0

Advanced IA:
Decision Making under Uncertainty

Hélène Fargier

SRI3, Autumn 2023

IRIT-CNRS, fargier@irit.fr

Autumn 22

Introduction

What is a decision ?
Decision = a choice between possible actions or available object, given (i) some knowledge about the context and (ii) objectives

Several types of decisions :

- low level : many small decisions (ex : scheduling tasks, configuring a product), combinatorial dimension, simplified criteria.
- High level : choosing between a few high level alternatives (ex : choosing a site for a plant) - not combinatorial -, complexe evaluation, several criteria

Dimensions of a decision problem

- One or several criteria
- Time, sequential decision
- Uncertainty about the state of the world.

Static decision making under uncertainty : basic definition

A one shot game between the DM and the nature, formalized by :

- $S = \{1, \ldots n\}$: a set of possible states of the word (or "situations")
- D : a set of decisions (actions) : $d, f \in D$
- X : the possible consequences of the decisions : $x \in X$
- Decision = mapping $d : S \mapsto X$
  $d(s) = x$ consequence of decision d in situation s

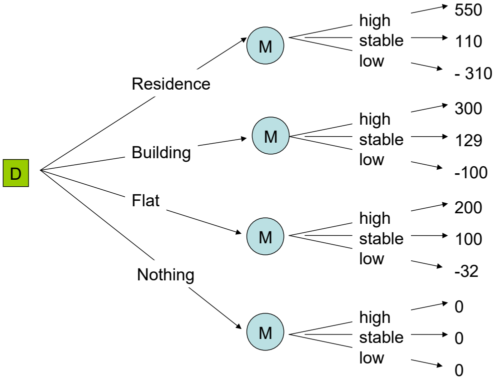
Decision problem : *Given some* **knowledge** *about a the real state of the world and a* **preference** *relation over X, built a preference relation over D, in order to compare / rank order / optimize decision(s).*

Example

Investment : Should the DM buy a residence, a building, a Flat, or keep his money ? ?
The rentability of the decision depends on the future state of the market : high, stable, low

States of the world (S) / Decisions (D)	high	stable	low
Residence	550	110	-310
Building	300	129	-100
Flat	200	100	-32
Nothing	0	0	0

Decision tree



A game with two players : the DM, then the "nature"

Representing the DM's preferences : the utility function

Utility function $X : u \mapsto \mathbb{R} : u(x) > u(x')$ iff x is preferred to x'

The utility function is a numerical but subjective function
$u(x)$ = attractivity of consequence x for the decision maker.

- "Qualitative"(ordinal) utility : just a way to represent an ordinal relation, numbers are just ranks
- "Quantitative" (cardinal) utility : encodes also the difference of preference intensity $(x - x') > (y - y')$ iff x is more prefered to x' than y is prefered to y'

Decision under probabilitistic risk : expected utility

Assume a rich information about the state of the world : a probability distribution $p : S \mapsto [0, 1]$ *numerical*

$Maximize\ UE(d) = \Sigma_s p(s)\ .\ u(d(s))$

When $u(x) = x$, $UE(d)$ is the expected gain
2 interpretations of p :

- statistics : describes the variability of precise observations (repeated observation of the system)
- subjective : describes the beliefs of the agent rather than statistics

Qualitative (subjective) probabilities

Assumption : the knowledge about the state of the world are representable a relation $\succeq$ on the events $A \subseteq S$, (= subjective probability) such that :

- $\succeq$ complete, reflexive, transitive (*weak order, "preordre" in french*)
- $A \subseteq B \implies B \succeq A$ (so $S \succeq A \succeq \emptyset, \forall A$)
- $\forall A, B, C$ disjoint , $A \succeq B \iff A \cup C \succeq B \cup C$ (preaditivity condition)

In most of the cases, there exist a probabilty distribution p capturing $\succeq$, i.e. such that : $P(A) \geq P(B) \iff A \succeq B$

Probabilities and Gambles

My belief in A is greater than my belief in B = I prefer betting on A than on B :

Gamble $yA0$: "win y if A occurs and 0 otherwise"

In probability theory, the attractivity of a gamble is evaluated by the expected utility of its payoff
$UE(yA0) = u(y)\ .\ P(A) + u(0)\ .\ P(\bar{A}) = u(y)\ .\ P(A)$

"I prefer betting on A than on B" : $UE((y)A(0)) > UE((y)B(0))$
"I prefer betting on A than on B" : $u(y).P(A) + u(0).P(\bar{A}) > u(y).P(B) + u(0).P(\bar{B})$

One can always set $u(0) = 0$
"I prefer betting on A than on B" : $P(A).u(y) > P(B)u(y)$

"I believe more in A than in B " = "I prefer betting on A than on B" = $\frac{P(A)}{P(B)} > 1$

To elicit a subjective probability, ask the decision maker to compare gambles - and deduce the numerical probability from the equations

Probabilities and Gambles

Do you prefer to bet on A or to get x for sure ?

- get x euros : $u(x)$
- bet on A : win y if A occurs, 0 otherwise

I prefer betting : $UE(yA0) = u(y).P(A)$
I prefer the sure payoff : $UE(x) = u(x)$

I prefer the bet on A to the sure payoff x (with $0 < x < y$) :
$UE((y)A(0)) > u(x)$

But I prefer the sure payoff x' to the bet on A ($0 < x < x' < y$) :
$u(x') > P(A).u(y) > u(x)$

$\frac{u(x')}{u(y)} > P(A) > \frac{u(x)}{u(y)}$

Probabitity of A : the (relative) amount of monney that makes you hesitate

Probabilities and Gambles

Pay x to bet on A?

- keep the x euros : $u(x)$
- bet on A : win y if A occurs, 0 otherwise

I pay x to bet on A : $UE(yA0) = u(y).P(A)$
I dont bet and keep my money : $UE(x) = u(x)$

I prefer betting $u(y).P(A) > u(x)$

$P(A) = \frac{u(x)}{u(y)}$, where x is the maximal price that the decision make will pay to bet on A

for a gain of y

Probability of an event A = the (relative) amount of money that you will pay to bet on A

Expected Utility

The knowledge is represented by a probability distribution

$$Maximize\ UE(d) = \Sigma_s p(s)\ .\ u(d(s))$$

- u is the attractivity of gain x
- $u(x) = x$, $UE(d)$ is the expected gain
- pessimism, risk aversion -> u concave
- optimism -> u convex

Properties

- Compact economical representation,
- Pareto efficient,
- Suits repeated decision :
  the good outcomes counterbalance the bad ones
- Elicitation of the data
  - When the (objective) probabilities are given
    $\mapsto$ elicit u by comparing gambles to constant acts .
  - Such methods also exist for subjective uncertainty : elicit both P and u

Properties

- Objective probabilities are assumed (statistics) ; otherwise, elicit subjective probabilities (= force de DM to choose, no "I dont know" option)
- Not pertinent when the preferences are non compensatory, when the decision is not repeated.
- Does not capture all the rational behaviour, even in the case of compensatory utility levels.

Ellsberg's paradox

Urn = 90 balls, 30 reds et 60 black or yellow.
Draw one ball.

- Option A : win 100 euros when the ball is red, otherwise 0.
- Option B : win 100 euros when the ball is black, otherwise 0.
- Option C : win 100 euros when the ball is red or yellow, otherwise 0.
- Option D : win 100 euros when the ball is black or yellow, otherwise 0.

Do you prefer the A gamble or the B gamble ?
Do you prefer the C gamble or the D gamble ? ?

Ellsberg's paradox

Urn = 90 balls, 30 reds et 60 black or yellow.
Draw one ball.

- Option A : win 100 euros when the ball is red, otherwise 0.
- Option B : win 100 euros when the ball is black, otherwise 0.
- Option C : win 100 euros when the ball is red or yellow, otherwise 0.
- Option D : win 100 euros when the ball is black or yellow, otherwise 0.

Decision makers are generally cautious :

- they prefer A to B
  ($1/3$ chances to win ; with B, the proba is between 0 et $2/3$)
- they prefer D to C
  ($2/3$ chances to win ; with C, the proba is between $1/3$ et 1)

Ellsberg's paradox

Urn = 90 balls, 30 reds and 60 black or yellow.

- Option A : win 100 euros when the ball is red, otherwise 0.
- Option B : win 100 euros when the ball is black, otherwise 0.
- Option C : win 100 euros when the ball is red or yellow, otherwise 0.
- Option D : win 100 euros when the ball is black or yellow, otherwise 0.

What ever u, there exist no probability distribution such that $EU(A) > EU(B)$ *et* $EU(C) < EU(D)$

Decision makers no not ever use an expeted utility

Ellsberg's Paradox : what really happens

The probabilities of R, B are not Y known with certainty.

A family of probability measures $\mathfrak{F} = \{P, P(R) = 1/3, P(B \text{ or } Y) = 2/3\}$.

In particular : $\begin{array}{lll} P_1(R) = 1/3 & P_1(Y) = 2/3 & P_1(B) = 0 \\ P_2(R) = 1/3 & P_2(Y) = 0 & P_2(B) = 2/3 \end{array}$

A : $EU(100\{R\}0\{Y\}0\{B\}) = 1/3 \cdot 100$: in all cases.
D : $EU(0\{R\}100\{Y\}100\{B\}) = 2/3 \cdot 100$: in all cases.
B : $EU(0\{R\}0\{Y\}100\{B\}) \in [0 \cdot 100, 2/3 \cdot 100]$: this gambles receive utility 0, *using* $P1$, cautiously
C : $EU(100\{R\}100\{Y\}0\{B\}) \in [1/3 \cdot 100, 1 \cdot 100]$: this gambles receive utility $1/3 \cdot 100$, *using* $P2$, cautiously

Upper and lower probabilities

"One third of the balls are red, the others are black or yellow ".

Question : what is the probability of getting a non black ball ? $\mapsto$at least $1/3$, at most 1
Work with a set of probability measures
$\mathfrak{F} = \{P, P(red) = 1/3, P(Black) + P(Yellow) = 2/3\}$
(all the probability distributions compatible with the information)

Compute upper and lower bounds of the probability
$P_*(yellow) = 0$, $P_*(black) = 0$,
$P_*(red) = 1/3$, $P_*(yellow\ or\ red) = 1/3$,

$P^*(yellow) = 2/3$, $P^*(black) = 2/30$,
$P^*(red) = 1/3$, $P^*(yellow\ or\ red) = 1$,

Belief functions

A belief mass function (or "basic probability assignment", bpa) is an application $m : 2^S \mapsto [0, 1]$ *such that* $\Sigma_{E \subseteq S} m(E) = 1$

Any $E \subseteq S$ such that $m(E) > 0$ is called a "focal element".

Example : 30 red balls and 60 black or yellow balls :
$m(\{R\}) = \frac{1}{3}$, $m(\{B, Y\}) = \frac{2}{3}$,

Mass $m(E)$: belief allocated to A (and to no strict subset).

Belief functions

Two measures :

- $Bel(A) = \Sigma_{E, E \subseteq A} m(E)$: total the belief mass supporting A. Estimates to what extend A is certain.
- $Pl(A) = \Sigma_{E, E \cap A \neq \emptyset} m(E)$: maximum part of the belief that can support A. Estimates to what extend A is plaisible.

Proposition $\forall A, Bel(A) = 1 - Pl(A^c)$
An event is certain when its opposite is impossible

Proposition : $\forall A, Bel(A) \leq Pl(A)$
An event is always more plausible than certain.

Belief Functions : Example

$m(\{R\}) = \frac{1}{3}$, $m(\{B, Y\}) = \frac{2}{3}$,

$Bel(\{R\}) = \Sigma_{E, E \subseteq \{R\}} m(E) = m((\{R\}) = \frac{1}{3}$
$Pl(\{R\}) = \Sigma_{E, E \cap \{R\} \neq \emptyset} m(E) = m((\{R\}) = \frac{1}{3}$

$Bel(\{R, B\}) = \Sigma_{E, E \subseteq \{R, B\}} m(E) = m((\{R\}) = \frac{1}{3}$
$Pl(\{R, B\}) = \Sigma_{E, E \cap \{R, B\} \neq \emptyset} m(E) = m((\{R\}) + m((\{B, Y\}) = 1$

$Bel(\{J\}) = \Sigma_{E, E \subseteq \{Y\}} m(E) = 0$,
$Pl(\{J\}) = \Sigma_{E, E \cap \{Y\} \neq \emptyset} m(E) = m((\{B, Y\}) = \frac{2}{3} = 1 - Bel(\{R, B\})$

Belief functions : representation of ignorance

Three horses : $c1$, $c2$, $c3$. Which will win the race ?
Expert 1 : Each of the three horses has the same chance of winning
Expert 2 : I don't know

Expert 3 : it's c_1 (but this expert is not reliable)

- Expert 1 : equiprobability - $m_1(\{c1\}) = m_1(\{c2\}) = m_1(\{c3\}) = \frac{1}{3}$;
  Pl ("c_1 wins") = Bel ("c_1 wins") = $\frac{1}{3}$
- Expert 2 : total ignorance - $m_2(\{c1, c2, c3\}) = 1$;
  Pl ("c_1 wins") = 1, Bel ("c_1 wins") = 0
  Ignorance and equiprobability are two different things
- Expert 3 : $m_3(\{c1\}) = \frac{2}{3}$ $m_3(\{c1, c2, c3\}) = \frac{1}{3}$; *Mass $m(S)$ = degree of total ignorance*
  Pl ("c_1 wins") = 1 ; Bel ("c_1 wins") = $\frac{2}{3}$

Belief functions : representing imprecise information

Consider a bottle containing at least as much water as wine, at most twice as much water as wine.
What is the probability that the bottle contains at most 1.5 times more water than wine ?
Imprecise but certain information $\frac{water}{wine} \in [1, 2]$

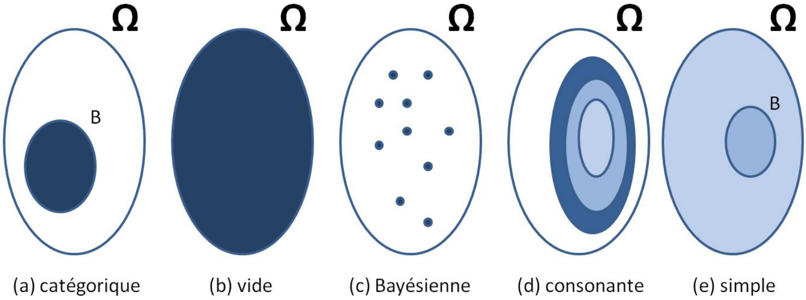
Two possible probabilistic models :

- $r_{\frac{water}{wine}}$ (water to wine ratio) : uniform distribution over $[1, 2]$. So $P(r_{\frac{water}{wine}} \leq 1.5) = \frac{1}{2}$
- $r_{\frac{wine}{water}}$ (wine to water ratio) : uniform distribution over $[\frac{1}{2}, 1]$. So $P(r_{\frac{wine}{water}} \geq \frac{2}{3}) = \frac{2}{3} \neq \frac{1}{2}$!!!

The belief function - based model :

- $r_{\frac{water}{wine}} \in [1, 2] : m([1, 2]) = 1$; so $Bel(r_{\frac{water}{wine}} \leq 1.5) = 0$, $Pl(r_{\frac{water}{wine}} \leq 1.5) = 1$
- $r_{\frac{water}{wine}} \in [0.5, 1] : m([\frac{1}{2}, 1]) = 1$; so $Bel(r_{\frac{wine}{water}} \geq \frac{2}{3}) = 0$, $Pl(r_{\frac{wine}{water}} \geq \frac{2}{3}) = 1$

Different Types of Belief Functions



- c All the focal elements are singletons,
  Bel $=$ *PL* and is a measure of probability
- d The focal elements are consonant,
  Bel is a necessity measure and *Pl* is a possibility measure
  (but *m* is not a possibility distribution stricto sensus)

Belief functions and capacity measures

- Bel and Pl are capacity measures
- Bel $\iff$ super-additive capacity measures :
  $\forall A, Bel(A \cup B) \geq Bel(A) + Bel(B) - Bel(A \cap B)$
- Pl $\iff$ sub-additive capacity measures :
  $\forall A, Pl(A \cup B) \leq Pl(A) + Pl(B) - Pl(A \cap B)$
- probability measures are super additive and sub additive capacities

Belief functions and imprecise probabilities

- *Bel (resp. Pl) is a lower (resp. upper) probabilities* :
  $\mathcal{F} = \{p, \forall A, Bel(A) \leq P(A) \leq PL(A)\}$;
  $Bel(A) \leq P(A) \leq PL(A)$: P a poorly known probability bounded by Bel and Pl

  $[0,0]$: A impossible ; $[0,1]$: nothing is known about A ; $[1,1]$ A is certain.

Belief mass = compact representation of a family of probabilities
We can recover m from $\mathcal{F}$ using the Moebius transforms of P_* and P^*

Decide with belief functions

Pessimistic global utility of f (to be maximised) :

$$U_{pes}(f) = \Sigma_{A \subseteq S} m(A).min_{s \in A} u(f(s))$$

On Ellsberg's example : $m(\{Rouge\}) = 1/3$, $m(\{Noir, Jaune\}) = 2/3$

$U_{pes}(100\{Rouge\}0) = m(\{Rouge\}).u(100) + m(\{Noir, Jaune\}).min(u(0), u(0)) = 1/3.u(100)$
. $U_{pes}(100\{Noir\}0) = m(\{Rouge\}).u(0) + m(\{Noir, Jaune\}).min(u(100), u(0)) = 0$
$U_{pes}(100\{Rouge, Jaune\}0) = m(\{Rouge\}).u(100) + m(\{Noir, Jaune\}).min(u(0), u(100)) = 1/3.u(100)$
$U_{pes}(100\{Noir, Jaune\}0) = m(\{Rouge\}).u(0) + m(\{Noir, Jaune\}).min(u(100), u(100)) = 2/3.u(100)$

$U_{pes}(100\{Rouge\}0) > U_{pes}(100\{Noir\}0)$ mais
$U_{pes}(100\{Rouge, Jaune\}0) > U_{pes}(100\{Noir, Jaune\}0)$

Extending expected utility to any capacity measure : Choquet's integral

How to compute a global utility given a utility function u and a capacity measure μ capturing your uncertainty (e.g. $\mu = P_*$) ?
Let us go back to the probabilistic case

Ex : 3 states, (non restrictive) assumption : $u(f(s_3)) \geq u(f(s_2)) \geq u(f(s_1)))$

$$UE(f) = \quad u(f(s_3)).p(s_3) \quad + \quad u(f(s_2)).p(s_2) \quad + \quad u(f(s_1)).p(s_1)$$

$$\begin{array}{rlcl} UE(f) = & u(f(s_1)) & . & (p(s_3) + p(s_2) + p(s_1)) \\ + & (u(f(s_2)) - u(f(s_1))) & . & (p(s_3) + p(s_2)) \\ + & (u(f(s_3)) - u(f(s_2))) & . & (p(s_3)) \end{array}$$

In the general case, with the scale $L = \lambda_m > \cdots > \lambda_1$ and denoting $F_{\lambda_j} = \{s, u(f(s)) \geq \lambda_j\}$

It can be shown that, what ever f, $UE(f) = \sum_{j=m...1}(\lambda_j - \lambda_{j-1}).P(F_{\lambda_j})$

Extending expected utility : Choquet's integral

With $\mu = P$ a probability measure :

Maximize $\quad UE(f) = \quad \sum_{j=m\ldots1}(\lambda_j - \lambda_{j-1}) \,.\, P(F_{\lambda_j})$

With $L = \lambda_m > \cdots > \lambda_1$ et $F_{\lambda_j} = \{s, u(f(s)) \geq \lambda_j\}$

What ever the capacity μ (e.g. $\mu = P_*, \mu = P^*$)

Maximize $\quad Ch(f)_\mu = \quad \sum_{j=m\ldots1}(\lambda_j - \lambda_{j-1}) \,.\, \mu(F_{\lambda_j})$

Note : $\sum_{j=m\ldots1}(\lambda_j - \lambda_{j-1}) \,.\, \mu(F_{\lambda_j}) = \sum_{j=m\ldots1} \lambda_{j-1} \,.\, (\mu(F_{\lambda_{j-1}}) - \mu(F_{\lambda_j}))$

Two equivalent expressions of the Choquet integral

Ellberg's example as a Choquet integral

$Ch(f) = \sum_{j=m\dots1}(\lambda_j - \lambda_{j-1}).\mu(F_{\lambda_j})$

where $\mu = P_*$ using $\mathfrak{F} = \{P, P(R) = 1/3, P(B \text{ or } Y) = 2/3\}$.

$$
\begin{array}{lclcl}
Ch(100\{R\}0\{Y\}0\{B\}) & = & (100-0)\,.\,1/3 + 0\,.\,1 & = & 1/3\,.\,100 \\
Ch(0\{R\}0\{Y\}100\{B\}) & = & (100-0)\,.\,0 + 0\,.\,1 & = & 0 \\
Ch(100\{R\}100\{Y\}0\{B\}) & = & (100-0)\,.\,1/3 + 0\,.\,1 & = & 1/3\,.\,100 \\
Ch(0\{R\}100\{Y\}100\{B\}) & = & (100-0)\,.\,2/3 + 0\,.\,1 & = & 2/3\,.\,100
\end{array}
$$

Choquet integrals : properties

- Suit any kind of capacity measre μ, includsing imprecise probabilities (probability families)
- Is compensatory : low utility on some states can be counterbalanced by high utility levels on others : a cardinal approach
- Encompasses expected utility as a special case (when μ is a classical, decomposable, probability measure)
- Encompasses the pessimistic utility as a special case (when μ is a *Bel* measure)
- Not always Pareto efficient, can violate the separability principle

In summary

Which decision rule will I use ? ? ?

IT DEPENDS

ON THE INFORMATION YOU HAVE

(TOTAL IGNORANCE, IMPRECISE PROBABILITY, STATISTICS)

ON YOUR ATTITUDE WITH RESPECT TO RISK

rule	type ok knoledge	utility function	Attitude
Expected Utility	Statistics objective ou subjective	quantitative	all (shape of the utility function)
Pessimistic utility	probabilities on sets	quantitative	all (shape of the utility function)
Choquet	probabilistic in the wide sense	quantitative	all (shape of the utility function)
Maximin	total ignorance	ordinal	pessimistic
Leximin	total ignorance	ordinal	pessimistic
Maximax	total ignorance	ordinal	optimistic
Leximax	total ignorance	ordinal	optimistic
Hurwicz	total ignorance	quantitative	all (depending on α)
Min Regret	total ignorance	quantitative	pessimistic
Laplace	ignorance =+	quantitative	neutram
Mean value of the utility	ignorance+	quantitative	all (shape of the utility function))
Sugeno	any capacity measure	quantitative	all