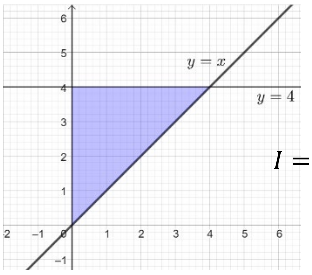
CÁLCULO II

RESOLUCIÓN DE LA PRÁCTICA CALIFICADA 3

Pregunta 1

Para D la región limitada por $y = x, y = 4$ y $x = 0$, calcular el valor de la integral doble siguiente:



$$\iint_D y^2 e^{xy} dA \qquad D = \{(x,y) \mid y \in [0,4]; 0 \le x \le y\}$$

$$I = \iint_D y^2 e^{xy} dA = \int_0^4 \int_0^y y^2 e^{xy} dx\, dy = \int_0^4 y \underbrace{\left(\int_0^y y e^{xy} dx\right)}_{(*)} dy = \int_0^4 \left(y e^{xy}\Big|_0^y\right) dy = \int_0^4 y\left(e^{y^2} - 1\right) dy$$

$$= \int_0^4 \underbrace{\left(y e^{y^2}\right.}_{(\#)} \left. - y\right) dy = \frac{1}{2} e^{y^2} - \frac{y^2}{2}\Big|_0^4 = \frac{1}{2} e^{16} - \frac{16}{2} - \left(\frac{1}{2} e^0 - 0\right) = \boxed{\frac{\boldsymbol{e^{16}}}{\mathbf{2}} - \frac{\mathbf{15}}{\mathbf{2}}}$$

Se usó:

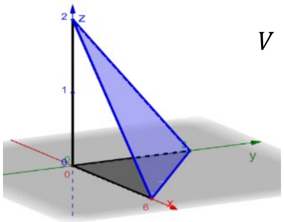
$$(*)\ \ u = xy \Rightarrow du = y dx \Rightarrow \int y e^{xy} dx = \int e^{xy} (y\, dy) = \int e^u du = e^u = e^{xy}$$

$$(\#)\ \ u = y^2 \Rightarrow du = 2y\, dy \Rightarrow \int y e^{y^2} dy = \int e^{y^2} (y\, dy) = \int e^u \left(\frac{du}{2}\right) = \frac{1}{2} \int e^u du = \frac{1}{2} e^u = \frac{1}{2} e^{y^2}$$

Pregunta 2

Usando integrales dobles, determine el volumen del sólido limitado por los planos coordenados y el plano

$$3x + 2y + z = 6 \ \Rightarrow \boxed{z = 6 - 3x - 2y} \qquad D = \left\{(x,y) \mid x \in [0,2]; 0 \le y \le \tfrac{6-3x}{2}\right\}$$



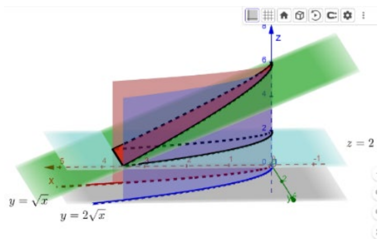
$$V = \iint_D (6 - 3x - 2y) dA = \int_0^2 \left(\int_0^{\frac{6-3x}{2}} (6 - 3x - 2y) dy\right) dx = \int_0^2 [(6-3x)y - y^2]_0^{\frac{6-3x}{2}} dx$$

$$= \int_0^2 \left[(6-3x)\frac{(6-3x)}{2} - \frac{(6-3x)^2}{4}\right] dx = \int_0^2 \frac{(6-3x)^2}{4} dx = \frac{1}{4}\int_0^2 (6-3x)^2 dx = -\frac{(6-3x)^3}{36}\Bigg|_0^2$$

$$= -\frac{1}{36}[0 - 6^3] = \boxed{\mathbf{6}}$$

Pregunta 3

Calcule el volumen del sólido limitado por las superficies $y = \sqrt{x},\ y = 2\sqrt{x},\ \ x + z = 6$ y el plano $z = 2$.



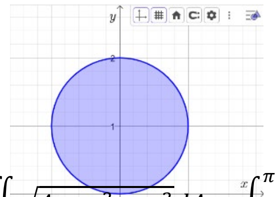
$$V = \iint_D (6 - x - 2) dA = \iint_D (4 - x) dA = \int_0^4 \left(\int_{\sqrt{x}}^{2\sqrt{x}} (4 - x) dy\right) dx$$

$$= \int_0^4 (4 - x)\left(2\sqrt{x} - \sqrt{x}\right) dx = \int_0^4 (4 - x)\sqrt{x}\, dx = \int_0^4 \left(4x^{\frac{1}{2}} - x^{\frac{3}{2}}\right) dx$$

$$= \frac{8}{3}\sqrt{x}^3 - \frac{2}{5}\sqrt{x}^5\Big|_0^4 = \frac{8}{3}\sqrt{4}^3 - \frac{2}{5}\sqrt{4}^5 - 0 = \frac{64}{3} - \frac{64}{5} = \boxed{\frac{\mathbf{128}}{\mathbf{15}}}$$

Pregunta 4

Para $D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\leq 2y\}$, calcular el valor de:



$$\iint_D \sqrt{4-x^2-y^2}\,dA$$

$x^2+y^2=2y$ en coordenadas polares es $r=2\operatorname{sen}(\theta)$, con $\theta\in[0,\pi]$. Así:

$$\iint_D \sqrt{4-x^2-y^2}\,dA=\int_0^{\pi}\int_0^{2\operatorname{sen}(\theta)}\sqrt{4-r^2}\;r\,dr\,d\theta=\int_0^{\pi}-\frac{1}{3}\sqrt{4-r^2}^{\,3}\Big|_0^{2\operatorname{sen}(\theta)}d\theta=-\frac{1}{3}\int_0^{\pi}\left(\sqrt{4-4\operatorname{sen}^2(\theta)}^{\,3}-8\right)d\theta$$

$$=-\frac{1}{3}\int_0^{\pi}\left(\sqrt{4(1-\operatorname{sen}^2(\theta))}^{\,3}-8\right)d\theta=-\frac{1}{3}\int_0^{\pi}\left(8\sqrt{1-\operatorname{sen}^2(\theta)}^{\,3}-8\right)d\theta=-\frac{8}{3}\int_0^{\pi}\left(\sqrt{\cos^2(\theta)}^{\,3}-1\right)d\theta$$

$$=-\frac{8}{3}\int_0^{\pi}(|\cos(\theta)|^3-1)d\theta=-\frac{8}{3}\left(\int_0^{\frac{\pi}{2}}(\underbrace{\cos^3(\theta)}_{(*)}-1)d\theta+\int_{\frac{\pi}{2}}^{\pi}(-\underbrace{\cos^3(\theta)}_{(*)}-1)d\theta\right)$$

$$=-\frac{8}{3}\left(\left[\operatorname{sen}(\theta)-\frac{\operatorname{sen}^3(\theta)}{3}-\theta\right]_0^{\frac{\pi}{2}}+\left[-\operatorname{sen}(\theta)+\frac{\operatorname{sen}^3(\theta)}{3}-\theta\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$=-\frac{8}{3}\left(\left[\operatorname{sen}\left(\frac{\pi}{2}\right)-\frac{\operatorname{sen}^3\left(\frac{\pi}{2}\right)}{3}-\frac{\pi}{2}-\left(\operatorname{sen}(0)-\frac{\operatorname{sen}^3(0)}{3}-0\right)\right]+\left[-\operatorname{sen}(\pi)+\frac{\operatorname{sen}^3(\pi)}{3}-\pi-\left(-\operatorname{sen}\left(\frac{\pi}{2}\right)+\frac{\operatorname{sen}^3\left(\frac{\pi}{2}\right)}{3}-\frac{\pi}{2}\right)\right]\right)$$

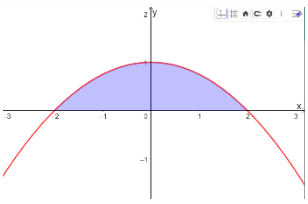
$$=-\frac{8}{3}\left(\left[1-\frac{1}{3}-\frac{\pi}{2}-0\right]+\left[0-\pi+1-\frac{1}{3}+\frac{\pi}{2}\right]\right)=-\frac{8}{3}\left(\frac{2}{3}-\frac{\pi}{2}+\frac{2}{3}-\frac{\pi}{2}\right)=-\frac{8}{3}\left(\frac{4}{3}-\pi\right)=\boxed{\mathbf{\frac{8}{3}\left(\pi-\frac{4}{3}\right)}}$$

Se usó:

$$(*)\ \int\cos^3(\theta)\,d\theta=\int\cos^2(\theta)\cos(\theta)\,d\theta=\int(1-\underbrace{\operatorname{sen}^2(\theta)}_{u})\underbrace{\cos(\theta)\,d\theta}_{du}=\int(1-u^2)du=u-\frac{u^3}{3}=\operatorname{sen}(\theta)-\frac{\operatorname{sen}^3(\theta)}{3}$$

Pregunta 5

Encontrar el centro de masa de la lámina homogénea cuya forma es la región acotada por el eje X y la parábola $x^2=4-4y$, considerando la densidad de masa como $\rho\ slugs/p^2$.



La parábola intercepta al eje X en $x=\pm 2$. Así, $R=\left\{(x,y)\mid x\in[-2,2];0\leq y\leq\frac{4-x^2}{4}\right\}$

$$m=\iint_R\rho\,dA=\rho\int_{-2}^{2}\left(\int_0^{\frac{4-x^2}{4}}dy\right)dx=\rho\int_{-2}^{2}\frac{4-x^2}{4}dx=\frac{\rho}{4}\left(4x-\frac{x^3}{3}\right)\Big|_{-2}^{2}=\frac{\rho}{4}\left(\frac{32}{3}\right)=\boxed{\mathbf{\frac{8}{3}\rho}}$$

$$m_x=\iint_R y\rho\,dA=\rho\int_{-2}^{2}\left(\int_0^{\frac{4-x^2}{4}}y\,dy\right)dx=\rho\int_{-2}^{2}\frac{y^2}{2}\Big|_0^{\frac{4-x^2}{4}}dx=\frac{\rho}{2}\int_{-2}^{2}\left(\frac{4-x^2}{4}\right)^2dx=\frac{\rho}{32}\int_{-2}^{2}(16-8x^2+x^4)dx$$

$$=\frac{\rho}{32}\left(16x-\frac{8}{3}x^3+\frac{x^5}{5}\right)\Big|_{-2}^{2}=\frac{\rho}{32}\left(32-\frac{64}{3}+\frac{32}{5}-\left(-32+\frac{64}{3}-\frac{32}{5}\right)\right)=\frac{\rho}{32}(32)\left(1-\frac{2}{3}+\frac{1}{5}\right)(2)=2\rho\left(\frac{8}{15}\right)=\boxed{\mathbf{\frac{16}{15}\rho}}$$

$$m_y=\iint_R x\rho\,dA=\rho\int_{-2}^{2}\left(x\int_0^{\frac{4-x^2}{4}}dy\right)dx=\rho\int_{-2}^{2}x\left(\frac{4-x^2}{4}\right)dx=\frac{\rho}{4}\int_{-2}^{2}(4x-x^3)dx=\frac{\rho}{4}\left(2x^2-\frac{x^4}{4}\right)\Big|_{-2}^{2}=\boxed{\mathbf{0}}$$

$$\therefore\ (\overline{x},\overline{y})=\left(\frac{0}{\frac{8}{3}\rho},\frac{\frac{16}{15}\rho}{\frac{8}{3}\rho}\right)=\boxed{\mathbf{\left(0,\frac{2}{5}\right)}}$$