Stability analysis for time-dependent nonlinear systems. An interval approach

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Abstract: One of the most important issues in control is determining the stability of a system. Since the 1960's, Lyapunov-based methods have been developed to determine the stability of linear and nonlinear systems. However, when the system is nonlinear, time-dependent and uncertain, in a set-membership context, stability analysis is challenging and no reliable methods have been developed. This paper proposes an original set-membership based approach for establishing the stability of non-linear, uncertain, time-dependent systems. Two new concepts *G-Stability* (which is the stability of nonlinear uncertain trajectories) and *capture tubes* (which is an invariant stability region for time-dependent systems) are introduced and illustrated for an autonomous, uncertain, robotic sailboat. Then, G-Stability is used to formulate and prove the *safety* of a squad of uncertain, autonomous robots (no collision among the robots). These results are then used to analyse the safety of a squad of uncertain, robotic sailboat moving in their environment.
Keywords: G-Stability, Capture Tube, Safety, V-Stability, Lyapunov Stability, Interval Analysis, Contractors.

I. INTRODUCTION

The concept of stability was initially formulated in late 19th century by the Russian mathematician Aleksandr Mikhailovich Lyapunov [1]. This came to the attention of the control engineering community in the 1960s and since then, rigorous stability methods have been developed for both linear [29] and nonlinear systems [30]. In particular, in the last 20 years, researchers have been developing methods for determining the stability of time-invariant uncertain systems, using set-membership techniques [2], [3].

A time-dependent, nonlinear, dynamic system can be described by a state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ or, if the system is uncertain, by a time-dependent differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$, [2], where $\mathbf{x}$ is the state vector and t represents time. Analyzing the stability properties of this system (or of the differential inclusion) is an important yet difficult problem. For some particular properties and for time invariant systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ or time invariant differential inclusions $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ (i.e. $\mathbf{f}$ or $\mathbf{F}$ do not depend on t), it has been shown that the V-Stability approach [2] (which is derived from Lyapunov theory for stability analysis of nonlinear systems) combined with interval analysis [6] can solve the stability problem. The main idea is to prove that the solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ (or the solutions of $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$) will always stay inside a bubble (i.e. inside an invariant set). There are two goals of this paper. Firstly, to show that proving V-Stability can be transformed into proving the inconsistency for a set of inequalities and secondly to introduce two new concepts, *capture tubes* and *G-Stability*, to deal with stability problem for time varying systems or time varying differential inclusions. To the best of our knowledge, when a system is nonlinear, time-dependent and uncertain, in a set-membership context, no reliable methods are available for determining its stability. To prove that $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$ is G-Stable amounts to proving that all solutions of $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$ will always stay inside a time varying bubble (i.e. a *capture tube*). It will be shown that if capture tube candidate can be calculated for a time-dependent system (or for a differential inclusion) then the system (or the differential inclusion) is G-Stable. Moreover, using these concepts, a new method is proposed to prove the *safety* for a squad of uncertain robots (no collisions between the robots).

The paper is organized as follows. Section II shows how the V-Stability problem can be transformed into proving the inconsistency of a set of inequalities. It will also be shown that interval analysis and contractors [10] can efficiently solve this problem in polynomial-time. In sections III and IV, two new concepts to analyse the stability problem, capture tubes and G-Stability, will be introduced. Based on the same idea proposed for V-Stability in section II, it will be shown that proving that a tube is a capture tube candidate can be transformed into proving the inconsistency of a set of inequalities. These new concepts will then be used in section V to establish the safety of a squad of autonomous, uncertain robots. In Section VI, two case-studies are described which illustrate the application of both G-Stability and safety. Section VII concludes the paper.

II. V-Stability for time-invariant dynamic systems

The idea of V-Stability is derived from Lyapunov stability theory [1] [19] and influenced by the book of Aubin and Frankowska [12]. Before introducing the V-Stability concept, recall the Lyapunov stability theorem for time-invariant systems.

Lyapunov Stability Theorem (1892): Let $\mathbf{x} = 0$ be an equilibrium point of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, and let $V: D \to \mathbb{R}$ be a continuously differentiable function defined in a domain $D \subset \mathbb{R}^n$, such that:
(i) $V(0) = 0$
(ii) $V(\mathbf{x}) > 0 \text{ in } D - \{0\}$ (1)
(iii) $\dot{V}(\mathbf{x}) \leq 0 \text{ in } D - \{0\}$
then $\mathbf{x} = 0$ is stable.

Lyapunov stability theory requires that $V(\mathbf{x})$ is always positive, except at the equilibrium point. V-Stability [2] [12] extends Lyapunov stability to the case where $V(\mathbf{x})$ is not always positive, i.e. it can also be negative for some $\mathbf{x}$.

Definition 1 (*V*-Stability) [2]: Consider a differentiable function $V: \mathbb{R}^n \to \mathbb{R}$. This system is said to be V-Stable if there exist $\varepsilon > 0$ such that:

If $V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) \leq -\varepsilon < 0$ (2)

$V(\mathbf{x})$ appears similar to a Lyapunov function candidate, but, in reality, it is not exactly the same. The concept of V-Stability is illustrated in Figure 1 where the black arrows represent the vector field of the dynamical system. When $V(\mathbf{x}) \geq 0$ then all system trajectories must be attracted by the grey region (bubble) limited by the level curve $V(\mathbf{x}) = 0$, which is similar to Lyapunov-stability, because $\dot{V}(\mathbf{x}) \leq -\varepsilon < 0$. The difference arises when the state lies in the grey bubble where $V(\mathbf{x})$ becomes negative. Then, the system trajectory can follow any shape, for instance a limit cycle, as illustrated in the figure. The V-Stability definition states that when the scalar differentiable function $V(\mathbf{x})$ is positive then it will strictly decrease along the level curves. In this case the trajectory will be attracted by the grey region which is an invariant set. Once inside the attraction zone, the trajectory will stay inside forever and can follow any behavior. Consequently, the notion of V-Stability is weaker than the stability in the sense of Lyapunov [30].

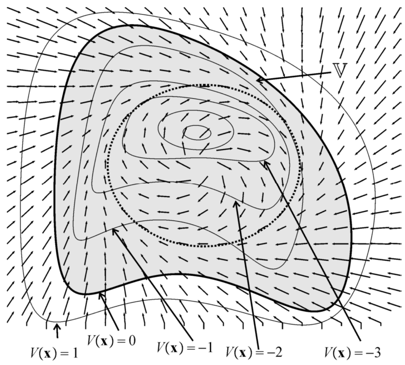

Figure 1. *V*-Stability for $V(\mathbf{x}) \in [0, \infty]$.

Consequence of the Lyapunov theorem: The stability problem of a dynamic system can be represented as a set of inequalities. This is very important because the set of inequalities can be easily solved using numerical methods. This paper proposes a similar idea, by showing that V-Stability can be determined from the inconsistency of a set of inequalities, as is proposed in theorem 1.

Theorem 1: Consider a nonlinear time independent system $\dot{\mathbf{x}} = f(\mathbf{x})$. If the set of constraints:

$$\begin{cases} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x}) \geq 0 \\ V(\mathbf{x}) \geq 0 \end{cases} \text{ is inconsistent } \Leftrightarrow \dot{\mathbf{x}} = f(\mathbf{x}) \text{ is } V\text{-Stable.} \quad (3)$$

Proof:
Assume that $\dot{\mathbf{x}} = f(\mathbf{x})$ is V-Stable. From Definition 1, $V(\mathbf{x}) \geq 0 \;\Rightarrow\; \dot{V}(\mathbf{x}) < 0$, where V is a scalar differentiable function. As the state vector $\mathbf{x}$ depends on time, this implies that the function V will also depend on time. Differentiating the function V with respect to time gives $\frac{\partial V(\mathbf{x})}{\partial t} = \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\dot{\mathbf{x}} = \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x})$.
From definition 1, $V(\mathbf{x}) \geq 0 \;\Rightarrow\; \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x}) < 0$, which is true for all x. Using the logical relationship $(A \Rightarrow B) \Leftrightarrow (B \text{ or } \neg A)$ where A is $V(\mathbf{x}) \geq 0$ and B is $\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x}) < 0$, the implication is equivalent with:

$V(\mathbf{x}) \geq 0 \;\; \Rightarrow \;\; \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x}) < 0 \Leftrightarrow \; \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x}) < 0 \; or \; V(\mathbf{x}) < 0$ (4)

Using De Morgan's law $\overline{A \vee B} = \bar{A} \wedge \bar{B}$, where $\vee$ and $\wedge$ represent the logical *OR* and *AND* operators, respectively, the V-Stability problem can be represented as:
$\neg\left(\exists \mathbf{x}, \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).f(\mathbf{x}) \geq 0 \; and \; V(\mathbf{x}) \geq 0\right)$ (5)
which proves the theorem.

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Example 1: Proving V-Stability for a simple time-invariant system using Theorem 1:
A simple example of the determination of V-Stability for the time-invariant linear system $\dot{x} = -x + 1$ is now described. Consider $V(x) = x^2 - r^2$ as a V function candidate. It can be easily observed that $V(x)$ is not always positive and, using Theorem 1, it will be shown that the system is V-Stable for $r = 1$. V-Stability is equivalent to finding r such that $V(x)$ always converges to a negative value, i.e. $V(x) = x^2 - r^2 \leq 0$. The rate of change of the scalar field V along the flow of the vector field $\dot{x}$ is $\frac{\partial V}{\partial x}(x) = 2x\dot{x} = 2x(-x+1) = -2x(x-1)$. The V-Stability problem becomes $\begin{cases} 2x(-x+1) < 0 \\ x^2 - r^2 \geq 0 \end{cases}$ $\overset{Theorem\ 1}{\Longleftrightarrow} \begin{cases} 2x(-x+1) \geq 0 \\ x^2 - r^2 \geq 0 \end{cases}$ which is inconsistent for $r \geq 1$ and therefore the system is V-Stable for $r \geq 1$. The disc $x^2 - r^2$ becomes negative for $r \geq 1$ which is an attractive disc for the system. Inside this disc it is easy to see that $\dot{V}$ is positive for $x \in [0,1]$ and negative for $x \in [-1,0]$, and as it was previously stated, the system behavior is unimportant inside the disc.

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III. V-Stability for time-invariant differential inclusions

As uncertainties are present in real systems, it is more realistic to represent the dynamical system as a differential inclusion [2]. This notion makes it possible to develop numerical algorithms to rigorously study stability. In this section, the concept of a differential inclusion is introduced and then it is shown that Theorem 1 can be extended to include uncertain systems. Furthermore, it is also shown, how V-Stability can be efficiently determined using interval methods and contractors.

Differential inclusions are a generalization of the concept of the state equation and are used to represent uncertain dynamic systems in a set-membership framework. A *differential inclusion* can be defined by the following inclusion:
$\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ (6)
where $\mathbf{F}$ is a multivalued function from $\mathbb{R}^n$ to $\mathbb{R}^n$. A multivalued function $\mathbf{F}$ from $\mathbb{R}^n \to \mathbb{R}^n$ associates each $\mathbf{x} \in \mathbb{R}^n$ with a subset $\mathbf{F}(\mathbf{x})$ of $\mathbb{R}^n$ ([12] and [13]).

Definition 2 (V-Stability for differential inclusion): Consider a differentiable function $V(\mathbf{x}): \mathbb{R}^n \to \mathbb{R}$. The differential inclusion (6) is V-Stable
if for $V(\mathbf{x}) \geq 0 \;\; \Rightarrow \;\; \dot{V}(\mathbf{x}) < 0$ (7)

The following theorem extends Theorem 1 for V-Stability when the system is uncertain, in a set-membership context, i.e. is represented by a time invariant differential inclusion.

Theorem 2 (*V*-Stability for time invariant differential inclusion): Consider an uncertain system represented by a time invariant differential inclusion (6). If the system of constraints
$\begin{cases} \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a} \geq 0 \\ \mathbf{a} \in \mathbf{F}(\mathbf{x}) \\ V(\mathbf{x}) \geq 0 \end{cases}$ is inconsistent $\Leftrightarrow \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x})$ is V-Stable. (8)
Observation: In practice there exists an analytical expression for $\mathbf{F}(\mathbf{x})$ and $\mathbf{a} \in \mathbf{F}(\mathbf{x})$ corresponds to an inequality of the form:
$\boldsymbol{\varphi}(\mathbf{a}, \mathbf{x}) \leq 0$ (9)
where $\dot{\mathbf{x}} = \mathbf{a}$. In this way, the set-membership relations can always be transformed into inequalities.
Proof of Theorem 2:
Assume that (6) is V-Stable. From Definition 2, $V(\mathbf{x}) \geq 0 \;\; \Rightarrow \;\; \dot{V}(\mathbf{x}) < 0$, where V is a scalar function. Differentiating V with respect to time:
$\frac{\partial V(\mathbf{x})}{\partial t} = \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a}$ where $\mathbf{a} \in \mathbf{F}(\mathbf{x})$. V-Stability becomes: $V(\mathbf{x}) \geq 0 \;\; \Rightarrow \;\; \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a} \; < 0$ for $\forall \mathbf{a} \in \mathbf{F}(\mathbf{x})$.

Using the logical rule $(A \Rightarrow B) \Leftrightarrow (B \text{ or } \neg A)$ the following relationship exists:

$$\forall x \in [\mathbf{x}], \forall \mathbf{a} \in \mathbf{F}(\mathbf{x}), \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a} < 0 \text{ or } V(\mathbf{x}) < 0 \tag{10}$$

and applying De Morgan's laws:

$$\neg\left(\exists x \in [\mathbf{x}], \exists \mathbf{a} \in \mathbf{F}(\mathbf{x}), \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}).\mathbf{a} \geq 0 \text{ and } V(\mathbf{x}) \geq 0\right) \tag{11}$$

which proves the theorem.

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Consequence: Theorems 1 and 2 show how the V-Stability problem can be transformed into a set of inequalities. Using interval analysis and *contractors* it is possible to prove that a time invariant differential inclusion is V-Stable by efficiently establishing the inconsistency of a set of inequalities.

This is illustrated in Example 2 where it is shown that a set of inequalities are inconsistent and the computational complexity is polynomial-time with respect to the number of variables [10].

Example 2: Consider the following set of inequalities:

$$\begin{cases} f(x) = x^2 - 1 \geq 0 \\ f(x) = -\frac{1}{x} \geq 0 \\ f(x) = 2x + 1 \geq 0 \end{cases} \tag{12}$$

where x is defined on the domain $[-\infty, \infty]$. Using interval propagation [10] it is required to show that the system of inequalities (12) is inconsistent, i.e. has no solution. A contractor based method is a very powerful tool to prove the inconsistency for a set of inequalities. A *contractor* $\mathcal{C}$ is an operator that uses a constraint satisfaction problem [10] to compute the subset $[\mathbf{x}']$ such that the solution set $\mathbb{S}$ remains unchanged, i.e., $\mathbb{S} \subseteq [\mathbf{x}'] \subseteq [\mathbf{x}]$. The formal definition of a contractor is a mapping from $\mathbb{IR}^n$ to $\mathbb{IR}^n$ that satisfies the contractance and correctness properties as shown in Equations (13) and (14), respectively.

$$\forall [\mathbf{x}] \in \mathbb{IR}^n,\ \mathcal{C}([\mathbf{x}]) \subset [\mathbf{x}] \qquad \text{contractance} \tag{13}$$

$$\mathcal{C}([\mathbf{x}]) \cap \mathbb{S} = [\mathbf{x}] \cap \mathbb{S} \qquad \text{correctness} \tag{14}$$

The inequality set (12) can be transformed into a constraint satisfaction problem by introducing a *slack variable* y. Using this slack variable (12) is transformed into the following set of constraints $\begin{cases} x^2 - 1 - y = 0 \\ -\frac{1}{x} - y = 0 \\ 2x + 1 - y = 0 \end{cases}$, where the domain for the slack variable y is $[0, \infty]$.

Using interval propagation [10] it is necessary to prove that the set of equations

$$\begin{cases} (C_1):\ y = x^2 - 1 \\ (C_2):\quad y = -\frac{1}{x} \\ (C_3):\ y = 2x + 1 \end{cases} \text{ has no solution.} \tag{15}$$

When several constraints are involved, the contractions are performed sequentially until no further contraction can be made [10]. The domains are contracted in the following order: C_1, C_2, C_3, C_1, C_2, until empty intervals for x and y are obtained. The resulting interval computation is as follows:

$$(C_1):\ \Rightarrow\ y \in [0, \infty] \cap [-\infty, \infty]^2 - 1 = [0, \infty] \cap [-1, \infty] = [0, \infty]$$

$$(C_2):\ \Rightarrow\ x \in \frac{-1}{[0, \infty]} = [-\infty, 0]$$

$$(C_3):\ \Rightarrow\ y \in [0.\infty] \cap (2 \cdot [-\infty, 0] + 1) = [0,1] \ \text{ and } \ x \in [-\infty, 0] \cap \left(\frac{[0,1]}{2}, -\frac{1}{2}\right) = \left[-\frac{1}{2}, 0\right]$$

$$(C_1):\ \Rightarrow\ y \in [0,1] \cap \left[-\frac{1}{2}, 0\right]^2 = \left[0, \frac{1}{4}\right]$$

$$(C_2):\ \Rightarrow\ x \in \left[0, \frac{1}{2}\right] \cap \frac{1}{\left[\frac{0,1}{4}\right]} = \emptyset$$

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Remark: The interval propagation method converges to a box that encloses all solutions (if any exist), but the box is not necessarily minimal. The box is said to be locally consistent because it is consistent with all constraints taken independently. The smallest box that encloses all solutions is said to be globally consistent. The problem of computing this smallest box is NP-hard, and can be solved using bisection methods only for problems involving a small number of variables. This is not necessary for the methods proposed in this paper where the aim is to establish inconsistency.

IV. G-Stability for time-dependent systems

In this section, the concept of V-Stability is extended to systems where $\mathbf{f}$ and $\mathbf{F}$ both depend on time. In section **A** the notion of capture tubes will be introduced and used in section **B** to define G-Stability for time dependent systems and differential inclusions.

A. Capture tubes

Consider a non-autonomous, nonlinear system described by the state equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ with a target condition. This can be described by:

$$\begin{cases} \mathcal{S}_{\mathbf{f}} : \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) & \text{(evolution equation)} \quad (16) \\ \mathbf{g}(\mathbf{x}, t) \leq 0 & \text{(target condition)} \quad (17) \end{cases}$$

where $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$ is the evolution function and $\mathbf{g}: \mathbb{R}^n \mathrm{x} \mathbb{R} \to \mathbb{R}^m$ is the target function. At every time instant, the target condition from equation (16) is a bubble similar to the limit cycle represented by a circle in figure 1. Since the target condition is time dependent, the bubble moves in time and will generate a *target tube*, represented by inequalities, $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ which associates with each value of $t \in \mathbb{R}$ a subset of $\mathbb{R}^n$. A tube (or interval of trajectories) [24] [25] is a set-membership vision of a random signal. In figures 2 and 3 it is shown two possible representations for a tube. Figure 2 shows a bubble that moves with respect to time is represented at 6 different time instants in state space. Another possible representation for a tube is in the time domain as the grey tube in figure 3. $\mathbb{G}(t_1)$ from figure 3 is the bubble at time instant t_1, as illustrated in figure 2.

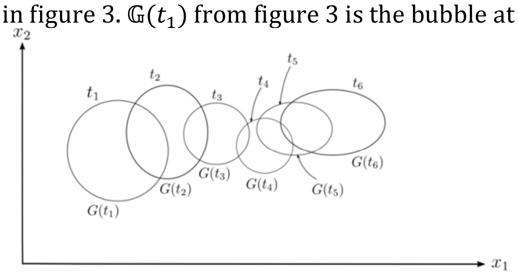

Figure 2: Representing a tube in state space as a time moving bubble.

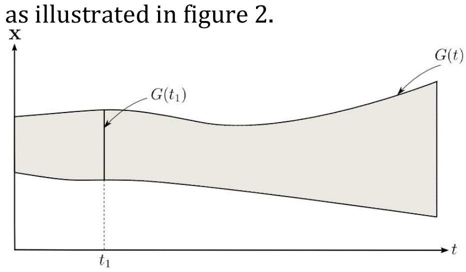

Figure 3: A tube in the time domain.

Definition 3: A trajectory is *stable* if it satisfies conditions (16) and (17). The system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is said to be *stable* if for all $\mathbf{x}(0) \in \mathbb{G}(0)$, the corresponding trajectory is stable.

Definition 4 (Capture tube): A target tube is said to be a *capture tube* if the fact that $\mathbf{x}(t) \in \mathbb{G}(t)$ implies that $\mathbf{x}(t + t_1) \in \mathbb{G}(t + t_1)$ for all $t_1 > 0$.

Figure 4 illustrates some feasible trajectories and a tube $\mathbb{G}(t)$ (in grey). All the trajectories are consistent with the assumption that $\mathbb{G}(t)$ is a capture tube, except for the trajectory which is represented by a dotted curve at the bottom which was able to escape from the tube.

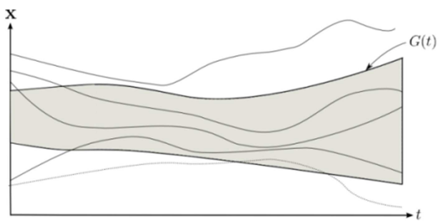

Figure 4: A target tube (coloured grey) and possible trajectories for different initial conditions.

Taking into account that a tube can be represented by inequalities, the following theorem shows that the problem of proving that $\mathbb{G}(t)$ is a capture tube can be reformulated as proving that a set of inequalities has no solution.

Theorem 3. (**Capture tubes).** Consider a tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ where $\mathbf{g}: \mathbb{R}^n \mathrm{x} \mathbb{R} \to \mathbb{R}^m$. If the *cross out* condition

$$\begin{cases} \text{(i)} \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t). \mathbf{f}(\mathbf{x}) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \geq 0 \\ \text{(ii)} \quad g_i(\mathbf{x}, t) = 0 \\ \text{(iii)} \quad \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases} \tag{18}$$

is inconsistent for all $\mathbf{x}$, all $t > 0$, and all $i \in \{1, \dots, m\}$ then $\mathbb{G}(t)$ is a capture tube for the time dependent system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$.

Proof. The proof is by contradiction. Assume that $\mathbb{G}(t)$ is not a capture tube. It means that there exists one trajectory which leaves the tube $\mathbb{G}(t)$, i.e., which crosses the ith boundary $g_i(\mathbf{x}, t) = 0$ from inside to outside. This means that there exists a time-space pair $(\mathbf{s}, t_s)$ on the boundary of $\mathbb{G}(t)$ (i.e., such that (ii) and (iii) are satisfied) and such that $\dot{g}_i(\mathbf{x}, t) \geq 0$ (otherwise the trajectory cannot leave the tube). Thus, if $\mathbb{G}(t)$ is not a capture tube, there exists one i such that

$g_i(\mathbf{x}, t_1) = 0$ and $\mathbf{g}(\mathbf{x}, t_1) \leq 0$, and $\dot{g}_i(\mathbf{x}, t_1) \geq 0$, where $\dot{g}_i(\mathbf{x}, t) = \frac{\partial g}{\partial \mathbf{x}}(\mathbf{x}, t). \mathbf{f}(\mathbf{x}, t_1) + \frac{\partial g}{\partial t}(\mathbf{x}, t)$. Equivalently, the system of constraints $\begin{cases} \frac{\partial g_i}{\partial x}(\mathbf{x}, t_1). \mathbf{f}(\mathbf{x}, t_1) + \frac{\partial g_i}{\partial t}(\mathbf{x}, t_1) \geq 0 \\ g_i(\mathbf{x}, t_1) = 0 \\ \mathbf{g}(\mathbf{x}, t_1) \leq 0 \end{cases}$ is consistent, which is in contradiction with the original assumption.

■

An illustrative example is now given to show how theorem 3 can be used to prove that a tube is a capture tube for a simple time dependent function.

Example 3: Proving a capture tube using Theorem 3

Problem formulation: Consider a time dependent, linear, first order system

$$\dot{x} = -x + t^2 \tag{19}$$

It is necessary to prove, using theorem 3, that the tube $g(x, t) = (x - t^2)^2 - 1 \leq 0$ is a capture tube candidate for the system. This will be done by proving the inconsistency of the following set of inequalities:

$$\begin{cases} (i) \ \frac{\partial g}{\partial x}(x, t). f(x, t) + \frac{\partial g}{\partial t}(x, t) \geq 0 \ \Leftrightarrow \ 2(x - t^2)(-x + t^2) + 2(x - t^2)(-2t) \geq 0 \\ (ii) \quad g(x, t) = 0 \ \Leftrightarrow \ (x - t^2)^2 - 1 = 0 \\ (iii) \quad g(x, t) \leq 0 \ \Leftrightarrow \ (x - t^2)^2 - 1 \leq 0 \end{cases} \tag{20}$$

The set of inequalities (20) becomes $\begin{cases} (i) \ (x - t^2)^2 + 2(x - t^2)t \leq 0 \\ (ii) \quad x - t^2 = 1 \end{cases} \Leftrightarrow \begin{cases} (i) \ 1 + 2t \leq 0 \\ (ii) \ x - t^2 = 1 \end{cases}$ which is inconsistent therefore $g(x, t)$ is a capture tube for the time dependent system (19). This means that if the system is initialized in the capture tube at $t = 0$, it will remain inside the capture tube, i.e. the cross out condition is not satisfied. For a complex, nonlinear, differential inclusion it is not easy to prove stability by solving by hand the set of inequalities. In this case, a contractor based method will be used, as was shown in example 2.

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If the system is both time dependent and uncertain, in a set-membership context, theorem 3 can be extended to include a time-dependent differential inclusion

$$\mathcal{S}_{\mathbf{F}} : \dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t) \tag{21}$$

as in the following theorem.

Theorem 4 (**Capture tubes for time dependent differential inclusion):** Consider a tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ where $\mathbf{g}: \mathbb{R}^n \mathrm{x} \mathbb{R} \to \mathbb{R}^m$. If the *cross out* condition

$$\begin{cases} \text{(i)} \ \frac{\partial g_i}{\partial \mathbf{x}}(\mathbf{x}, t). \mathbf{a} + \frac{\partial g_i}{\partial t}(\mathbf{x}, t) \geq 0 \\ (ii) \quad \mathbf{a} \in \mathbf{F}(x) \\ (iii) \quad g_i(\mathbf{x}, t) = 0 \\ (iv) \quad \mathbf{g}(\mathbf{x}, t) \leq 0 \end{cases} \tag{22}$$

is inconsistent for all $\mathbf{x}$, all $\mathbf{a}$, all $t > 0$, and all $i \in \{1, \dots, m\}$ then $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ is a capture tube for the time dependent differential inclusion $\dot{\mathbf{x}} \in \mathbf{F}(\mathbf{x}, t)$.

Proof.

The proof is by contradiction. Assume that $\mathbb{G}(t)$ is not a capture tube. It means that there exists one trajectory $\mathbf{a} \in \mathbf{F}(\mathbf{x})$ which leaves the tube $\mathbb{G}(t)$, i.e., which crosses the ith boundary $g_i(\mathbf{x}, t) = 0$ from inside to outside. This means that there exists a time-space pair $(\mathbf{s}, t_s)$ on the boundary of $\mathbb{G}(t)$ (i.e., such that (iii) and (iv) are satisfied) and such that $\dot{g}_i(\mathbf{x}, t) \geq 0$ (otherwise the trajectory cannot leave the tube).
Thus, $\mathbb{G}(t)$ is not a capture tube, there exists one i such that $g_i(\mathbf{x}, t_1) = 0$ and $\mathbf{g}(\mathbf{x}, t_1) \leq 0$, and $\dot{g}_i(\mathbf{x}, t_1) \geq 0$, where $\dot{g}_i(\mathbf{x}, t) = \frac{\partial g}{\partial \mathrm{x}}(\mathbf{x}, t).\mathbf{a} + \frac{\partial g}{\partial t}(\mathbf{x}, t)$ and $\mathbf{a} \in \mathbf{F}(x)$. Equivalently, the system of constraints $\begin{cases} \frac{\partial g_i}{\partial x}(\mathbf{x}, t_1).\mathbf{a} + \frac{\partial g_i}{\partial t}(\mathbf{x}, t_1) \geq 0 \\ g_i(\mathbf{x}, t_1) = 0 \\ \mathbf{g}(\mathbf{x}, t_1) \leq 0 \end{cases}$ is consistent, which is in contradiction with the original assumption.

■

Consequence: A capture tube is a tube where, for a dynamic system, once the state is inside the tube, it will stay inside forever. From Theorems 3 and 4, checking that a tube, defined by a set of inequalities, is a capture tube candidate amounts to checking the inconsistency of a set of inequalities. This can easily be performed using contractor based methods as was shown in example 2.

Example 4. Consider again Figure 4 where it is assumed that the grey tube corresponds to $\mathbb{G}(t) = \{x, g_1(x, t) \leq 0\}$. The dotted trajectory leaves the tube at a time-space point (s, t_s), such that $g_1(s, t) = 0$ and $\dot{g}_1(s, t) > 0$. In this case, the tube is not a capture tube candidate since there exist a trajectory which leaves the tube, i.e. the tube is not an invariant set.

B. G-Stability

The following definition introduces the new concept of G-Stability:

Definition 5 (G-Stability for time-dependent systems): Consider a tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ where $\mathbf{g}: \mathbb{R}^n \mathrm{x} \mathbb{R} \to \mathbb{R}^m$. The time dependent system (16) is said to be G-Stable if:

$$\mathbf{g}(\mathbf{x}, t) \leq 0 \Rightarrow \dot{\mathbf{g}}(\mathbf{x}, t) < 0 \tag{23}$$

Theorem 5 (G-Stability): If a target tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ for a time dependent nonlinear system (16) is a capture tube candidate then the system (16) is *G-Stable*.

Proof.

Assume that $\dot{\mathbf{x}} = f(\mathbf{x}, t)$ is G-Stable. Definition 5 gives $\mathbf{g}(\mathbf{x}, t) \leq 0 \Rightarrow \dot{\mathbf{g}}(\mathbf{x}, t) < 0$, where $\mathbf{g}(\mathbf{x}, t)$ is a differentiable target tube.

Now, using the logical rule $(A \Rightarrow B) \Leftrightarrow (B \text{ or } \neg A)$ where A is $\mathbf{g}(\mathbf{x}, t) \leq 0$, B is $\dot{\mathbf{g}}(\mathbf{x}, t) < 0$ and applying De Morgan's laws, relation (23) becomes equivalent with the cross out condition (18). Finally, using Theorem 3 it can be concluded that $\mathbb{G}(t)$ is a capture tube for (16).

■

Definition 6 (G-Stability for time-dependent differential inclusion): Consider a tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ where $\mathbf{g}: \mathbb{R}^n \mathrm{x} \mathbb{R} \to \mathbb{R}^m$. The time dependent differential inclusion (21) is said to be G-Stable if:

$$\mathbf{g}(\mathbf{x}, t) \leq 0 \Rightarrow \dot{\mathbf{g}}(\mathbf{x}, t) < 0 \tag{24}$$

Theorem 6 (G-Stability for differential inclusion): If a target tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ for the time dependent differential inclusion (21) is a capture tube candidate then the differential inclusion (21) is *G-Stable*.

Proof.

The proof is a direct consequence of Theorems 4 and 5.

Consequence: A capture tube is a capture set which depends on time. From Theorems 5 and 6, it can be concluded that when there exists a capture tube candidate for $\mathcal{S}_\mathbf{f}$ or for $\mathcal{S}_\mathbf{F}$, then the system or the differential inclusion is G-Stable in the sense of definitions 5 or 6.

Figure 5 illustrates the notions of a capture tube and G-Stability for a two dimensional target tube. The capture tube is represented in the figure, at a particular time instant t, by a two dimensional capture bubble $\mathbb{G}(t)$ (grey bubble), as the intersection of the two bubbles, $g_1(\mathbf{x}, t) \leq 0$ and $g_2(\mathbf{x}, t) \leq 0$, at the same time instant. According to theorem 4, $\mathbb{G}(t)$ is a capture bubble for $\mathcal{S}_\mathbf{f}$ or $\mathcal{S}_\mathbf{F}$, (i.e. $\mathcal{S}_\mathbf{f}$ or $\mathcal{S}_\mathbf{F}$ are G-Stable) if for every state inside $\mathbb{G}(t)$, it will remain there forever. The system trajectories can cross out of the boundaries of $\mathbb{G}(t)$ only through two regions, i.e. either through the frontier $\begin{cases} g_1(\mathbf{x}, t) \leq 0 \\ g_2(\mathbf{x}, t) = 0 \end{cases}$ or through the frontier $\begin{cases} g_1(\mathbf{x}, t) = 0 \\ g_2(\mathbf{x}, t) \leq 0 \end{cases}$ (see figure 5). Theorems 3 and 4 answer the following question: Is it possible to leave the grey region, $\mathbb{G}(t)$? This only happens when $\dot{g}$ goes from negative to positive as in the case of the trajectories represented in the figure with 1 and 2, i.e. $\dot{g}_1(\mathbf{x}_1, t) > 0$ and $\dot{g}_2(\mathbf{x}_2, t) > 0$, respectively. In this case the cross-out condition from theorems 3 or 4 becomes consistent and $\mathcal{S}_\mathbf{f}$ or $\mathcal{S}_\mathbf{F}$ is not G-Stable, i.e. $\mathbb{G}(t)$ is not a capture bubble. If the cross-out condition from theorems 3 or 4 is inconsistent, this means that the trajectories 1 and 2 are not feasible and the only feasible trajectories are the ones represented by arrows oriented towards the centre of the grey region, $\dot{g}_1(\mathbf{x}, t) < 0$. In this case, the trajectories are always inside the bubble, which implies that $\mathcal{S}_\mathbf{f}$ (or $\mathcal{S}_\mathbf{F}$) is G-Stable and $\mathbb{G}(t)$ is a capture bubble. This reasoning is valid for any time $t > 0$.

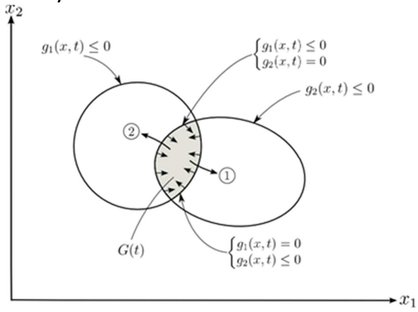

Figure 5. The cross out condition for a two dimensional capture tube.

V. Safe capture tubes

Definition 7 (Safe capture tubes): For a set of n G-Stable uncertain systems $\dot{\boldsymbol{x}}^a \in \mathbf{F}^a(x)$, the afferent capture tubes $\mathbb{G}^a(t) = \{\mathbf{x}, \mathbf{g}^\mathbf{a}(\mathbf{x}, t) \le 0\}$, a = 1,…,n are *safe* if any trajectory in $\mathbb{G}^a(t)$ does not intersect any other trajectory in $\mathbb{G}^b(t)$, $a, b \in \{1, \dots, n\}$, $a \neq b$.

Definition 8 (Safe capture tubes in a squad of robots): Consider a squad of p robots $\mathcal{R}^1, \dots, \mathcal{R}^P$ moving inside their environment. The movement of each robot $\mathcal{R}^a$ is described by a state equation $\dot{\mathbf{x}}^\mathbf{a} = \mathbf{f}(\mathbf{x}^a)$. For each robot there is an associated capture tube $\mathbb{G}^a(t) = \{\mathbf{x}, \mathbf{g}^a(\mathbf{x}, t) \le 0\}$. The squad is said to be *safe* if for each pair of robots $\mathcal{R}^a$, $\mathcal{R}^b$ with afferent capture tubes $\mathbb{G}^a(t) = \{\mathbf{x}, \mathbf{g}^a(\mathbf{x}^a, t) \le 0\}$ and $\mathbb{G}^b(t) = \{\mathbf{x}, \mathbf{g}^b(\mathbf{x}^b, t) \le 0\}$ and for all $t \ge 0$ we have $\mathbb{G}^a(t) \cap \mathbb{G}^b(t) = \emptyset$, i.e. $\begin{cases} \mathbf{g}^a(\mathbf{x}^b, t) > 0 \\ \mathbf{g}^b(\mathbf{x}^a, t) > 0 \end{cases}$.

The following theorem proves the safety of a squad of autonomous G-Stable robots moving in their environments by proving the inconsistency of a set of inequalities.

Theorem 7 (Safety of a squad of robots). Assume that in a squad of robots, each robot $\mathcal{R}^1, \dots, \mathcal{R}^P$ is G-Stable. The associated capture tubes for each robot are: $\mathbb{G}^1(t), \dots, \mathbb{G}^p(t)$. If for all $t \ge 0$, for all $a, b \in \{1, \dots, p\}$, and for all $\mathbf{x}^a, \mathbf{x}^b$, $a \neq b$, and for $\mathbf{h}(\mathbf{x}^a, \mathbf{g}^b) = \mathbf{g}^a(\mathbf{x}^a, t) \cdot \mathbf{g}^b(\mathbf{x}^a, t)$ the system of constraints

$$\begin{cases} (i)\ \mathbf{g}^a(\mathbf{x}^a, t) \le 0 \\ (ii)\ \mathbf{g}^b(\mathbf{x}^b, t) \le 0 \\ (iii)\ \mathbf{h}(\mathbf{x}^a, \mathbf{g}^b) \ge 0 \end{cases} \quad \text{is inconsistent, then the squad is safe.} \tag{25}$$

Proof:

The proof is by contradiction. Assume that the squad is not safe. From the theorem, each robot $\mathcal{R}^1, \dots, \mathcal{R}^P$ is G-Stable and using definition 4, for all $\mathbf{x}^a, \mathbf{x}^b$, $a \neq b$ the following relations hold:

$\mathbf{x}^a(t) \in \mathbb{G}^a(t)$ implies that $\mathbf{x}^a(t + t_1) \in \mathbb{G}^a(t + t_1)$ for all $t_1 > 0$, and

$\mathbf{x}^b(t) \in \mathbb{G}^b(t)$ implies that $\mathbf{x}^b(t + t_1) \in \mathbb{G}^b(t + t_1)$ for all $t_1 > 0$.

If the squad is not safe, this implies that there exists a $t_2 > 0$ such that $\mathbf{x}^a(t + t_2) \in \mathbb{G}^a(t + t_2)$ and $\mathbf{x}^a(t + t_2) \in \mathbb{G}^b(t + t_2)$. From definition 8, this implies that $\mathbf{g}^b(\mathbf{x}^a, t + t_2) \le 0$ and $\mathbf{h}(\mathbf{x}^a, \mathbf{g}^b)$ becomes positive, i.e. $\mathbf{h}(\mathbf{x}^a, \mathbf{g}^b) = \mathbf{g}^a(\mathbf{x}^a, t) \cdot \mathbf{g}^b(\mathbf{x}^a, t) \ge 0$. Equivalently, the system of constraints (25) is consistent which is in contradiction with the original assumption.

■

Consequence: Using G-Stability it is possible to establish the safety of a squad of autonomous uncertain robots. To our knowledge, no such algorithm exists. This result will be illustrated in section VI for a squad of three robotic sailboats.

VI. **TEST-CASES**

A. Proving stability for trajectory-tracking for a sailboat robot

In section A1, a controller for a time invariant uncertain sailboat robot will be designed and then using theorem 2 it will be proven that the controlled uncertain sailboat is V-Stable. In section A2, the sailboat robot will track a time dependent trajectory and, in this case, the theorem 4 will be used to prove that a target tube is a capture tube candidate for the uncertain time dependent sailboat robot, i.e. it is G-stable.

A1. Proving V-Stability for time-invariant sailboat robot

Consider a simple uncertain sailboat robot described by the following *velocity made good* model:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \in \begin{pmatrix} \sqrt{1+\cos(\psi-u)}\cdot\cos u \\ \sqrt{1+\cos(\psi-u)}\cdot\sin u \end{pmatrix} + \begin{pmatrix} [-\varepsilon,\varepsilon] \\ [-\varepsilon,\varepsilon] \end{pmatrix} \tag{26}$$

with $\varepsilon = 0.01$.

The input corresponds to the heading, θ, of the sailboat. In this model, ψ, is the measurable wind direction and $v = \sqrt{1+\cos(\psi-u)}$ corresponds to the boat's speed. Figure 6 provides an illustration of the corresponding polar speed diagram. Note that for $u \simeq \psi \pm \pi$ (which means that the robot is against the wind) the speed vanishes.

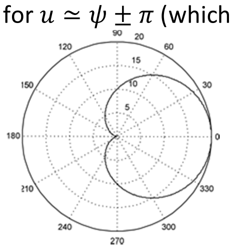

Figure 6. Polar speed diagram of the robot with respect to $\psi - \theta$.

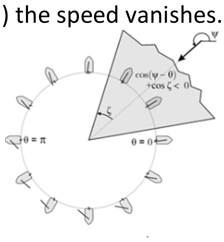

Figure 7. Some directions for the sailboat are not feasible and they form the grey no-go zone.

The proposed control law is influenced by the line following controller described in [2] where the sailboat follows the closed hauled angle taken as $\zeta = \frac{\pi}{4}$. Denote the nominal angle by θ^*, which represents the desired direction. Due to the wind, θ^* may not be feasible (see Figure 7). Hence define $\bar{\theta}$ as the corrected angle. If θ^* is feasible, then $\bar{\theta}$ = θ^*, and when θ^* is not feasible, $\bar{\theta}$ is the nearest feasible angle. The proposed controller is given in Algorithm 1.

```
Algorithm 1 (The controller for the sailboat robot): (in X, X̂, ψ;   inout: θ*)
if ‖X̂ − X‖ > r/2 then θ* = atan2(X̂ − X);
if cos(ψ − θ*) + cos ζ < 0
        then u = π + ψ + sign(sin(ψ − θ*))ζ
        else u = θ*
return u
```

When ψ is unknown, then

$$\begin{cases} u \in \operatorname{atan2}(\hat{\mathbf{X}} - \mathbf{X}) + [-\zeta,\zeta] & \text{if } \|\boldsymbol{X} - \hat{\boldsymbol{X}}\| > \frac{r}{2} \\ u \in [-\pi,\pi] & \text{otherwise} \end{cases} \tag{27}$$

and the control becomes the following set-valued map, i.e. an interval vector field

$$\mathbb{U}: \begin{cases} \mathbb{R}^2 \to \mathbb{IR}^2 \\ \bar{\mathbf{x}} \to \begin{cases} \begin{pmatrix} \cos(\operatorname{atan2}(\bar{\mathrm{x}})+[-\zeta,\zeta]) \\ \sin(\operatorname{atan2}(\bar{\mathrm{x}})+[-\zeta,\zeta]) \end{pmatrix} & \text{if} \|\hat{\boldsymbol{X}}\| > \frac{r}{2} \\ \begin{pmatrix} \cos([-\pi,\pi]) \\ \sin([-\pi,\pi]) \end{pmatrix} & \text{otherwise} \end{cases} \end{cases} \tag{28}$$

Here $\bar{\mathbf{x}}$ corresponds to $\hat{\mathbf{X}} - \mathbf{X}$. Figure 8 illustrates the vector field for the controlled sailboat (with the controller proposed in Algorithm 1 and for one particular wind direction) and the bubble (or the disk) with radius r, together with the sphere with radius $\frac{r}{2}$. This controller corresponds to the velocity good model $\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sqrt{1+\cos(\psi-u)}\cdot\cos u \\ \sqrt{1+\cos(\psi-u)}\cdot\sin u \end{pmatrix}$. The model is time invariant, so theorem 1 is used to prove the V-Stability for the controlled sailboat, where r is the radius of the bubble. Figure 9 shows the vector field for the differential inclusion (27) associated with the controlled sailboat (with the controller proposed in Algorithm 1 for all wind directions). In this case, V-Stability was proved using Theorem 2. Figure 10 shows the vector field for the differential inclusion (27) associated with the controlled sailboat with the controller proposed in (28) for one particular wind direction. The V-Stability for this control law was proved again using Theorem 2.

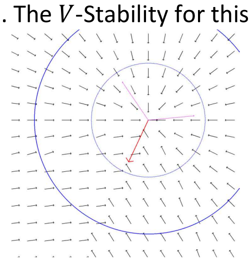

Figure 8. Controlled sailboat with controller from Algorithm 1 for one particular wind direction (the red arrow).

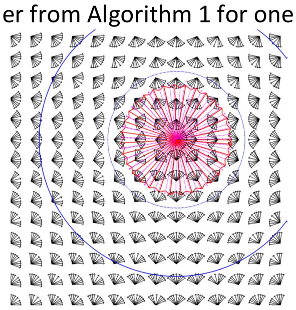

Figure 9. Differential inclusion associated with the controlled sailboat using the controller from Algorithm 1 for all wind directions (the red arrows).

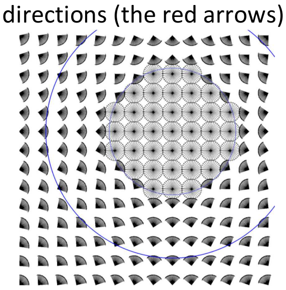

Figure 10. Differential inclusion associated with the controlled sailboat using controller (28) for one wind.

A2. Proving G-stability for time dependent sailboat robot

Consider the target tube $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ for the differential inclusion (26), with $\mathbf{g}(\mathbf{x}, t) = (x_1 - \hat{x}_1(t))^2 + (x_2 - \hat{x}_2(t))^2 - r^2$, which corresponds to a time dependent bubble (or disk) with center $\hat{\mathbf{x}} = (\hat{\mathrm{x}}_1(t), \hat{\mathrm{x}}_2(t))$**.** The center is assumed to vary with time according to the following equation

$$\begin{cases} \hat{x}_1(t) = 10\cos t \\ \hat{x}_2(t) = 5\sin t \end{cases} \tag{29}$$

which corresponds to an ellipsoidal trajectory.

In this situation, the system becomes time dependent and the time dependent bubble becomes a target tube for the sailboat robot. It is therefore necessary to prove that the target tube is a capture tube candidate for the sailboat robot. As the sailboat is time dependent and uncertain, Theorem 6 is used to prove G-Stability. It is necessary to find a capture tube candidate for the differential inclusion. Since the tube is periodic with a period equal to 2π, the analysis can be restricted to $t \in [0, 2\pi]$. Theorem 4 is used to prove that $\mathbb{G}(t)$ is a capture tube candidate, i.e. to show that the system of constraints

$$\begin{cases} \text{(a)} & (x_1 - 10\cos(t))^2 + (x_2 - 5\sin(t))^2 - r^2 = 0 \\ \text{(b)} & \left|u - \operatorname{atan2}(\hat{\mathrm{X}} - \mathrm{X})\right| \leq \zeta \\ \text{(c)} & u = \sqrt{1 + \cos(\psi - u)} \\ \text{(d)} & g_t = -20(x_1 - 10\cos t)\sin t + 10(x_2 - 5\sin t)\cos t \\ \text{(e)} & g_x = -2\dot{x}_1 v(x_1 - 10\cos t)\cos t + 2\dot{x}_2 v(x_2 - 5\,sin\,t)\sin t \\ \text{(f)} & g_x + g_t \geq 0 \end{cases} \tag{30}$$

has no solution for all $\mathrm{x} = (x_1, x_2) \in \mathbb{R}^2$, for all $\psi \in [-\pi, \pi]$, for all $u \in [-\pi, \pi]$ and for all $t \in [0, 2\pi]$. Equation (a) corresponds to Condition (iii) of Theorem 4. Equation (c) defines an intermediate variable v which corresponds to the speed of the robot. Equations (d) and (e) introduce two intermediate variables g_t and g_x which correspond to $\frac{\partial \mathbf{g}}{\partial t}(\mathbf{x}, t)$ and

$\frac{\partial \mathbf{g}}{\partial x}(\mathbf{x}, t)\ \mathbf{f}\ (\mathrm{x})$, respectively. Inequality (f) corresponds to Condition (i) of Theorem 4. If the system of constraints (30) is inconsistent, then $\mathbb{G}(t)$ is a capture tube and according with Theorem 6 the uncertain sailboat robot is G-Stable. Figure 11 shows the G-Stability for the sailboat robot for the ellipsoidal capture tube (29).

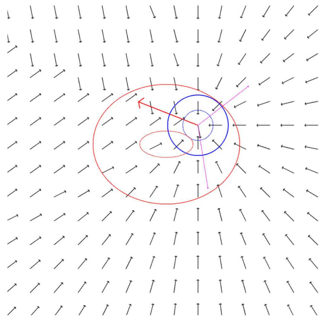

Figure 11. G-Stability for the sailboat robot for the ellipsoidal capture tube.

B. Proving the safety for a squad of three G-Stable sailboat robots

Consider the problem of three sailboat robots $\mathcal{R}^1, \mathcal{R}^2, \mathcal{R}^3$, each moving on a trajectory in their environment. The target trajectory has the form $\hat{\mathbf{x}}^i = \begin{pmatrix} h_1(t+\varphi_i) \\ h_2(t+\varphi_i) \end{pmatrix}$, where **h** is periodic of period T and $\phi_i = \frac{T(i-1)}{3}$ is the time shift between the three targets. The target tube has the form $\mathbb{G}(t) = \{\mathbf{x}, \mathbf{g}(\mathbf{x}, t) \leq 0\}$ with $g(\mathbf{x}, t) = (x_1^i - \hat{x}_1^i(t))^2 + (x_2^i - \hat{x}_2^i(t))^2 - r^2$. If, for instance, the target trajectory is an ∞-shape trajectory, we could have $\hat{\mathbf{x}}^i = \begin{pmatrix} \mathrm{k}{*}\cos(\mathrm{t}+\varphi_i) \\ \mathrm{k}{*}\sin(2(\mathrm{t}+\varphi_i)) \end{pmatrix}$, where $k > 0$. It will be shown, using Theorem 7, that there is no collision among the robots in the squad, i.e. the squad is safe. It is therefore necessary to establish that for all $t > 0$ each robot is G-Stable and the afferent capture tubes are safe (Definition 7). Moreover, for all $,j \in \{1, 2, 3\}$, $i \neq j$, it will be established that, using Theorem 7, for all $t > 0$, the set of inequalities

$$\begin{cases} (a)\ \ (x_1^i - \hat{x}_1^i(t))^2 + (x_2^i - \hat{x}_2^i(t))^2 - 4 \leq 0 \\ (b)\ \ (x_1^j - \hat{x}_1^j(t))^2 + (x_2^j - \hat{x}_2^j(t))^2 - 4 \leq 0 \\ (c)\ \ (x_1^i - \hat{x}_1^j(t))^2 + (x_2^j - \hat{x}_2^j(t))^2 - r_\infty \geq 0 \end{cases}$$ has no solution.

where r_∞ is the dimension of the ∞-shape, Inequality *(a)* corresponds to Condition *(i)* of Theorem 7, Inequality *(b)* corresponds to Condition *(ii)* of Theorem 7, and Inequality *(c)* Corresponds to Condition (iii) of Theorem 7, respectively.

Figures 12 and 13 show the squad of three sailboat robots following an ∞-shape capture tube with two different values of r_∞. In figure 12 the squad is not safe since the stability regions for the sailboat robots intersect for some time instants, i.e. the condition of theorem 7 is not satisfied. In figure 13, for a larger r_∞, the squad is safe since the stability regions for the sailboat robots never intersect.

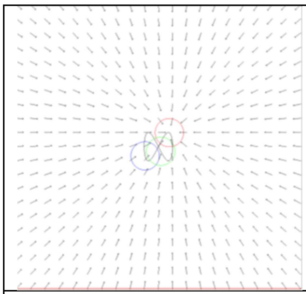

Figure 12. The three capture tubes intersect.

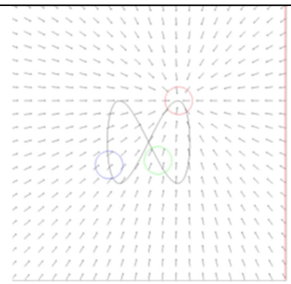

Figure 13. The three capture tubes never intersect.

Observation: The capture tubes are shown at a particular time instant in figures 12 and 13, and hence are represented by capture bubbles for the three sailboats. The vector fields in figures 12 and 13 correspond to a single sailboat identified with red color, i.e. the sailboat in the top of the figure.

VI. Conclusions

This paper has considered the problem of determining the stability of nonlinear, time-dependent and uncertain systems, in a set-membership context. Moreover, it was proposed a simple and efficient method to solve the V-Stability problem by proving the inconsistency of a set of inequalities. A new concept of G-Stability was developed, as an extension of V-Stability for time-dependent systems, which is based on representing the uncertain system as a differential inclusion. In order to prove that a time dependent differential inclusion is G-Stable, it is necessary to find a capture tube candidate for the differential inclusion. To prove that a target tube is a capture tube candidate is transformed into proving the inconsistency of a set of inequalities which can be efficiently solved in polynomial-time using a contractor based method. To illustrate the concept of G-Stability, a trajectory tracking application of an uncertain robotic sailboat was presented.

An important and yet unsolved problem for a squad of autonomous uncertain robots is to be able to prove that no collision occurs between the robots within the squad. To solve this problem, the concept of safety in a squad of autonomous uncertain robots has been proposed for the first time. This concept of safety is based on G-Stability and is established by proving the inconsistency of a set of inequalities. It is then used to analyze the stability of a squad of uncertain robotic sailboats.

For many systems, such as non-holonomic systems, potential candidates for a capture tube do not exist. Current work is focusing on developing methods that can help to find such capture tubes. Moreover, it is of great interest to extend the consensus control method [28] for uncertain robots. The new concepts introduced in this paper, G-Stability and safety, will be used for consensus control for a squad of autonomous robots.

REFERENCES

[1] LYAPUNOV, A. M. The general problem of motion stability, (1892), in Russian. Translated in French, Ann Fac. Sci. Toulouse 9, pp. 203-474 (1907). Reprinted in Ann. Math. Study No. 17, Princeton Univ. Press (1949).

[2] L. JAULIN AND F. L. BARS. An interval approach for stability analysis; Application to sailboat robotics. *IEEE Transaction on Robotics* 27(5) (2012).

[3] M. Lhommeau, L. Jaulin and L. Hardouin (2011) Capture basin approximation using interval analysis. *International Journal of Adaptive Control and Signal Processing*. Volume 25, Issue 3, pp 264-272.

[4] L. Jaulin (2012). Solving set-valued constraint satisfaction problems. *Computing*. Volume 94, Issue 2, Page 297-311.

[5] A. Neumaier. Complete search in continuous global optimization and constraint satisfaction. Acta Numerica, 13:271—369, 2004.

[6] R. E. MOORE, R. B. KEARFOTT, M. J. CLOUD, Introduction to Interval Analysis, SIAM, Philadelphia, 2009.

[10] L. Jaulin, M. Kieffer, O. Didrit and E. Walter (2001), Applied Interval Analysis with Examples in Parameter and State Estimation, Robust Control and Robotics, Springer-Verlag, ISBN: 1-85233-219-0.

[11] L. Jaulin, I. Braems, M. Kieffer and E. Walter (2001), Nonlinear State Estimation Using Forward-Backward Propagation of Intervals, In W. Kraemer and J. W. Gudenberg (Eds), Scientific Computing, Validated Numerics, Interval Methods, pp 191-204, Kluwer Academic Publishers, Boston/Dordrecht/London.

[12] J. AUBIN AND H. FRANKOWSKA. "Set-Valued Analysis". Birkhäuser, Boston (1990).

[13] C. BERGE. "Topological Spaces: Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity". Dover Books on Mathematics (1963).

[14] P. Herrero, P. Georgiou, C. Toumazou, B. Delaunay and L. Jaulin (2012). An Efficient Implementation of SIVIA Algorithm in a High-Level Numerical Programming Language, *Reliable computing*, pp. 239-251.

[16] P. SAINT-PIERRE. Approximation of the viability kernel. *Applied Mathematics and Optimization* 29(2) (1994).

[17] E. CRUCK, R. MOITIE, AND N. SEUBE. Estimation of basins of attraction for uncertain systems with affine and lipschitz dynamics.
Dynamics and Control 11(3), 211–227 (2001).

[18] M. QUINCAMPOIX. Differential inclusions and target problems. *SIAM journal on control and optimization* 30(2), 324–335 (1992).

[19] JEAN-JACKUES E. SLOTINE, WEIPING LI, Applied nonlinear control, Prentice Hall, 1991.

[20] A. Kurzhanski and I. Valyi, *Ellipsoidal Calculus for Estimation and Control*. Boston, MA: Birkhäuser, 1997.

[21] M. Milanese, J. Norton, H. Piet-Lahanier, and E. Walter, Eds., *Bounding Approaches to System Identification*. New York, NY: Plenum Press, 1996.

[22] B. A. Davey and H. A. Priestley, *Introduction to Lattices and Order*. (ISBN 0521784514): Cambridge University Press, 2002.

[24] A. Kurzhanski and I. Valyi, Ellipsoidal Calculus for Estimation and Control. Boston, MA: Birkhäuser, 1997.

[25] M. Milanese, J. Norton, H. Piet-Lahanier, and E. Walter, Eds., Bounding Approaches to System Identification. New York, NY: Plenum Press, 1996.
[26] B. A. Davey and H. A. Priestley, Introduction to Lattices and Order. (ISBN 0521784514): Cambridge University Press, 2002.
[27] P. Herrero, L. Jaulin, J. Vehi, M and A. Sainz (2010). Guaranteed Set-point Computation with Application to the Control of a Sailboat. International Journal of Control Automation and Systems. Vol. 8, No 1, pp 1-7.
[28] Wenwu Yu, Guanrong Chen, Ming Cao (2010). Some necessary and sufficient conditions for second-order consensus in multi-agent dynamical systems. Automatica, Volume 46, Issue 6, June 2010, Pages 1089-1095.
[29] T. Khailat (1980), Linear Systems, Prentice Hall.
[30] H. K. Khalil (2002), Nonlinear systems Third Edition, Prentice Hall.