearize THEN discretize so you don't assume state constant through timestep*

2 Nonlinear Continuous Model

$$\tilde{\mathbf{x}} = \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta \theta \end{bmatrix} \Rightarrow \dot{\mathbf{x}} = f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} = \begin{bmatrix} v \cdot \cos(\theta) \\ v \cdot \sin(\theta) \\ v \cdot \tan(\delta)/L \end{bmatrix}$$

3 Linearization

Nutshell 3. Use first order Taylor expansion and Jacobians

Remark 1. The **subscript O** means evaluated at **previous state/control at (t-1)**, ie. operating point

$$\dot{\mathbf{x}} \approx f(x_o, u_o) + \left. \frac{\partial f}{\partial \tilde{\mathbf{x}}} \right|_{u_o, x_o} \cdot (x - x_o) + \left. \frac{\partial f}{\partial \tilde{\mathbf{u}}} \right|_{u_o, x_o} \cdot (u - u_o)$$

3.1 Jacobians

$$\frac{\partial f}{\partial \tilde{\mathbf{x}}} = \begin{bmatrix} \partial f_1 / \partial x & \partial f_1 / \partial y & \partial f_1 / \partial \theta \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & -v \cdot \sin(\theta) \\ 0 & 0 & v \cdot \cos(\theta) \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{\partial f}{\partial \tilde{\mathbf{u}}} = \begin{bmatrix} \partial f_1 / \partial v & \partial f_2 / \partial \delta \\ \vdots & \vdots \\ \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \cos(\theta) & 0 \\ \sin(\theta) & 0 \\ \tan(\delta)/L & v \cdot \sec^2(\delta)/L \end{bmatrix}$$

3.2 Plugging into Taylor Expansion

$$\dot{\mathbf{x}} \approx \begin{bmatrix} v_o \cdot \cos(\theta_o) \\ v_o \cdot \sin(\theta_o) \\ v_o \cdot \tan(\delta_o)/L \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & -v_o \cdot \sin(\theta_o) \\ 0 & 0 & v_o \cdot \cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix}}_{\partial \tilde{\mathbf{x}}} \cdot \underbrace{\begin{bmatrix} x - x_o \\ y - y_o \\ \theta - \theta_o \end{bmatrix}}_{\partial \tilde{\mathbf{x}}} + \underbrace{\begin{bmatrix} \cos(\theta_o) & 0 \\ \sin(\theta_o) & 0 \\ \tan(\delta_o)/L & v_o \cdot \sec^2(\delta_o)/L \end{bmatrix}}_{\partial \tilde{\mathbf{u}}} \cdot \underbrace{\begin{bmatrix} v - v_o \\ \delta - \delta_o \end{bmatrix}}_{\partial \tilde{\mathbf{u}}}$$

$$\partial \tilde{\mathbf{x}} \Rightarrow (x - x_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (y - y_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + (\theta - \theta_o) \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix}$$

$$\partial \tilde{\mathbf{u}} \Rightarrow (v - v_o) \cdot \begin{bmatrix} \cos(\theta_o) \\ \sin(\theta_o) \\ \tan(\delta_o)/L \end{bmatrix} + (\delta - \delta_o) \cdot \begin{bmatrix} 0 \\ 0 \\ v_o \cdot \sec^2(\delta_o)/L \end{bmatrix}$$

3.3 Simplifying

Variable Terms

$$\theta \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix} + \begin{bmatrix} \cos(\theta_o) & 0 \\ \sin(\theta_o) & 0 \\ \tan(\delta_o)/L & v_o \cdot \sec^2(\delta_o)/L \end{bmatrix} \cdot \begin{bmatrix} v \\ \delta \end{bmatrix}$$

Constant Terms

Nutshell 4. Do I want steering third row or no? That would involve control vector, which wasn't in Hadis' derivation

$$\underbrace{\begin{bmatrix} v_o \cdot \cos(\theta_o) \\ v_o \cdot \sin(\theta_o) \\ v_o \cdot \tan(\delta_o)/L \end{bmatrix} - v_o \cdot \begin{bmatrix} \cos(\theta_o) \\ \sin(\theta_o) \\ \tan(\delta_o)/L \end{bmatrix}}_{cancel} - \theta_o \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix} - \delta_o \cdot \begin{bmatrix} 0 \\ 0 \\ v_o \cdot \sec^2(\delta_o)/L \end{bmatrix}$$

3.4 Putting Together

$$\underbrace{\begin{bmatrix} 0 & 0 & -v_o \cdot \sin(\theta_o) \\ 0 & 0 & v_o \cdot \cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix}}_{A_{continuous}} \cdot \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} + \underbrace{\begin{bmatrix} \cos(\theta_o) & 0 \\ \sin(\theta_o) & 0 \\ \tan(\delta_o)/L & v_o \cdot \sec^2(\delta_o)/L \end{bmatrix}}_{B_{continuous}} \cdot \begin{bmatrix} v \\ \delta \end{bmatrix} - \underbrace{\begin{bmatrix} -\theta_o \cdot v_o \cdot \sin(\theta_o) \\ \theta_o \cdot v_o \cdot \cos(\theta_o) \\ \delta_o \cdot v_o \cdot \sec^2(\delta_o)/L \end{bmatrix}}_{\tilde{\mathbf{d}}_c}$$

3.5 Finished Linearization

$$\dot{\mathbf{x}} = A_c \cdot \tilde{\mathbf{x}} + B_c \cdot \tilde{\mathbf{u}} + (-\tilde{\mathbf{d}}_c)$$

4 Discretization

Nutshell 5. *Zero order hold*

4.1 Original Full Form

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= A_d(k) \cdot \tilde{\mathbf{x}}(k) + B_d(k) \cdot \tilde{\mathbf{u}}(k) + \tilde{\mathbf{d}}_d \\ A_d &= e^{A_c \Delta t} = I + A_c \cdot (\Delta t) + \frac{1}{2!} A_c^2 \cdot (\Delta t^2) + \frac{1}{3!} A_c^3 \cdot (\Delta t^3) \\ B_d &= B_c \cdot (\Delta t) + \frac{1}{2!} A_c B_c \cdot (\Delta t^2) + \frac{1}{3!} A_c^2 B_c \cdot (\Delta t^3) + \frac{1}{4!} A_c^3 B_c \cdot (\Delta t^4) \\ \tilde{\mathbf{d}}_d &= \tilde{\mathbf{d}}_c \cdot (\Delta t) + \frac{1}{2!} A_c \cdot \tilde{\mathbf{d}}_c \cdot (\Delta t^2) + \frac{1}{3!} A_c^2 G_c \cdot (\Delta t^3) + \frac{1}{4!} A_c^3 G_c \cdot (\Delta t^4) \end{aligned}$$

4.2 $A_c^n = 0$

Nutshell 6. *Zeros out because no term in third row of third column, so nothing multiplies rightmost basis vector*

$$\begin{bmatrix} 0 & 0 & -v_o \cdot \sin(\theta_o) \\ 0 & 0 & v_o \cdot \cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & -v_o \cdot \sin(\theta_o) \\ 0 & 0 & v_o \cdot \cos(\theta_o) \\ 0 & 0 & 0 \end{bmatrix} = -v_o \cdot \sin(\theta_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + v_o \cdot \cos(\theta_o) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix} \boxed{= 0}$$

4.3 Short Form

$$A_d = I + A_c \cdot \Delta t \quad B_d = B_c \cdot \Delta t + 0.5 \cdot \Delta t^2 \cdot A_c B_c \quad \tilde{\mathbf{d}}_d = \tilde{\mathbf{d}}_c \cdot \Delta t + 0.5 \cdot \Delta t^2 \cdot A_c \cdot \tilde{\mathbf{d}}_c$$

4.4 Short Form Further Derived

$A_c \cdot B_c$:

$$\left[\frac{\tan(\delta_o)}{L} \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix} \right] \cdot \frac{v_o \cdot \sec^2(\delta_o)}{L} \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix} = \frac{v_o}{L} \begin{bmatrix} -\sin(\theta_o) \cdot \tan(\delta_o) & -v_o \cdot \sin(\theta_o) \cdot \sec^2(\delta_o) \\ \cos(\theta_o) \cdot \tan(\delta_o) & v_o \cdot \cos(\theta_o) \cdot \sec^2(\delta_o) \\ 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_c \cdot \tilde{\mathbf{d}}_c$$

$$-\delta_o \cdot v_o \cdot \sec^2(\delta_o)/L \cdot \begin{bmatrix} -v_o \cdot \sin(\theta_o) \\ v_o \cdot \cos(\theta_o) \\ 0 \end{bmatrix} = \delta_o \cdot v_o^2 \cdot \sec^2(\delta_o)/L \cdot \begin{bmatrix} \sin(\theta_o) \\ -\cos(\theta_o) \\ 0 \end{bmatrix}$$

4.5 Expanded Short Form

$$A_d = \begin{bmatrix} 1 & 0 & -v_o \cdot \sin(\theta_o) \cdot \Delta t \\ 0 & 1 & v_o \cdot \cos(\theta_o) \cdot \Delta t \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_d = \Delta t \cdot \begin{bmatrix} \cos(\theta_o) & 0 \\ \sin(\theta_o) & 0 \\ \tan(\delta_o)/L & v_o \cdot \sec^2(\delta_o)/L \end{bmatrix} + 0.5 \cdot \Delta t^2 \cdot \frac{v_o}{L} \begin{bmatrix} -\sin(\theta_o) \cdot \tan(\delta_o) & -v_o \cdot \sin(\theta_o) \cdot \sec^2(\delta_o) \\ \cos(\theta_o) \cdot \tan(\delta_o) & v_o \cdot \cos(\theta_o) \cdot \sec^2(\delta_o) \\ 0 & 0 \end{bmatrix}$$

$$\tilde{\mathbf{d}}_d = \Delta t \cdot \begin{bmatrix} \theta_o \cdot v_o \cdot \sin(\theta_o) \\ -\theta_o \cdot v_o \cdot \cos(\theta_o) \\ -\delta_o \cdot v_o \cdot \sec^2(\delta_o)/L \end{bmatrix} + 0.5 \cdot \Delta t^2 \cdot \delta_o \cdot v_o^2 \cdot \sec^2(\delta_o)/L \cdot \begin{bmatrix} \sin(\theta_o) \\ -\cos(\theta_o) \\ 0 \end{bmatrix}$$

4.6 Linearized State Space Model

$$\tilde{\mathbf{x}}(k+1) = A_d(k) \cdot \tilde{\mathbf{x}}(k) + B_d(k) \cdot \tilde{\mathbf{u}}(k) + \tilde{\mathbf{d}}_d$$