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HW2

3.2-1.) For max{f(n), g(n)} = Θ(f(n) + g(n)): c_1(f(n)+g(n)) $\leq$ max{f(n), g(n)} $\leq$ c_2(f(n)+g(n)). Max{f(n), g(n)} = f(n) or g(n) $\leq$ c_2(f(n)+g(n)) = O(f(n)+g(n)), however f(n) or g(n) $\neq$ Ω(f(n)+g(n)). Therefore max{f(n), g(n)} $\neq$ Θ(f(n) + g(n))

3.2-2.) The O-notation is only an upper bound not a tight bound essentially meaning at most. The statement is meaningless because it contradicts itself with a runtime "at least at most n^2".

3.2-3.) If 2^{n+1} = O(2^n), there must be c, n_0 > 0 such that 0 $\leq$ 2^{n+1} $\leq$ c2^n. Since 2^{n+1} can be expressed as 2*2^n, therefore 2^{n+1} = O(2^n) when c = 2 for all n $\geq$ 1.
2^{2n} = O(2^n) if c, n_0 > 0 exists and 0 $\leq$ 2^{2n} $\leq$ c2^n. Since 2^{2n} = 2^n*2^n, there is no such c that satisfies the definition, and 2^{2n} $\neq$ O(2^n).

3.2-4.) The definition of Θ-notation states for f(n) = Θ(g(n)), c_1g(n) $\leq$ f(n) $\leq$ c_2g(n). When f(n) $\geq$ c_1g(n), f(n) = Ω(g(n)), and when f(n) $\leq$ c_2g(n), f(n) = O(g(n)). Therefore when c_1g(n) $\leq$ f(n) $\leq$ c_2g(n), f(n) = Ω(g(n)) = O(g(n)) thus f(n) = Θ(g(n)).

3.3-1.) Since f(n) and g(n) are monotonically increasing, m $\leq$ n, f(m) $\leq$ f(n), and g(m) $\leq$ g(n). f(m) + g(m) $\leq$ f(n) + g(n), f(g(m)) $\leq$ f(g(n)), and if also nonnegative, f(m)*g(m) $\leq$ f(n)*g(n).

3.3-2.) When $\alpha \neq 1$, the term $\lfloor \alpha n \rfloor$ = $\lfloor \alpha \rfloor$*n = 0*n reduces to 0, and $\lceil (1-\alpha)n \rceil$ = $\lceil (1-\alpha) \rceil$*n = 1*n. When α = 1, the term $\lfloor \alpha n \rfloor$ = $\lfloor 1 \rfloor$*n = 1*n, in which the term $\lceil (1-\alpha)n \rceil$ = $\lceil (1-1) \rceil$*n = 0 reduces. Therefore $\lfloor \alpha n \rfloor + \lceil (1-\alpha)n \rceil$ = n for all real numbers α in the range $0 \leq \alpha \leq 1$.

3.3-4b.) Applying lg to Stirling's approximation lg(n!) = lg($\sqrt{2\pi n}(\frac{n}{e})^n$(1+$\Theta(\frac{1}{n})$)) $\Rightarrow$ lg($\sqrt{2\pi n}$) + nlg($\frac{n}{e}$) + lg(1+$\Theta(\frac{1}{n})$) $\Rightarrow$ lg($\sqrt{n}$) + nlgn + lg($\Theta(\frac{1}{n})$) $\Rightarrow$ lg(n!) = Θ(nlgn).

3-1a.) If p(n) = O(n^k), there must be c, n_0 > 0 such that 0 $\leq$ p(n) $\leq$ cn^k. Degree d polynomial p(n) can be expressed as n^d, and k $\geq$ d. Since 0 $\leq$ n^d $\leq$ cn^k satisfies the definition of O-notation, the expression p(n) = O(n^k) is true.

3-1b.) If p(n) = Ω(n^k), there must be c, n_0 > 0 such that 0 $\leq$ cn^k $\leq$ p(n). Degree d polynomial p(n) can be expressed as n^d, and k $\leq$ d. Since 0 $\leq$ cn^k $\leq$ n^d satisfies the definition of Ω-notation, the expression p(n) = Ω(n^k) is true.

3-1c.) If $p(n) = \Theta(n^k)$, there must be $c_1, c_2, n_0 > 0$ such that $0 \leq c_1n^k \leq p(n) \leq c_2n^k$. Degree d polynomial p(n) can be expressed as n^d, and k = d. Since $0 \leq c_1n^k \leq n^d \leq c_2n^k$ satisfies the definition of the Θ-notation, the expression $p(n) = \Theta(n^k)$ is true.

3-2.)

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	Yes	Yes	No	No	No
b.	n^k	c^n	Yes	Yes	No	No	No
c.	$\sqrt{n}$	$n^{\sin(n)}$	No	No	No	No	No
d.	2^n	$2^{\frac{n}{2}}$	No	No	Yes	Yes	No
e.	$n^{\lg(c)}$	$c^{\lg(n)}$	No	No	Yes	Yes	No
f.	lg(n!)	$\lg(n^n)$	Yes	Yes	No	No	No

3-4a.) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$, not $O(f(n))$, due to transpose symmetry.

3-4b.) $f(n) + g(n) = \Omega(\min\{f(n), g(n)\})$, not Θ, because there exists c, n_0 such that $0 \leq c\min\{f(n), g(n)\} \leq f(n) + g(n)$.

3-4e.) $f(n) = O((f(n))^2)$ because there exists c, n_0 such that $0 \leq f(n) \leq cf(n)^2$.

3-4f.) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$ due to transpose symmetry.

3-4g.) $f(n) = \Theta(f(n/2))$ because there is no such c_1, c_2, n_0 that satisfies the expression $0 \leq c_1f(n/2) \leq f(n) \leq c_2f(n/2)$.

4.2-1.) 1: Matrix A B and C are partitioned into 1x1 matrices. 2: S_1=8-2=6, S_2=1+3=4, S_3=7+5=12, S_4=4-6=-2, S_5=1+5=6, S_6=6+2=8, S_7=3-5=-2, S_8=4+2=6, S_9=1-7=-6, S_{10}=6+8=14.
3: P_1=1*6=6, P_2=4*2=8, P_3=12*6=72, P_4=5*-2=-10, P_5=6*8=48, P_6=-2*6=-12, P_7=-84.
4: C_{11} = 48-10-8-12 = 18. C_{12} = 6+8 = 14. C_{21} = 72-10 = 62, C_{22} = 48+6-72+84 = 66.

4.2-5.) The multiplication of complex numbers (a + bi)*(c + di) can be done with an algorithm taking a b c and d as inputs, multiplying real numbers ac, ad, bc and imaginary numbers bd. The real component ac-bd and the imaginary component ad+bc would each be returned.

4.3-1a.)

4.3-1b.)

4.3-1c.)

4.5-1a.) When T(n) = 2T($\frac{n}{4}$)+1, a=2, b=4, and f(n)=1. This implies $n^{\log 4(2)}$ = Θ($\sqrt{n}$), f(n) = 1 = O($n^{(½)-\epsilon}$) with $\epsilon = \frac{1}{2}$, the first case of the master theorem is applied, and T(n) = Θ($\sqrt{n}$).

4.5-1b.) When T(n) = 2T($\frac{n}{4}$)+$\sqrt{n}$, a=2, b=4, and f(n)=$\sqrt{n}$. This implies $n^{\log 4(2)}$ = Θ($\sqrt{n}$), f(n) = $\sqrt{n}$ = Θ($\sqrt{n}$*$\lg^0$n), the second case of the master theorem is applied, and T(n) = Θ($\sqrt{n}$*lgn).

4.5-1d.) When T(n) = 2T($\frac{n}{4}$)+n, a=2, b=4, and f(n)=n. This implies $n^{\log 4(2)}$ = Θ($\sqrt{n}$), f(n) = n = Ω($n^{(½)+\epsilon}$) with $\epsilon = \frac{1}{2}$, the third case of the master theorem is applied, thus the regularity condition af($\frac{n}{b}$) ≤ cf(n) must be satisfied. The term af($\frac{n}{b}$) = 2*$\frac{n}{4} = \frac{2n}{4}$ and when c = $\frac{2}{4}$, cf(n) = $\frac{2n}{4}$, and T(n) = Θ(n).

4.5-1e.) When T(n) = 2T($\frac{n}{4}$)+n^2, a=2, b=4, and f(n)=n^2. This implies $n^{\log 4(2)}$ = Θ($\sqrt{n}$), f(n) = n^2 = Ω($n^{(½)+\epsilon}$) with $\epsilon = \frac{3}{2}$, the third case of the master theorem is applied, thus the regularity condition af($\frac{n}{b}$) ≤ cf(n) must be satisfied. The term af($\frac{n}{b}$) = 2*$(\frac{n}{4})^2 = \frac{2n}{8}$ and when c = $\frac{2}{8}$, cf(n) = $\frac{2n}{8}$, and T(n) = Θ(n^2).

4.5-3.) When T(n) = T($\frac{n}{2}$)+Θ(1), a=1, b=2, and f(n)=Θ(1). This implies $n^{\lg(1)}$ = Θ(n^0) = Θ(1), f(n) = Θ(1) = Θ(1*$\lg^0$n), the second case of the master theorem is applied, and T(n) = Θ(1*$\lg^1$n) = Θ(lgn).

Proof1.) If a non-empty finite set S of positive integers has only 1 element n, it is the least element and the base case holds. If a non-empty finite set S of positive integers has at least n+1 elements, for some x ∈ S, x is either the least element of S, or there exists some y ∈ S, where y < x, in such case y is the least element of S. The base and inductive cases show every non-empty finite set contains a least element.

Proof2.) If a non-empty set S of positive integers does not contain a least element, then for any positive integer n ∈ S there must be an x ∈ S such that x < n, creating a sequence of positive integers in S continuously smaller than the previous. This will continue indefinitely, however an infinite sequence of positive integers must reach 1, with no other positive integer smaller than it. Therefore any non-empty set of S of positive integers must contain a least element, even if infinite.