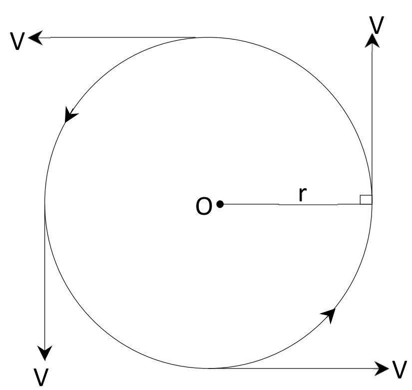
MODULE 4: CIRCULAR MOTION

TOPIC 4: CIRCULAR MOTION

CIRCULAR MOTION - Is the motion of the body moving in the circular path with constant or different speed.
The motion of anybody which can be described in the form of:-
(i) Curve.
(ii) Hyperbolic.
(iii) Parabolic.
(iv) Elliptical in path. This represent circular motion.

TYPES OF CIRCULAR MOTION:
Circular motion can be classified into two types namely;
1. Uniform circular motion.
2. Non-uniform circular motion.

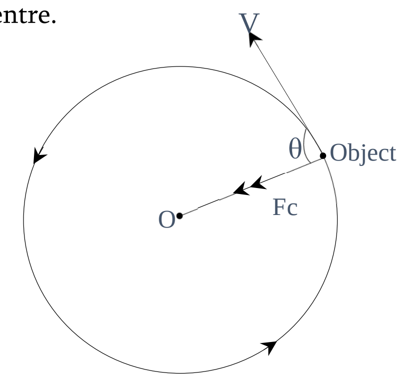
UNIFORM CIRCULAR MOTION
This describes the motion of a body that moves in a circular path at a constant speed. When a point object is moving in a circular path with a constant speed (i.e. it covers equal distance on the circumference of the circle in equal interval of time), then the motion of the object is said to be a uniform circular motion.



Examples of uniform circular motion:-
1. Motion of an electron around the nucleus of an atom.
2. Motion of planets around the sun.

NON-UNIFORM CIRCULAR MOTION
Is the motion of body moves in a circular path with different speeds.
Examples:
1. Motion of a stone attached at the end of a string moving in a vertical circle.
2. Rotor motion.

CONCEPT OF UNIFORM CIRCULAR MOTION
In uniform circular motion, the speed of the object remains constant but its direction is constantly changing. For this to happen the force must not act along the direction of motion, but towards the centre.

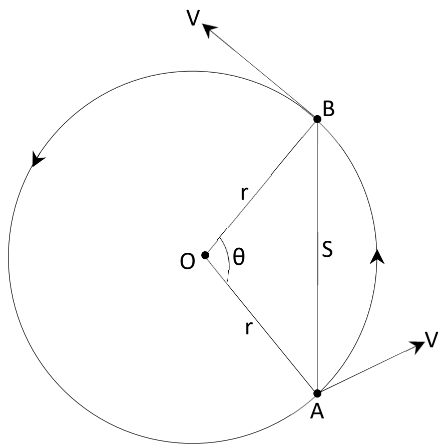


The force does not act along V, but it has a component $F\cos\theta$ along vector, V. This component will change the speed, consequently for uniform circular motion. $F\cos\theta = 0$. Since $F \neq 0$, then $\cos\theta = 0$, hence $\theta = 90^{\circ}$.
Therefore, centripetal force is perpendicular to the velocity and directed towards to the centre of the circular path.

- There are two necessary conditions for the body to move in circular motion:-
  (i) The body must be given some initial velocity. The velocity vector is always tangential to the path of the object and perpendicular to the radius of the circular path.
  (ii) The force which is always directed at right angle to the velocity vector must act on the body towards the centre.

PARAMETERS OF UNIFORM CIRCULAR MOTION

Consider the particle of mass M moving in a circular path as shown on the figure below:



Let, r = radius of the circular path

O = centre of the circular path

θ = Angular displacement

S = Linear displacement

1. LINEAR DISPLACEMENT, S

Is the distance covered by the particle move in circular path in specific direction. The S.I unit of displacement is metre (M).

2. ANGULAR DISPLACEMENT, θ

Is the angle turned through by an object moving along a circular path of radius, r. i.e. Angular displacement in a given time of the object, moving around a circular path is the angle traced out by the radius vector at the centre of the circular path in the given time.

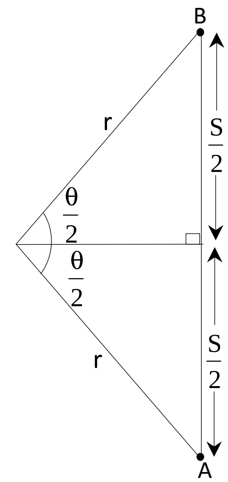
S.I Unit of angular displacement is radian (rad)

$$\theta = \frac{\text{arc length AB}}{\text{length of OB}} = \frac{s}{r}$$

Relationship between S and θ

From the figure below

If θ is very small angle measured in radian.



$$\text{Sin}\left(\frac{\theta}{2}\right) = \left(\frac{S}{2r}\right), \text{Sin}\left(\frac{\theta}{2}\right) \approx \frac{\theta}{2}$$

$$\frac{\theta}{2} = \frac{S}{2r}, \theta = \frac{S}{r}$$

3. ANGULAR VELOCITY (ω)

Is defined as the rate of change of angular displacement of the object

$$\omega = \frac{d\theta}{dt}$$

Also, $\omega = \dfrac{\text{Angular displacement}}{\text{time}}$

$$\omega = \frac{\theta}{t}$$

S.I unit of angular velocity is rad S^{-1}

Dimensional formula of ω is $[M^{\circ}L^{\circ}T^{-1}]$

Relationship between ω and V

Since S = rθ

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = r\frac{d\theta}{dt}$$

V = ωr

Note that: $\omega(t) = \dfrac{d\theta}{dt}$

$$d\theta = \omega(t)\, dt$$

$$\theta = \int w(t)dt$$

4. PERIODIC TIME (T)

The time period of an object in circular motion is defined as time taken to complete one revolution.

S.I unit of periodic time is second.

Periodic time is the reciprocal of frequency.

$$T = \frac{1}{f}$$

Relationship between T and ω

Since, $\omega = \frac{\theta}{t}$

when t =T, θ =2π

$$\omega = \frac{2\pi}{T} \text{ or } T = \frac{2\pi}{\omega}$$

5. FREQUENCY, *f*

Is defined as the number of revolutions complete per second.

$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{Or } \omega = 2\pi f$$

S.I unit of frequency is Hertz (Hz) or S^{-1}

Note;

$$1 \text{ r.p.s} = \frac{1 \text{ revolution}}{1 \text{ second}} = 1 \text{ Hz}$$

$$1 \text{ r.p.m} = \frac{1 \text{ revolution}}{1 \text{ minute}} = \frac{1}{60} \text{ Hz}$$

Also, 1 revolution $\longrightarrow$ 2π rad

N revolution $\longrightarrow$ θ

θ = 2Πn

6. ANGULAR ACCELERATION (α)

Is defined as the rate of change of angular velocity.

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta} = \frac{d^2\theta}{dt^2}$$

Also, $\alpha = \frac{\text{change in regular velocity}}{\text{time taken}}$

$$\alpha = \frac{\omega - \omega_o}{t}$$

ω_o = initial angular velocity

ω = final angular velocity

t = time taken

S.I unit of angular acceleration is rad S^{-2}

Dimensional formula of angular acceleration is $[M^{\circ} L^{\circ} T^{-2}]$

Relationship between α and *a*

α = Angular acceleration

a = Linear acceleration

Since, V = ωr

$$\frac{dv}{dt} = \frac{d}{dt}(\omega r) = r\frac{dw}{dt}$$

$$a = \alpha r$$

Note that, $\alpha = \frac{d\omega}{dt}$

$$d\omega = \alpha dt$$

$$\omega = \int \alpha(t) dt$$

7. EQUATION OF THE UNIFORM LINEAR MOTION APPLIED TO THE UNIFORM CIRCULAR MOTION

Equations of the linear uniform motion.

V = u + at ………………. (i)

$S = ut + \frac{1}{2}at^2$ …………..(ii)

$V^2 = u^2 + 2as$ …………..(iii)

Each symbols have usual meaning, comparing with equations of uniform linear motion, equations of uniform circular motions are:

$\omega = \omega_o + \alpha t$ ………………..(1)

$\theta = \omega_o + \frac{1}{2}\alpha t^2$ ……………..(2)

$\omega^2 = \omega_o^2 + 2\alpha\theta$……………….(3)

Where, θ = Angular displacement

$\omega_o t$ = Initial angular velocity

ω = final angular velocity

α = angular acceleration.

Derivation: $\omega = \omega_o + \alpha t$

Method 1. From the first principle

$$\alpha = \frac{\omega - \omega_o}{t}$$

$\omega - \omega_o = \alpha t$

$\omega = \omega_o + \alpha t$

Method 2: From the relationship between linear and circular motion.

$U = \omega_o r$, $V = \omega r$, $a = \alpha r$

Since V = U + at

$\omega r = \omega_o r + \alpha r t$

Derivation: $\theta = \omega_o t + \frac{1}{2}\alpha t^2$

Method 1: For the first principle

$$\theta = \overline{\omega} t\left(\frac{\omega + \omega_o}{2}\right)t = \left(\frac{\omega + \alpha t + \omega_o}{2}\right)t$$

$\theta = \omega_o t + \frac{1}{2}\alpha t^2$

Method 2: Since $S = Ut + \frac{1}{2}\alpha t^2$

$$\theta r = \omega_o r + \frac{1}{2}\alpha r t^2$$

$$\theta = \omega_o t + \frac{1}{2}\alpha t^2$$

Derivation: $\omega^2 = \omega_o^2 + 2\alpha\alpha$

Method 1. Since $\omega = \omega_o + \alpha t^2$

$\omega^2 = (\omega_o + \alpha t)^2$

$= \omega_o^2 + 2\alpha\omega_o t + \alpha^2 t^2$

$= \omega_o^2 + 2\alpha\left(\omega_o t + \frac{1}{2}\alpha t^2\right)$

$\omega^2 = \omega_o^2 + 2\alpha\theta$

Method 2. Since $V^2 = U^2 + 2as$

$(\omega r)^2 = (\omega_o r)^2 + 2\alpha r(\theta r)$

$\omega^2 = \omega_o^2 + 2\alpha\theta$

8. CENTRIPETAL (RADIAL) ACCELERATION (a_c)

Is the acceleration of the body moving around a circle and it is always directed along the radius towards the centre of the circle.

This acceleration is also known as radial or normal acceleration.

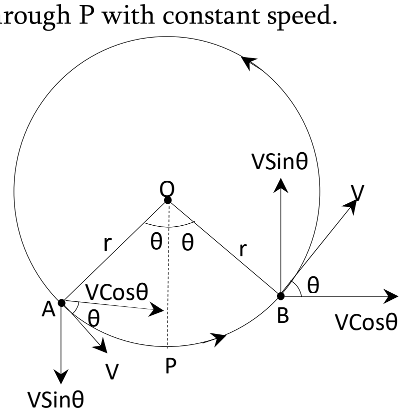
Expression of centripetal acceleration.

$a_c = \frac{V^2}{r} = \omega^2 r = \omega V$

V= speed, r = radius, ω = angular velocity

Derivation of expression of the centripetal acceleration

Consider a body moving on a circular path of radius r such that it passed from point A to B through P with constant speed.



The horizontal acceleration a_x of the body along x-direction is given by

$a_x = \frac{V_x}{t} = \frac{V_{BX} - V_{AX}}{t}$

$a_x = \frac{V\cos\theta - V\cos\theta}{t} = 0$

It follows that acceleration along x– direction is zero. Acceleration of the body along y – direction.

$a_y = \frac{V_y}{t} = V_{BY} - V_{AY}$

$= \frac{V\sin\theta - (-V\sin\theta)}{t}$

$a_y = \frac{2V\sin\theta}{t}$

If t is the time taken by body to move from A to B.

$V = \frac{\text{arc length AB}}{t}$

$= \frac{2\theta r}{t}$

$t = \frac{2\theta r}{v}$

$a_y = \frac{2V\sin\theta}{2\theta\frac{r}{v}} = \frac{V^2}{r}\bullet\frac{\sin\theta}{\theta}$

If θ is very small measured in radian.

$\underset{\Delta\theta\to o}{1cm}\left(\frac{\sin\theta}{\theta}\right) \approx 1$

$a_c = \frac{V^2}{r} = \omega^2 r = \omega V$

TANGENTIAL ACCELERATION

Is the linear acceleration along the tangent. It acts along the tangent to the circular path and is in the plane of the circular path.

$a_r = \alpha r$

DIFFERENT MODES OF ACCELERATION

Acceleration is the rate of change of velocity.

There are three different modes of the acceleration: -

1. By changing speed leaving the direction constant.
2. By changing direction leaving the speed constant e.g. circular motion.

3. By changing both direction and speed.

   The acceleration experienced by the body moving in the circular motion is by changing direction leaving the speed constant.

TANGENTIAL ACCELERATION AND CENTRIPETAL ACCELERATION

Although the speed is constant for an object moving in the circular path but the object has a linear acceleration. This is due to the change of velocity of the object with changes of the direction when the object is under different point on the circular path with constant speed.

When particle moving in a circular motion have two types of accelerations: -

1. Centripetal acceleration

   $$a_c = \frac{V^2}{r} = \omega V$$

   It acts along the radius and is directed towards the centre of the circular path.

2. Tangential acceleration. $a_T = \alpha r$. It acts along path and is in the plane of circular path.

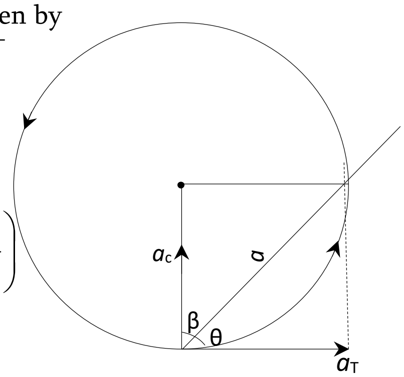
   Since a_c and a_T are perpendicular to each other. The total or magnitude of acceleration of the particle is given by

   $$a = \sqrt{a_c^2 + a_T^2}$$

Direction of a.

$$\tan\theta = \frac{a_c}{a_T}$$

$$\theta = \tan^{-1}\left(\frac{a_c}{a_T}\right)$$



Special cases

Case 1: If $a_T = 0$, the particle will have only centripetal acceleration. the particle will move on a circular path with constant angular velocity. The magnitude of the centripetal acceleration, $a_c = \frac{V^2}{r}$

Case 2: If $a_c = 0$, the particle will have only tangential acceleration. Now the particle will accelerate along the tangent to the circular path.

9. CENTRIPETAL (RADIAL) FORCE

Is the force experienced by an object moving in the circular motion and always is directed towards to the centre of the circular path. In order to maintain a body to move along a circular path an external force is required, which will deflect the body from its straight path to the circular path at every point of the path. Always centripetal force is balanced with centrifugal force in order to maintain the body to move in the circular motion.

Expression of the centripetal force

According to the Newton's second law of motion.

$$F_c = \frac{MV^2}{r} = M\omega^2 r = M\omega V$$

CENTRIFUGAL FORCE – Is the force experimented by the body moving in the circular path act away from the centre and always is counter balanced with centripetal force.

In magnitude centripetal force is equal to the centrifugal force.

$$F_c = \frac{MV^2}{r} = M\omega^2 r = M\omega V$$

The centrifugal force is not a real force. As we cannot identify its source; it is called "fictitious or pseudo force".

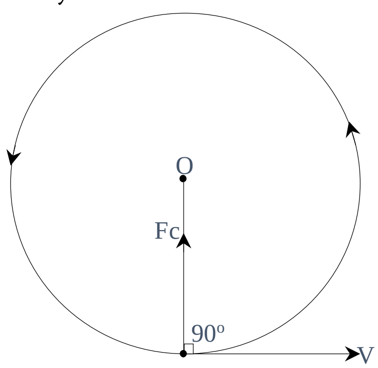
Examples of centripetal forces

TYPES OF MOTION	SOURCE OF CENTRIPETAL FORCE
1. Centrifugal device	Reaction at the walls
2. Planetary orbit	Gravitation force
3. Vehicle concerning	Friction force between the wheels and road
4. Electron orbit	Electrostatic force
5. Aircraft banking	Lifts force on the wings.
6. Object on the string	Tension on the string
7. Rotating liquid surface	Gravity
8. Variation of g with Latitude	Gravity

9. Gramophone needle	Frictions with grooves.
10. Rotor motion	Reaction force on the walls.
11.Motion of charged particle in uniform magnetic field	Magnetic force

Note that:

1. The direction of the centripetal force is perpendicular to the direction of the velocity of the body move in the circular motion.



2. Work done by the centripetal force.
   $W = FSCos\theta$ but $\theta = 90^o$
   $W = FSCos90^o = 0$
   Therefore, there is no work done by the centripetal force because centripetal force is perpendicular to the direction of the velocity.
3. Power needed to maintain the body to move in the circular path.
   $P = FVCos\theta = FVCos90^o = 0$
   Thus, there is no any power needed to maintain the body to move in circular path.
4. The centripetal force does not change the speed of the particle moving in the circular path. So the kinetic energy of the particle remains constant.
5. The acceleration of the particle is always perpendicular to the velocity.

TYPE: A SOLVED EXAMPLES

Example 1

(a) During uniform circular motion name of the quantities which remain constant.

(b) A motor car travelling at 30 m/s on a circular road of radius 500 m. it is increasing its speed at the rate of 2ml/s². What is its acceleration?

Solution

(a) Speed, kinetic energy and angular velocity.

(b) Here $V = 30$ m/s, $r = 500$ m. $a_T = 2$ m/s².

Centripetal acceleration

$$A_c = \frac{V^2}{r} = \frac{30\times 30}{500} = 1.8 \text{ m/s}^2.$$

The magnitude of acceleration

$$a = \sqrt{a_c^2 + a_T^2} = \sqrt{(1.8)^2 + 2^2}$$

$a = 2.7$ m/s².

Example 2.

(a) Can a body move on a curved path without having acceleration?

(b) A stone tied to the end of a string 2 m long is whirled in a horizontal circle with constant speed. If the stones makes 10 revolution in 20s. calculate the magnitude and direction of acceleration.

Solution

(a) No. it is so because while moving on a curved path, the velocity of the body (which is represented by tangent to the curved path at a point) changes with time and the body is in accelerated state.

(b) $a_c = \omega^2 r = (2\pi f)^2 r \bullet \left(\frac{f = 10\,S^{-1}}{20}\right)$

$$a_c = \left(2\times\frac{22}{7}\times\frac{1}{2}\right)^2 \times 2$$

$a_c = 19.74$ m/s².

Example3

Calculate the magnitude of linear acceleration of a particle moving in a circle of radius 0.5 m at the instant when its angular velocity is 2.5 rad S^{-2} and its angular acceleration is 6 rad S^{-2}.

Solution

Tangential acceleration $a_T = \theta r = 0.5 \times 6 = 3$m/s².

Centripetal acceleration

$a_c = \omega^2 r = 0.5 \times (2.5)^2 = 3.125$ m/s².

Magnitude of the total linear acceleration

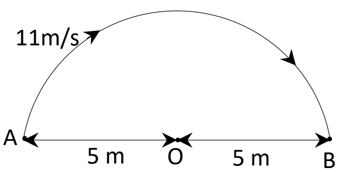
$$a = \sqrt{a_T^2 + a_c^2} = \sqrt{3^2 + (3.125)^2}$$

$a = 4.33$ m/s².

Example 4

(a) A particle moves in a circle of radius 20 cm. its linear speed is given by
V = 2t, where t is in second and V in m/s. Find the radial and tangential acceleration at t = 3 seconds.

(b) A particle moves in a semi-circular path of radius 5.0 m with constant speed 11 m/s as shown on the figure below. Calculate:-



(i) The time taken to travel from A to B $\left(\pi = \frac{22}{7}\right)$.

(ii) The average velocity.

(iii) The average acceleration.

Solution

(a) Linear speed at t = 3 sec. V= 2t
= 2 × 3 = 6 m/s². Radial acceleration at this instant

$$a_c = \frac{V^2}{r} = \frac{6^2}{0.2}$$

a_c = 180 m/s².

Tangential acceleration

$$A_T = \frac{dV}{dt} = \frac{d}{dt}(2t) = 2 \text{ m/s}^2$$

A_T = 2 m/s².

(b) (i) Time = $\frac{\text{arc AB}}{\text{speed}} = \frac{\pi \times 5}{11}$

$$T = \frac{10}{7} \text{sec.}$$

(ii) Average velocity = $\frac{\text{displacement}}{\text{time}}$

The displacement is diameter AB to the right

$$\overline{V} = \frac{10}{10/7} = 7 \text{ m/s}^2 \text{ to the right}$$

$$\overline{V} = 7 \text{ m/s}^2$$

(iii) Average acceleration = $\frac{\text{change of velocity}}{\text{time}}$

$$= \frac{11-(-11)}{10/7} = \frac{22}{10/7}$$

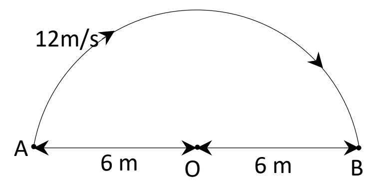
$$a = \frac{154}{10} \text{m/s}^2 = 15.4 \text{ m/s}^2$$

Example 5. NECTA 2019/P1/5

(a) (i) In which aspect does circular motion differ from linear motion?

(ii) Why there must be a force acting on a particle moving with uniform speed in a circular path?

(b) (i) Figure 1. below shows a particle moving in a semi-circular path AB of radius 6 m with constant speed of 12 m/s². Calculate the average velocity.



(ii) A stone tied to the end of string 80 cm long, is whirled in a horizontal circle with constant speed making 25 revolutions in 14 seconds. Determine the magnitude of its acceleration.

Solution

(a) (i) For uniform circular motion the direction of centripetal acceleration is perpendicular to the direction of the velocity while for linear motion the direction of acceleration is on the same as the direction of the velocity.

- For circular motion the magnitude of accelerations remains constant but direction of acceleration continuously with time while for the linear motion, magnitude of acceleration changes with time.

(ii) There must be a centripetal force for the particle moving in a circular path with constant speed to maintain it to move in a circular path and always centripetal force is directed towards to the centre of the path.

(b) (i) Time, $t = \frac{\text{arc AB}}{\text{speed}} = \frac{\pi \times 6}{12}$

$t = \frac{\pi}{2}\text{sec} = 1.57 \text{ sec}$

Average velocity = $\frac{\text{displacement}}{\text{time}}$

$\overline{V} = \frac{12}{1.57}$

$\overline{V} = 7.643 \text{ m/s}$

(ii) $r = 80 \text{ cm}, T = \frac{25}{14} \text{ sec}$

$\omega = \frac{2\pi}{T} = \frac{2\pi}{25/14} = \frac{28\pi}{25} \text{ rad S}^{-1}.$

The acceleration of uniform circular motion is given by

$a = \omega^2 r = \left(\frac{28\pi}{25}\right)^2 \times 80$

$a = 990.4 \text{ cm/s}.$

The acceleration is directed along the radius of the circular path and towards the centre of the circle.

Example 6

A cyclist is riding with a speed of 27 kmhr^{-1}. As he approached a circular turn on the road of radius 80 m, he applies brakes and reduces his speed at the constant rate of 0.5 m/s. What is the magnitude and direction of the net acceleration of the cyclist on the circular turn?

Solution

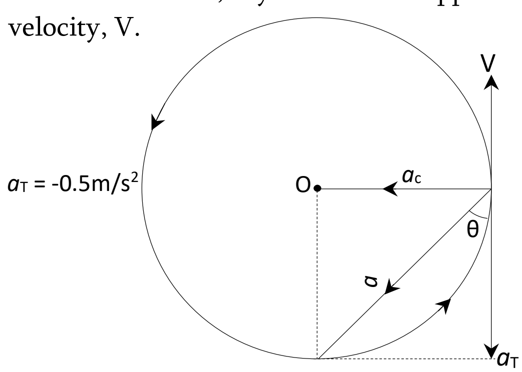
Here V = 27 kmh^{-1} = 7.5 m/s.

r = 80 m.

Centripetal acceleration

$ac = \frac{V^2}{r} = \left(\frac{7.5}{80}\right)^2 = 0.7 \text{ m/s}^2.$

Suppose that the cyclist applies brakes at the point A of the circular turn. Then retardation produced due to the brakes, say a_T will acts opposite to the velocity, V.



Total acceleration is given by

$a = \sqrt{ac^2 + a_T^2} = \sqrt{(0.7)^2 + (-0.5)^2}$

$a = 0.86 \text{ m/s}^2.$

Direction of a

$\tan\theta = \frac{a_c}{a_T} = \frac{0.7}{0.5} = 1.4$

$\theta = 54^o\ 28^1$

Example 7

A race car negotiates a curve of radius 50 m on a flat road. What its angular displacement when its linear distance covered by the car is 78.5 m?

Solution

$\theta = \frac{s}{r} = \frac{78.5}{50} = \frac{25}{14}$

$\theta = 1.54 \text{ rad} = 90^o.$

Example 8

A satellite moving in a circular orbit at an altitude of 1000 km complete one revolution round the earth is 105 minutes. What is

(i) its angular velocity and

(ii) speed? Radius of the earth, R = 6400 km.

Solution

(i) $\omega = \frac{2\pi}{T} = \frac{2 \times 3.14}{105 \times 60}$

$\omega = 9.97 \times 10^{-4} \text{ rad S}^{-1}$

(ii) $V = \omega r = \omega(R+h)$

$= 9.9 \times 10^{-4} \times (64000 + 1000) \times 10^3$

$V = 7{,}326 \text{ m/s}.$

Example 9

(a) Is any work done by the centripetal force?

(b) Why centripetal force cannot do work?

(c) What is the power needed to maintained object in circular motion?

Solution

(a) No work done by the centripetal force. W = FSCos90^o = 0

(b) The centripetal force cannot do work on the object because it is always perpendicular to the direction of the velocity.

(c) No power is needed to maintain uniform circular motion.

P = FVCosθ = FSCos90^o = 0

Example 10

(a) Why do we express the angular displacement in radian?

(b) The angular displacement of a body is given by $\theta = 2t^2 + 5t - 3$. Find the value of the angular velocity and angular acceleration when t = 2 second.

Solution

(a) The length of an arc $L = r\theta$, in this equation θ is in radian and not in degree. Using this equation angular displacement can be converted into a linear displacement.

(b) $\theta = 2t^2 + 5t - 3$

$$\omega = \frac{d\theta}{dt} = 4t + 5 \text{ when } t = 2$$

$\omega = 13$ rad/s.

Angular acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d(4t+5)}{dt}$$

$\alpha = 4$ rad/s^2.

Example 11

A wheel starting from rest rotates with an angular acceleration $\alpha = (10 + 6t)$ rad/s^2. Where t is time in seconds. Determine the angle in radian through which the wheel has turned in first four seconds.

Solution

Since,

$$\alpha = \frac{d\omega}{dt} = 10 + 6t$$

$d\omega = (10 + 6t)dt$

$$\int d\omega = \int (10 + 6t)dt$$

$\omega = 10t + 3t^2 + C$

When $t = 0$, $\omega = 0$ $C = 0$

$\omega = 10t + 3t^2$

$$\omega = \frac{d\theta}{dt} = (10t + 3t^2)$$

$d\theta = (10t + 3t^2)dt$

$$\theta = \int_0^4 (10t + 3t^2)dt$$

$\theta = 144$ rad.

Example 12

A motor revolving at 1800 r.p.m slows down uniformly to 120 r.p.m in 2 seconds. Calculate: -

(i) Angular acceleration of the motor.

(ii) Number of revolutions it makes in this time.

Solution

Initial frequency, $f_1 = \frac{1800}{60} = 30$ Hz

Final frequency, $f_2 = \frac{1200}{60} = 20$ Hz

(i) Let α = Angular acceleration

$$\alpha = \frac{2\pi(f_2 - f_1)}{t} = \frac{2\pi(20-30)}{2}$$

$\alpha = -31.43$ rad S^{-2}

(ii) Let θ total angle turned in 2 sec.

Since $\omega_2^2 - \omega_1^2 = 2\alpha\theta$

$$\theta = \frac{4\pi^2(f_2^2 - f_1^2)}{2\alpha} = \frac{2\times(3.14)^2(20^2 - 30^2)}{-2\times 31.43}$$

$\theta = 314.28$ rad.

Number of revolutions

$$N = \frac{\theta}{2\pi} = \frac{314.28}{2\times 3.14}$$

N= 50 revolutions.

Example 13

Angular frequency of a particle is increased by 100 rev min^{-1} in one minute when it moves in a circular path of radius 200 cm. find its

(i) Angular acceleration

(ii) Tangential acceleration

Solution

(i) $\alpha = \frac{\Delta\omega}{t} = \frac{2\pi\Delta f}{t}$

$$= \frac{2\times 3.14}{60} \times \frac{100}{60}$$

$\alpha = 0.1746$ rads^{-2}

(ii) $a = \alpha r = 2 \times 0.1746$

$a = 0.3492$ m/s^2.

Example 14

The angular velocity of a fly wheel decreases uniformly from 1200 r.p.m to 900 r.p.m in 5 seconds. Find:-

(i) the angular acceleration.

(ii) the angular displacement in 5 seconds.

(iii) the number of rotations made by the wheel in 5 second.

(iv) further time taken by the wheel to come to rest.

Solution

$F_o = 1200$ r.p.m = $\dfrac{1200}{60}$ = 20 HZ

f = 900 r.p.m = $\dfrac{900}{60}$ = 15 HZ

(i) $\alpha = \dfrac{2\pi(f - f_o)}{t} = \dfrac{2\pi(15-20)}{5}$

α = -6.284 rads^{-2}.

(ii) $\theta = \dfrac{2\pi(f + f_o)}{2} = \dfrac{2\pi(20+15)\times 5}{2}$

θ = 175π rad.

(iii) Let N be the number of rotations made by the wheel

$N = \dfrac{\theta}{2\pi} = \dfrac{175\pi}{2\pi}$

N = 87.5 rotations.

(iv) $t = \dfrac{\omega = \omega_o}{\alpha} = \dfrac{0 - 30\pi}{-2\pi}$

t = 15 sec.

Example 15

The centripetal force acting on a body moving with speed in circular path is 10N. Without changing the path if the speed of the body is doubled, what will be the centripetal force?

Solution

Let $F_1 = \dfrac{MV_1^2}{r}$, $F_2 = \dfrac{MV_2^2}{r}$

$\dfrac{F_2}{F_1} = \left(\dfrac{V_2}{V_1}\right)^2 = \left(\dfrac{2V}{V}\right)^2 = 4$

$F_2 = 4F_1 = 4 \times 10$

F_2 = 40N.

Example 16

A metal bob of mass 0.6 kg is attached to the end of a string of length 1 m and the bob is whirled in a horizontal circle with a uniform speed of 12 m/s. what is the centripetal force acting on the bob? If the speed of the bob is 5 m/s. calculate the tension in the string.

Solution

- $F = \dfrac{MV^2}{r} = \dfrac{0.6\times 12^2}{1}$

  F = 86.4N.
- Tension in the string = centripetal force

  $T = \dfrac{MV^2}{r} = \dfrac{0.6\times 5^2}{1}$

  T = 15N.

Example 17

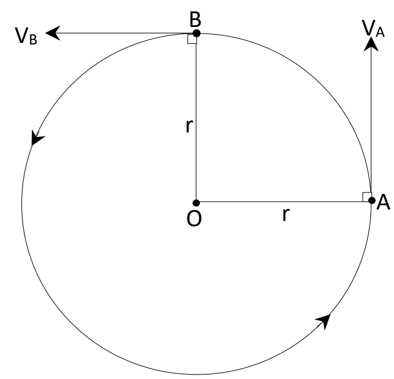
Consider the object of mass, M moving in a circular path with a constant speed.

(a) What is the direction of the velocity at some instant and quarter of revolution later.

(b) Explain why although the speed is constant, we say the velocity is changing?

(c) What would happen if there were no force acting on the object?

Solution



(a) The velocity is tangential to the circular path and thus the two velocities are at right angle to each other after revolution later.

(b) Speed can be constant if the direction is changing velocity can only be constant if the direction does not change i.e speed is the scalar quantity while the velocity is the vector quantity. Therefore in the circular motion velocity changes with the change when object moves in the circular path.

(c) If there is no centripetal force acts on the object then object continue to move in the linear motion i.e straight line with a constant speed.

Example 18

What provides the centripetal force in the following cases:

(i) Electron revolving around the nucleus.

(ii) Earth revolving around the sun.

(iii) Car taking a curve.

(iv) A coin placed on a horizontal turn table.

(v) A train travelling round a horizontal circular arc.
(vi) A stone tied to a string and whirled in a horizontal circle.
(vii) Motion of charged particle in uniform magnetic field.

Solution
(i) Electrostatic force.
(ii) Gravitational force.
(iii) A component of the reaction force of the road.
(iv) Friction force between the coin and the plane.
(v) The reaction force between the flage of the outer rail and the inner rail.
(vi) Tension in the string.
(viii)Magnetic force.

Example 19
What is the effect on the direction of the centripetal force when the revolving body reverses its direction of the motion?

Solution
The centripetal force will be directed towards the centre of the circle. This fact does not depend upon the sense of rotation of the particle.

Example 20
Why are passengers sitting in a vehicle thrown outwards, when the vehicle rounds a curve suddenly? OR Passengers are thrown outwards, when the bus takes a circular turn. Why?

Solution
When the bus takes a turn, the centripetal acceleration of the bus acts towards the centre of the circular turn and along its radius. Due to inertia, the passengers are thrown outwards. The force experienced by the passengers is called the centrifugal force.

Example 21
A car is taking a sudden turn to the left. A passenger in the front seat finds himself sliding towards the door. Explain, indication the forces acting on the passenger and on the car at this instant.

Solution
The passenger in the front seat slides towards door i.e away from the centre of the circular turn. It is due to the centrifugal force acting on the passenger.

Example 22
Why does a child in a merry-go-round press the side of his seat radially outward?

Solution
When the child presses the side of his seat radially outward in a merry-go-round, the side of the seat presses the child radially inwards in accordance with Newton's third law of motion. This force, which comes as reaction, provides the child the necessary centripetal force to move along a circular path.

TYPE A: TYPICAL PROBLEM

1. The wheel of an automobile is rotating with 4 rotations per second. Find its angular velocity. If the radius of the flywheel is 50 cm. find the linear velocity of a point on its circumference. (Ans. 8π rads^{-1}, 400π cms^{-1})
2. A body of mass 5 kg is revolving in a circle of diameter 0.30 cm making 2000 revolution in 2 minutes. Calculate the linear velocity and centripetal acceleration. (Ans. 15.71m/s, 1646.3m/s^2)
3. Find the magnitude of the centripetal acceleration of a particle on the tip of a blade, 0.30 m in diameter rotating at 1200 rev/minute. (Ans, 2370.6m/s^2)
4. Calculate the linear acceleration of a particle moving in a circle of radius 0.5 m at the instant when its angular velocity is 2 rads^{-1} and its angular acceleration is 6 rads^{-2}. (Ans. 3.6m/s^2, $33^o\ 42'$ with tangential acceleration)
5. Give explanation for each of the following observations:-
   (a) If there is a net force on a particle in uniform circular motion, why does the particle's linear speed not change?
   (b) As a car rounds a banked circular curve at constant speed, several forces are acting on it, for example, air resistance towards the rear, friction from the pavement in the forward direction, gravity, and the normal force from the tilted road surface.

In what direction does the net force point?

6. What does the word "uniform" indicate in uniform circular motion?

CHARACTERISTICS OF UNIFORM CIRCULAR MOTION

1. Angular speed of the object is constant.
2. Angular acceleration of the object is zero.
3. Linear velocity of the object has constant magnitude but its direction keeps on changing continuously.
4. Uniform circular motion is an accelerated motion, in which acceleration is directed and towards its centre.
5. It is a periodic motion having a definite period and frequency.

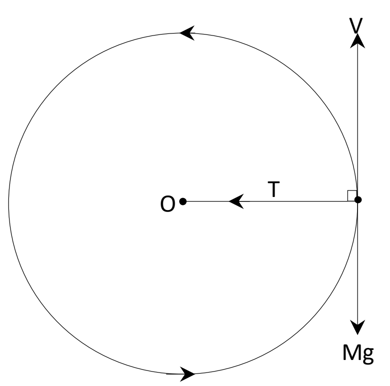
EXAMPLES OF THE CIRCULAR MOTION

Circular motion can be described into two ways:-

1. Motion in a horizontal circle.
2. Motion in a vertical circle.

1. MOTION IN A HORIZONTAL CIRCLE

Case 1. Consider an object of mass m tied to one end of a string and whirled in a circular path on a horizontal plane as shown in the figure below:



Force acting on an object are weight (W = Mg) and tension, T on the string. The weight has no component towards to the centre, 0 of the circle. The force which direct object towards to the centre is only tension on the string. Thus the tension will provide necessary centripetal force to keep the object in a circular path.

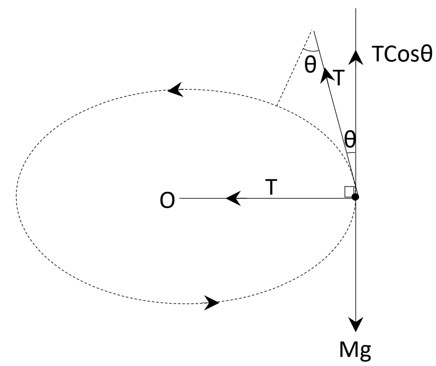
$$T = \frac{MV^2}{r}$$

Also, $T = M\omega r$

Note that:

If the motion of the object is in air its is not possible for the string to be perfectly horizontally. Why?

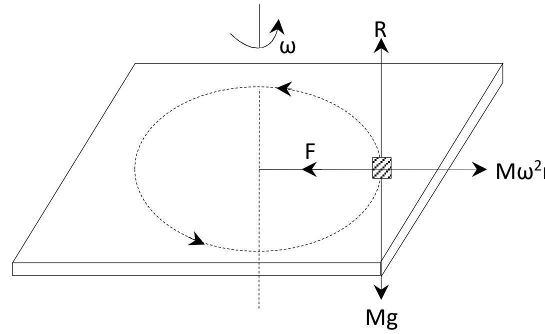
To balance the weight of the object the string is inclined, so that the vertical component of tension supports the weight.



However, the angle θ may be made nearly 90° so that the motion is nearly horizontal circle.

Case 2. Particle at rest on a turn table

Consider a particle of mass, M resting on a turn table which is rotating at a constant angular speed, ω at a distance r from the centre. The coefficient of friction force between the table and the particle is μ. The friction force between table and the practical procedures necessary centripetal force for the particle to rotate with the turntable.



R is the normal reaction force of the table (plane) on the particle R = Mg _____(i) For the particle to rotate with the turntable

$\mu R \geq M\omega^2$

$\mu Mg \geq M\omega^2 r$

$$\mu \geq \frac{\omega^2 r}{g} \qquad \text{\_\_\_\_\_\_\_\_\_\_ (ii)}$$

The minimum coefficient friction for the particle to rotate with the table can be obtained when the particle is at equilibrium.

$\mu r = M\omega^2 r$

$\mu Mg = M\omega^2 r$

$$\omega = \sqrt{\frac{\mu g}{r}} \text{ or } \mu = \frac{\omega^2 r}{g}$$

$$2\pi f = \sqrt{\frac{\mu g}{r}},\ f = \frac{1}{2\pi}\sqrt{\frac{\mu g}{r}}$$

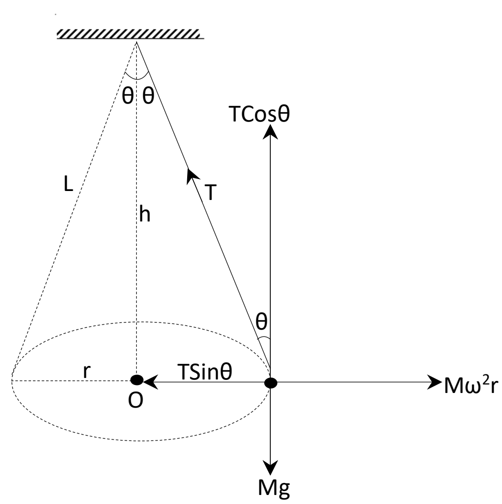
∴ The maximum angular speed for the coin for rotate with the turn table is given by:

$$\omega = \sqrt{\frac{\mu g}{r}}$$

- If the friction force is less than the centripetal force the particle will slide towards the centre of rotation.

Case 3. Conical pendulum

A conical pendulum is a simple pendulum consisting of a small heavy bob attached to a light inextensible string, the bob is made to revolve in a horizontal circle with uniform circular motion about a vertical axis such that the string traces out a cone. Consider a body of mass M revolving in a horizontal circle of radius r at a constant speed V as shown in the figure below:



The centripetal force can be provided by the horizontal component of the tension on the string i.e TSinθ.

- Expression of the tension on the string.

At the equilibrium

$$TSin\theta = \frac{MV^2}{r} = M\omega^2 r \text{ \_\_\_\_\_\_\_ (i)}$$

$$TCos\theta = Mg \text{ \_\_\_\_\_\_ (ii)}$$

$$\text{Takes } (TCos\theta)^2 + (TSin\theta)^2 = \left(\frac{MV^2}{r}\right)^2 + (Mg)^2$$

$$T^2(Cos^2\theta + Sin^2\theta) = \left(\frac{MV^2}{r}\right)^2 + (Mg)^2$$

$$T = \sqrt{\left(\frac{MV^2}{r}\right)^2 + (Mg)^2} \text{ \_\_\_\_\_\_\_ (iii)}$$

$$\text{Also } Sin\theta = \frac{r}{L},\ r = LSin\theta$$

$$TSin\theta = M\,\omega LSin^2\theta$$

$$T = M\omega^2 L \text{ \_\_\_\_\_\_\_ (iv)}$$

Where T = tension on the string, M = mass of the body, V = speed, r = radius, L = length of the string, ω = angular velocity, θ = Angle between the string with the vertical.

$$\text{Again, } \frac{TSin\theta}{TCos\theta} = \frac{MV^2}{r}/mg$$

$$\tan\theta = \frac{V^2}{rg}$$

$$\text{Also } \tan\theta = \frac{r}{h} = \frac{V^2}{rg}$$

$$V^2 = \frac{r^2 g}{h}$$

From the equation (iii)

$$T = \sqrt{\left(\frac{MV^2}{r}\right)^2 + (Mg)^2}$$

$$= \sqrt{\left(\frac{M}{r}\bullet\frac{r^2 g}{h}\right)^2 + (Mg)^2}$$

$$T = Mg\bullet\sqrt{1 + \left(\frac{r}{h}\right)^2}$$

- Expression of velocity.

$$\tan\theta = \frac{V^2}{rg},\ V^2 = rg\tan\theta$$

$$V = \sqrt{rg\tan\theta} = \sqrt{gLSin\theta\tan\theta}$$

$$\text{From } V = \omega r = \sqrt{rg\tan\theta}$$

$$\omega = \sqrt{\frac{g\tan\theta}{r}},\ \tan\theta = \frac{r}{h}$$

$$\omega = \sqrt{\frac{g}{h}}.\ \text{Also } \cos\theta = \frac{h}{L}$$

$$\omega = \sqrt{\frac{g}{L\cos\theta}}$$

Expression of the periodic time

$$\omega = \sqrt{\frac{g\tan\theta}{r}} = \sqrt{\frac{g}{L\cos\theta}}$$

but $\omega = \dfrac{2\pi}{T}$

$$T = 2\pi\sqrt{\frac{r}{g\tan\theta}} = 2\pi\sqrt{\frac{L\cos\theta}{g}}$$

Expressions of the frequency of rotations of the bob.

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{L\cos\theta}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g\tan\theta}{r}} = \frac{1}{2\pi}\sqrt{\frac{g}{h}}$$

Note that:

(i) Expression of te tension on the string.

$$T = \frac{Mg}{\cos\theta}\ \frac{MgL}{h} = \frac{MgL}{\sqrt{L^2 - r^2}}$$

(ii) The principle of conical pendulum used in construction of the centrifugal governer used for regulating automatically the speed of engines.

TYPES B: SOLVED EXAMPLES

Example 1

A 500 g stone attached to a string is whirled in a horizontal circle at a constant speed of 10m/s. the length of the string is 1.0 m. Neglecting the effect of gravity, find:

(a) The centripetal acceleration of the stone and

(b) The centripetal force acting on the stone.

Solution

(a) The centripetal acceleration

$$a_c = \frac{V^2}{r} = \frac{10^2}{1}$$

$a_c = 100\text{m/s}^2$.

(b) The centripetal force, $F_c = Ma_c$

$F_c = 0.5 \times 100 = 50\text{N}$

$F_c = 50\text{N}$.

Example 2

A rubber stopper 13g is attached to a 0.93 m string. The stopper is swung in a horizontal circle, making 10 revolutions in 31.4 seconds. Find the tension in a sting.

Solution

Using the relation $T = M\omega^2 r$

$$T = M(2\Pi f)^2 r = \frac{0.013 \times 4\pi^2 \times 10^2 \times 0.93}{(31.4)^2}$$

$T = 0.043\text{N}$.

Example 3

A small mass of 1 kg is attached to the lower end of a string 1 m long whose upper end is fixed. The mass is made to rotate in a horizontal circle of radius 0.6 m. if the circular speed of the mass is constant. Find:

(a) Tension, F in the string.

(b) The period of the motion.

Solution

(a) Consider vertical forces

$$F\cos\theta = Mg,\ F = \frac{Mg}{\cos\theta}$$

$$F = \frac{MgL}{\sqrt{L^2 - r^2}} = \frac{1 \times 9.8 \times 1}{\sqrt{1^2 - (0.6)^2}}$$

$F = 12.25\text{N}$.

(b) $T = 2\pi\sqrt{\dfrac{L\cos\theta}{g}}$

$$\cos\theta = \sqrt{\frac{L^2 - r^2}{L}}$$

$$T = 2\pi\sqrt{\frac{1 \times 0.8}{9.8}}$$

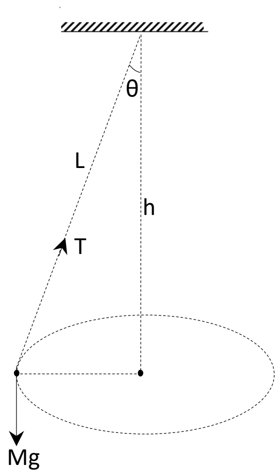
$T = 1.8\text{sec}$.

Example 4

A ball of mass 200 gm attached to the end of a cord of length 1 m and is whirled in a horizontal circle as shown in the figure below, with $\theta = 10^{\circ}$. What is:

(i) The speed of the ball and

(ii) The time period of revolution of the bell?



Solution

(i) $V = \sqrt{gL\text{Sin}\theta\tan\theta}$

$= \sqrt{9.8\times 1\times \text{Sin}10\times\tan 10^{\circ}}$

$V = 0.55\text{m/s}.$

(ii) $T_1 = 2\pi\sqrt{\frac{L\text{Cos}\theta}{g}} = 2\pi\sqrt{\frac{1\times\text{Cos}10^{\circ}}{9.8}}$

$T_1 = 1.9\text{sec}.$

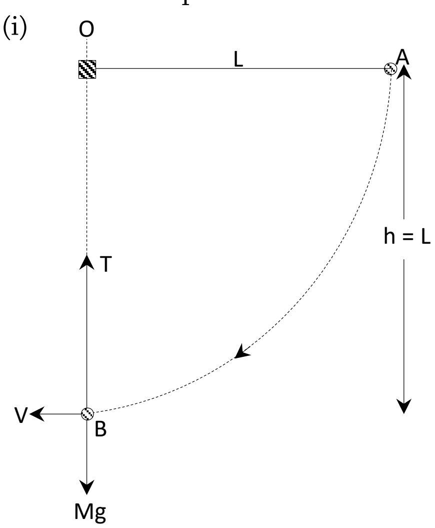
Example 5

A pendulum bob of mass, M held out in the horizontal position and then released. If the length of the string is L. Calculate

(i) The velocity of the bob and

(ii) The force on the string when the bob reaches the lowest position.

(i)



Solution

Apply the Law of conservation of energy.

Decrease in P.E of the bob in falling through height L =Increase in K.e

$MgL = \frac{1}{2}MV^2$

$V = \sqrt{2gL}$

(ii) At the lowest position.

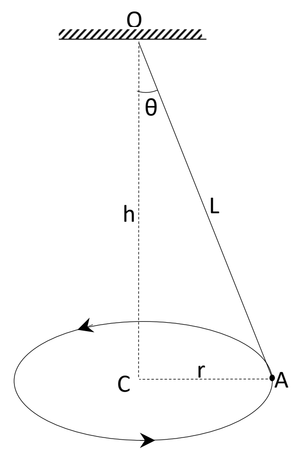
$$\frac{MV^2}{L} = Mg + \frac{2MgL}{L}$$

$T = 3Mg.$

Example 6

A ball of mass 0.1 kg is suspended by a string 30 cm long. Keeping the strings always taut, the ball describes horizontal circle of radius 15 cm. find the angular speed

Solution



$\omega = \frac{g}{h}$

$h = \sqrt{\overline{OA}^2 - \overline{CA}^2}$

$= \sqrt{0.3^2 - 0.15^2}$

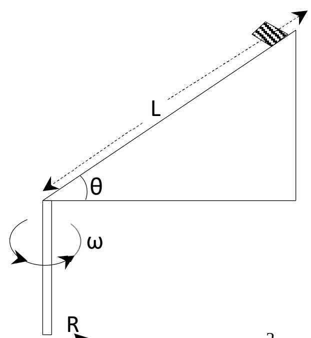
$h = 0.26\text{ m}.$

$\omega = \sqrt{\frac{9.8}{0.26}}$

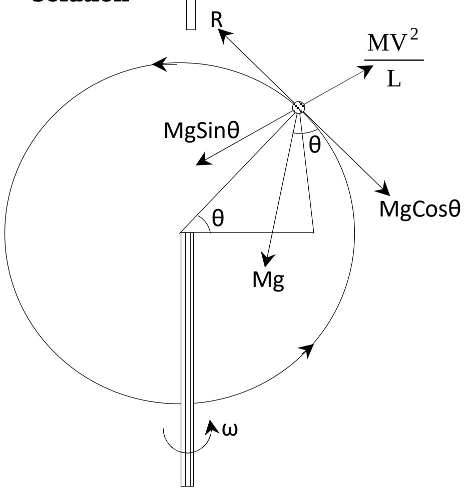
$\omega = 6.14\text{ rad/s}$

Example 7

In the figure below shows a child's toy made of a small wedge which has an acute angle, θ. The slopping side is frictionless. The wedge is attached to a rod. The rod is rotated and the wedge is spun at a constant speed. Show that when the mass, M rises up the wedge at a distance L the speed of mass is $V = \sqrt{gLSin\theta}$



Solution



When the mass M is at the top of the wedge, the component of the weight parallel to the sloping side provide necessary centripetal force.

$$MgSin\theta = \frac{MV^2}{L}$$

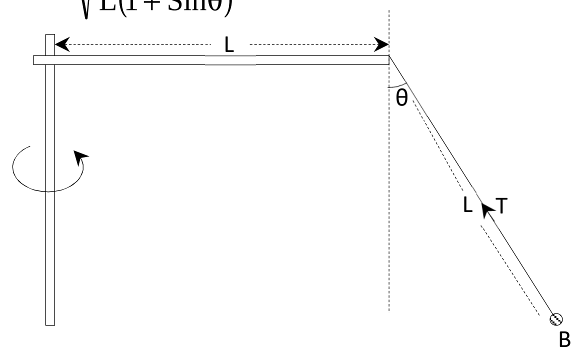
$V^2 = gLSin\theta$

$V = \sqrt{gLSin\theta}$ Hence shown.

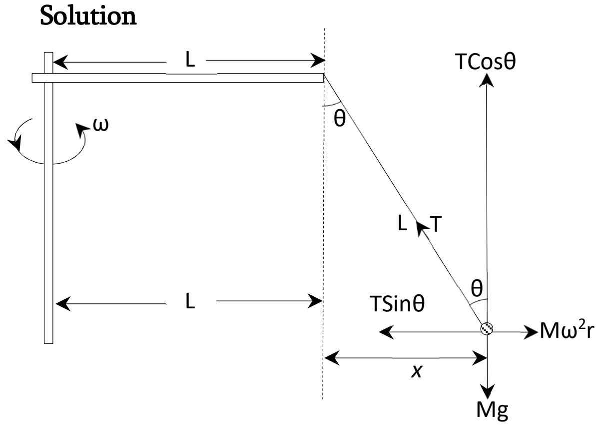
Example 8

Instead of a wedge in the figure below, a uniform rod of length L is attached to the central rod. The free end of the rod of length L a simple pendulum of length is suspended. The mass of the bob is M. When the system is rotated with an angular velocity ω, the angle made by the string with the vertical is θ. Show that the angular velocity of rotation is

$$\omega = \sqrt{\frac{g\tan\theta}{L(1+Sin\theta)}}$$



Solution



TSinθ provides the centripetal force.

Since $Sin\theta = \frac{x}{L}$, $x = LSin\theta$

$r = x + L = L(1 + Sin\theta)$

At the equilibrium

$TSin\theta = M\omega^2 r$

$TSin\theta = M\omega^2 L(1 + Sin\theta)$

$TCos\theta = Mg$__________(2)

Dividing equation (1) and (2)

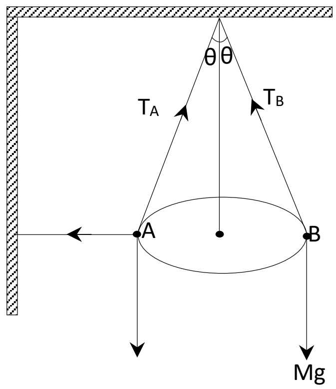
$$\frac{TSin\theta}{TCos\theta} = \frac{M\omega^2(1+Sin\theta)L}{Mg}$$

$$\tan\theta = \frac{\omega^2 L(1+Sin\theta)}{g}$$

$$\omega = \sqrt{\frac{g\tan\theta}{L(1+Sin\theta)}}$$

hence shown

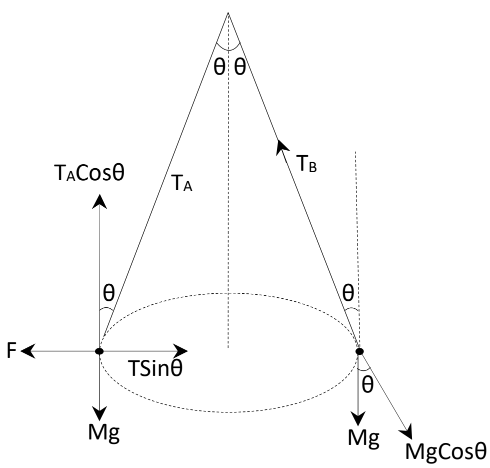
Example 9



Using two light cords, a bob is held in equilibrium at A. The horizontal cord is cut and the bob is allowed to swing as a pendulum. Calculate the ratio of the tension in the supporting cord in position B to that in position A

Solution

Consider the FBD as shown below



At the equilibrium

$T_A Cos\theta = Mg$

$T_B = MgCos\theta$

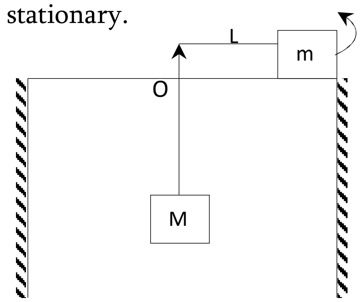
$T_B = (T_A Cos\theta)(Cos\theta)$

$T_B = T_A Cos^2\theta$

$\frac{T_B}{T_A} = Cos^2\theta$

Example 10

Two masses m and M are connected by a light string that passes through a smooth hole O at the centre of a table. Mass m lies on the table and M hangs vertically m is moved round in a horizontal circle with O as the centre. If L is the length of the string from O to m then, find the frequency with which m should revolve so that M remains stationary.



Solution

Let T = tension in the string. At the equilibrium.

$T = Mg$ ____________ (i)

T provides the necessary centripetal force for the revolution of m.

$T = m\omega^2 L$ __________(ii)

(i) = (ii)

$m\omega^2 L = Mg$

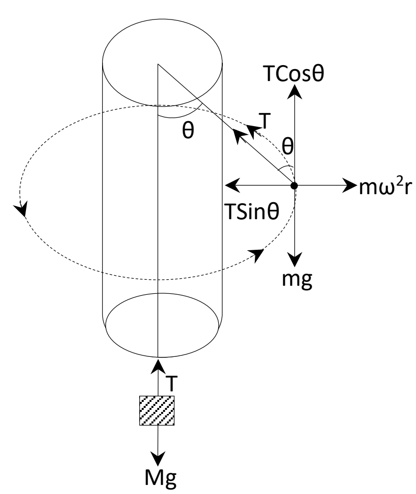
$\omega = \sqrt{\frac{Mg}{mL}}$ but $\omega = 2\pi f$

$2\pi f = \sqrt{\frac{Mg}{mL}}$

$f = \frac{1}{2\pi}\sqrt{\frac{Mg}{mL}}$

Example 11

A large mass M and small mass m hang at the two ends of a string that passes through a smooth tube as shown in figure below.



The mass m moves round in a circular path which lies in the horizontal plane. The length of the string from the mass m to the top of the tube is L and θ is the angle this length makes with the vertical. What should be the frequency of the mass m so that M remains stationary?

Solution

At the equilibrium

$T\sin\theta = m\omega^2 L = m\omega^2 L\sin\theta$

$T = m\omega^2 L$ _____(i)

Also $T = Mg$ _____(ii)

(i) = (ii)

$m\omega^2 L = Mg,\ \omega = \sqrt{\frac{Mg}{mL}}$

$2\pi f = \sqrt{\frac{Mg}{mL}}$

$f = \frac{1}{2\pi}\sqrt{\frac{Mg}{mL}}$

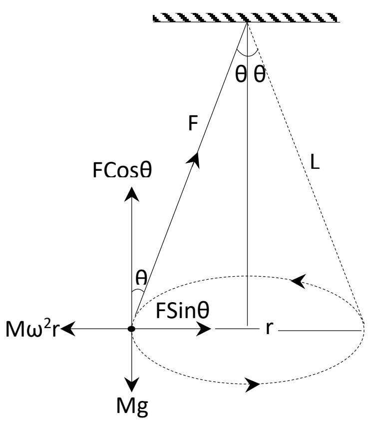
Example 12

A sphere of mass 200 gm is attached to an inextensible string of length 130 cm whose upper end is fixed to the celling the sphere is made to describe a horizontal circle radius 50 cm.

(i) Calculate the time period of revolution.

(ii) What is the tension in the string.

Solution



(i) $T = 2\pi\sqrt{\frac{L\cos\theta}{g}} = 2\pi\sqrt{\frac{130\times120}{130\times980}}$

$T = 2.2$ sec.

(ii) $F\cos\theta = Mg,\ F = \frac{Mg}{\cos\theta}$

$F = \frac{0.2\times9.8}{120/130}$

$F = 2.12N.$

Example 13

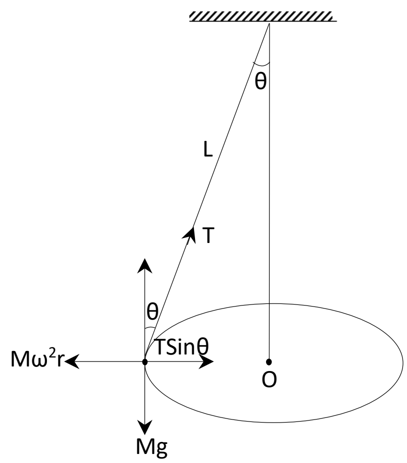
A string of length 1 m is fixed at one end and carried a mass of 100 gm at the other end. The string makes $\frac{2}{\pi}$ revolution per second around a vertical axis passing through its fixed end. Calculate:-

(a) The angle of inclination of the string with the vertical.

(b) The tension in the string.

(c) The linear velocity of the mass.

Solution



(a) At the equilibrium

$T\cos\theta = Mg$

$\cos\theta = \frac{Mg}{T} = \frac{0.1\times9.8}{1.6}$

$\cos\theta = 0.6125.$

$\theta = 52^\circ 14'$

(b) $T = M\omega^2 L = 0.1 \times 4\pi^2 \times \frac{4\times1}{\pi^2}$

$T = 1.6N$

(c) $V = \omega r = 2\pi f L\sin\theta$

$= 2\pi \times \frac{2}{h} \times 1 \times \sin 52^\circ 14'$

$V = 3.6\ m/s^2.$

Example 14

A special prototype model aeroplane of mass 400 gm long attached to its body. The other end of the control line is attached to a fixed point. When the aeroplane flies with its wings horizontal in a horizontal circle, making one revolution in every 4S, the control wire is elevated 30° above the horizontal. Draw a diagram showing the forces exerted on the plane and determined:-

(a) The tension in the control wire.

(b) The lift on the plane.

Solution

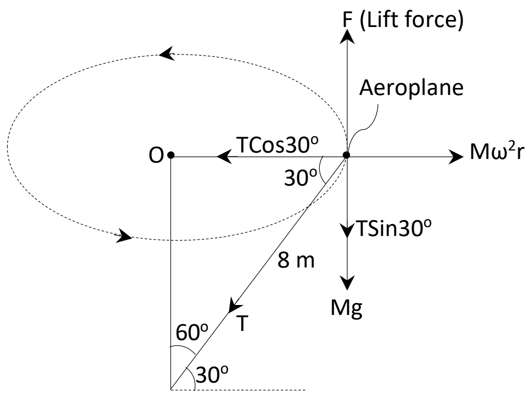


Diagram of the forces

(a) Let T = tension in the control wire.

At the equilibrium

$TCOS30^o = M\omega^2 r$

$$T = \frac{M\omega^2 r}{Cos30^o} = \frac{0.4\times(\pi/2)^2\times 8Sin60^o}{Cos30^0}$$

T = 8.0N.

(b) At the equilibrium of Aeroplane.

$F = Mg + TSin30^o$

$= 0.4 \times 10 + 8Sin30^o$

F = 8.0N.

Example 15

A particle of mass 80 g rests at 16 cm from the centre of a turn table. If the maximum frictional force between the particle and the turn table is 0.72N. What is the maximum angular velocity at which the turn table could rotate without the particle slipping?

Solution

Since $f = \mu Mg$, $\mu = \dfrac{F}{Mg}$

$$\mu = \frac{0.72}{0.08\times 9.8} = 0.91837$$

Maximum angular speed

$$\omega = \sqrt{\frac{\mu g}{r}} = \sqrt{\frac{0.91837\times 9.8}{0.16}}$$

$\omega = 7.5$ rad/s.

Example 16

A small button placed on a horizontal rotating plat form with diameter 0.320 m will revolve with the plat form when it is brought up to a speed of 40 rev/min, provided. The button is more than 0.150 m from the axis.

(a) What is the coefficient of static friction between the button and the plat form?

(b) How far from the axis can the button be placed, without slipping, if the plat form rotates at 60rev/min?

Solution

(a) $$\mu = \frac{\omega^2 r}{g} = \frac{(2\pi f)^2 r}{g}$$

$$= \frac{(2\pi\times 40/60)^2\times 0.15}{9.8}$$

$\mu = 0.27$

(b) $$r = \frac{\mu g}{(2\pi f)^2} = \frac{0.2686\times 9.8}{(2\pi\times 1)^2}$$

r = 0.067 m.

Example 17

A string 0.6 m long can just sustain a weight of 18 kg, without breaking. A mass of 1.8 kg is attached to one end of the string and revolves uniformly on a smooth table. Find the greatest number of complete revolutions the mass can make, in a minute without breaking the string

Solution

Since $T_{max} = M\omega^2 r$

$$2\pi f = \omega = \sqrt{\frac{T_{max}}{rM}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{Mg}{rM}} = \frac{1}{2\pi}\sqrt{\frac{18\times 9.8}{1.8\times 0.6}}$$

f = 2.03 rev/s = 122 rev/min.

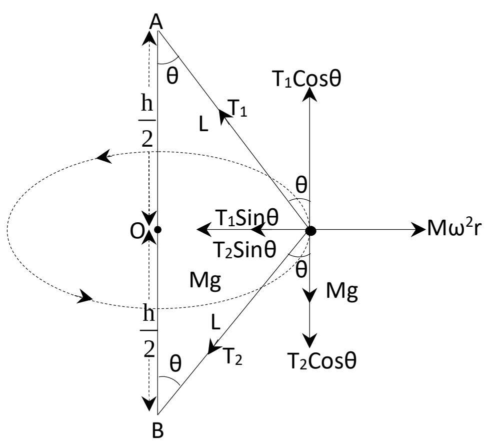
Example 18

A particle is attached by means of two equal strings to two points A and B in the same vertical line and describes a horizontal circle with uniform angular speed.

(a) Prove that, in order that both strings may remain stretched, the angular speed must exceed $2\sqrt{\frac{2g}{h}}$ where h = AB.

(b) If the angular speed is $2\sqrt{\frac{2g}{h}}$, prove that the ratio of the tensions of the strings is 5:3

Solution



(a) At the equilibrium

Vertical components of the forces.

$(T_1 - T_2)\cos\theta = Mg$

Horizontal component of forces

$(T_1 + T_2)\sin\theta = M\omega^2 r$

But $\cos\theta = \frac{h}{2L}$, $\sin\theta = \frac{r}{L}$

Now $(T_1 - T_2)\frac{h}{2L} = Mg$

$T_1 - T_2 = 2\frac{MgL}{h}$ ______(i)

Also

$(T_1 + T_2)\sin\theta = M\omega^2 L\sin\theta$

$T_1 + T_2 = M\omega^2 L$ _______(ii)

On solving simultaneously equation (i) and (ii)

$$T_1 = \frac{1}{2}ML\left(\omega^2 + \frac{2g}{h}\right)$$

$$T_2 = \frac{1}{2}ML\left(\omega^2 - \frac{2g}{h}\right)$$

For both string to remain stretched, $T_2 > 0$

$$\frac{1}{2}ML\left(\omega^2 - \frac{2g}{h}\right) > 0$$

$\omega > \sqrt{\frac{2g}{h}}$ proved

(b) Given that $\omega = 2\sqrt{\frac{2g}{h}}$

$$\omega^2 = \frac{8g}{h}$$

$$T_1 = \frac{ML}{2}\left(\omega^2 + \frac{2g}{h}\right) = \frac{ML}{2}\left(\frac{8g}{h} + \frac{2g}{h}\right)$$

$$T_1 = \frac{5MgL}{h}$$

Also

$$T_2 = \frac{ML}{2}\left(\omega^2 - \frac{2g}{h}\right) = \frac{ML}{2}\left(\frac{8g}{h} - \frac{2g}{h}\right)$$

$$T_2 = 2\frac{MgL}{h}$$

Takes $\frac{T_1}{T_2} = \frac{5MgL}{h} / \frac{3MgL}{h}$

$$\frac{T_1}{T_2} = \frac{5}{3} = 5:3$$

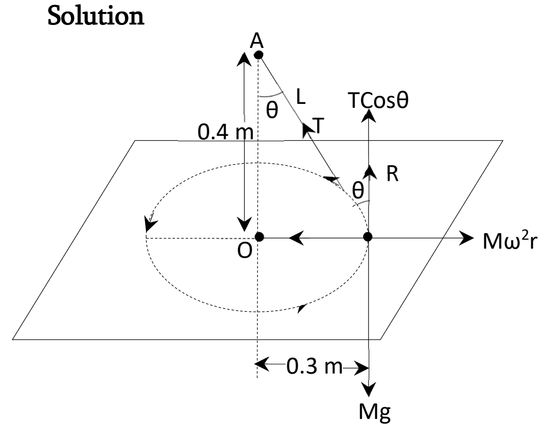
Example 19

A particle is attached by means of a light inextensible string to a point 0.40 m above a smooth table in a circle of radius 0.30 m with angular velocity ω.

(a) Find the reaction on the table in terms of ω.

(b) Hence, find the maximum angular velocity for which the particle can remain on the table.

Solution



By using Pythagoras theorem

$L = \sqrt{0.4^2 + 0.3^2} = 0.5\ M$

From the figure above

$\text{Sin}\theta = \dfrac{0.3}{0.5} = \dfrac{3}{5},\ \text{Cos}\theta = \dfrac{0.4}{0.5} = \dfrac{4}{6}$

$\tan\theta = \dfrac{3}{4}$

(a) At the equilibrium vertical component of the forces.

$T\text{Cos}\theta + R = Mg$

$R = Mg - T\text{Cos}\theta$

$= Mg - T \bullet \dfrac{4}{5}$

$R = Mg - \dfrac{4}{5}T$ ______(i)

Horizontal component of force

$T\text{Sin}\theta = M\omega^2 r$

$T = \dfrac{M\omega^2 r}{\text{Sin}\theta} = \dfrac{0.3M\omega^2}{3/5}$

$T = 0.5\ M\omega^2$ _________(ii)

Putting equation (ii) into (i)

$R = Mg - \dfrac{4}{5}\left(0.5M\omega^2\right)$

$R = M(g - 0.4\omega^2)$.

(b) For the maximum speed of particle to remain on the table, $R = 0$.

$0 = m(g - 0.4\omega^2)$

$0 = g - 0.4\omega^2$

$\omega = \sqrt{\dfrac{g}{0.4}} = \sqrt{\dfrac{10}{0.4}}$

$\omega = 5$ rad/s.

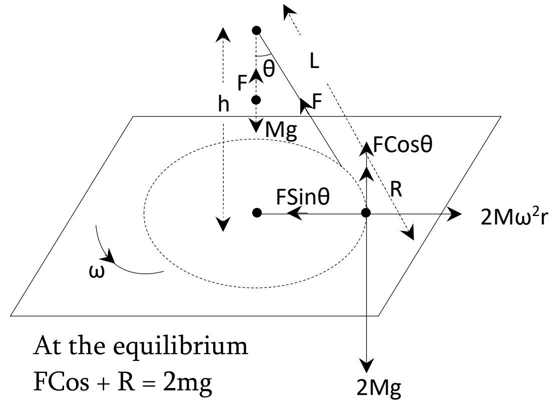
Example 20

A mass 2 m rests on a smooth horizontal table and is connected to a mass m by a light inextensible string passing through a small ring fixed at a height h above the table. If the mass 2 m is made to describe a circle of radius $\dfrac{1}{2}h$ having its centre on the table vertically below the ring show that the time it takes to describe the circle once.

$T = 2\pi\sqrt{\dfrac{h\sqrt{5}}{g}}$

Solution

Let R be normal reaction force on the table and F is the tension throughout the string.



At the equilibrium

$F\text{Cos} + R = 2mg$

$F = mg$

$R = 2mg - F\text{Cos}\theta$

$= 2mg - mgF\text{Cos}\theta$________(i)

$F\text{Sin}\theta = 2M\omega^2 r$

$\omega = \sqrt{\dfrac{F\text{Sin}\theta}{2Mr}} = \sqrt{\dfrac{Mg\dfrac{h/2}{\sqrt{h^2 + (h/2)^2}}}{2M(h/2)}}$

$\dfrac{2\pi}{T} = \sqrt{\dfrac{g}{2\sqrt{5/4h^2}}} = \sqrt{\dfrac{g}{h\sqrt{5}}}$

$T = 2\pi\sqrt{\dfrac{h\sqrt{5}}{g}}$ Hence shown

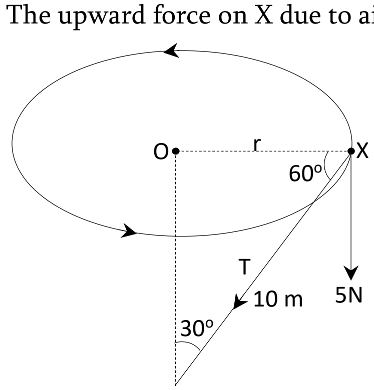
Example 21

A model aeroplane X has a mass of 0.5 kg and has a control wire OX of length 10 m attached to it when it flies in a horizontal circle with its wings horizontal, figure below. The wire OX is then inclined at 60°to the horizontal and fixed to a point O and X takes 2 seconds to fly once round its circular path.

Calculate

(a) The tension T in the control wire.

(b) The upward force on X due to air.



Solution

(a) $\omega = \frac{2\pi}{2} = \pi rads^{-1}$

$F = M\omega^2 r = TCos60^o$

but $r = LSin30^o = 10Sin30^o = 5$ m

$TCos60^o = 0.5 \times 5\pi^2$

$T = \frac{2.5\pi^2}{0.5}$

$T = 50N$ (approx)

(b) Upward force due to the air = weight of X + $TCos30^o$

$\mu = 48N$ (approx)

Example 22

A long playing record revolves with speed of $33\frac{1}{3}$ rev/min and has a radius of 15 cm. two coins are placed at 4 cm and 14 cm always from the centre of the record. If the coefficient of friction between the coins and the record is 0.15, which of the two coin, if any will revolve with record?

Solution

For a coin to revolve with the record the force of friction must be sufficient to provide the necessary centripetal force.

$M\omega^2 r \leq \mu Mg$ or $r \leq \frac{\mu g}{\omega^2}$

$r \leq \frac{0.15 \times 9.8 \times 60^2}{4\pi^2 (100/3)^2}$

$r \leq 0.12$ m

$r \leq 12$ cm.

Therefore, the coin at 4 cm distance from the centre will revolve with the record since it fill condition above.

Example 23 NECTA 1995

What is the speed of an aircraft moving in a horizontal circle with radius 300 m. if the acceleration experienced by the path is 4g.

Solution

The centripetal acceleration

$a = \frac{V^2}{r} = 4g$

$V^2 = 4gr$, $V = \sqrt{4gr}$

$V = \sqrt{4 \times 9.8 \times 300}$

$V = 108.44$ m/s.

Example 24

A man stands on the Earth at latitude 60°N. Calculate:-

(a) Its angular velocity.

(b) It linear velocity.

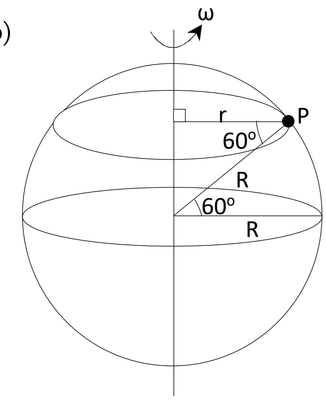
(c) It acceleration due to the rotation of the Earth about its axis. ($g = 9.8$ m/s², $R = 6400$ km)

Solution

(a) $\omega = \frac{2\pi}{T} = \frac{2\pi}{24 \times 3600}$

$\omega = 7.3 \times 10^{-5}$ rads^{-1}

(b)



$Cos60^o = \frac{r}{R}$

$r = RCos60^o$

$V = \omega r$

$= \omega RCos60^o$

$= 7.3 \times 10^{-5} \times 6.4 \times 10^6 \times \frac{1}{2}$

$V = 233.6$m/s

(c) $a = \omega^2 r = (7.3 \times 10^{-5})^2 \, 6.4 \times 10^6 \times \frac{1}{2}$

$a = 0.017$ m/s².

Example 25

One end of an elastic string 0.6 m long, is attached to a fixed point on a smooth table and the other end to a mass of 1.8 kg resting on the table. If the 1.8 kg mass were suspended vertically by the string the extension would be 10 cm. The mass is made to describe a circle round the fixed point at 40 r.p.m. calculate the extension of the string.

Solution

Elastic constant of the string

$K = \frac{Mg}{e} = \frac{1.8 \times 9.8}{0.1}$

$K = 176.4Nm^{-1}$.

Tension in the string when the mass describes the circle

$T = M\omega^2 r = M(2\pi f)^2 r$

$= 1.8\ (2\pi \times 40/60)^2\ 0.6$

$T = 18.95N$.

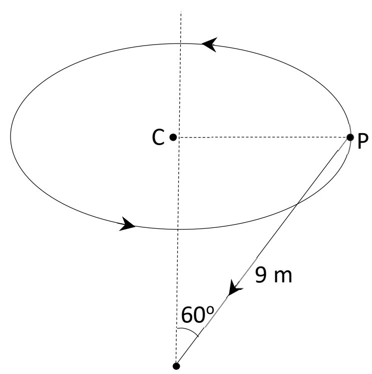
Extension produced is given by

$e = \frac{T}{K} = \frac{18.95}{176.4}$

$e = 0.107$ m.

Example 26

A toy aircraft of mass 0.5 kg is attached to one end of a light inextensible string of length 9 m. The other end of the string is attached to a fixed point O. The aircraft moves with constant speed in a horizontal circle. The string is taut, and makes an angle of 60° with the upward vertical at O as show in figure below.



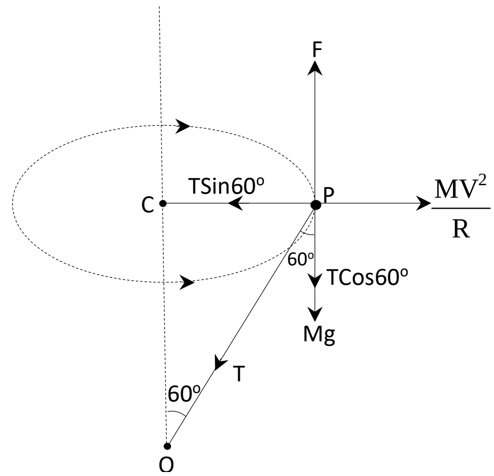
In a simplified model of the motion, the aircraft is treated as a particle and the force of the air on the aircraft is taken to act vertically upward with magnitude 8N. Find

(a) The tension in the string.

(b) The speed of the aircraft

Solution

Consider the FBD as shown below.



(a) At the equilibrium of an aircraft.

$TCos60^o + Mg = F$

$0.5T + 0.5 \times 10 = 8$

$T = 6N$.

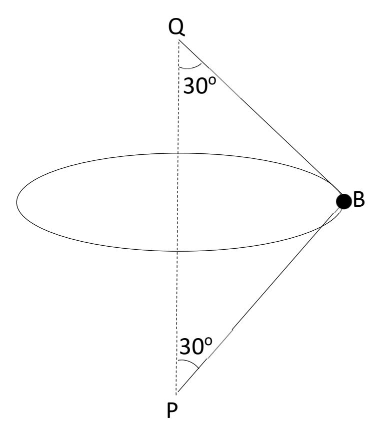
(b) $\frac{MV^2}{R} = TSin60^o$

$\frac{0.5V^2}{9Sin60^o} = 6Sin60^o$

$V = 9$ m/s.

Example 27

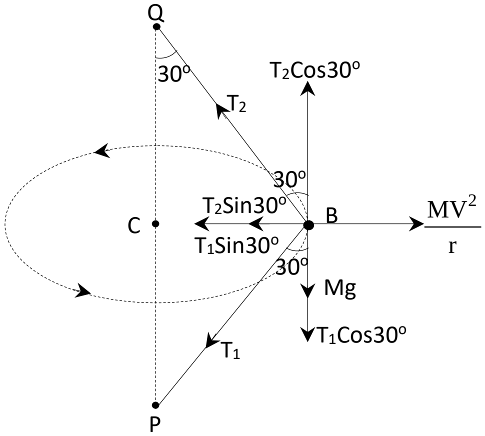
A small ball B of mass 0.4 kg is attached to fixed points P and Q on a vertical axis by two light inextensible of equal length both strings are taut and each is inclined at 30° to the vertical. The ball moves in a horizontal circle as shown in figure below.



(a) It is given that when the ball moves with a speed 6m/s, the tension in the string QB is three times the tension in the string PB. Calculate the radius of the circle. The ball now moves along this circular path with the minimum possible speed.

(b) State the tension in the string PB in this case and find the speed of the ball.

Solution



(a) At the equilibrium for the ball.

$T_2Cos30^o - T_1Cos30^o = Mg$

$3T_1Cos30^o - T_1Cos30^o = Mg$

$2T_1Cos30^o = 0.4 \times 10$

$T_1 = 2.31N$

$T_2 = 3T_1 = 6.93N$

Also

$$\frac{MV^2}{r} = T_1Sin30^o + T_2Sin30^o$$

$$\frac{0.4 \times 6^2}{r} = (2.31 + 6.93)Sin30^o$$

r = 3.12 m.

(b) When the ball moves along this circular path with the minimum possible speed, the tension T_1 in the string is zero

Then $T_2Cos30^o = Mg$

$$T_2 = \frac{0.4 \times 10}{0.866} = 4.62N$$

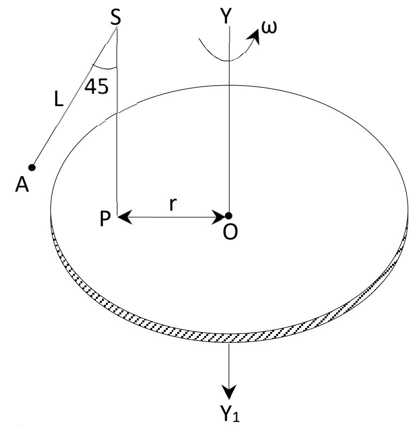
Also $\frac{MV^2}{r} = T_2Sin30^o$

$$\frac{0.4\ V^2}{3.12} = 4.62 \times 0.866$$

V = 4.25 m/s.

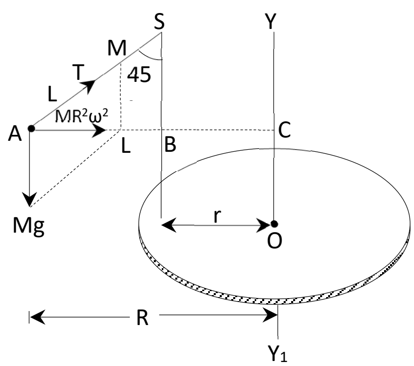
Example 28

A ball of mass, m is attached to the end of a thread fastened to the top of a vertical rod which is secured to the horizontally revolving round the table shown below. With what angular velocity will the table rotate, if the thread forms an angle of 45^o with the vertical? Given that the length of the thread is 6 cm and the distance of the rod from the axis of rotation is 10 cm.



Solution

Consider the FBD as shown on the figure below.



Now R = AC = AB + BC

= $ASSin45^o$ + PO

$$R = \frac{L}{\sqrt{2}} + r$$

L = 6 cm = 0.06 m, r = 10 cm = 0.1 m

$$R = \frac{0.06}{\sqrt{2}} + 0.1$$

R = 0.1424 m.

The tension T_1 weight Mg and the centripetal force $F = M\omega^2R$ can be represented respectively by the sides AM, ML and AL of the right angles ΔALM

Now, $\tan 45^o = \dfrac{AL}{ML} = \dfrac{M\omega^2 R}{Mg}$

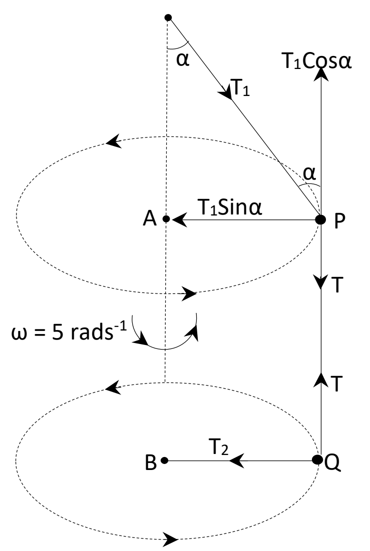
$\tan 45^o = \dfrac{R\omega^2}{g}$

$\omega = \sqrt{\dfrac{g \tan 45^o}{R}} = \sqrt{\dfrac{9.8}{0.1424}}$

$\omega = 8.3\ rads^{-1}$.

Example 29

The particles P and Q have masses 0.8 kg and 0.4 kg respectively. P is attaced to a fixed point A by a light inextensible string , which is inclined at an angle α to the vertical. Q is attached to a fixed point B, which is vertically below A, by a light inextensible string of length 0.3 m. the string BQ is horizontal. P and Q are joined to each other by a light inextensible string which is vertical. The particles rotate in horizontal circle of radius 0.3 m about the axis through A and B with constant angular speed 5 rad/s (See figure below)



(a) By consider the motion of Q, find the tension in the string PQ and BQ.
(b) Find the tension in the string AP and the value of α.

Solution

$M_P = 0.8$ kg $M_Q = 0.4$ kg

$\varpi = 5\ rads^{-1}$ $R = 0.3$ m

(a) The tension in the string PQ

$T = M_Q g = 0.4 \times 10$

$T = 4N$.

The tension in the string BQ

$T_2 = M_Q R\omega^2 = 0.4 \times 0.3 \times 5^2$

$T_2 = 3N$.

(b) At the equilibrium at P

$T_1 Sin\alpha = M_P \omega^2 R$

$T_1 Sin\alpha = 0.8 \times 0.3 \times 5^2$

$T_1 Sin\alpha = 6N$.

Also

$T_1 Cos\alpha = (M_P + M_Q)g$

$= (0.8 + 0.4) \times 10$

$T_1 Cos\alpha = 12N$.

Takes $\dfrac{T_1 Sin\alpha}{T_1 Cos\alpha} = \dfrac{6}{12} = \dfrac{1}{2}$

$\alpha = \tan^{-1}\dfrac{1}{2}$

$\alpha = 26.56^o$.

Again

$T_1 = \sqrt{6^2 \times 12^2}$

$T_1 = 13.4N$.

Example 30

Two blocks of mass $M_1 = 10$ kg and $M_2 = 5$ kg connected to each other by a mass less inextensible string of length 0.3 m are places along a diameter of a turn table. The coefficient of friction between the table and M_1 is 0.5, while there is no friction between M_2 and the table. The table is rotating with an angular velocity of $10 rads^{-1}$ about a vertical axis passing through its centre O. The masses are placed along the diameter of the table on the either side of the centre O, such that the mass M_1 is at a distance of 0.124 m from O. The masses are observed to be at rest w.r.t to an observer on the turn table.

(a) Calculate the frictional force on M_1.
(b) What should be the minimum angular speed of the turn table so that the masses will slip from this position.

Solution

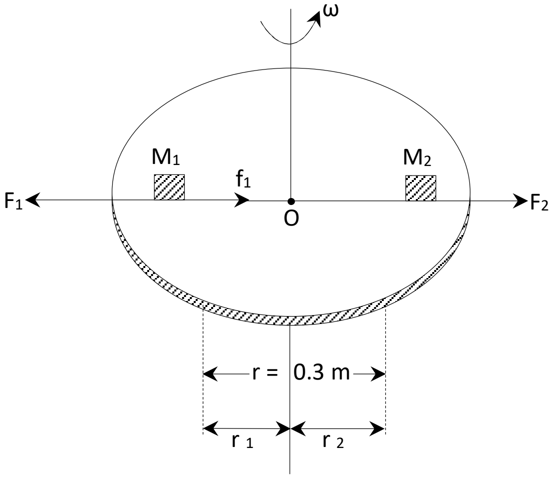
(a) Here $M_1 = 10$ kg, $M_2 = 5$ kg

$\omega = 10\ rads^{-1}$, $\mu = 0.5$

$r = 0.3$ cm, $r_1 = 0.124$ m

$r_2 = r - r_1 = 0.3 - 0.124 = 0.176$ cm.

As the turn table rotates, the centrifugal forces F_1, and F_2 act on the masses M_1 and M_2 in the directions away from the centre O of the turn table.



Force f_1 due friction acts on mass M_1 towards the centre O i.e in a direction opposite to that of F_1.

Now

$F_1 = M_1r_1\omega^2$; $F_2 = M_2r_2\omega^2$

$f_1 = \mu M_1g$.

As the two masses are in equilibrium

$F_1 - f_1 = F_2$

$f_1 = F_1 - F_2 = (M_1r_1 - M_2r_2)\omega^2$

$f_1 = (10 \times 0.124 - 5 \times 0.176) \times 10^2$

$f_1 = 36N$.

(b) The masses will slip from their positions on the turn table, if $f_1 \leq F_1 - F_2$. If ω_{min} is the angular velocity of the turn table which f_1 is just equal to $F_1 - F_2$.

$$f_1 = (M_1r_1 - M_2r_2)\,\omega_{mm}^2$$

$$\omega_{min} = \sqrt{\frac{f_1}{M_1r_1 - M_2r_2}}$$

$$= \sqrt{\frac{\mu M_1 g}{M_1r_1 - M_2r_2}}$$

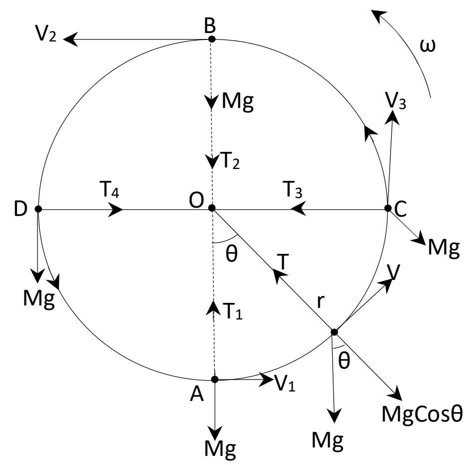
$$= \sqrt{\frac{0.5\times10\times9.8}{10\times0.124-5\times0.176}}$$

$$\omega_{min} = 11.67\ \text{rads}^{-1}$$

2. MOTION IN A VERTICAL CIRCLE

Consider a body of mass tied at one end of a string and whirled in a vertical circle of radius r as shown in figure below. The motion is Non-uniform circular motion because the body have different speeds in the different points on the circular path.

As the body moves from the lowest point A of the vertical circle to the highest point B, the speed of the body decreases because gravity oppose the motion. As the body moves from B to A, speed of the body increase because the gravity helps the motion of the body. Therefore the speed of the body is maximum at lowest position (point A) and minimum at highest position (point B). The original of the centripetal force for the object attached on the string and moves in a vertical circle is the tension on the string.



1. (i) EXPRESSIONS OF TENSIONS IN THE STRING

The resultant force on the string is the centripetal force

- At the point A (i.e Lowest position)

$$T_1 - Mg = \frac{MV_1^2}{r}$$

$$T_1 = M\left(\frac{V_1^2}{r} + g\right) \_\_\_\_\_\_\_\_\_\_ (1)$$

- At the point B (Highest point)

$$T_2 + Mg = \frac{MV_2^2}{r}$$

$$T_2 = M\left(\frac{V_2^2}{r} - g\right) \_\_\_\_\_\_\_\_\_\_ (2)$$

- At the point C (The string when is in horizontal position)

$$T_3 = \frac{MV_3^2}{r} \_\_\_\_\_\_\_\_\_\_ (3)$$

At the point, E

$$T - MgCos\theta = \frac{MV^2}{r}$$

$$T = \frac{MV^2}{r} + MgCos\theta$$

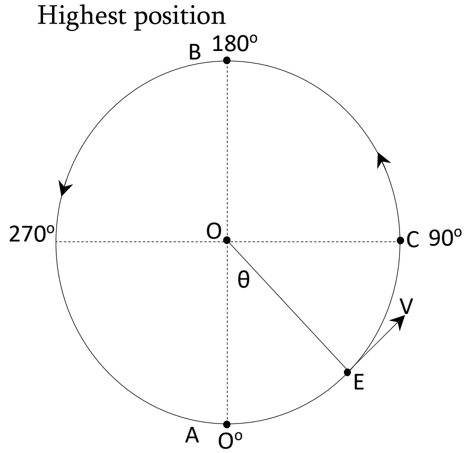
$$T = M\left(\frac{V^2}{r} + gCos\theta\right) \_\_\_\_\_ (4)$$

Equation (4) represent the general equation of the tension on the string.

Verification of the equation

$$T = M\left(\frac{V^2}{r} + gCos\theta\right)$$

Highest position



Lowest position

- At the point A i.e Lowest position $\theta = 0^o$, $V = V_1$, $T = T_1$

$$T_1 = M\left(\frac{V_1^2}{r} + gCos\theta\right)$$

$$T_1 = M\left(\frac{V_1^2}{r} + g\right) \_\_\_\_\_ (i)$$

- At the point B i.e Highest position $\theta = 180^o$, $V = V_2$, $T = T_2$

$$T_2 = M\left(\frac{V_2^2}{r} + gCos180^o\right)$$

$$T_2 = M\left(\frac{V_2^2}{r} - g\right) \_\_\_\_\_ (ii)$$

- At point C: $\theta = 90^o$, $V = V_3$, $T = T_3$

$$T_3 = M\left(\frac{V_3^2}{r} + gCos90^o\right) \_\_\_\_\_ (iii)$$

The equation (1), (2) and (3) are corresponding to the equation (i), (ii) and (iii) respectively.

Therefore, the general equation of the tension on the string is given by

$$T = M\left(\frac{V^2}{r} + gCos\theta\right)$$

The tension in the string is maximum at the LOWEST POSITION and tension in the string is minimum at the HIGHEST POSITION.

Expression of the maximum tension on the string

- For uniform circular motion.

$$T_{max} = M\left(\frac{V^2}{r} + g\right)$$

- For Non – uniform circular motion.

$$T_{max} = M\left(\frac{V_1^2}{r} + g\right)$$

Expression of minimum tension on the string

- For uniform circular motion.

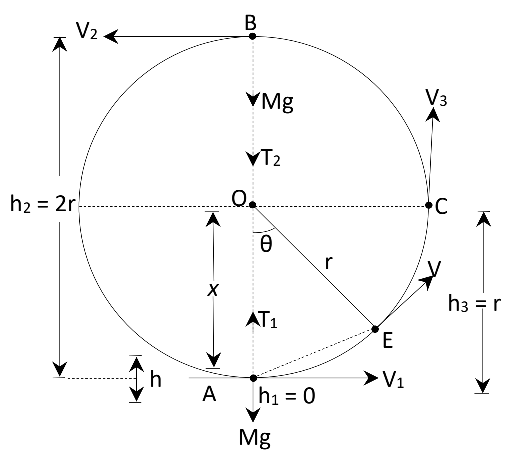
$$T_{min} = M\left(\frac{V^2}{r} - g\right)$$

- For Non – uniform circular motion.

$$T_{min} = M\left(\frac{V_2^2}{r} - g\right)$$

EXPRESSION OF DIFFERENCE IN TENSIONS: $T_1 - T_2$

Consider the figure below



(ii) Expression of $T_1 - T_2$ for Non-uniform circular motion

At point A

$$T_1 = M\left(\frac{V_1^2}{r} + g\right)$$

At point B

$$T_2 = M\left(\frac{V_2^2}{r} - g\right)$$

Now $T_1 - T_2 = \frac{M}{r}\left(V_1^2 - V_2^2\right) + 2\,Mg$

Apply the law of conservation of mechanical energy.

$(P.e + K.e)_A = (P.e + K.e)_B$

$$Mgh_1 + \frac{1}{2}MV_1^2 = Mgh_2 + \frac{1}{2}MV_2^2$$

$$V_1^2 = V_2^2 + 2g(h_2 - h_1)$$

but $h_2 = 2r$, $h_1 = 0$

$$V_1^2 - V_2^2 = 2g(2r - 0)$$

$$V_1^2 - V_2^2 = 4gr$$

Now $T_1 - T_2 = \frac{M}{r}(4gr) + 2Mg$

$T_1 - T_2 = 6Mg.$

(iii) Expression of $T_1 - T_2$ for uniform circular motion

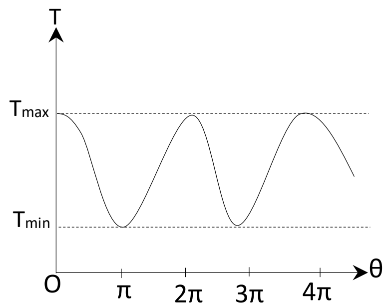
$$T_1 = M\left(\frac{V^2}{r} + g\right) = \frac{MV^2}{r} + Mg$$

$$T_2 = M\left(\frac{V^2}{r} - g\right) = \frac{MV^2}{r} - Mg$$

$T_1 - T_2 = 2Mg.$

GRAPH OF TENSION AGAINST θ

$$T = M\left(\frac{V^2}{r} + gCos\theta\right)$$



2. EXPRESSION ON VELOCITY OF THE PARTICLE IN A VERTICAL CIRCLE

(a) At the highest point i.e B CRITICAL VELOCITY (V_c)

Is the velocity possessed by the body at the highest position when the tension on the string is reduced and becomes equal to zero.

Since

$$T_2 = M\left(\frac{V_2^2}{r} - g\right)$$

When $T_2 = 0$, $V_2 = V_c$

$$0 = M\left(\frac{V_c^2}{r} - g\right)$$

$$V_c = V_2 = \sqrt{rg}$$

Therefore the body must be reached on the highest point B with at least speed. $V_2 = V_c = \sqrt{rg}$ remain in the circular path. The speed at B is less than V_c, the body will fail to loop the circle. It is because the required centripetal force would be less than the weight of the body which will therefore fall down (i.e the string will slack).

(b) At the point B i.e Lowest point

Apply the law of conservation of mechanical energy

$(P.e + K.e)_A = (P.e + K.e)_B$

$$Mgh_1 + \frac{1}{2}MV_1^2 = Mgh_2 + \frac{1}{2}MV_2^2$$

$$V_1^2 = V_2^2 + 2gh_2$$

$h_1 = 0$, $h_2 = 2r$, $V_2^2 = rg_2$

$$V_1^2 = 5gr$$

$$V_1 = \sqrt{5gr}$$

(c) At the point C i.e position in which the string is in horizontal position.

$(P.e + K.e)_A = (P.e + K.e)_C$

$$\frac{1}{2}MV_1^2 = \frac{1}{2}MV_3^2 + Mgr$$

$$V_3^2 = V_1^2 - 2gr$$

$$= 5gr - 2gr = 3gr$$

$$V_3 = \sqrt{3gr}$$

(d) At the point E

$\cos\theta = \frac{x}{r}$, $X = r\cos\theta$

$h = r - x = r(1 - \cos\theta)$

Apply the principle of the conservation of mechanical energy.

$(P.e + K.e)_E = (P.e + K.e)_A$

$$\frac{1}{2}MV^2 + Mgh = \frac{1}{2}MV_1^2$$

$$V^2 = V_1^2 - 2gh$$

$$V^2 = 5gr - 2g(r - r\cos\theta)$$

$$V = \sqrt{gr(3 + 2\cos\theta)}$$

This represent general expression of the velocity of particle in the vertical circle.

Additional concepts

1. Tension on the string will be minimum if $\theta = 180^o$ or $\cos\theta = -1$, the body is at highest position.

$$T_{min} = \frac{MV_2^2}{r} - Mg$$

The body will move in the vertical circle only when

$T_{min} \geq 0$. If $T_{min} < 0$, the string will slack and the body will fall down from B instead of moving in the circle. Hence for completing the vertical circle i.e for Looping the loop,

$$T_{min} = \frac{MV_2^2}{r} - Mg \geq 0$$

$$V_2 \geq \sqrt{gr}$$

Thus minimum value of velocity at the highest point is $\sqrt{gr}$. Also, the tension in the string is maximum when $\cos\theta = 1$ or $\theta = 0^o$

$$T_1 = \left(\frac{MV_2^2}{r} - Mg\right) \geq 0$$

$$V_1 \geq \sqrt{5gr}$$

2. Condition for oscillation over the arc of vertical circle, $0 < \theta \leq 90^o$ and

$0 < V_1 \leq \sqrt{2gr}$

This follows from energy conservation.

$$\frac{1}{2}MV_1^2 \leq Mgr$$

$$V_1 \leq \sqrt{2gr}$$

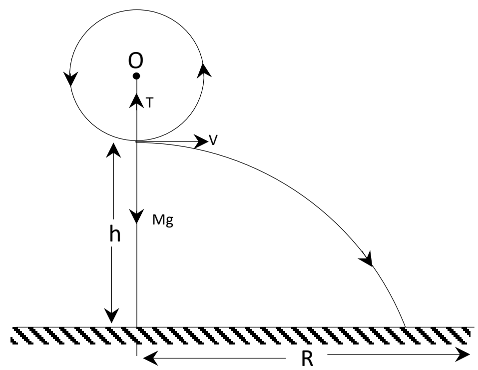
Obviously, velocity become zero before T vanishes. That is the body oscillates.

3. Condition for the body leaving the vertical circle somewhere between $90^o < \theta < 180^o$ i.e

$\sqrt{2gr} < V_1 < \sqrt{5gr}$

Note that

- The string will break down at point A when the tension is maximum at the lowest position. In this case circular motion is converted into projectile motion as shown below.



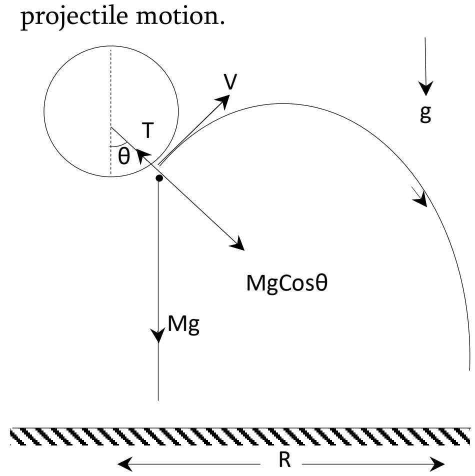
Flight time t

$$t = \sqrt{\frac{2h}{g}}$$

Horizontal range

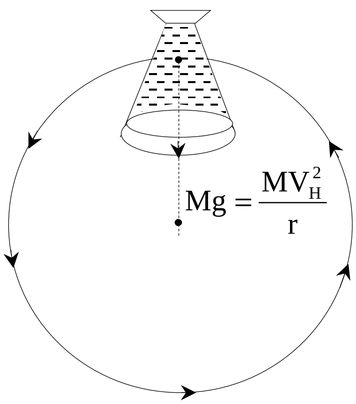
$$R = Vt = V = \sqrt{\frac{2h}{g}}$$

- If the string break down at any point between A and C also circular motion changed into projectile motion.



PRACTICAL APPLICATIONS OF MOTION IN A VERTICAL CIRCLE

1. Bucket of water whirled in the vertical circle. Consider the figure below which shows the whirling of the bucket which consist water.



When a bucket containing water is rotated, water shall not spil with a velocity at the lowest point, $V_L \geq \sqrt{5gr}$, water shall not spil even at the highest point, when the bucket is upside down. If the bucket is whirled slowly, so that $Mg > \frac{MV_H^2}{r}$, and the rest of the weight of water $\left(Mg - \frac{MV_H^2}{r}\right)$ will spil. Only this much water shall leave the bucket.

2. A pilot of an aircraft can successfully loop a vertical loop without falling at the top of the loop (being without belt), when its velocity at the bottom of the loop is $V_L \geq \sqrt{5gr}$.

3. Motor cyclist in a globe of death. The figure below shows a motor cyclist in a circus driving the motor cycle along a vertical circle in a cage. If the speed of the motor cyclist at the lowest point is greater than $\sqrt{5gr}$, then the speed at the highest point will be great than $\sqrt{gr}$ and the motor cyclist shall remain safe.

Diagram

TYPE C: SOLVED EXAMPLES

Example 1.

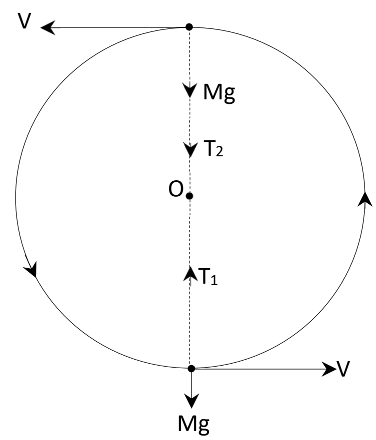
A string 0.5 m in length is used to whirled a 1.0 kg stone in a vertical circle at uniform speed of 5 m/s. determine the tension in the string. When the stone is

(i) At the top of the circle

(ii) At the bottom of the circle.

Solution

Diagram



(i) At the top

$$T_2 = \left(\frac{V^2}{r} - g\right) = 1\left(\frac{5^2}{0.5} - 9.8\right)$$

$T_2 = 40.2$ N.

(ii) At the bottom

$$T_1 = M\left(\frac{V^2}{r} + g\right)$$

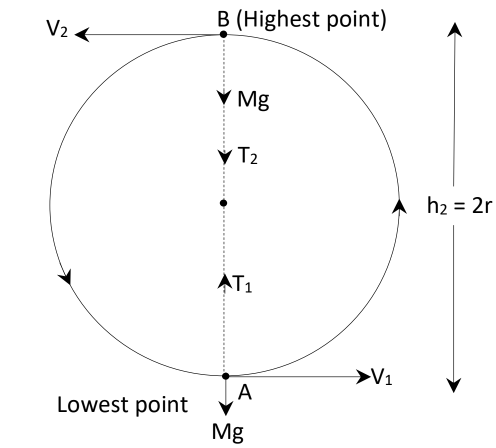
$$= 1 \times \left(\frac{5^2}{0.5} + 9.8\right)$$

$T_1 = 59.8$N.

Example 2

One end of a string of length 1.5 m is tied to a stone of mass 0.4 kg and the other end to a small pivot on a smooth vertical board. What is the minimum speed of the stone required at its lowest most point that the string does not slack at any point in its motion along vertical circle?

Solution



At the point B: $V_2 = \sqrt{gr}$

Apply the law of conservation of mechanical energy.

$(P.e + K.e)_A = (P.e + K.e)_B$

$$\frac{1}{2}MV^2 = Mgh_2 + \frac{1}{2}MV_2^2$$

$$V_1^2 = V_2^2 + 2gh_2$$

$$= gr + 2g(2r)$$

$$V_1^2 = 5gr$$

$$V_1 = \sqrt{5gr} = \sqrt{5\times9.8\times1.5}$$

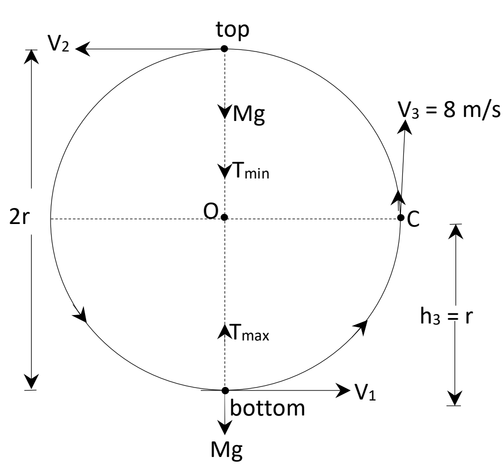
$V_{min} = 8.573$ m/s.

Example 3

A body of 50 g is whirled in a vertical circle of radius 60 cm. determine the maximum and minimum tensions in the string when its velocity at horizontal position is 8 m/s.

Solution

Consider the motion of a body in a vertical circle as show below:



Apply the principle of the conservation of mechanical energy.

$P.e_1 + K.e_1 = P.e_3 + K.e_3$

$$Mgh_1 + \frac{1}{2}MV_1^2 = Mgh_3 + \frac{1}{2}MV_3^2$$

$$2gh_1 + V_1^2 = 2gh_3 + V_3^2$$

$h_1 = 0$, $V_3 = 8$ m.s, $h_3 = r$

$$V_1 = \sqrt{V_3^2 + 2gr}$$

$$= \sqrt{8^2 + 2\times9.8\times0.6}$$

$V_1 = 8.7$ m.s.

$$\text{Since } T_{max} = \frac{MV_1^2}{r} + Mg$$

$$= \frac{50\times10^{-3}\times8.7^2 + 50\times10^{-3}\times9.8}{0.6}$$

$T_{max} = 6.8$N.

Again

$$\frac{1}{2}MV_2^2 + Mgh_2 = \frac{1}{2}MV_3^2 + Mgh_3$$

$$V_2 = \sqrt{V_3^2 - 2gr}$$

$$= \sqrt{82 - 2\times9.8\times0.6}$$

$V_2 = 7.2$ m/s.

Now

$$T_{min} = \frac{MV_2^2}{r} - Mg$$

$$= 50\times10^{-3}\left(\frac{(7.2)^2}{0.6} - 9.8\right)$$

$T_{min} = 3.8$N.

Example 4

A stone is tied to a weightless string and revolved in a vertical circle of radius 5 m. what should be the minimum speed of the stone at the highest point of the circle so that the string does not slack? What should be the speed of the stone at the lowest point of vertical circle?

Take $g = 98$ m/s².

Solution

$r = 5$m $V_H = ?$

$V_L = ?$ $g = 9.8$ m/s².

The string will not slack at the highest point when

$$V_H = \sqrt{gr} = \sqrt{9.8\times5}$$

$V_H = 7$ m/s

At the bottom of vertical circle

$V_L = \sqrt{5gr} = \sqrt{5\times 9.8\times 5}$

$V_L = 15.65$ m/s

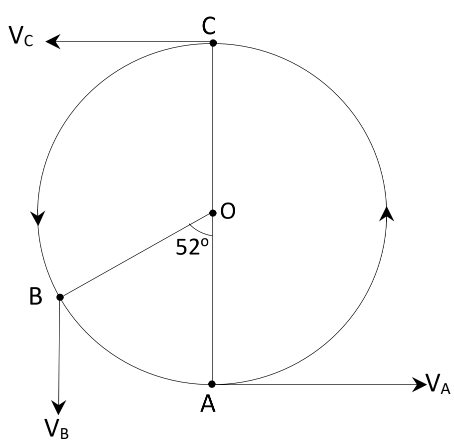
Example 5

Consider the figure below which shows the particle of mass M = 5 kg moving in a circle as shown in the figure below. If the (radius) length of the string is 1.5 m and the body is then pulled to the position B from C such that the string makes an angle of 52° with the vertical

(i) With what velocity should the body be realised from B so that its reaches at C with a velocity of 2.20m/s?

(ii) With what velocity will the body pass the lowest point A?

(iii) Find the tension in the string at the lowest point, A.



Solution

(i) $V_B = ?$

Apply the principle of conservation of the mechanical energy.

$(P.e + K.e)_B = (P.e + K.e)_C$

$$Mgh + \frac{1}{2}MV_B^2 = Mg(2r) + \frac{1}{2}MV_C^2$$

$$V_B^2 = V_C^2 + 4gr - 2gh$$

but $h = r(1 - \cos\theta)$

$$V_B = \sqrt{V_C^2 + 4gr - 2gr(1-\cos\theta)} =$$

$$\sqrt{2.2^2 + 4\times 9.8\times 1.5 - 2\times 9.8\times 1.5(1-\cos 52^\circ)}\ V_B$$

= 7.23 m/s.

(ii) $V_A = ?$

Apply principle of conservation of mechanical energy.

$$\frac{1}{2}MV_A^2 = \frac{1}{2}MV_C^2 + 2Mgr$$

$$V_A = \sqrt{V_C^2 + 4gr}$$

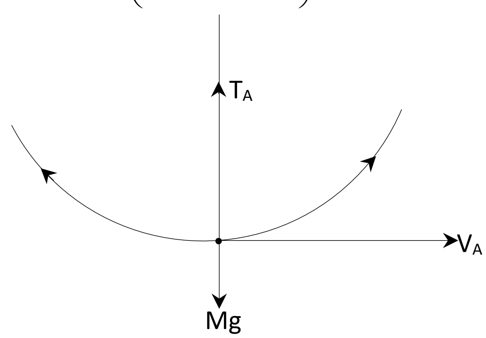
$$= \sqrt{(2.2)^2 + 4\times 9.8\times 1.5}$$

$V_A = 7.97$ m/s.

(iii) Tension on the string at the lowest position.

$$T_A = M\left(\frac{V_A^2}{r} + g\right)$$

$$= 5\left(\frac{(7.97)^2}{1.5} + 9.8\right)$$



$T_A = 260.736$N.

Example 6. NECTA 2007/P1/4

(a) (i) What is meant by centripetal force.

(ii) Derive the expression of $a = \frac{V^2}{r}$ where a, V and r stand for the centripetal acceleration, linear velocity and radius of a circular path respectively.

(b) A ball of mass 0.5 kg attached to a light inextensible string rotates in a vertical circle of radius 0.75 m such that its has a speed of 5 m/s. When the string is in horizontal. Calculate:-

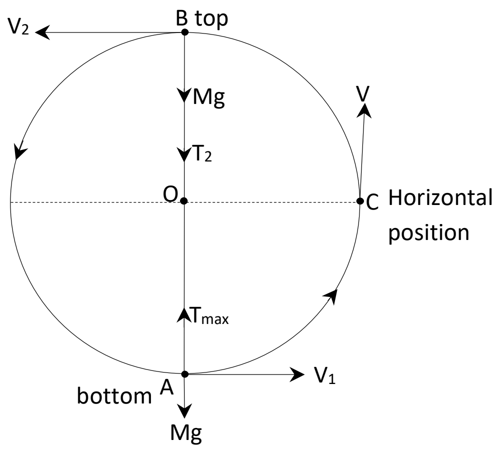
(i) The speed of the ball and the tension in the string at the lowest point of its circular path.

(ii) Evaluate the work done by Earth's gravitational force and by the tension in the string as the ball moves from its highest to its lowest point?

Solution

(a) Refer to your notes.

(b) Consider the figure below.



(i) Apply the law of conservation of energy.

$(P.e + K.e)_A = (P.e + K.e)_C$

$$\frac{1}{2}MV_1^2 = \frac{1}{2}MV^2 + Mgh \text{ but } h = r$$

$$V_1 = \sqrt{V^2 + 2gr}$$

$$V_1 = \sqrt{5^2 + 2\times9.8\times0.75}$$

$$V_1 = 6.30 \text{ m/s.}$$

Tension of the string at the bottom.

$$T_{max} = M\left(\frac{V_1^2}{r} + g\right)$$

$$= 0.5\left(\frac{6.3^2}{0.75} + 9.8\right)$$

$T_{max} = 31.36$N.

(ii) The work done by the gravitational force as the ball moves from its highest point to its lowest point is the increase in the kinetic energy of the body. By the law of the conservation of energy increase in K.e = decreases in P.e

$\Delta K.E = 2Mgr$

By the work energy theorem.

$W = \Delta K.E = 2Mgr$

$= 2 \times 0.5 \times 9.8 \times 0.75$J

The tension in the string does not do any work done on the ball as the tension in the string acts perpendicular to the ball.

$W = 0$

Example 7

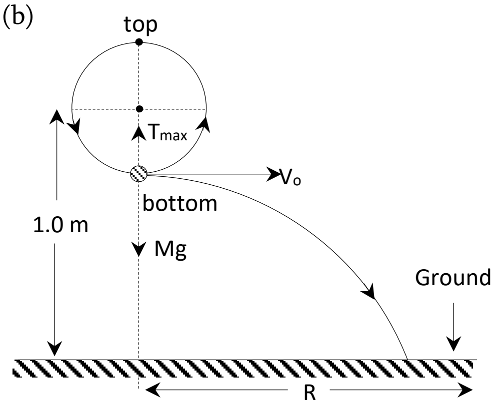
(a) Explain what is meant by angular velocity. Derive an expression for the force required to keep a particle of mass M move in a circle of radius r with uniform angular velocity, ω.

(b) A stone of mass 500 gm is attached to a string of length 50 cm which will break if the tension in it exceeds 20N. The stone is whirled in a vertical circle, the axis of rotation being at a height of 100 cm above the ground. The angular speed is very slowly increased until the string breaks. In what position is this break most likely to occur and at what angular speed? Where will the stone hit the ground?

Solution

(a) Refer to your notes.

(b)



The position in which this likely to occur is at the LOWEST POSITION since the tension in the string is maximum at this point.

$$T_{max} = M\left(\frac{V_1^2}{r} + g\right) \text{ but } V = \omega r$$

$$\omega = \sqrt{\frac{1}{r}\left(\frac{T_{max}}{M} - g\right)}$$

$$= \sqrt{\frac{1}{0.5}\left(\frac{20}{0.5} - 9.8\right)}$$

$\omega = 7.77$ rads^{-1}.

Let R = Horizontal range

Flight time

$$h = \frac{1}{2}gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}}$$

$$t = \sqrt{\frac{2\times 0.5}{9.8}} = 0.319 \text{ sec}$$

Now

$R = V_o t = \omega r t$

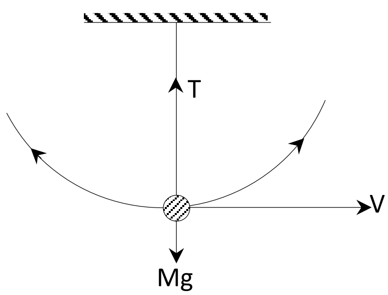
$= 0.5 \times 7.77 \times 0.319$

$R = 1.2410$ m.

Example 8

A boy weighing 55 kg swings on a rope 4.5 m long. If he passes through the lowest position with a velocity of 4 m/s. What is the corresponding tension in the string?

Solution



$$T = M\left(\frac{V_1^2}{L} + g\right)$$

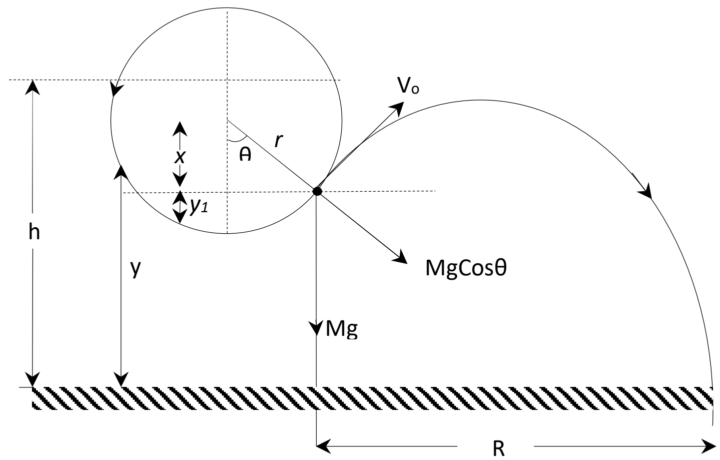
$$= 55\left(\frac{4^2}{4.5} + 9.8\right)$$

T = 734.6N.

Example 9

A stone of mass 2.0 kg and attached to the end of a string 1.0 m long is whirled round vertical circle with constant speed. The horizontal axis of rotation is 10 m above the ground. As the stone moves upwards from its lowest position the string snaps just when its makes an angle of 60° to the vertical. Calculate the horizontal distance travelled by the stone before striking the ground. (Assume tension T is 810N before its snaps).

Solution



$\cos\theta = x/r \Rightarrow x = r\cos\theta$

$y_1 = r = x = r - r\cos\theta$

$X = 1 \times \cos 60^o = 0.5$ m.

Now

$y = h - x = 10 - 0.5 = 9.5$ m.

Let V_o = velocity of a projectile.

$$\frac{MV_o^2}{r} = T - Mg\cos\theta$$

$$V_o = \left(\frac{r}{M}(T - Mg\cos\theta)\right)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}(810 - 2\times 9.8\cos 60^o)\right)^{\frac{1}{2}}$$

$V_o = 20$ m/s.

Flight time, t

$$-y = (V_o\sin\theta)t - \frac{1}{2}gt^2$$

$-9.5 = (20\sin 60^o(t - 4.9t^2.$

On solving t quadratically and omitting negative time.

t = 4.01 sec.

horizontal range

$R = (V_o\cos\theta)t$

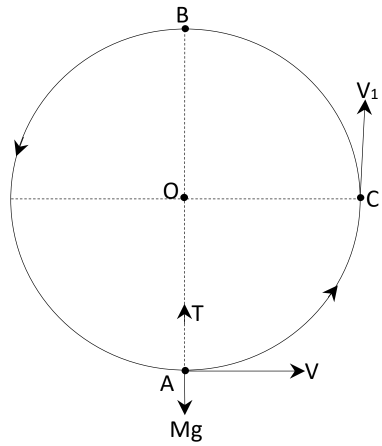
$= (20\cos 60^o) \times 4.01$

R = 40.17 m.

Example 10

A stone of mass 0.5 kg tied to a rope of length 0.5 m revolves along a circular path in a vertical plane. The tension in the rope at the bottom point of the circle is 35N. To what height the stone will rise if the rope breaks the moment the velocity is directed upwards. (Take g = 10 m/s²).

Solution



At the point A

$$\frac{MV^2}{L} = T - Mg$$

$$V^2 = \frac{L}{M}(T - Mg)$$

$$= \frac{0.5}{0.5}(45 - 0.5\times10)$$

$V^2 = 40\ m^2/s^2$.

When the rope breaks, the stone is rises to height h until its entire kinetic energy is converted into potential energy.

$$\frac{1}{2}MV^2 = Mgh$$

$$h = \frac{V^2}{2g} = \frac{40}{2\times10}$$

h = 2 m.

Example 11

A bucket containing water is tied to one end of a rope of length 2.5m and rotated about the other end in a vertical circle so that water does not spill even when bucket is upside down. What is the minimum velocity of the bucket at which this happens? How many revolution per minute is it making? ($g = 10\ m/s^2$)

Solution

Water in the bucket will not spill when weight of water is balanced by centrifugal force.

$$Mg = \frac{MV^2}{r},\ V = \sqrt{rg}$$

$V = \sqrt{2.5\times10} = 5$ m/s.

This is the velocity at the highest point.

Angular velocity, $\omega = \frac{V}{r} = \frac{5}{2.5}$

$\omega = 2\ rads^{-1}$.

Frequency of rotation

$$f = \frac{\omega}{2\pi}\ \text{r.p.s} = \frac{60\ \text{r.p.m}}{\pi}$$

$$f = \frac{60}{\pi}\ \text{r.p.m}$$

Example 12

In circus the diameter of globe of death is 20 m. from what minimum height must a motor cyclist start in order to go around the globe successfully?

Solution

Here $r = \frac{20}{2} = 10$ m, h = ?

On rolling down the incline loss in P.e = gain in K.e

$$Mgh = \frac{1}{2}MV^2$$

$$V = \sqrt{2gh}$$

For looping the loop minimum velocity at the lowest point.

$$V_{min} = \sqrt{5gr}$$

$$\sqrt{5gr} = \sqrt{2gh}$$

$$h = \frac{5}{2}r = \frac{5}{2}\times10$$

h = 25 m.

Example 13

A small stone of mass 200 g is tied to one end of a string of length 80 cm. holding the other end in hand, the stone is whirled into a vertical circle. What is the minimum speed that needs to be imparted at the lowest point of the circular path, so that the stone is just able to complete the vertical circle? What would be the tension at the lowest point of the circular path? (Take $g = 10\ m/s^2$).

Solution

Given that: M = 200 g = 0.2 kg

L = r = 80 cm = 0.8 m, V_L = ?

T_L = ?

Since $V_L = \sqrt{5gr} = \sqrt{5\times10\times0.8}$

$V_L = 6.32$ m/s.

$T_L = 6\ Mg = 6 \times 0.2 \times 10 = 12N.$

Example 14

A mass less string of length 1.2 m has a breaking strength of 2 kgwt. A stone of mass 0.4 kg tied to one end of the string is made to move in a vertical circle by holding the other end in hand. Can the particle describe the vertical circle? (Take g = 10 m/s²).

Solution

Breaking strength = 2 kgwt

F = 2 kgwt = 20N.

Now $T_{max} = T_L = 6\ Mg$

$T_{max} = 6 \times 0.4 \times 10 = 24N$, which is more than breaking strength of string. Therefore, the string will break and the particle cannot describe the vertical circle.

Example 15

A small stone of mass 0.2 kg tied to a massless, inextensible string rotated in a vertical circle of radius 2 m. If the particle is just able to compete the vertical circle, what is its speed at the highest point of the circular path? How would the speed get affected if the mass of the stone is increased by 50%? (Take g = 10 m/s²)

Solution

When the particle is just able to complete the vertical circle,

$V_L = \sqrt{5gr}$ and

$V_H = \sqrt{gr} = \sqrt{10\times 2} = 4.47$ m/s.

As V_H does not depend upon mass, M of the stone, therefore, the value of V_H will not be affected by any change in mass of the stone.

Example 16

A particle of mass 150 g is attached to one end of a massless inextensible string. It made to describe a vertical circle of radius 1 m. when the string is making an angle of 48.2° with the vertical, its instantaneous speed is 2 m/s. what is the tension in the string in this position? Would this particle be able to complete its circular path?

Solution

From the equation

$$T = \frac{MV^2}{L} + MgCos\theta$$

$$= \frac{0.15\times 2^2}{1} + 0.15\times 10Cos48.2^o$$

T = 1.6N.

To be able to complete the vertical circle

$V_H = \sqrt{gr} = \sqrt{10\times 1} = 3.16$m/s.

As instantaneous velocity is 2 m/s, it cannot increase (to 3.16 m/s) on reaching the highest point as gravity is oppositing the motion. Therefore, the particle cannot complete its circular path.

Example 17

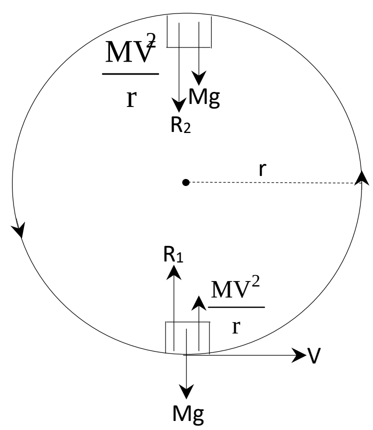
An aeroplane flying in the sky dives with a speed of 360 km/hr in a vertical circle of radius 200 m. the weight of pilot sitting in its is 75 kg. Calculate the force with which the pilot presses his seat when the aeroplane is (i) at the lowest position (ii) at the highest position. (g = 10 m/s²)

Solution

R_1 is normal reaction at the lowest position and R_2 is normal reaction at the highest position of vertical circle.

r = 200 cm, M = 75 kg

V = 360 km/hr = 100 m/s



(i) At the lowest position

$$R_1 - Mg = \frac{MV^2}{r}$$

$$R_1 = M\left(\frac{V^2}{r} + g\right) = 75\left(\frac{100^2}{20} + 10\right)$$

R_1 = 4500N = 450 kgwt.

(ii) At the highest position

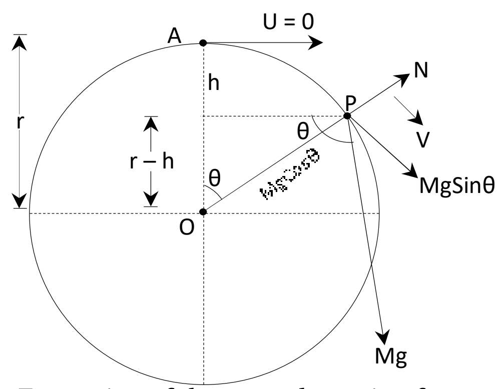
$$R_2 + Mg = \frac{MV^2}{r}$$

$$R_2 = M\left(\frac{V^2}{r} - g\right) = 75\left(\frac{100^2}{20} - 10\right)$$

$R_2 = 3000N = 300$ kgwt.

3. MOTION OF AN OBJECT OUTSIDE A SMOOTH VERTICAL CIRCULAR ROAD

Consider an object of mass M being released from the top of smooth vertical circular road (sphere) radius r as shown on the figure below.



- Expression of the normal reaction force

$$MgCos\theta - N = \frac{MV^2}{r}$$

$$N = M\left(gCos\theta - \frac{V^2}{r}\right)$$

Since $V^2 = 2gh$, $h = r($

$V^2 = 2gr(1 - Cos\theta)$

$$\frac{V^2}{r}\ 2g(1 - Cos\theta)$$

Now $N = M(gCos\theta - (2g - 2gCos\theta))$

$N = Mg(3Cos\theta - 2)$

From the diagram above

$$Cos\theta = \frac{r-h}{r},\ Mg = W$$

$$N = W\left(\frac{3(r-h)-2}{r}\right)$$

$$N = W\left(\frac{r-3h}{r}\right)$$

- Expression of the speed of the particle at point, P.

$$V = \sqrt{2gh} = \sqrt{2gr(1-Cos\theta)}$$

- Condition for the particle to leave the circular road.

$N = 0$

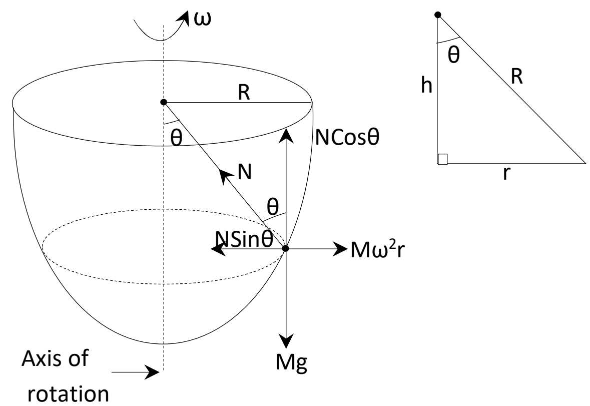
$N = Mg(3Cos\theta - 2)$

$0 = 3Cos\theta - 2$

$Cos\theta = \frac{2}{3}$, $\theta = 48^o$ (approx)

4. ROTATION OF A PARTILCE INSDIE A SMOOTH HEMISPHERICAL BOWL

Consider a particle of mass M, revolving around inside a hemispherical bowl of radius R at constant angular frequency, ω. The radius of the transverse circle is r. the horizontal component of normal reaction force of the bowl on the particle provides the necessary centripetal force for the particle to take the curve.



Where θ is the inclination of the normal reaction force N to the vertical. At the equilibrium of the particle.

$NSin\theta = M\omega^2 r$______(i)

$NCos\theta = Mg$________(ii)

Dividing equation (i) by (ii)

$$\frac{NSin\theta}{NCos\theta} = \frac{\omega^2 r}{g},\ \left(\tan\theta = \frac{\omega^2 r}{g}\right)$$

- Expression of angular velocity and frequency of the rotation.

$$\text{Since } \tan\theta = \frac{\omega^2 r}{g}$$

$$\omega = \sqrt{\frac{g\tan\theta}{r}} \text{ but } r = RSin\theta$$

$$\omega = \sqrt{\frac{gSin\theta}{RSin\theta Cos\theta}} = \sqrt{\frac{g}{RCos\theta}}$$

$\omega = \sqrt{\dfrac{g}{RCos\theta}}$

From the diagram above

$h = \sqrt{R^2 - r}$ Pythagoras theorem

$\tan\theta = \dfrac{r}{h} = \dfrac{r}{\sqrt{R^2 - r^2}}$

$\omega^2 = \dfrac{g}{\sqrt{R^2 - r^2}}$

Frequency of the rotation

$\omega = 2\pi f = \dfrac{g}{\sqrt{RCos\theta}}$

$f = \dfrac{1}{2\pi}\dfrac{g}{\sqrt{RCos\theta}}$

- Expression of the normal reaction force on the particle

$NCos\theta = Mg$

$N = \dfrac{Mg}{Cos\theta} = \dfrac{MgR}{\sqrt{R^2 - r^2}}$

TYPE D: SOLVED EXAMPLES

Example 1. NECTA 2014/P1/3(C)

(i) Define the tem radial acceleration.

(ii) An insert is realised from rest at the top of the smooth bowling ball such that it slides over the ball. Prove that it will loose its footing with the ball at an angle of about 48° with the vertical.

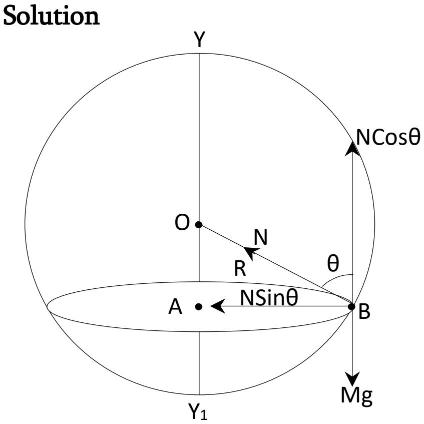
Example 2

A thin circular wire of radius R rotates about its vertical diameter with an angular frequency, ω. Show that a small bead on the wire remains at its lower most point for

$\omega \geq \sqrt{\dfrac{g}{R}}$. What is the angle made by the radius vector joining the centre to the bead with the vertical downward direction for

$\omega = \sqrt{\dfrac{2g}{R}}$ Neglect friction.

Solution



A bed is placed at the point B, $r = RSin\theta$

At the equilibrium

$NCos\theta = Mg$ ______(i)

$NSin\theta = M\omega^2 r$ ________(ii)

Also

$NSin\theta = M\omega^2 RSin\theta$

$N = M\omega^2 R$______(iii)

Putting equation (iii) into (ii)

$M\omega^2 RCos\theta = Mg$

$Cos\theta = \dfrac{g}{R\omega^2}$

Now $(1Cos\theta) \leq 1$

$\dfrac{g}{R\omega^2} \leq 1$

$\omega^2 \geq \dfrac{g}{R}, \omega \geq \sqrt{\dfrac{g}{R}}$

Hence shown

If $\omega^2 \geq \dfrac{2g}{R}, Cos\theta \geq \dfrac{r}{R\omega^2}$

$Cos\theta = \dfrac{g}{\frac{2g}{r}R} = \dfrac{1}{2}$

$\theta = Cos^{-1} = \left(\dfrac{1}{2}\right) = 60^o$

Example 3

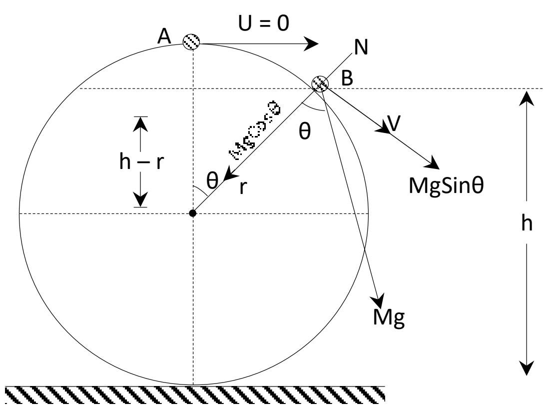
A small mass M kg is balanced at the top of the large spherical smooth globe of radius r placed on the ground. It now given a very small push so that it starts to slide down a frictionless globe. The mass is observed to lose contact with globe at a height that is greater than the radius of the globe. Calculate:-

(i) The initial velocity just when the mass lose contact.

(ii) At what height from the ground will the mass lose contact with the globe?

Solution

Let h be the height from the ground which the mass lose contact and V be its initial velocity at that instant.



At the point B

$$\frac{MV^2}{r} = MgCos\theta - N$$

For the particle loses contact

$$N = 0, \ \frac{MV^2}{r} = MgCos\theta$$

$V^2 = rgCos\theta$ ________(i)

Apply the principle of the conservation of mechanical energy.

$(P.e + K.e)_A = (P.e + K.e)_B$

$$2Mgr = \frac{1}{2}MV^2 + Mgh$$

$4gr = V^2 + 2gh$

$$Cos\theta = \frac{h-r}{r}, \ rCos\theta = h - r$$

$h = r(1 + Cos\theta)$

Now

$4gr = V^2 + 2gh(1 + Cos\theta)$

$V^2 = 2gr\ (1 - Cos\theta)$ ________(ii)

(i) = (ii)

$rgCos\theta = 2gr\ (1 - Cos\theta)$

$3grCos\theta = 2gr$

but $rCos\theta = h - r$

$3g(h - r) = 2gr$

$3h = 5r$

$$h = \frac{5}{3}r$$

From the equation

$4gr = V^2 + 2gh$

$$4gr = V^2 + 2g \bullet \frac{5}{3}r$$

$$V = \sqrt{\frac{2}{3}rg}$$

(i) $V = \sqrt{\frac{2}{3}rg}$

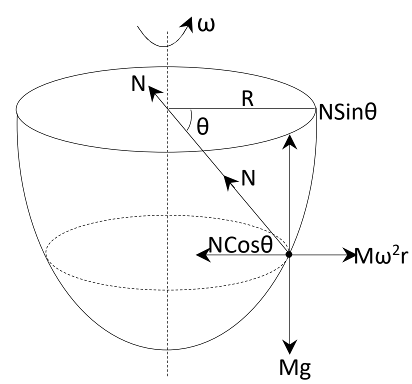
(ii) $h = \frac{5}{3}r$

Example 4

A particle of mass 0.3 kg moves with angular velocity of 10 $rads^{-1}$ in a horizontal circle of radius 20 cm inside a smooth hemispherical bowl. Find the reactions of the bowl on the particle and the radius of the bowl.

Solution

Consider the figure below.



Inclination of the normal to the horizontal is given by

$$tan\theta = \frac{g}{\omega^2 r} = \frac{9.8}{10^2 \times 0.2}$$

$\theta = tan^{-1}(0.49) = 26.1°$

Normal reaction force

$$N = \frac{Mg}{\cos\theta} = \frac{0.3\times 9.8}{\sin(26.1^o)}$$

N = 6.68N.

Radius of the bowl.

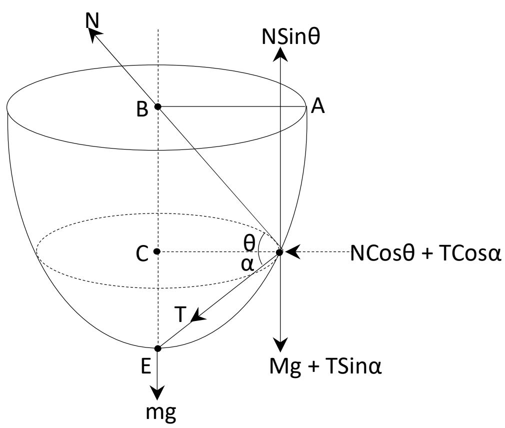
$$R = \frac{r}{\cos\theta} = \frac{20}{\cos(26.1^o)}$$

R = 22.3 cm.

Example 5

A particle of mass M describes a horizontal circle of radius 24 cm inside a smooth hemispherical bowl of internal radius 25 cm which is held with its rim horizontal. A fine weightless thread tied to the particle passes through a small smooth hole at the bottom of the bowl and supports another particle of mass m which hangs at rest. Calculate the speed of the first particle.

Solution



At the equilibrium by balancing the forces.

T = Mg, and Mg + TSinα = NSinθ

Mg + MgSinα = NSinθ ______(i)

The centripetal force is given by

$N\cos\theta + T\sin\alpha = M\omega^2 r$

$N\cos\theta + Mg\cos\alpha = M\omega^2 r$______(ii)

From the figure above

$$\overline{CE} = \overline{BE} - \overline{BC} = 25 - \sqrt{25^2 - 24^2}\ ,$$

$$\overline{CE} = 18$$

$$\cos\theta = \frac{24}{25},\ \sin\theta = \frac{7}{25}$$

$$\sin\alpha = \frac{3}{5},\ \cos\alpha = \frac{4}{5}$$

$$\tan\alpha = \frac{18}{24} = \frac{3}{4}$$

Now putting these values into equation (i) and (ii)

$$Mg + \frac{3}{5}Mg = \frac{7}{25}N$$

$$N = \frac{40}{7}Mg$$

$$\text{Also } \frac{40}{7}\times\frac{24}{25}Mg + \frac{4}{5}Mg = 0.24M\omega^2$$

$$\frac{44g}{7} = 0.24\omega^2$$

$\omega = 16.02\ \text{rads}^{-1}$.

Speed of the first particle

$V = \omega r$

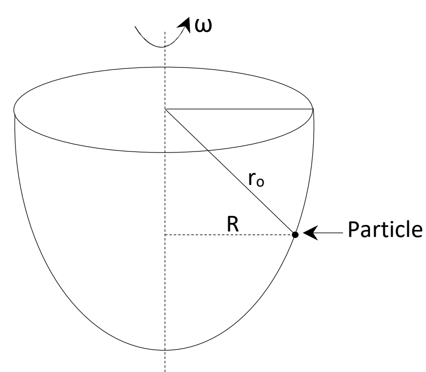
$\ \ = 16.02 \times 0.24$

V = 3.84 m/s.

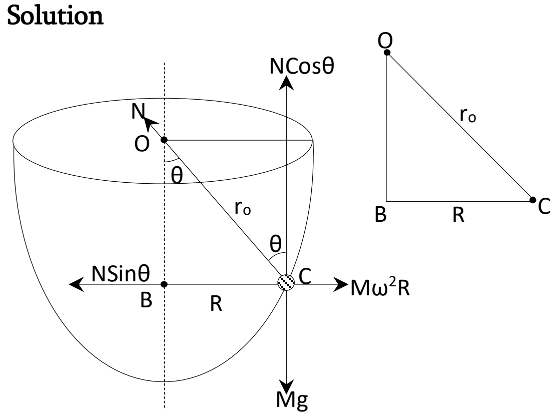
Example 6

A hemispherical bowl of radius r_o is rotating with constant angular velocity ω about its vertical axis of symmetry. If a particle placed on the inside surface of the bowl is at rest relative to the surface at a distance R from the axis, show that

$$R = \sqrt{r_o^2 - \frac{g^2}{\omega^4}}$$



Solution



At the equilibrium of the particle

NSinθ = Mω²r_______(i)

NCosθ = Mg________(ii)

(i) (ii)

$$\frac{NSin\theta}{NCos\theta} = \frac{M\omega^2 R}{g}$$

$$\tan\theta = \frac{\omega^2 R}{g}$$

But $\tan\theta = \dfrac{R}{\sqrt{r_o^2 - R}}$

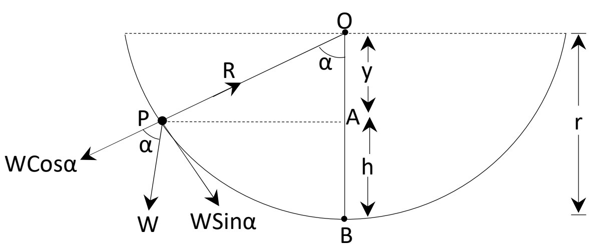
$$\frac{\omega^2 R}{g} = \frac{R}{\sqrt{r_o^2 - R}}$$

On simplifying we get

$$R = \sqrt{r_o^2 - \frac{g^2}{\omega^4}}$$

Example 7

Calculate the height up to which an insect can crawl up a fixed bowl in the form of a hemisphere of radius r. given coefficient of friction = $\dfrac{1}{\sqrt{3}}$.



Solution

The insect can crawl up the bowl from B to P through a height BA = h.

At the equilibrium

R = ωCosα, F = ωSinα

F = friction force

$$\mu = \frac{F}{R} = \frac{\omega Sin\alpha}{\omega Cos\alpha} = \tan\alpha$$

In a ΔOPA, $\tan\alpha = \dfrac{\overline{PA}}{OA}$

$$\tan\alpha = \sqrt{\frac{r^2 - y^2}{y}} = \frac{1}{\sqrt{3}}$$

$$\frac{r^2 - y^2}{y^2} = \frac{1}{3}$$

$$y^2 = \frac{3r^2}{4},\ y = \frac{\sqrt{3}r}{2}$$

h = BA = OB – OA = r – y

$$h = r - \frac{\sqrt{3}}{2} r = 0.134r$$

h = 13.4% of r.

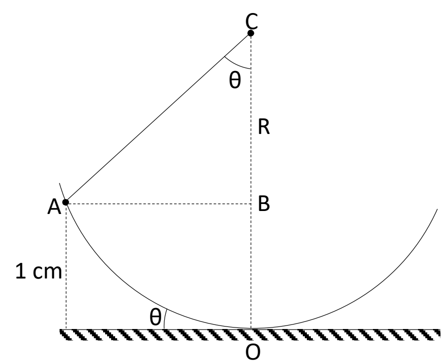
Example 8

A particle of mass 1 g executes an oscillatory motion on the concave surface of a spherical dish of radius 2m placed on a horizontal plane, shown on the figure below. If the motion of the particle begins from a point on the dist at a height of 1 cm; from the horizontal plane and coefficient of friction is 0.01, find the total distance covered by the particle before coming to rest

Solution

In the figure

CO = CA = 200 cm, 0B = 1 cm BC = CO – OB = 200 – 1 = 199 cm



In ΔABC, $Cos\theta = \dfrac{BC}{AC} = \dfrac{199}{200}$

Treating the dist as an inclined plane and using principle of conservation of energy.

Loss in P.e = work done against friction.

Mgh = μR × S = μMgCosθ × S

$$S = \frac{Mgh}{\mu MgCos\theta} = \frac{h}{\mu Cos\theta}$$

$$= \frac{\frac{1}{100}}{0.01 \times 199/200} = \frac{200}{199}$$

S = 1.005 m.

Example 9

A hemispherical bowl of radius R is rotating about its axis of symmetry which is kept vertical. A small block is kept in the bowl and rotates with it without slipping. If the surface of the bowl is smooth and the angle made by the radius through, the block with the vertical is θ, show that the angular speed of the bowl is given by

$$\omega = \sqrt{\frac{g\cos\theta}{R}}$$

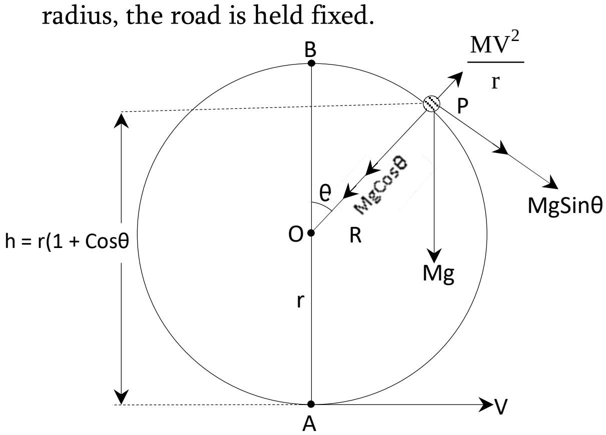
Example 10

An object of mass M is resting on top of a hemispherical mound of ice whose radius of curvature is R. The object is given a small push and start slidding down the mound. Show that the object will lose contact with the surface of ice at a vertical height of $\frac{2R}{3}$

5. MOTION OF A PARTICLE THREADED ON A SMOOTH VERTICAL CIRCULAR TRACK

Consider a particle of mass M which is projected with initial velocity μ from the lowest point A and reached at a point P as shown on the figure below.

Let O be the centre of the circular road, A is the lowest point, B is the highest point and r is the radius, the road is held fixed.



By the principle of conservation of energy

Loss in K.e = gain in P.e

$$\frac{1}{2}MU^2 - \frac{1}{2}MV^2 = Mgh$$

$U^2 - V^2 = 2gh$

At the point P, h = r(1 + Cosθ)

$U^2 - V^2 = 2gr(1 + \cos\theta)$ ______(1)

(i) Expression of the speed of the particle at any position.

$V^2 = U^2 - 2gr(1 + \cos\theta)$

$V = \sqrt{U^2 - 2gr(1+\cos\theta)}$ ____(2)

(ii) The minimum speed U for the particle to reach at B. For the particle to come to rest at B, V = 0 and θ = O°.

$U^2 = O^2 + 2gr(1 + \cos O^o)$

$U^2 = 4gr$, $U = \sqrt{4gr}$ ____(3)

(iii) Reaction force on the particle at the point P.

$$Mg\cos\theta + R = \frac{MV^2}{r}$$

$$R = M\left(\frac{V^2}{r} - g\cos\theta\right)$$ ______(4)

Deductions from the equation (4).

When point P is below the centre, the reaction R is directed inward into the road. Here θ > 90° and Cosθ < 1, Cosθ is negative.

1. Check at A: θ = 180°

$$R = M\left(\frac{V^2}{r} - g\cos 180^o\right)$$

$$R = M\left(\frac{V^2}{r} + g\right)$$

2. When θ = 90°; the particle is in the horizontal level with the centre, O.

$$R = M\left(\frac{V^2}{r} - g\cos 90^o\right)$$

$$R = \frac{MV^2}{r}$$

(iv) The minimum speed U for looping the circle.

From the equations

$U^2 - V^2 = 2gr(1 + \cos\theta)$ and

$$R = M\left(\frac{V^2}{r} - g\cos\theta\right)$$

For the minimum speed U to loop the track, the particle is about to lose contact with track at B. When θ = 0° and so R = 0.

$$0 = M\left(\frac{V^2}{r} - g\cos\theta\right)$$

$V^2 = rg$, $V = \sqrt{gr}$

Putting $V^2 = rg$ and $\theta = 0^o$ on the equation.

$U^2 - V^2 = 2gr(1 + Cos\theta)$

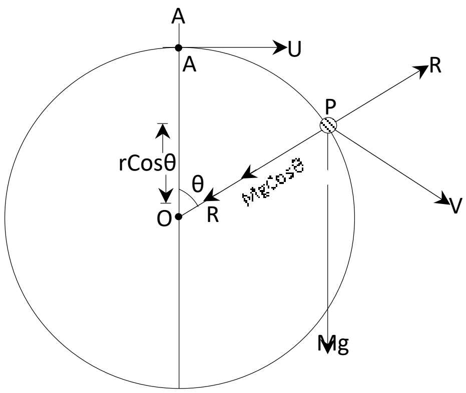
$U^2 - gr = 2gr(1 + Cos0^o)$

$U^2 = 5gr$, $U = \sqrt{5gr}$

Example 11

A small bead of mass M is threaded on a smooth circular of radius, r and centre O, and which is fixed in a vertical plane. The bead is projected with speed U from the highest point A of the wire. Find the reaction on the bead due to the wire when the bead is at P, in terms of M, g, r, U and θ where $\theta = A\hat{O}P$

Solution



Apply the principle of conservation of energy increase in K.e = decrease in P.e

$$\frac{1}{2}MV^2 - \frac{1}{2}MV^2 = Mg(r - rCos\theta)$$

At the equilibrium when the particle is at OP

$$MgCos\theta - R = \frac{M}{r}\left[U^2 + 2gr(1 - Cos\theta)\right]$$

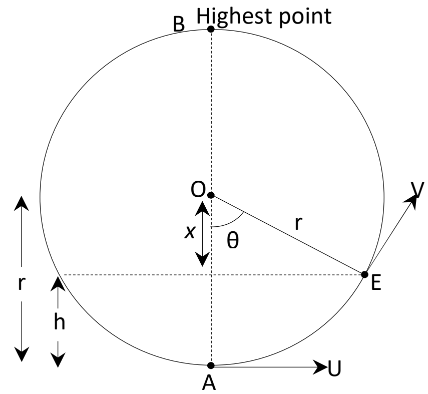
$$R = Mg(3Cos\theta - 2) - \frac{MU^2}{r}$$

Example 12

A small bead is thread on a smooth circular wire of radius r which is fixed in a vertical plane. The bead is projected from the lowest point of the wire with speed $\sqrt{6gr}$. Find the speed of the bead when it turned through

(a) 60^o (b) 90^o (c) 180^o (d) 300^o

Solution



From the figure above

$X = rCos\theta$, $h = r - x = r - rCos\theta$

Apply the principle of the conservation of mechanical energy.

$(P.e + K.e)_E = (P.e + K.e)_A$

$$Mgh + \frac{1}{2}MV^2 = \frac{1}{2}MU^2$$

$V^2 = U^2 - 2gh$

$V^2 = U^2 - 2gh(r - rCos\theta)$

but $U = \sqrt{6gr}$, $U^2 = 6gr$

$U^2 = 6gr + 2grCos\theta - 2gr$

$V = \sqrt{2gr(2 + Cos\theta)}$

(a) At $\theta = 60^o$

$V = \sqrt{2gr(2 + Cos60^o)}$

$V = \sqrt{5gr}$

(b) $\theta = 90^o$, $V = \sqrt{2gr(2 + Cos90^o)}$

$V = \sqrt{4gr}$

(c) $\theta = 180^o$, $V = \sqrt{2gr(2 + Cos180^o)}$

$V = \sqrt{2gr}$

(d) $\theta = 300^o$, $V = \sqrt{2gr(2 + Cos300^o)}$

$V = \sqrt{5gr}$

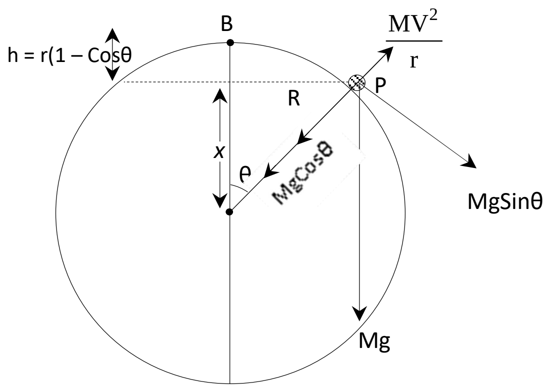
Example 13

A smooth circular tube is held fixed in a vertical plane. A particle of mass, M which can slide inside the tube is slightly displaced from rest at the highest point of the tube. Find the reaction between the particle and the tube when it is at an angular distance θ from the highest point of the tube. Also find the vertical component of the acceleration of the particle when $\theta = 120^\circ$.

Solution

Consider the figure below:

$h = r(1 - \cos\theta)$



Apply the principle of conservation of energy.

Gain in K.e = Loss in P.e

$$\frac{1}{2}MU^2 = Mgh$$

$U^2 = 2gh$

At the point, P $h = r(1 - \cos\theta)$

$U^2 = 2gr(1 - \cos\theta)$

Reaction on the particle at P.

$$Mg\cos\theta + R = \frac{MU^2}{r}$$

$$R = \frac{MU^2}{r} - Mg\cos\theta$$

$$R = \frac{M}{r}\left[2gr(1 - \cos\theta)\right] - Mg\cos\theta$$

$R = Mg(2 - 3\cos\theta)$ in ward.

Acceleration of the particle at $\theta = 120^\circ$

$Ma = Mg\cos\theta + R$

$\quad = Mg\cos\theta + Mg(2 - 3\cos\theta)$

$Ma = Mg\cos 120^\circ + Mg(2 - 3\cos 120^\circ)$

$Ma = 3Mg$

$a = 3g.$

Vertical component of the acceleration.

$a_y = a\cos\theta = 3g\cos 120^\circ.$

$$a_y = \frac{3g}{2}$$

Example 14

Prove that the velocity V with which a body must be projected at the lowest part of a loop apparatus of radius R in the vertical plane so that it passes at the highest position with minimum velocity, is given by

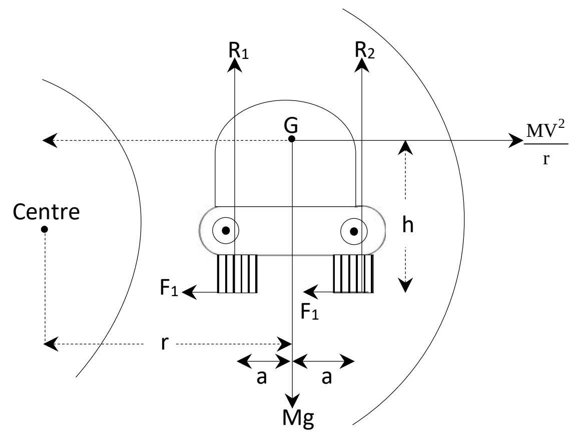
$V = \sqrt{5gr}$

6. MOTION OF A CAR ON THE FLAT CURVED ROAD

Case1: OVER TURNING or TOPPLING

When a car moves on a flat horizontal circular road the friction force between the tyres and the road provides the necessary centripetal force.

Consider a car of the total mass, M moving with constant speed, V on the flat circular road of radius, the tyres of the car ten to leave the road and move away from the centre. So the friction force (F_1 or F_2) acts inward to the two tyres.



Let F_1, F_2 = Friction forces on the inner and outer tyres respectively.

R_1, R_2 = Normal reaction forces on the inner and outer tyres respectively.

G = position of the centre of gravity

h = height above the ground

g = Acceleration due to gravity.

2a = d = distance between two tyres.

μ = coefficient of friction force

At the equilibrium

$R_1 + R_2 = Mg$

$R_2 = Mg - R_1$ ____(i)

$R_1 = Mg - R_2$ ____(ii)

Horizontal forces

$$F_1 + F_2 = \frac{MV^2}{r} \quad \text{\_\_\_\_\_(iii)}$$

Takes the moment of forces about G.

Sum of clockwise moment = Sum of anticlockwise.

$(F_1 + F_2)h + R_1a = R_2a$

$$\frac{MV^2h}{r} + R_1a = (Mg - R_1)a$$

$$\frac{MV^2h}{r} + R_1a = Mga - R_1a$$

$$2R_1a = Mga - \frac{MV^2h}{r}$$

$$R_1 = \frac{M}{2}\left(g - \frac{V^2h}{ra}\right) \text{\_\_\_\_\_\_\_\_(iv)}$$

Again

$$\frac{MV^2h}{r} + a(Mg - R_2) = R_2a$$

$$R_2 = \frac{M}{2}\left(g + \frac{V^2h}{ra}\right) \text{\_\_\_\_\_\_\_\_(v)}$$

Conclusion

1. From the equation

$$R_2 = \frac{M}{2}\left(g + \frac{V^2h}{ra}\right)$$

Always $R_2 > 0$. Therefore the outer tyres does not losses contact with the road.

2. From the equation

$R_1 = \frac{M}{2}\left(g - \frac{V^2h}{ra}\right)$, the following points may be noted:-

(i) When $R_1 > 0$, the inner tyres are still on the contact with the road.

(ii) When $R_1 = 0$, the inner tyres are at the point of losing contact with the ground (road).

(iii) When $R_1 < 0$, the inner tyres are completely in air. The car has "overturned"

- **CONDITION FOR OVERTURNING**

$R_1 < 0$

$$\frac{M}{2}\left(g - \frac{V^2h}{ra}\right) < 0$$

$$g - \frac{V^2h}{ra} < 0$$

$$V > \sqrt{\frac{rag}{h}}$$

Note that: Condition for no overturned

$R_1 \geq 0$.

- Expression of the maximum speed of the car on the flat curved road.

The maximum speed V_C at which the vehicle can take the curve without toppling is obtained when $R_1 = 0$.

Since $M \neq 0$, then

$$0 = \frac{M}{2}\left(g - \frac{V_c^2h}{ra}\right)$$

$$V_C = V_{max} = \sqrt{\frac{rag}{h}} = \sqrt{\frac{rdg}{2h}}$$

A vehicle is least likely to occur when

(i) $V < V_{max}$

The driver should reduces the speed of the vehicle as it takes the curve.

(ii) h is small

The position of C.G depends on the distribution of mass. Putting more mass at the bottom reduces h.

(iii) A is large

The vehicle should have a wide base. Racing cars are designed in a such a way that they have a low of centre of gravity and wide base.

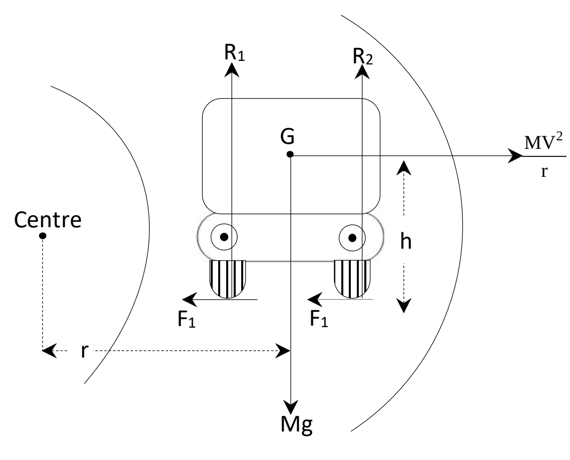
Case 2. SKIDDING OF A CAR

A vehicle is skid when available centripetal forces is not enough to balance with the centrifugal force. The vehicle fails to negotiate the curve and goes of f track outwards.

- Condition for skidding

Centripetal force < centrifugal force

Consider the motion of the car along curved flat road as shown below.



Since

$$F_1 + F_2 < \frac{MV^2}{r}$$

$$\mu(R_1 + R_2) < \frac{MV^2}{r}$$

$$\mu Mg < \frac{MV^2}{r},\ V^2 > \mu rg$$

$$v > \sqrt{\mu rg}$$

- Expression of the maximum speed of the car without skidding.

$$F_1 + F_2 = \frac{MV^2_{max}}{r}$$

$$\mu(R_1 + R_2) = \frac{MV^2_C}{r} = \mu Mg \quad V_C = V_{max} = \sqrt{\mu rg}\ .$$

- The value of V_C depends on the radius r of the curve and on coefficient of friction (μ) between the tyres and the road.
- When the road is unbanked and friction is small, the vehicle has to round the curve at a much lower speed to avoid overturning.
- A smooth road offers no friction to the wheels of the vehicle. Therefore the vehicle taking a corner on such road will skid outwards away from the road. For this reason, most of the roads are banked at the corners so that the vehicle will not depend on friction only.

7. BANKING OF ROADS

The maximum permissible velocity with which a vehicle can go round a level curved road without skidding depends on μ, the coefficient of friction between the tyres and the road. The value of μ decreases when vehicle are worn out or the road is wet and so on. Thus force of friction is not a reliable for providing the required centripetal force to the vehicle.

A safer course of action would be to raise outer edge of the curved road above the inner edge. By doing so, a component of normal reaction of the road shall be shared to provide the centripetal force. This would reduces considerably the wear and tear of the tyres.

BANKING OF ROAD – Is the phenomenon of raising outer edge of the curved road above the inner edge.

ANGLE OF BANKING – Is the angle through which the outer edge is raised above the inner edge.

Note: Roads and rail lines are baked in order to provide centripetal force which will prevent the tear and war or overturned of the car or train on the track.

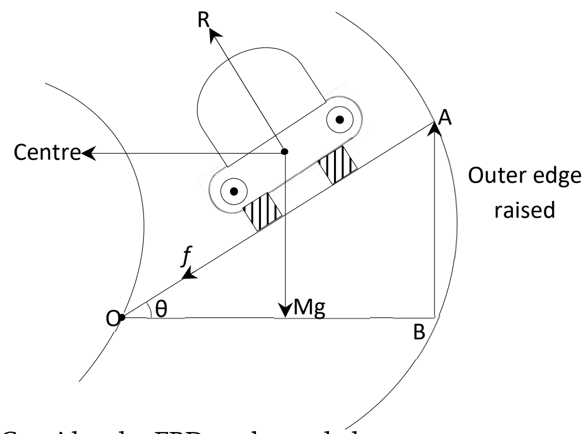
ADVANTAGE OF ROAD BANKING

1. It's prevent the tear and wear of the tyres of the car.
2. It's avoid the overturning or skidding of the car.

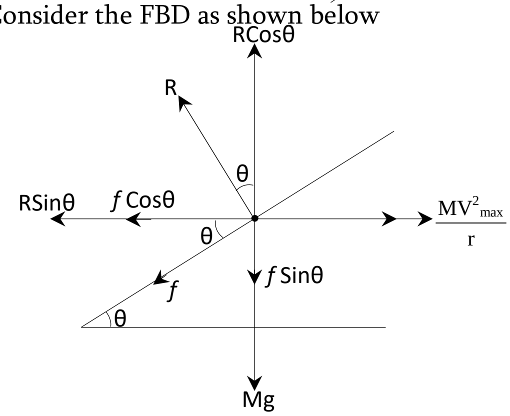
Therefore, three different cases for the motion of the car on the banked curved road.

Case 1. Expression of the maximum speed of the car on the banked curved road.

Consider the motion of the car along the curved banked road as shown on the figure below.



Consider the FBD as shown below



$R = R_1 + R_2$

$f = f_1 + f_2$

$f = \mu R$

At the equilibrium

Horizontal forces

$$RSin\theta + fCos\theta = \frac{MV_{max}^2}{r}$$

$$R(Sin\theta + \mu Cos\theta) = \frac{MV_{max}^2}{r} \_\_\_\_\_\_\_\_(i)$$

Vertical forces

$$RCos\theta - fSin\theta = Mg$$

$$R(Cos\theta - \mu Sin\theta) = Mg \_\_\_\_\_\_\_\_(ii)$$

Dividing equation (i) and (ii) and simplifying we can get

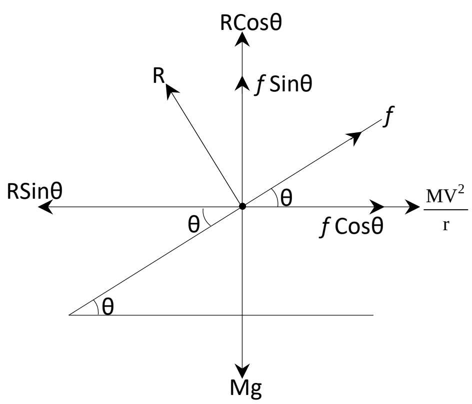
$$V_{max} = \sqrt{\frac{rg(\mu Cos\theta + Sin\theta)}{Cos\theta - \mu Sin\theta}} \text{ or}$$

$$V_{max} = \left(\frac{rg(\mu + \tan\theta)}{1 - \mu\tan\theta}\right)^{\frac{1}{2}}$$

This is the maximum safes velocity of vehicle on a banked road.

Case2. Expression of the minimum speed of the car on the curved banked road.

When the car move with minimum speed, the car tend to accelerated down the banked road, so the friction force acting in upward direction on the banked road as shown on the FBD below.



At the equilibrium

$$R(Sin\theta - \mu Cos\theta) = \frac{MV_{min}^2}{r} \_\_\_\_\_\_\_(i)$$

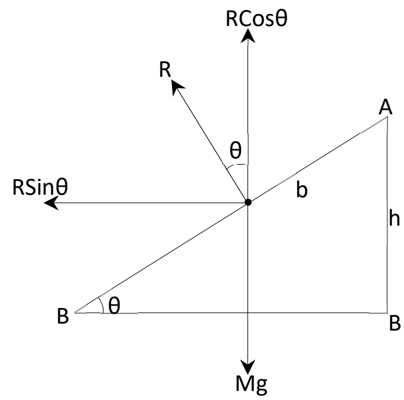
$$R(Cos\theta + \mu Sin\theta) = Mg \_\_\_\_\_(ii)$$

On dividing eqn (i) and (ii) and simplifying.

$$V_{min} = \sqrt{\frac{rg(\tan\theta - \mu)}{1 + \mu\tan\theta}}$$

Case 3. A car on a banked smooth curved road.

A smooth surface offers no friction, but when such a surface is inclined, it can provide the necessary centripetal force for a vehicle to successfully take a corner.



Vertical forces: $RCos\theta = Mg \_\_\_\_\_\_\_\_\_(i)$

Horizontal force:

$$RSin\theta = \frac{MV^2}{r} \_\_\_\_\_\_(ii)$$

Dividing equation (i) by (ii)

$$\tan\theta = \frac{V^2}{rg} = \frac{h}{\sqrt{b^2 - h^2}}$$

$$V = \sqrt{rg\tan\theta}$$

Also

$$\tan\theta \approx Sin\theta = \frac{h}{b} = \frac{V^2}{rg}$$

$$h = \frac{V^2 b}{rg}$$

Note that

On the same basis curved railway tracks are also banked. The level of the outer rail is raised a little above the level of inner rail, while laying a curved railway track.

TYPES OF BANKED ROAD (RAIL LINES)

There are two types of the banked roads:-

1. Perfect banked road.
2. Imperfect banked road.

PERFECT BANKED ROAD

Is the kind of banked road where by the horizontal component of normal reaction force must be balanced with the centrifugal force. i.e

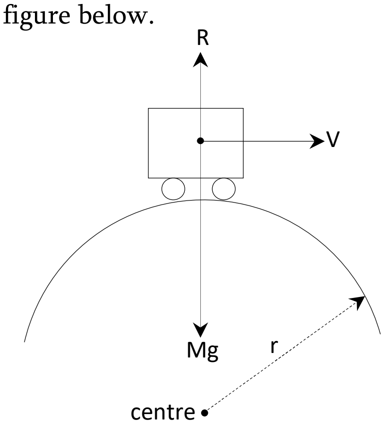
$$RSin\theta = \frac{MV^2}{r}$$

IMPERFECT BANKED ROAD

Is the kind of banked road where by the horizontal component of normal reaction force does not balanced with the centrifugal force. i.e $RSin\theta \neq \frac{MV^2}{r}$

8. MOTION OF THE CAR OVER HUMP BRIDGE OR CURVED BRIDGE

Consider a car of mass, M moving on a circular bridge of radius r at the highest point as shown on the figure below.



Resultant force on the car

$$Mg - R = \frac{MV^2}{r}$$

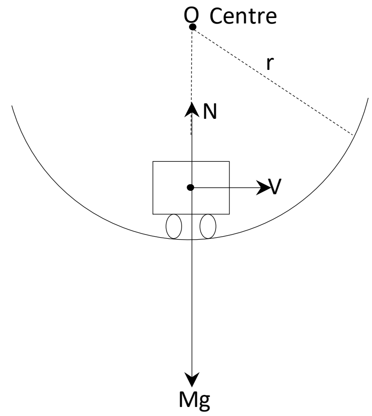
$$R = M\left(g - \frac{V^2}{r}\right)$$

Expression of the maximum speed of the car at the highest point of bump road. At the maximum speed of the car at the top of bump road, the car losses constant with the road, i.e R = 0.

$$0 = M\left(g - \frac{V_{max}^2}{r}\right)$$

$$V_{max} = \sqrt{gr}$$

Special case. Motion of a car passing through a dip.



A vehicle passing through a dip experienced a normal reaction greater than its weight. The different N – Mg provides the necessary centripetal force.

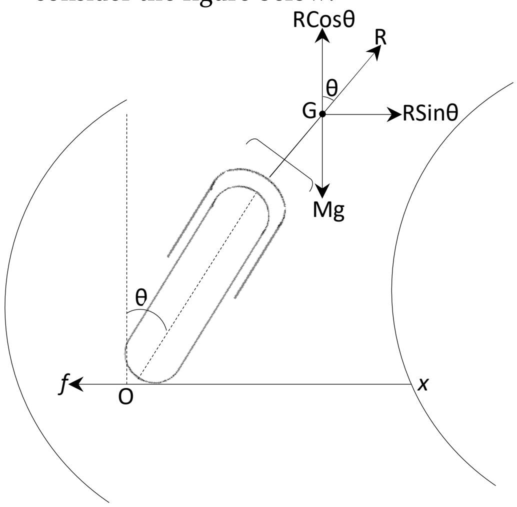
$$N - Mg = \frac{MV^2}{r}$$

$$N = M\left(g + \frac{V^2}{r}\right)$$

9. BENDING OF A CYCLIST

A cyclist (such as of motor cycle) taking a corner on a curved rough level road must bend inwards towards the centre of the curved road so as to create the friction between the tyres and the road which is required in order to provide necessary centripetal force. As force of friction is small and uncertain, dependence on it is not safe. To avoid dependence on force of friction for obtaining centripetal force, the cyclist has to bend a little inwards from his vertical position while turning. By doing so, a component of normal reaction in the horizontal direction provides the necessary centripetal force.

To calculate the angle of bending with vertical, consider the figure below:



Let M = Mass of the cyclist
V = Velocity of the cyclist while turning
R = radius of the circular path
θ = angle of bending with vertical

At the equilibrium for the cyclist

$$RSin\theta = \frac{MV^2}{r} \text{ \_\_\_\_\_\_\_\_(i)}$$

Since RSinθ provides the necessary centripetal force.

$$RCos\theta = Mg \text{ \_\_\_\_\_\_\_\_\_\_(ii)}$$

Taking (i) (ii)

$$\tan\theta = \frac{V^2}{rg} \text{ \_\_\_\_\_\_\_\_\_\_\_\_\_(1)}$$

The maximum angle through which the cyclist can lean without "skidding"

$RSin\theta = f = \mu Mg$

$RCos\theta = Mg$

$$\frac{RSin\theta}{RCos\theta} = \frac{\mu Mg}{Mg}$$

$$\tan\theta = \mu \text{ \_\_\_\_\_\_\_\_\_(2)}$$

$$\frac{V^2}{rg} = \mu$$

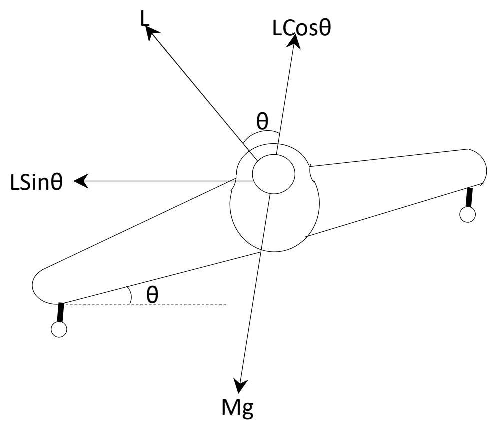
$$V_{max} = \sqrt{\mu rg} = \sqrt{rg\tan\theta}$$

Note:

$\tan\theta = \frac{V^2}{rg}$ clearly θ would depend on V and r. for a safe turn, θ should be small, for which V should be small and r should be large i.e *turning should be at a slow speed and along a track of large radius.* This means that, a safe turn should neither be fast nor sharp.

10. AIRCRAFT BANKING

The wings of the Aircraft are banked in order to produce the lifting forces and the horizontal component of the lift force provides the necessary centripetal force when aeroplane taking a curved track.



At the equilibrium, when Aeroplane (Aircraft) move with constant speed.

$$LSin\theta = \frac{MV^2}{r} \text{ \_\_\_\_\_\_\_\_\_(i)}$$

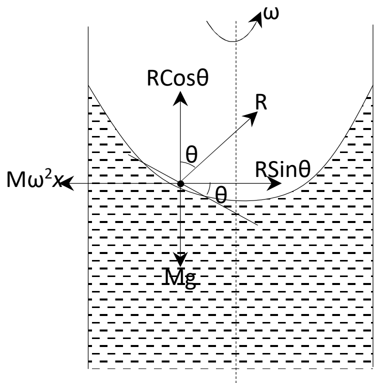
$$LSin\theta = Mg \text{ \_\_\_\_\_\_\_\_\_\_\_(ii)}$$

On dividing equation (i) by (ii)

$$\tan\theta = \frac{V^2}{r},\ V = \sqrt{rg\tan\theta}$$

11. ROTATING LIQUIED SURFACE

This is another example of the circular motion. Consider a point on the liquid surface at a distance x from the axis of rotation as shown in the figure below.



At this point the surface of liquid makes angle θ with the horizontal.

$RSin\theta = M\omega^2 x$ ______(i)

$RCos\theta = Mg$ ________(ii)

On dividing equation (i) by (ii) $\tan\theta = \frac{\omega^2 x}{g}$

$$\omega = \sqrt{\frac{g\tan\theta}{x}}$$

Frequency of oscillation

$$2\pi f = \sqrt{\frac{g\tan\theta}{x}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g\tan\theta}{x}}$$

We required to show that the oscillation of the liquid surface is the circular motion.

$$\tan\theta = \frac{\omega^2 x}{g} = \frac{dy}{dx}$$

$$dy = \frac{\omega^2}{g} \times dx$$

$$\int dy = \frac{\omega^2}{g}\int \times dx$$

$$y = \frac{\omega^2 x^2}{2g} + C$$

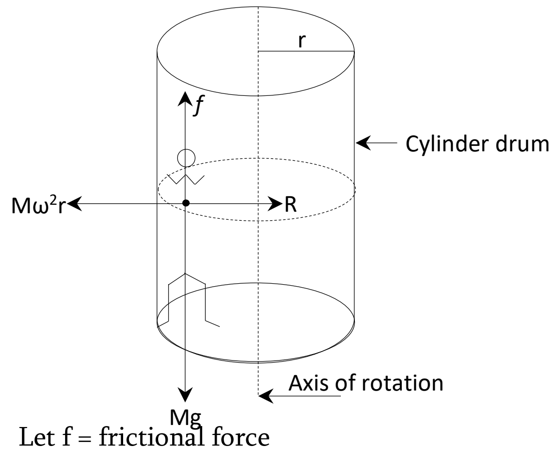
When $y = 0$, $X = 0$, $C = 0$

$$y = \frac{\omega^2 x^2}{2g} = KX^2$$

This represent equation of the parabola. This is an example of circular motion.

12. THE ROTOR MOTION

This is an example of the circular motion. It is consist of an upright drum inside in which people stand with their backs against the wall. The drum is spun at increasing speed about its centre vertical axis and at a certain speed the floor is pulled downward. The occupants as not fall but remain pinned against the wall of the rotor.



Let f = frictional force

R = Normal reaction force

μ = coefficient of friction force

r = radius of drum

g = acceleration due to gravity

At the equilibrium

$R = M\omega^2 r$, $\mu R = Mg$

$$\frac{\mu}{R} = \frac{Mg}{M\omega^2 r}$$

$$\mu = \frac{g}{\omega^2 r}$$

Angular velocity (ω) and frequency

$$\omega^2 = \frac{g}{\mu r},\ \omega = \sqrt{\frac{g}{\mu r}}$$

but $\omega = 2\pi f$

$$2\pi f = \sqrt{\frac{g}{\mu r}}$$

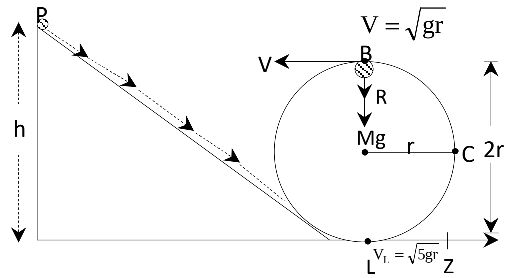
$$f = \frac{1}{2\pi}\sqrt{\frac{g}{\mu r}}$$

11. LOOPING OF THE LOOP

This is an example of the circular motion. When the model car (particle) of mass M is released from the point P to L, B and finally to the point Z is associated with the energy changes.

- At the point P

Initially the particle is at the rest when is at the point P, then the total energy possessed by the particle is only.



Gravitational potential energy.

$E_1 = Mgh$ ________(1)

When the particle is released and moves down the plane, the total energy is sum of potential energy and kinetic energy.

At the point B

The total energy possessed by the particle at the point B.

$E_2 = K.e + P.e$

$E_2 = 2Mgr + \frac{1}{2}MV^2$

At the highest point B

$\frac{MV^2}{r} = Mg$

$V^2 = rg$, $V = \sqrt{rg}$

Now

$E_2 = \frac{1}{2}Mgr + 2Mgr$

$E_2 = \frac{5}{2}Mgr$ _______(2)

Apply the law of conservation of mechanical energy.

(1) = (2)

$Mgh = \frac{5}{2}Mgr$, $h = \frac{5}{2}r$

When the particle reached at the point L and z, its possessing only kinetic energy.

APPLICATIONS OF CIRCULAR MOTION

There are several applications of circular motion in daily life. These include the following:-

1. CENTRIFUGAL GOVERNOR

By making use of centrifugal force, the speed of an engine can be automatically controlled. Such an automatic device is called "speed governor".

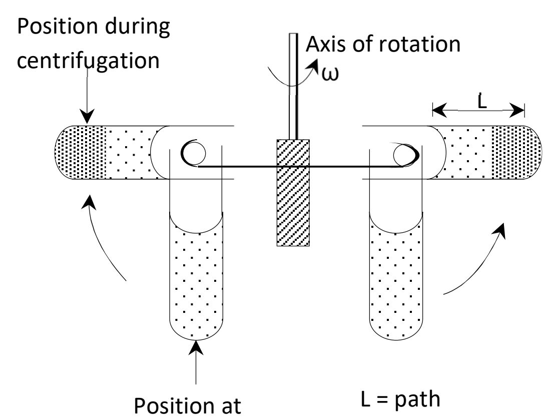
2. CENTRIFUGE DEVICE

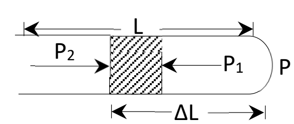
A centrifuge is a device that is used for separating mixture. Centrifuge device is the device which is used to separate the immiscible liquids of the different densities.

The centrifuge works using the sedimentation principle. The sample of liquid mixture is spanned at relatively high speed, creating a strong centripetal force on the liquid and its content. This force will make denser particle to accelerate outward in the radial direction. When the centrifuge is settled, the heavier (denser) particles settle to the bottom while lighter (less dense particles rise to the top.

Expression of centrifugal force for separation of two immiscible liquid

Consider the figure below which shows the motion of centrifuge device.





Let $P_2 = P$, $P_1 = P_2 + \Delta P$

The change in pressure, $\Delta P = P_1 - P_2$

Force on the test tube.

$F = A\Delta D$

Centripetal force corresponding to ΔL, $F = M\omega^2\Delta L$.

$M\omega^2 \Delta L = A\Delta P$

$$dp = \frac{M\omega^2}{A} dL$$

$$\int_o^L dP = \frac{M\omega^2}{A}\int_o^L dL$$

$$P - P_O = \frac{Mw^2L}{A}$$

$$\boxed{F = (P - P_O)A = M\omega^2 L}$$

Note that

1. Expression of centripetal force required to separates denser liquid and higher liquid.

$$F = (M - M')\omega^2 L$$

M = Mass of denser liquid.

M' = Mass of the higher liquid.

2. The centrifuge device can be applied to separates the following :-

   (i) Sugar crystals from molasses.
   (ii) Cream is separated from the milk in cream – separators.
   (iii) Bee wax from honey.
   (iv) Constituents of blood and urine samples.
   (v) The wet clothes are dried by dry – cleaners in the drying machines.
   (vi) Ultra centrifuge (high speed centrifuge are used to separate finest from water or from even highly viscous liquids).

3. **CENTRIFUGAL PUMP.**
   By making uses of centrifugal force , it is possible to transfer larger quantities of liquid against low back pressure. Such device is called centrifugal pump. These pumps are used as exhaust fans and blowers.

TYPE E: SOLVED EXAMPLES

EXAMPLE 01.

A small toy car moves round a circular track of radius 4m. the car makes one revolution in 10 second calculate:-

(a) The speed of the car.
(b) It centripetal acceleration
(c) The centripetal force exerted by the track mass of the car is 200gm.
(d) The safe speed with which the car can move around without toppling , it the distance between wheel is 4cm and the height of the centre of gravity of the car from the horizontal is 2cm.

Solution

(a) $V = \frac{2\pi r}{T} = \frac{2 \times 3.14 \times 4}{10}$

$V = 2.51mfs$

(b) $a = \frac{v^2}{r} = \frac{(2.51)^2}{4}$

$a = 1.57mfs^2$

(c) $F = Ma = 0.2 \times 1.57$

$F = 0.314N$

(d) $V_{max} = \sqrt{\frac{rag}{h}} = \sqrt{\frac{9.8 \times 0.02 \times 4}{0.02}}$

$V_{max} = 6.26mfs$

Example 02.

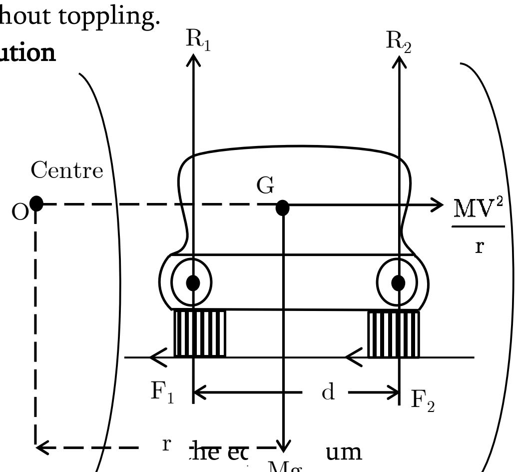
A car whose wheels are 1.4m apart laterally and whose centre of gravity is 0.5m above the ground moves round a curve of radius 60m. assuming no slipping of the wheels on the road find the highest speed at which the car can round the curve without overturning.

Solution

$$V_{max} = \sqrt{\frac{rdg}{2h}} = \sqrt{\frac{1.4 \times 60 \times 9.8}{2 \times 0.5}}$$

$$V_{max} = 2.69m/s$$

Example 03.
A car whose wheels are 1.5m apart laterally and whose centre of the gravity is 1.5m above the ground round a curve of radius 250m. find the maximum speed at which the car can travel without toppling.
Solution



$$R_1 + R_2 = Mg = F_1 + F_2$$

At the maximum speed when the car is moving without toppling , $R_1 = 0$, $R_2 = Mg$.
Take the moment of force about G.

$$Fh = \frac{R_2 d}{2}$$

$$\frac{MV_{max}^2 h}{r} = \frac{Mgd}{2}$$

$$V_{max} = \sqrt{\frac{rgd}{2h}} = \sqrt{\frac{9.8\times1.5\times250}{2\times1.5}}$$

$$V_{max} = 35m/s$$

Example 04.
Find the maximum speed at which car can take turn round a curve of 30m radius on a level road if the coefficient of friction between the tyres and the road is 0.4.
Solution

$$r = 30m \ , \ \mu = 0.4 \ , \ g = 10mfs^2$$

$$V_{max} = \sqrt{\mu rg} = \sqrt{0.4\times30\times10}$$

$$V_{max} = 11m/s$$

Example 05
A car travels on a flat , circular track of radius 200m at 30m/s and has a centripetal acceleration = 4.5m/s².

(a) If the mass of the car is 1000kg , what frictional force is required to provide the acceleration.
(b) If the coefficient of static friction is 0.8 , what is the maximum speed at which the car can circle the track?

Solutions.
(a) Frictional force required

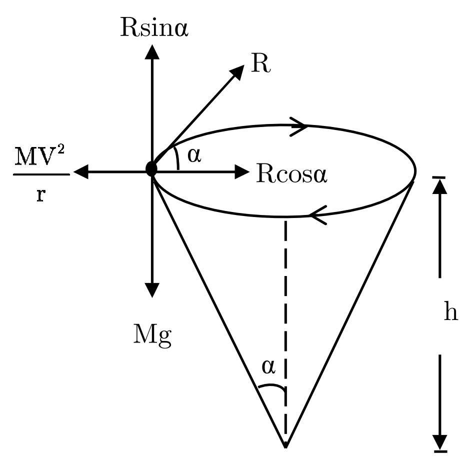
$$F = Ma = 1000\times4.5$$
$$F = 4,5000N$$

(b) $$V = \sqrt{\mu rg} = \sqrt{0.8\times200\times9.8}$$
$$V = 39.6m/s$$

Example 06.
A car is moving at 30km/hr in a circle of radius. Find the minimum value of μ for the car to make the turn without skidding.
Solution

$$\text{Since } \mu = \frac{v^2}{rg} = \frac{(8.3)^2}{60\times9.8}$$
$$\mu = 0.12$$

Example 07
A particle describes a horizontal circle on the smooth surface of an inverted cone. The height of the plane of the circle above the vertex is 9.8cm. Find the speed of the particle take g = 9.8m/s².
Solution.
h = 9.8cm , V = ?



$$R\cos\alpha = \frac{mv^2}{r} \ , \ R\sin\alpha = mg$$

Dividing , we get

$$\tan\alpha = \frac{rg}{v^2} \text{ or } \frac{r}{h} = \frac{rg}{v^2}$$

$$V = \sqrt{gh} = \sqrt{9.8\times9.8\times10^{-2}}$$

$$V = 0.98\text{m}/\text{s}$$

Example 08

What is the maximum speed with which a cyclist can turn around curved path of radius 10m , if the coefficient of friction force between the tyres and road is 0.5? If the cyclist wants to avoid overturning by what angle the cyclist must lean from the vertical?

Solution.

$$V_{max} = \sqrt{\mu rg} = \sqrt{0.5\times10\times9.8}$$

$$V_{max} = 7\text{m}/\text{s}$$

For no overturning

$$\tan\theta = \mu \ , \ \theta = \tan^{-1}(\mu)$$

$$\theta = \tan^{-1}(0.5) = 26.8$$

Example 09.

A cyclist speeding at 18km/hr on a level road takes a sharp circular turn of radius 3m without reducing the speed. The coefficient of static friction between the tyres and the road is 0.1, will the cyclist slip while taking the turn?

Solution

The condition for the cyclist not to slip

$$V^2 \le \mu rg$$

$$\le 0.1\times3\times9.8$$

$$V^2 \le 2.94\text{m}^2/\text{s}^2$$

Thus, for the cyclist not to slip , V^2 should be less than $2.94m^2/s^2$.

$$V = 18\text{km}/\text{hr} = 5\text{m}/\text{s}$$

$$V^2 = 25\text{m}^2/\text{s}^2$$

It's clear that the condition above is not obeyed. Hence the cyclist will slip.

Example 10.

A highway road designed for an overage speed of 72km/hr has a turn of radius 50m. to what angle must the road be banked so that cars travelling at 72km/hr may not overturn.

$$V = \frac{72\times10^3}{3600} = 20\text{m}/\text{s}$$

$$\text{Since } \tan\theta = \frac{v^2}{rg} \ , \ \theta = \tan^{-1}\left[\frac{v^2}{rg}\right]$$

$$\theta = \tan^{-1}\left(\frac{20^2}{50\times9.8}\right)$$

$$\theta = 39^\circ$$

Example 11

A circular race track of radius 400m is banked at an angle of 10°. If the coefficient of friction between the wheels of a race car and the road is 0.2 , what is the (i) optimum speed of the car to avoid wear and tear on its tyres. (ii) maximum permissible speed to avoid slipping .

Solution

Here r = 400m , θ = 10° , μ = 0.2

(i) Optimum speed of the car to avoid tear and wear

$$V = \sqrt{rg\tan\theta} = \sqrt{400\times9.8\tan10^\circ}$$

$$V = 26.29\text{m}/\text{s}$$

(ii) Maximum permissible speed

$$V_{max} = \sqrt{\frac{rg(\mu+\tan\theta)}{1-\mu\tan\theta}}$$

$$= \sqrt{\frac{400\times9.8(0.2+1763)}{1-0.2\times0.1763}}$$

$$V_{max} = 39.10\text{m}/\text{s}$$

Example 12

A cyclist speeding at 6m/s in a circle of 36m diameter makes an angle θ with the vertical. What is the value of θ? Also determine the maximum possible value of the coefficient of friction between the tyres and the road.

Solution

Here , V = 6m/s , $r = 36/2 = 18\text{m}$

$$\tan\theta = \frac{v^2}{rg} = \frac{6\times6}{18\times9.8} = 0.2040$$

$$\theta = \tan^{-1}(0.2040) = 11^\circ32'$$

$$\theta = 11^\circ32'$$

Also $\mu = \tan\theta = 0.2040$

Example 13

A car is speeding on a horizontal road curving round with a radius 60m. The coefficient of friction between the wheels and the road is 0.5 the height of centre of gravity of the car from the road level is 0.3m and the distance between the wheels is 0.8m. Calculate the maximum safe velocity for negotiating the curve will the car skid or topple if this velocity is exceeded?

Solution

$r = 60m$, $\mu = 0.5$, $h = 0.3m$, $d = 0.8m$

For no skidding

$$\tan\theta = \mu = \frac{V^2}{rg}$$

$$V = \sqrt{\mu rg} = \sqrt{0.5\times60\times9.8}$$

$$V = 17.15m/s$$

For toppling, $\frac{mv^2h}{r} = mga\left\{a = \frac{d}{2}\right\}$

$$V = \sqrt{\frac{rdg}{2h}} = \sqrt{\frac{9.8\times60\times0.8}{2\times0.3}}$$

$$V = 28m/s$$

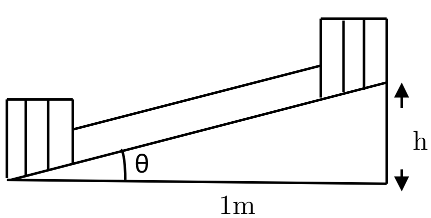
Hence the maximum safe velocity for negotiating the curve is 17.15m/s. Beyond this speed , skidding starts until the car topples at V = 28m/s.

Example 14.

A train has to take a circular turn of radius 500m with a speed of 36km/hr. By how much the outer rail should be raised above the inner rail so that there is no side pressure on the rails. The distance between the rails is 1m ($g = 10m/s^2$).

Solution

Let the outer rail be raised by h metre.



$$\tan\theta = \frac{h}{1} = \frac{V^2}{rg}$$

$$h = \frac{V^2}{rg} = \left(\frac{36\times1000}{3600}\right)^2\times\frac{1}{500\times10}$$

$$h = 0.02m = 2cm$$

Example 15

A racing – car of 1000kg moves round a banked track at a constant speed of 108km/hr. Assuming the total reaction on at the wheels is normal to the track and the horizontal radius of the track is 100m calculate the angle of inclination of the track to the horizontal and the reaction at the wheels ($g = 10m/s^2$)

Solution

$$V = \frac{108\times1000}{3600} = 30m/s$$

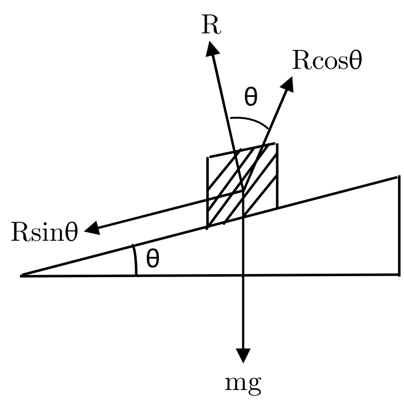
Since $\tan\theta = \frac{V^2}{rg}$, $\theta = \tan^{-1}\left(\frac{V^2}{rg}\right)$

$$\theta = \tan^{-1}\left[\frac{30\times30}{100\times10}\right]$$

$$\theta = 42^\circ \text{ (approx.)}$$

Let $R_1 + R_2 = R$ = total reactions

Forces on the wheels



At the equilibrium of the car

$$R\cos\theta = mg \;,\; R = \frac{mg}{\cos\theta}$$

$$R = \frac{1000\times10}{\cos42^\circ} = 13456.33N$$

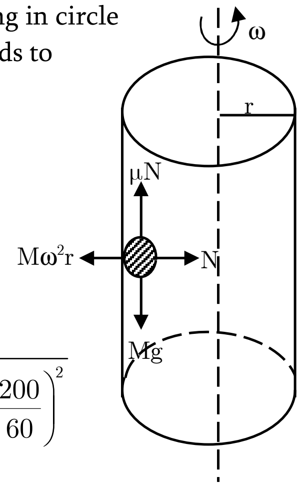
$$R = 13456.33N$$

Example 16

A hallow cylinder of radius 10cm rotates about its axis which is vertical. A small body remains in constant with the inner wall if the frequency of rotation is 200rev per minutes but falls at lower frequency. Find the coefficient of friction between the body and the cylinder.

Solution

Since the body is moving in circle and N is directed towards to the centre.



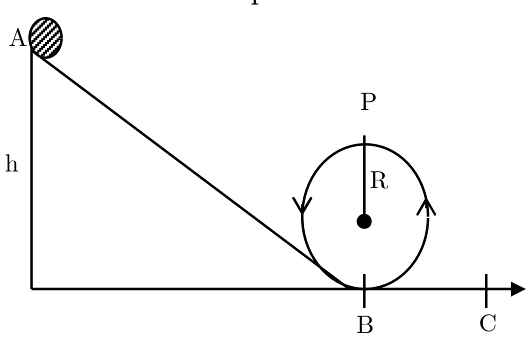
$$N = m\omega^2 r$$
$$\mu N = mg$$
$$\mu m\omega^2 r = mg$$
$$\mu = \frac{g}{\omega^2 r}$$
$$\mu = \frac{9.8}{0.1\times 4\pi^2\left(\frac{200}{60}\right)^2}$$
$$\mu = 0.225$$

Example 17 **NECTA 2001/P1/3**

(a) (i) Why are road beds and high ways banked on curve.

(ii) What is the smallest radius of a circle at which a cyclist can travel , if his speed is 30km/hr and the coefficient of friction is 0.32? (g = 10m/s²).

(b) Figure below shows a toy runway after release from a point A a small model car runs down the slope loops the loop and travels on towards c. The radius of the loop is 0.25m.



(i) Outline the energy changes as the model car moves from A to C ignore frictional effect.

(ii) What is the minimum speed with which the car must pass point P at the top of the loop if it is the remain in constant with the runway?

Solution

(a) (i) Refer to your notes

(ii) since $\frac{mv^2}{r} = \mu mg$

$$r = \frac{v^2}{\mu g} = \frac{8.33\times 8.33}{0.32\times 10}$$
$$r = 21.7\text{m}$$

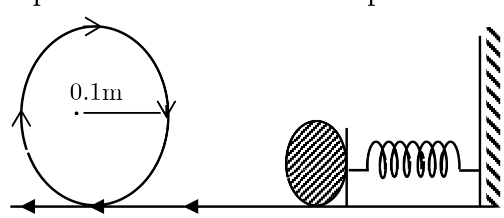
(b) (i) initially , the car is at the rest when is at the point A , then its possessing only gravitational potential energy from point A to point B the car loses gravitational potential energy (g.p.e) and gains kinetic energy (k.e). From point B to point P its loses some of its kinetic energy and gain gravitational potential energy. From point B to P its loses some of its kinetic energy and gain gravitational potential energy , it gained and regains its original kinetic energy at point B from B to C kinetic energy is constant.

(ii) At the highest point P

$$Mg = \frac{MV^2}{r}$$
$$V = \sqrt{rg} = \sqrt{10\times 0.25}$$
$$V = 1.581\text{m}/\text{s}$$

Example 18.

A compressed spring is used to propel a ball bearing along a track which contains a circular loop of radius 0.1m in a vertical plane below.



The spring obeys Hooke's law (i.e F = KX). It requires a force of 0.2N to compress it by 1mm. the spring is compressed by 30mm. calculate:-

(a) The energy stored in the spring(E = ½ Fe)

(b) A ball bearing of mass 25g is placed at the end of the spring which is then released , calculate the:-

(i) Speed with which the ball bearing leave the spring.
(ii) Speed of the ball at the top of the loop.
(iii) Force exerted by the track on the ball at the top of the loop. (Ignore friction).

Solution

(a) Force constant on the spring , $\frac{F}{e} = K$

$$K = \frac{0.2}{1\times10^{-3}} = 200\text{Nm}^{-1}$$

Energy stored in the spring

$$E = \frac{1}{2}KX^2$$

$$= \frac{1}{2}\times200\times\left(30\times10^{-3}\right)^2$$

$$E = 9.0\times10^{-2}\text{J}$$

(b) (i) Apply the law of conservation of energy

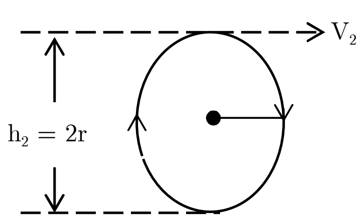
Loss in elastic p.e on the spring = gain in k.e of the ball

$$9.0\times10^{-2} = \frac{1}{2}MV^2$$

$$V = \left[\frac{2\times9\times10^{-2}}{0.025}\right]^{1/2}$$

$$V = 2.7\text{m}/\text{s}$$

(ii) Let V_2 be speed of the ball reached at the top of loop



Apply the law of conservation of mechanical energy

p.e + k.e = Elastic p.e on spring

$$Mgh_2 + \frac{1}{2}MV_2^2 = \frac{1}{2}KX^2$$

$$Mgh_2 + \frac{1}{2}MV_2^2 = 9\times10^{-2}$$

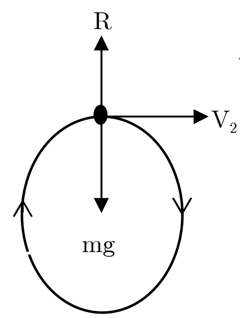
$$V_2^2 = \frac{2\times9\times10^{-2}}{M} - 2gh_2$$

$$h_2 = 2r$$

$$V_2 = \left[\frac{2\times9\times10^{-2}}{0.02} - 4\times10\times0.1\right]^{1/2}$$

$$V_2 = 1.8\text{m}/\text{s}$$

(iii) Let R = force exerted on the ball at the top of the loop



$$\frac{MV_2^2}{r} = Mg - R$$

$$R = M\left[g - \frac{V_2^2}{r}\right]$$

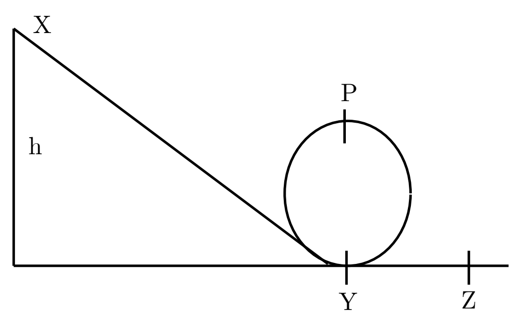
$$R = 0.025\left[10 - \frac{(1.8)^2}{0.1}\right]$$

$$R = 0.55\text{N}$$

Example 19

(a) Write down an expression for the force required to maintain the motion of a body of mass , M moving with constant speed V in a circle of radius R. in which direction does the force a it?

(b) Figure below shows a toy runway after release from a point such as X , a small model car runs down the slope , loops the loop , and travels on towards Z the radius of the loop is 0.25m.

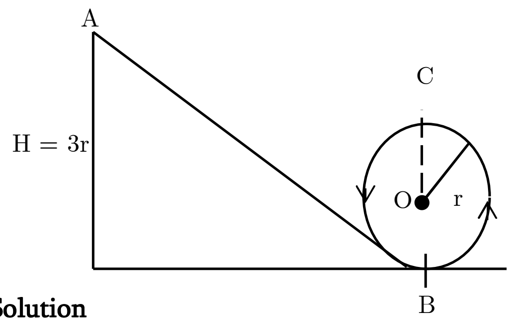


(i) Ignoring the effect of friction outline the energy changes as the model moves from X to Z.
(ii) What is the minimum speed with which the car must pass point P at the top of the loop if it is the remain in constant with the runway?
(iii) What is the minimum value of h which allows the speed calculated in (ii) to achieved? The effect of friction can gain be

ignored (Assume that g = $10m/s^2$)
Ans: (b) (ii) 1.58m/s (iii) 0.625m.

Example 20

Figure below shows a smooth looping the loop' track A particle of mass M is released from point A as shown. If H = 3r , would the particle loop the loop? What is the force on the circular track when the particle is at point (i) B (ii) C ?



Solution

Here H = 3r

When the particle of mass M is released from A , velocity acquired by it on reaching B is $V_B = \sqrt{2gH} = \sqrt{6gr}$ which is greater than $\sqrt{5gr}$, the minimum velocity required at B for looping the loop. Hence the particle will loop the loop.

(i) Force exerted by the circular track on the particle at B.

$$N_1 = Mg + \frac{MV_B^2}{r}$$

$$= Mg + \frac{M6gr}{2}$$

$$\underline{N_1 = 7Mg}$$

(ii) Force exerted by circular track on the particle at C.

$$N_2 = \frac{MV_C^2}{r} - Mg$$

$$\text{Where } V_C^2 = V_B^2 - 4gr = 6gr - 4gr$$

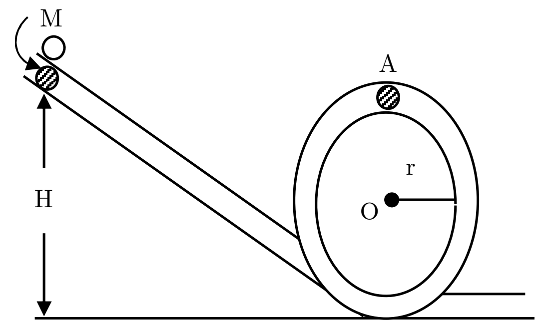
$$V_C^2 = 2gr$$

$$N_2 = \frac{M.2gr}{r} - Mg$$

$$\underline{N_2 = Mg}$$

Example 21

The figure below shows a frictionless track. A block of mass M starts from rest on at O. The block on reaching A pushes a track with a force equal to twice its weight. Obtain the value of H in terms of r.



Solution

Total energy at point , O

$$E_1 = MgH$$

At the point A

$$E_2 = k.e + p.e = \frac{1}{2}MV^2 + 2Mgr$$

Apply the law of conservation of mechanical energy

$$E_1 = E_2$$

$$MgH = \frac{1}{2}MV^2 + 2Mgr$$

$$V^2 = 2g(H - 2r) \ldots\ldots\ldots(i)$$

At the point A

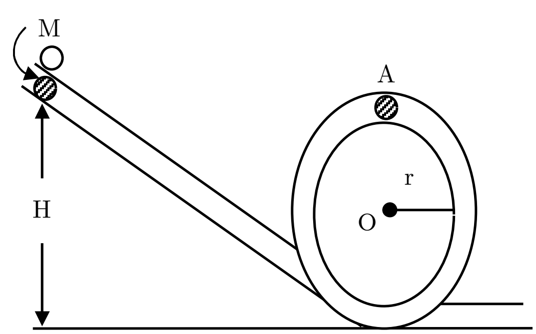
$$\frac{MV^2}{r} = Mg = 2Mg$$

$$\frac{MV^2}{r} = 3Mg$$

$$V^2 = 3gr \ldots\ldots\ldots\ldots\ldots\ldots(ii)$$

$$(i) = (ii)$$

Its weight. Obtain the value of H in terms of r.



Solution

Total energy at point , O

$$E_1 = MgH$$

At the point A

$$E_2 = k.e + p.e = \frac{1}{2}MV^2 + 2Mgr$$

Apply the law of conservation of mechanical energy

$$E_1 = E_2$$

$$MgH = \frac{1}{2}MV^2 + 2Mgr$$

$$V^2 = 2g(H - 2r) \ldots\ldots\ldots\text{(i)}$$

At the point A

$$\frac{MV^2}{r} = Mg = 2Mg$$

$$\frac{MV^2}{r} = 3Mg$$

$$V^2 = 3gr \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$\text{(i)} = \text{(ii)}$$

Which is greater than $\sqrt{5gr}$, the maximum velocity required at B for looping the loop. Hence the particle will loop the loop.

(i) Force exerted by the circular track on the particle at B.

$$N_1 = Mg + \frac{MV_B^2}{r}$$

$$= Mg + \frac{M6gr}{r}$$

$$\underline{N_1 = 7Mg}$$

(ii) Force exerted by circular track on the particle at C.

$$N_2 = \frac{MV_C^2}{r} - Mg$$

Where $V_C^2 = V_B^2 - 4gr = 6gr - 4gr$

$$V_C^2 = 2gr$$

$$N_2 = \frac{M \cdot 2gr}{r} - Mg$$

$$\underline{N_2 = Mg}$$

Example 22

The figure below show a frictionless track. A block of mass M starts from rest on at O. The block on reaching A pushes a track with a force equal to twice.

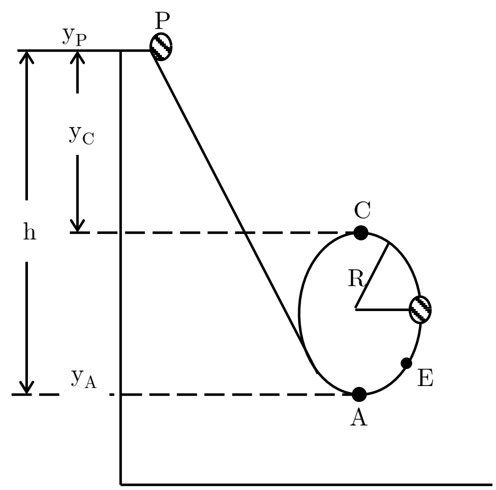
$$3gr = 2g(H - 2r)$$

$$3gr = 2gH - 4gr$$

$$H = \frac{7}{2}r$$

Example 23

The figure below shows a toy run away after being released from a point X a small model car run down the slope loops the loop as illustrated on the figure below.



(i) Show that $V_C^2 = 2g(h - 2R)$ and to complete loops of the loop track $V_C^2 > O$, $h > 2R$

(ii) Show that $V_A^2 = 2gh$ and reaction at the point A is $R_A = Mg\left[1 + \frac{2h}{R}\right]$

(iii) Show that $R_C = Mg\left[\frac{2h}{R} - 5\right]$ and $b \geq 2.5R$ and $V_C \geq \sqrt{gR}$.

Solution

(i) Apply the principle of conservation of mechanical energy.

$$(k.e + p.e)_p = (k.e + p.e)_c$$

$V_p = O$ Since the body starts from the rest

$$Mgh = \frac{1}{2}MV_C^2 + 2MgR$$

$$V_C^2 = 2g(h-2R)$$

Hence shown

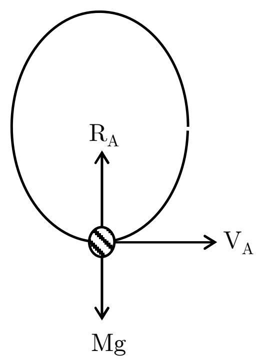
$$V_C = \sqrt{2g(h-2R)} \text{ , for } V_C^2 > O$$

$$2g(h-2R) > O$$

$$h > 2R$$

Hence shown.

(ii) At the point , A



$$R_A - Mg = \frac{MV_A^2}{R}$$

$$R_A = M\left[g + \frac{V_A^2}{R}\right]$$

Apply the principle of conservation of mechanical energy.

$$(p.e + ke)_A = (p.e + k.e)_P$$

$$Y_A = 0 \text{ , } V_P = 0$$

$$\frac{1}{2}MV_A^2 = \frac{1}{2}MV_P^2 + Mgh$$

$$V_A^2 = 2gh$$

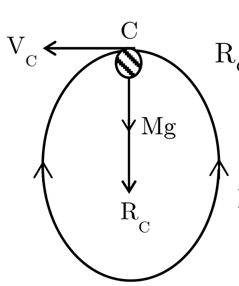
Now,

$$R_A = Mg + \frac{2Mgh}{R}$$

$$R_A = Mg\left[1 + \frac{2h}{R}\right]$$

Hence shown.

(iii) At the point C



$$R_C + Mg = \frac{MV_C^2}{R}$$

$$R_C = \frac{MV_C^2}{R} - Mg$$

$$\text{But } V_C^2 = 2g(h-2R)$$

$$R_C = 2g(h-2R)\frac{M}{R} - Mg$$

$$R_C = Mg\left[\frac{2h}{R} - 5\right]$$

For the body to be contact with the track , $R_C \geq O$

$$Mg\left[\frac{2h}{R} - 5\right] \geq O \text{ , } h \geq 2.5R$$

$$\text{Also } V_C^2 = 2g(h-2R)$$

$$\frac{V_C^2}{2g} = h - 2R$$

$$h = \frac{V_C^2}{2g} + 2R$$

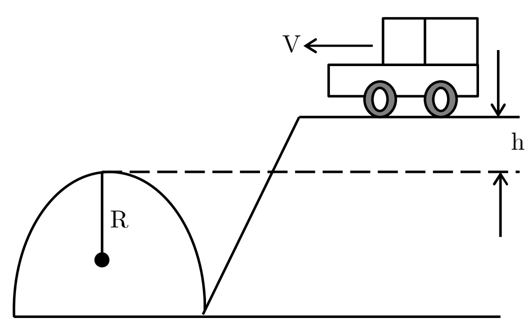
$$\frac{V_C^2}{2g} + 2R \geq 2.5R$$

$$V_C \geq \sqrt{gR}$$

Hence shown

Example 24

Figure below shows a loops the loop track radius , R a car (without engine) starts from a plat form at a distance h above the top of the loop and goes around the loop and goes around the loop without falling off the track. Find the minimum value of h for a successful looping (Neglecting friction)



Solution

Let the gravitational potential energy be zero at the platform and the car starts with negligible speed. Suppose the speed of the car at the top most of the loop is V.

Apply the law of conservation of energy.

$$\frac{1}{2}MV^2 = Mgh \text{ , } V^2 = 2gh$$

At the top of circular path

$$Mg + N = \frac{MV^2}{R}$$

$$Mg + N = \frac{2Mgh}{R}$$

Where $h = h_{min}$, $N = 0$

$$h_{min} = \frac{R}{2}$$

Example 25 **NECTA 2006/P1/3**

(a) (i) Explain what is meant by angular velocity and centripetal acceleration.

(ii) Why does a motorbike rider bend while going around a corner?

(b) (i) What is the maximum speed at which the car can safety go round a circular curve of radius 48m on a horizontal road if the coefficient of static friction between tyres and road is 0.8?

(ii) Will the car overturn or skid if it just exceeds the speed started in (b) (i) above? Assume the width between the wheels is 1.5 metre and the centre of gravity of the car is 0.6m above the road.

Solution

(a) Refer to your notes

(b) (i) For the maximum safe speed.

$$\mu Mg = \frac{MV^2}{r}$$

$$V_{max} = \sqrt{\mu rg} = \sqrt{0.8 \times 48 \times 9.8}$$

$$V_{max} = 19.40 m/s$$

(ii) For no overturning of the car

$$V \leq \sqrt{\frac{rag}{h}}$$

$$\leq \sqrt{\frac{0.75 \times 48 \times 9.8}{0.6}}$$

$$V \geq 24.24 m/s$$

$\therefore$ The car will not overturn , it will just skid.

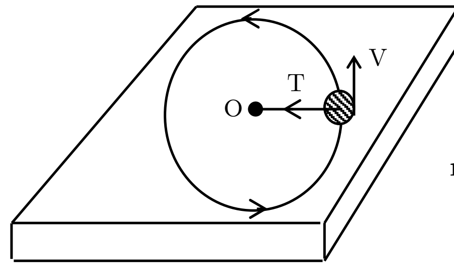
Example 26

A 200gm mass lying on a smooth table is attached to 100cm length of string whose other end is fixed at the point O on the table calculate:-

(i) The initial tension in the string if the mass is pushed with an initial velocity of 3m/s in a direction perpendicular to the string.

(ii) The new velocity of the mass if the string is shorted to 50cm while the mass rotates.

Solution



(i) $T = \frac{MV_1^2}{r_1} = \frac{0.2 \times 3 \times 3}{1}$

$T = 1.8N$

(ii) Apply the principle of conservation of angular momentum.

$$MVr = \text{Constant}$$

$$MV_1r_1 = MV_2r_2$$

$$V_2 = V_1\left[\frac{r_1}{r_2}\right] = \frac{3 \times 100}{50}$$

$$\underline{V_2 = 6m/s}$$

Example 27

A curve in a road has 60m radius. The angle of bank of the road is 47°. Find the maximum speed a car have without skidding if the coefficient of static friction between the tyres and road is 0.8.

Solution

Since

$$V_{max} = \sqrt{\frac{gr(\mu + \tan\theta)}{1 - \mu\tan\theta}}$$

$$= \sqrt{\frac{9.8 \times 60(0.8 + \tan 47^\circ)}{1 - 0.8\tan 47^\circ}}$$

$$V_{max} = 88m/s$$

Example 28

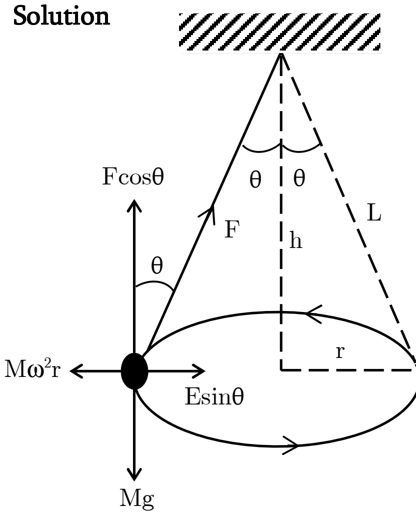
A thin inextensible string 200cm long has its upper end attached to a fixed point and a small lead sphere of mass 75gm is attached to the lower end of the string. If the sphere is made to describe a horizontal surface of radius 100cm with the uniform speed. Find the:

(i) Tension in the string

(ii) Time taken to describe a complete circle.

(iii) The angular velocity

Solution



M = 75 × 10⁻³kg

r = 100cm = 1m

g = 9.8m/s²

L = 200cm = 2m

(i) F = tension in the string

$$L^2 = h^2 + r^2$$

$$h = \sqrt{L^2 - r^2}$$

$$\cos = \frac{\sqrt{L^2 - r^2}}{L}$$

At the equilibrium

$$F\cos\theta = Mg$$

$$F = \frac{Mg}{\cos\theta} = \frac{MgL}{\sqrt{L^2 - r^2}}$$

$$F = \frac{75\times10^{-3}\times9.8\times2}{\sqrt{2^2 - 1^2}}$$

$$F = 0.84957\text{N}$$

(ii) Periodic time of conical pendulum

$$T = 2\pi\sqrt{\frac{L\cos\theta}{g}} = 2\pi\sqrt{\frac{h}{g}}$$

$$T = 2\times3.14\times\sqrt{\frac{\sqrt{3}}{9.8}}$$

$$T = 2.638\text{ sec}$$

(iii) $$\omega = \frac{2\pi}{T} = \frac{2\times3.14}{2.63}$$

$$\omega = 2.379\text{ rad / sec}$$

Example 29

A car travels over a hump back bridge of radius of curvature 45m. Calculate the maximum speed of the car if the wheels are to remain in contact with the bridge.

Solution

$$V_{max} = \sqrt{gR} = \sqrt{9.8\times45}$$

$$V_{max} = 21\text{m / s}$$

Example 30

A road curve of 200m radius is banked at the correct angle for a speed of 15m/s. if a car rounds this curve at 30m/s , what is the minimum coefficient of friction between the tyres and the road so that the car will not skid?

Solution.

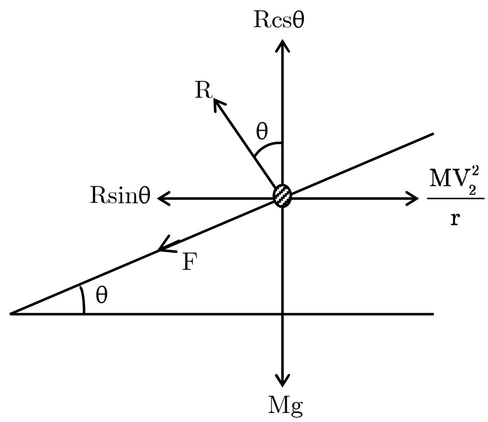
Let θ = correct angle of banking

$$\tan\theta = \frac{V_1^2}{rg}$$

$$\theta = \tan^{-1}\left[\frac{V_1^2}{rg}\right] = \tan^{-1}\left[\frac{15\times15}{9.8\times200}\right]$$

$$\theta = 6.5^\circ$$

For the car take the curve without skidding at a higher speed , there must be friction between the road and tyres.



Resolving horizontally.

$$F\cos6.5^\circ + R\sin6.5^\circ \geq \frac{MV_2^2}{r} \text{.......(i)}$$

Resolving vertically

$$R\cos6.5^\circ = Mg \text{..........(ii)}$$

$$\frac{\mu R\cos6.5^\circ + R\sin6.5^\circ}{R\cos6.5^\circ} \geq \frac{MV_2^2}{r} / Mg$$

$$\frac{\mu\cos6.5^\circ + \sin6.5^\circ}{\cos6.5^\circ} \geq \frac{V_2^2}{rg}$$

$$\frac{\mu\cos 6.5^\circ + \sin 6.5^\circ}{\cos 6.5^\circ} = \frac{30\times 30}{200\times 9.8}$$

$$\mu = 0.345$$

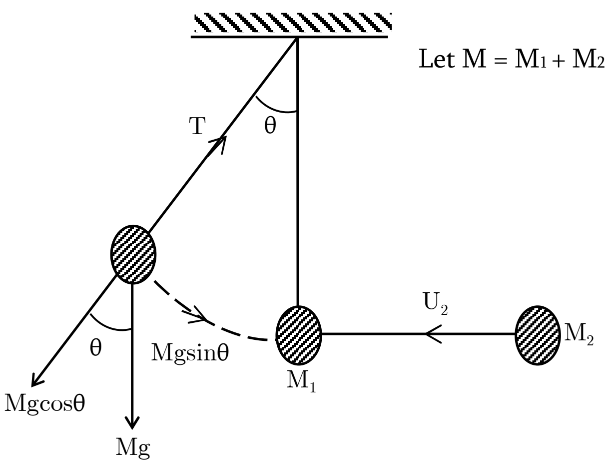
Example 31.

A mass of 2.9kg is suspended from a string 50cm and is at rest another body of mass 100gm moving horizontally with a velocity of 150m/s , strikes and sticks to it.

(a) What is the tension in the string when it makes an angle of 60° with the vertical ?

(b) Will it complete the circle ?

Solution



Let $M = M_1 + M_2$

(a) Apply the principle of conservation of linear momentum

$$M_1U_1 + M_2U_2 = (M_1 + M_2)V$$

$$V = \frac{M_1U_1 + M_2U_2}{M_1 + M_2} = \frac{0.1\times 150}{2.9 + 0.1}$$

$$V = 5\text{m/s}$$

Tension on the string at that point

$$T = M\left[\frac{V^2}{r} - g\cos\theta\right]$$

$$= 3\left[\frac{5^2}{0.5} - 9.8\cos 60^\circ\right]$$

$$T = 135.3\text{N}$$

At the lowest point the velocity is 5m/s in order to complete the circle the minimum velocity at the bottom of the circle.

$$V_1 = \sqrt{5gr}$$

$$V_1 = \sqrt{5\times 9.8\times 0.5} = 4.95\text{m/s}$$

Since $V_1 < V$, hence the body will complete the circle.

Example 32 **NECTA 2015/P1/3**

(a) (i) Mention three effects of looping the loop.

(ii) Why there must be a force acting on the particle moving with uniform speed in a circular path ? Write down an expression for its magnitude.

(b) A driver negotiating a sharp bend usually tend to reduce the speed of the car.

(i) What provides the centripetal force on the car ?

(ii) Why is it necessary to reduce its speed ?

(c) A ball of mass 0.5kg is attached to the end of a cord whose length is 1.5m then whirled in the horizontal circle. If the cord can withstand a maximum tension of 50N, calculate the :-

(i) Maximum speed the ball can have before the cord breaks

(ii) Tension in the cord if the ball speed is 5m/s

Solution

(a) (i) water stays in the bucket as the bucket is made to loops the loop.

- A pilot does not fall as an aeroplane execute a vertical loop.
- Passenger of a roller coaster do not fall over as it loops the loop.

(ii) A particle moving with uniform speed in the circular path has an acceleration. According to the Newton's second law , there must be a force associated with that acceleration and is the force which maintain the particle moving in a circular path. The magnitude of the force , $F = MV^2/r$

(b) (i) Friction force between the road and the tyres.

(ii) It is necessary to reduce speed as to avoid overturning or skidding of the car.

(c) (i) $V = \sqrt{\frac{T_{max}r}{M}} = \sqrt{\frac{50\times 1.5}{0.5}}$

$V = 12.25\text{m/s}$

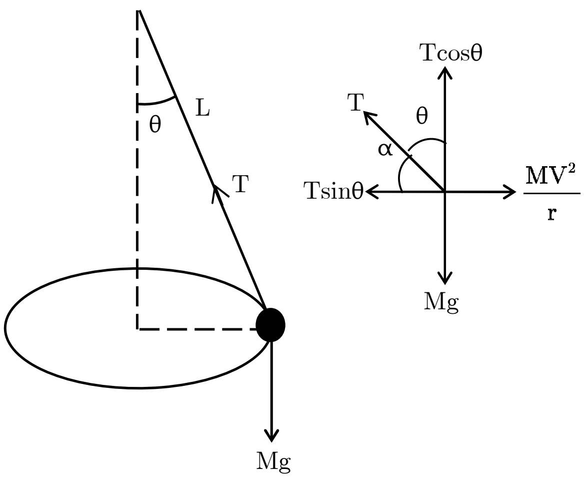
(ii) $$T = \frac{0.5\times5\times5}{1.5}$$

$$T = 8.33\text{N}$$

Example 33.

A man whirls a stone round his bead on the end of a string 4.0m long. Can the string be in a horizontal circle ? if the stone has a mass of 0.4kg and the string will break if the tension in it exceeds 8N, what is the smallest angle the string can make with the horizontal ? What is the speed of the stone ? (Take g = 10m/s²).

Solution



At the equilibrium

$$T\cos\theta = Mg \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$T\sin\theta = \frac{MV^2}{r} \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

From equation (i)

$$T = \frac{Mg}{\cos\theta}$$

When the string is in horizontal position

$$\theta = 90^\circ \text{ , } \cos 90^\circ = 0$$

$$T = \frac{Mg}{0} = \alpha$$

Thus , the tension must be infinite which it is impossible so the string cannot be in the horizontal plane. The maximum angle θ is given by the breaking tension of the string in the equation.

$$T\cos\theta = Mg$$

$$8\cos\theta = 0.4\times10$$

$$\cos\theta = \frac{1}{2} \text{ , } \theta = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta = 60^\circ$$

The angle with horizontal

$$\alpha = 90^\circ - \theta = 90^\circ - 60^\circ$$

$$\theta = 30^\circ$$

$$\text{Since } T\sin\theta = \frac{MV^2}{r} \text{ , } r = L\sin\theta$$

$$T\sin\theta = \frac{MV^2}{L\sin\theta}$$

$$V = \sqrt{\frac{TL\sin^2\theta}{M}} = \sqrt{\frac{8\times4\left(\sin 60^\circ\right)^2}{0.4}}$$

$$V = 7.7\text{m}/\text{s}$$

Example 34

A 0.6kg and a 1.2kg model Airplane fly in horizontal circles on guidelines that a parallel to the ground. The speeds of the plane are the same and the same type of cord is used for each guideline. The smallest circle on which the 0.6kg plane can fly without breaking its guideline has a radius of 3.5m. what is the radius of the smallest circle on which the 1.2kg plane can fly ?

Solution

$$\text{For the 0.6kg plane: } T = \frac{M_1V^2}{r_1} \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\text{For the 1.2kg plane: } T = \frac{M_2V^2}{r^2} \ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$\text{(i)} = \text{(ii)}$$

$$\frac{M_1V^2}{r_1} = \frac{M_2V^2}{r_2}$$

$$r_2 = \frac{M_2r_1}{M_1} = \frac{1.2\times3.5}{0.6}$$

$$r_2 = 7\text{m}$$

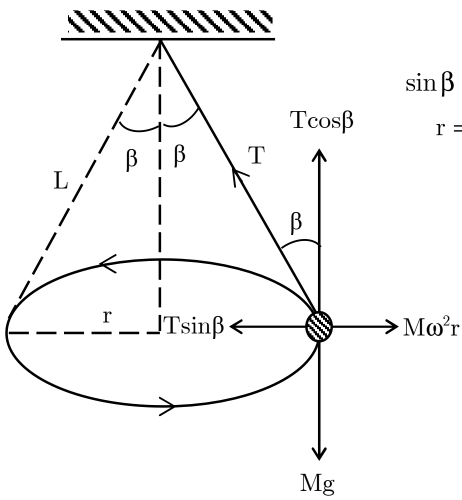
Example 35

A stone 0.5kg is tied to string of the length 0.25m and is rotated in a horizontal plane at a constant rat of 1.5 r.p.s.

(i) Find the tension in the string if the string makes an angle β with the vertical.

(ii) Calculate the value of β

Solution



$\sin\beta = r/L$

$r = L\sin\beta$

(i) At the equilibrium resolves horizontally

$$T\sin\beta = M\omega^2 r = M\omega^2 L\sin\beta$$

$$T = M\omega^2 L = M(2\pi f)^2 L$$

$$= 0.5\left[2\times3.14\times1.5\right]^2\times0.25$$

$$\underline{T = 11.09205\text{N}}$$

(ii) $T\cos\beta = Mg$

$$\cos\beta = \frac{Mg}{T}$$

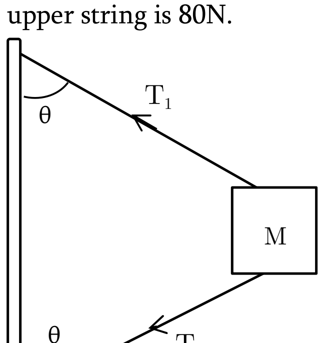
$$\beta = \cos^{-1}\left[\frac{Mg}{T}\right]$$

$$= \cos^{-1}\left[\frac{0.5\times9.8}{11.09205}\right]$$

$$\beta = 63.8^\circ$$

Example 36

The 4kg block in the figure below is attached to a vertical rod by means of the two strings when the system rotates about the axis of the rod, the string are extended as shown in the figure below and the tension in the upper string is 80N.

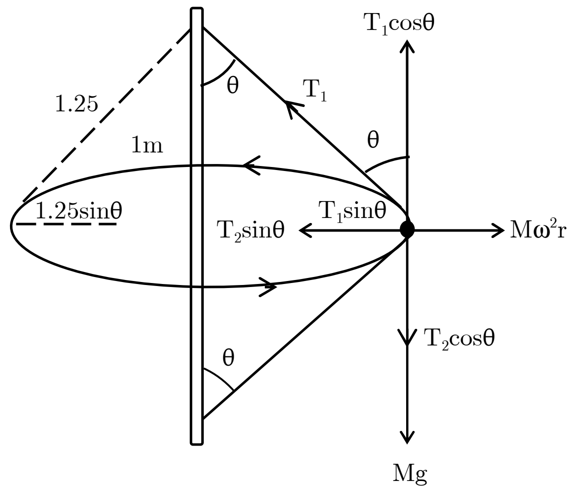


(a) What is the tension in the lower cord ?

(b) How many revolutions per minute does the system make ?

(c) Find the number of revolutions per minute at which the lower cord just goes slack.

Solution



(a) At the equilibrium

$$T_1\cos\theta = T_2\cos\theta + Mg$$

$$T_2 = T_1 - \frac{Mg}{\cos\theta}$$

$$= 80 - \frac{4\times9.8}{1/1.25}$$

$$T_2 = 31\text{N}$$

(b) Also

$$T_1\sin\theta + T_2\sin\theta = M\omega^2 r$$

$$(T_1 + T_2)\sin\theta = M\omega^2\times1.25\sin\theta$$

$$T_1 + T_2 = 1.25M\omega^2$$

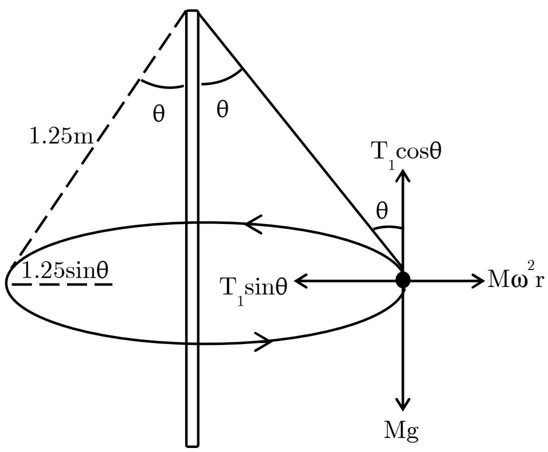
But $\omega = 2\pi f$

$$f = \frac{1}{2\pi}\sqrt{\frac{T_1+T_2}{1.25M}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{80+31}{1.25\times4}}$$

$$f = 0.75\text{Hz} = 45\text{r.p.s}$$

(c) If the lower cord just g slack



$$T_1 \sin\theta = M\omega^2 r$$

$$T_1 \cos\theta = Mg$$

$$\tan\theta = \frac{\omega^2 r}{g} = \frac{\omega^2 1.25}{g}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{1.25\cos\theta}} = \frac{1}{2\pi}\sqrt{\frac{9}{1.25}}$$

$$f = 0.45\text{Hz} = 30\text{r}$$

Example 37

Two particles, each of mass M are attached to the two ends of a light string of length L which passes through a hole at the centre of a smooth table. One particle describes a circle on the table angular velocity ω_1 and the other describes a circle as a conical pendulum with angular velocity ω_2 below the table. If L1 and L2 are the lengths of the position of the string above and below the table , then show that.

(i) $\dfrac{L_1}{L_2} = \dfrac{\omega_2^2}{\omega_1^2}$ and

(ii) $\dfrac{1}{\omega_1^2} + \dfrac{1}{\omega_2^2} < \dfrac{L}{g}$

Solution

(i) Let T = tension on the string

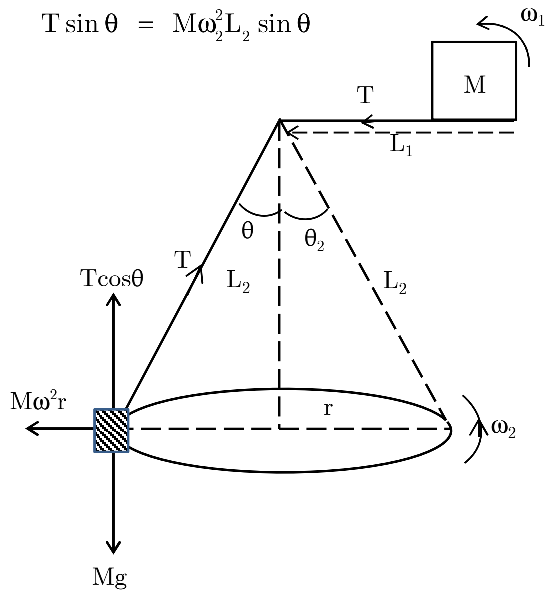
For the particle on the table

$$T = M\omega_1^2 L_1 \,\text{..........(i)}$$

For the particle below the table

$$T\sin\theta = M\omega_2^2 r \quad , \quad r = L_2 \sin\theta$$

$$T\sin\theta = M\omega_2^2 L_2 \sin\theta$$



$$T = M\omega_2^2 L_2 \,\text{............(2)}$$

Also $T\cos\theta = Mg \,\text{..............(3)}$

(1) = (2)

$$ML_1\omega_1^2 = ML_2\omega_2^2$$

$$\frac{L_1}{L_2} = \frac{\omega_2^2}{\omega_1^2}$$

Hence shown.

(ii) Now.

$$L_1 = \frac{T}{M\omega_1^2} \quad , \quad L_2 = \frac{T}{M\omega_2^2}$$

Total length of the string

$$L = L_1 + L_2$$

$$= \frac{T}{M\omega_1^2} + \frac{T}{M\omega_2^2}$$

$$= \frac{T}{M}\left[\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right]$$

$$\frac{ML}{T} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}$$

But $T = \frac{Mg}{\cos\theta}$, $\frac{1}{T} = \frac{\cos\theta}{Mg}$

$$ML \cdot \frac{\cos\theta}{Mg} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}$$

$$\cos\theta = \frac{g}{L}\left[\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right]$$

Since $|\cos\theta| < 1$

$$\frac{g}{L}\left[\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2}\right] < 1$$

$$\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} < \frac{L}{g}$$

Hence shown.

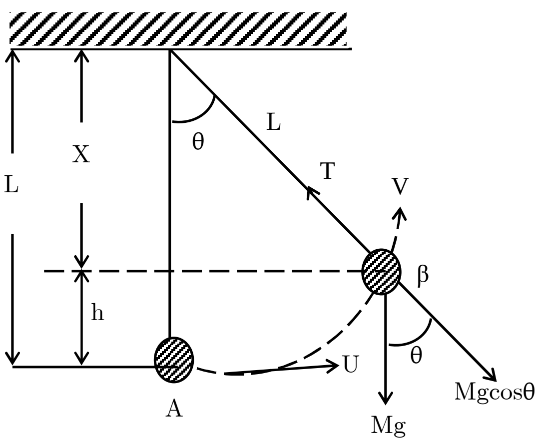
Example 38

A store of mass 0.1kg is suspended from the end weightless string of length 1.0m and is allowed to swing in the vertical. The speed of the mass is 2m/s when the string make an angle θ , with the vertical . calculate the tension in the string at $\theta = 60^o$. calculate the speed of the store when it is in the lowest position ($g = 9.8m/s^2$).

Solution

$M = 0.1kg$, $r = L = 1.0m$

$V = 2m/s$, $\theta = 60^o$, $g = 9.8m/s^2$.



At the equilibrium when the store is at point B

$$T - Mg\cos\theta = \frac{MV^2}{r}$$

$$T = Mg\cos\theta + \frac{MV^2}{r}$$

$$= 0.1 \times 9.8\cos 60^\circ + \frac{0.1 \times 2 \times 2}{1}$$

$$T = 0.89N$$

From the figure above

$h = L - X = L - L\cos\theta$

$h = 1 - \cos 60^\circ = 0.5m$

Apply the principle of conservation of mechanical energy.

$$(p.e + ke)_A = (p.e + ke)_B$$

$$\frac{1}{2}MU^2 = \frac{1}{2}MV^2 + Mgh$$

$$U = \sqrt{V^2 + 2gh}$$

$$= \sqrt{2^2 + 2 \times 9.8 \times 0.5}$$

$$U = 3.715m/s$$

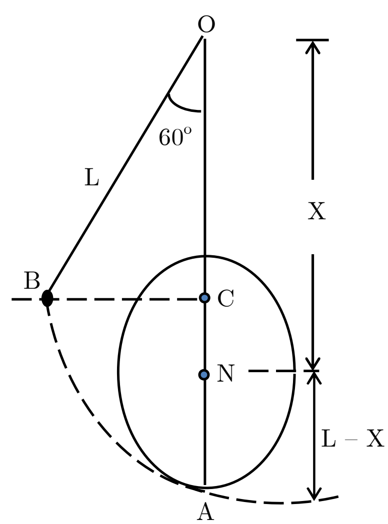
Example 39

A nail is located at a certain distance vertically below the point of suspension of simple pendulum. The pendulum bob is release from the position where the string makes an angle of 60^o with the vertical. Calculate the distance the nail from point of suspension such that the bob will just perform revolution with the nail as centre. Assume the length of the pendulum to be 1m.

Solution

Let OB = length of pendulum

L. N be the nail L at a distance X from O



NA = radius of the vertical circle.

NA = $L - X$, $\theta = 60^o$.

$$Co = OB\cos 60^\circ = \frac{1}{2}$$

Height through which the bob is raised

$$h = L - \frac{L}{2} = \frac{L}{2}$$

Apply the law of conservation of energy

Loss in k.e = gain in p.e

$$\frac{1}{2}MV^2 = Mgh$$

$$V^2 = 2gh = 2g\frac{L}{2}$$

$$V^2 = gL \ldots\ldots\ldots\ldots\ldots(i)$$

For bob to perform revolution about the nail, the minimum speed of the bob at the point A is $\sqrt{5gr}$

$$V = \sqrt{5gr} = \sqrt{5g(L-X)}$$

$$V^2 = 5g(L-X)$$

$$X = \frac{4}{5}L = 0.8L = 0.8\times 1$$

$$X = 0.8m$$

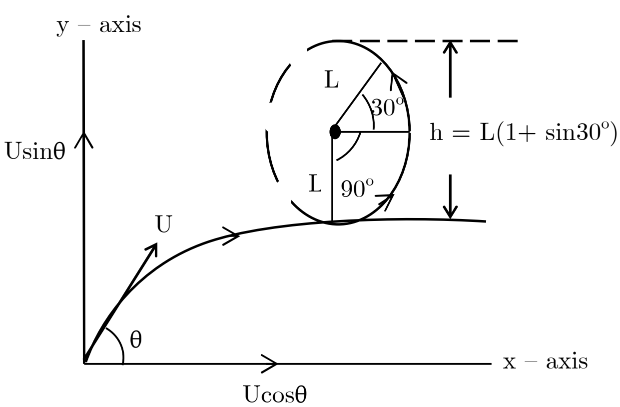
Example 40.

A bullet of mass M is fired with a velocity of 50m/s at an angle θ with the horizontal at the highest point of its trajectory it collides lead on with a bob of mass 3M suspended by a massless string of length $^{10}/_{3}$ m and gets embedded in the bob after collision of the string moves to an angle of 120° find.

(a) The angle , θ

(b) The length of the string so that the system could just complete a vertical circle.

Solution



(a) At the highest point of the bullet motion , its velocity is only the horizontal component.

Apply the principle of conservation of linear momentum.

$$MV_x = (M+3M)V$$

$$MU\cos\theta = 4MV$$

$$V = 12.5\cos\theta$$

Apply the principle of conservation of energy.

k.e of bullet bob system = p.e of the systemat h

$$\frac{1}{2}(M+3M)V^2 = (M+3M)gh$$

$$V^2 = 2gh$$

$$V = \sqrt{2gL(1+\sin 30^\circ)}$$

$$\text{Now , } 12.5\cos\theta = \sqrt{3gL}$$

$$\cos\theta = \frac{\sqrt{3gL}}{12.5}$$

$$\theta = \cos^{-1}\left[\frac{\sqrt{3\times 9.8\times {}^{10}/_{3}}}{12.5}\right]$$

$$\theta = 37.6^\circ$$

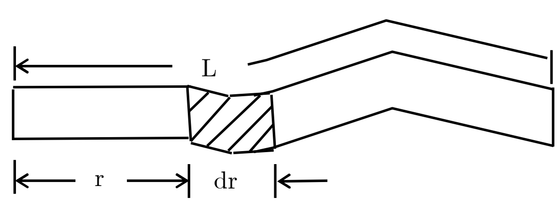
(b) $V^2 = 5gr$, $r = \frac{V^2}{5g}$

$$r = \frac{(12.5\cos 37.6^\circ)}{5\times 9.8}$$

$$r = 2m$$

Example 41

A wire of mass 0.45kg and length 2.0m is pun in a horizontal circle. The breaking stress of the wire is 8×10^9Pa and the cross – sectional area is 8×10^{-6}m². Calculate the maximum frequency of rotation before wire breaks and find at which point it will break



Solution

Let, (BADO)? $= \frac{M}{L}$, M = ?ρϑ

L = Length of the wire

Consider the forces acting on the short length of the wire.

$$T + dT - T = -\omega^2 r dm$$
$$dT = -\omega^2 r dm$$

But $\quad \int = \frac{dm}{dr}\ ,\ dm = \int dr$

$$dT = -\omega^2 r \int dr$$
$$\int dT = -\omega^2 \int r dr$$
$$T = \frac{-\omega^2 \int r^2}{2} + C$$

When r = L , T = 0 , $c = \frac{\int \omega^2 L^2}{2}$

Now $T = \frac{1}{2} M\omega^2 \left(L - \frac{r^2}{L}\right)$

When $T = T_{max}\ ,\ r = 0$

$$T_{max} = \frac{1}{2} M\omega^2 L$$
$$\omega = \sqrt{\frac{2T_{max}}{ML}}\ ,\ \omega = 2\pi f$$
$$f_{max} = \frac{1}{2\pi}\sqrt{\frac{2T_{max}}{ML}}$$

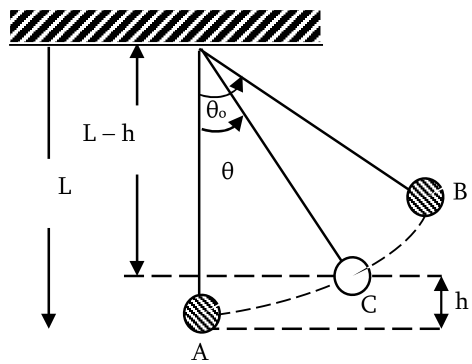
$$\text{Breaking stress} = \frac{T_{max}}{A}$$
$$T_{max} = A \times \text{breaking stress}$$
$$f_{max} = \frac{1}{2\pi}\sqrt{\frac{2A \times \text{breaking stress}}{ML}}$$
$$= \frac{1}{2}\sqrt{\frac{2 \times 8 \times 10^{-6} \times 8 \times 10^{9}}{0.4 \times 2}}$$
$$f_{max} = 63.6 \text{ r.p.s}$$

∴ The wire will break at the point of attachment.

Example 42

A 40kg mass , hanging at the end of a rope of length L oscillates in a vertical plane with an angular amplitude of θ₀. What is the tension in the rope when it makes an angle θ with the vertical? If the breaking strength of the rope is 80kg , what is the maximum angular amplitude with which the mass can oscillate without the rope breaking?

Solution



Let B be the position of the mass when the angular amplitude is θ_o.

Let C be the position of the mass when the angular displacement is θ.

The height of C above A is given by

$$h = L - L\cos\theta = L(1 - \cos\theta)$$

The height of B above A is given by

$$h_o = L(1 - \cos\theta_o)$$

Apply the principle of conservation of mechanical energy

Total energy at point C = Total energy at point B

$$(p.e + ke)_C = (p.e + k.e)_B$$

$$\frac{1}{2}MV^2 + MgL(1 - \cos\theta) = MgL(1 - \cos\theta_o)$$

$$MV^2 = 2MgL(\cos\theta - \cos\theta_o)$$

At the point C

$$T - Mg\cos\theta = \frac{MV^2}{L}$$

Now, $\frac{MV^2}{L} = 2Mg(\cos\theta - \cos\theta_o)$

Then

$$T - Mg\cos\theta = 2Mg(\cos\theta - \cos\theta_o)$$
$$T = Mg(3\cos\theta - \cos\theta_o)$$

$$T = M\left(3\cos\theta - \cos\theta_{\circ}\right)\text{kgwt}$$
$$T = 40\left[3\cos\theta - \cos\theta_{\circ}\right]\text{kgwt}$$

The tension will be maximum at A when θ = 0o using the maximum value of tension in the above equation.

$$80 = 40\left[3\cos 0^{\circ} - 2\cos\theta_{\circ}\right]$$
$$2 = 3 - 2\cos\theta_{\circ}$$
$$\cos\theta_{\circ} = \tfrac{1}{2}\ ,\ \theta_{\circ} = 60^{\circ}$$

CONCEPTUAL PROBLEMS

Example 43

(a) A pilot does not fall down when his aeroplane loops the loop why ?

(b) A cyclist negotiates a curve at high speed and bends more than a cyclist negotiating the same curve at low speed why ?

Solution

(a) This is because when the pilot is at highest point of the loop , the weight of the pilot is balanced by the centrifugal force.

(b) Since $\tan\theta = V^2/rg$, each symbol have usual meaning greater the value of V , more will be the value of θ so , cyclist negotiating a curve at high speed shall have to bend more.

Example 44

Is it correct to say that the banking of road reduces the wear and tear of the tyres of automobile? If yes , explain.

Solution

If the road is not banked , then the necessary centripetal force will be provided by the force of friction between the tyres and the road. On the other hand, a component of the normal reaction provide the necessary centripetal force this reduces the wear and tear of the tyres.

Example 45

Explain the following observations:-

(a) When the car takes turn round a curve , passenger sitting in the car tends to slide. To which side does the passenger slide ?

(b) Comment on the statement 'sharper the curve , more is the bending'

Solution

(a) The passenger slides away from the centre of the curve due to the centrifugal force.

(b) The angle θ which a cyclist should make with the vertical while taking circular turn of radius r with velocity , V is given by $\tan\theta = V^2/rg$. Sharper the curve , smaller is the radius and greater is the value of θ.

Example 46

(a) A stone tied to the end of a string is whirled in a horizontal circle when the strings breaks the store flies away tangentially. Why ?

(b) Passengers are thrown outwards , when the bus takes a circular turn why?

Solution

(a) When a store moves in a circular path , its instanteneous velocity is always tangential to the circle when the string breaks , the centripetal force ceases to act. According to the inertia , store continues its motion along the tangent to the circular path. Hence it flies away tangentially.

(b) When the bus takes a turn , the centripetal acceleration of the bus acts towards the centre of the circular turn and along its radius due to inertia , the passengers are thrown outwards the force experienced by the passengers is called the centrifugal force.

Example 47

Why is the outer rail of a curved railway track generally raised over the inner?

Solution

When the outer rail of curved track is raised , weight of the train provides a force component along the radius of the curved track towards the centre. This components of the weight provides the necessary centripetal force to the train to enable it to move along the curved path.

Example 48

Why does an aeroplane tilt while making a curved flight?

Solution

The plane tilts to provide the necessary centripetal force. If θ is the angle of the tilt then $\tan\theta = V^2/rg$ where V is the speed of the plane and r is the radius of the circular path.

Example 49

Explain why it is describe to bank a road at corners and calculate the optimum angle of banking for a curve of radius r , to be transvered at a speed , V.

Solution

Roads are banked along corners so as to reduce the dependency of friction on providing the necessary centripetal force to take it. This enables cars to take it with greater speed without overturning as well as reducing the wear and tear of tyres due to friction.

Example 50

Explain the following observation:-

(a) When athletes run around a curve , they usually lean inward.

(b) A cyclist leans inward while taking a turn.

Solution

They lean inward to provide the necessary centripetal force to take the curve.

Example 51

Explain why racing cars can travels very fast on banked round track than on flat track.

Solution

On flat track , the centripetal force is provided by friction between the tyres and truck. If limiting friction is exceeded then the car will overturn. On banked truck, the centripetal force can be provided wholly by the normal reaction on the car. This enables the car to take the curve with greater speed without overturning.

Example 52

Explain the following observations:-

(a) What is the need for banking of roads?

(b) Why are wheels of an automobile made circular.

(c) Why does a child in a merry – go – round press the side of his seat radially outward?

Solution

(a) When a curved roads are banked , horizontal component of normal reaction of the road provides the necessary centripetal force to move the vertical along the curved path. This reduces wear and tear of tryres.

(b) Circular wheels roll on the road. Therefore motion of the vehicle is opposed by rolling friction in which is much smaller than the sliding friction.

(c) This is because of centrifugal force acting radially outwards on the child rotating actually in the merry go round.

Example 53

How does banking of roads reduce wear and tear of the tyres ?

Solution

When a curved round is unbaked , force of friction between the types and the roads provides the necessary centripetal force. Friction has to be increased suitably that will cause wear and tear. However , when the curved road is banked , a component of normal reaction of the ground provides the necessary centripetal force. The role of friction becomes secondary. Therefore, wear and tear of the tyres is reduced.

REVISION

EXERCISE 4. **NECTA 2017/P1/4**

1. (a) (i) Justify the statement that if no external torque act on a body, its angular velocity will not conserve.

   (ii) A car is moving with a speed of 30m/s on a circular track of radius 500m. If its speed is increasing at the rate of $2m/s^2$, find its resultant linear acceleration.

   (b) An object of mass 1kg is attached to the lower end of string 1m long whose upper end is fixed and made to rotate in a horizontal circle of radius 0.6m. if circular speed of the mass is constant , find the;

   (i) Tension in the string

   (ii) Period of motion

2. JECAS 2017/P1/5

   (a) Briefly explain the following observations:-

(i) Why are passengers of a car rounding a curve thrown outwards ?
(ii) Why are wheels mad circular?
(iii) Why are fast moving vehicles given streamlines shape?
(iv) When does the body start moving?

(b) A small box is placed on the surface of a horizontal disc at a point 5cm from the centre of the disc. The box is on the point of shipping when the disc rotates at 1.4rev/sec. find the coefficient of friction between the box and the surface of the disc.

(c) (i) Define the term 'Banking of road'
(ii) At what angle must a truck with a bend of radius 1.5m banked for safe running of the trains at a speed of 48km/hr.

3. NECTA 2016/P1/3(C)
A boy ties a string around a stone of mass 0.15kg and then whirls it in a horizontal circle at constant speed. If the period of rotation of the stone is 0.4sec and the length between the stone and boy's hand is 0.5m
(i) Calculate the tension in the string.
(ii) State one assumption taken to each the answer in 3(c)(i)

4. NECTA 2013/P1/3
(a) Why is technically advised to bank a road at corners?
(b) A wheel rotates at a constant rate of 10revolution per second. Calculate the centripetal acceleration at a distance of 0.80m from the centre of the wheel.

5. NECTA 2012/P1/3
(b) A child is whirling a 0.012kg ball on a string in a horizontal circle whose radius I s0.10cm. if the ball travels once round the circle in 0.5sec.
(i) Determine the centripetal force acting on the ball.
(ii) Why does such a force do not work in a circular motion
(c) If the speed of the ball in (b) is doubled , by what factor does the centripetal force increases ans.

(a) (i) 0.00189N
(b) Factor of 4

6. NECTA 2011/P1/3
(a) Mention the sources of centripetal force in each of the following cases:-
(i) An object at the end of a string is whirled in a horizontal circle.
(ii) A car travelling round a banked road.
(iii) The moon orbiting the earth

(b) (i) An object of mass 4kg is whirled in a vertical circle of radius 2m with a speed of 5m/s calculate the maximum and minimum tension in the string connecting the object and the centre of the circle.
(ii) Find the angle of banking necessary for a corner of radius 80m if cars are to travel round this corner at a speed of 120km/hr without relying on the frictional force ($g = 10m/s^2$).
Ans: (b) (i) 90N , 10N
(ii) 54.2°

7. NECTA 2010/P1/3
(a) What is the origin of centripetal force for:-
(i) A satellite orbiting around the each other.
(ii) An electron in the hydrogen atom?
(b) A small mass of 0.15kg is suspended from a fixed point by a thread of fixed length. The mass is give a push so it moves along a circular path of radius 1.82m in a horizontal plane at a steal speed, taking 18sec to make 10 complete revolutions calculate:-
(i) The speed of the small mass
(ii) The centripetal acceleration.
(iii) The tension in the thread
Ans (b) (i) 6.35m/s
(ii) 22.18m/s2
(iii) 3.64N

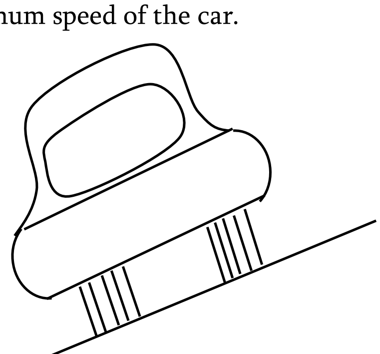
8. A particle of mass 5kg is slightly disturbed from rest on the top of a smooth hemisphere , radius 4m and centre O resting with its plane

face on horizontal ground. Show that the particle leaves the surface of the hemisphere at the point B , where the angle between the radius BO and the upward vertical is $\cos^{-1}\left(\frac{2}{3}\right)$

9. NECTA 2000/P1/2
   (a) Show that the period of a body of mass M revolving in a horizontal circle with constant velocity V at the end of a string of length , L is independent of mass of the object.
   (b) A ball of mass 100gm is attached to the end of a string and is swung in a circle of radius 100cm at a constant velocity of 200cm/s while in motion the string is shortened to 50cm. calculate:-
   (c) A car travels over a humpback bridge of radius of curvature 45m. calculate the maximum speed of the car if the wheels are to remain in constant with the bridge.

10. NECTA 2005/P2/2a
    (i) What is the centripetal force?
    (ii) Why does a centripetal force not do any work in a circular orbit?
    (iii) A conical pendulum is an example on which a force acts centripetal show that the period T is given by $T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$

11. NECTA 2008/P1/3(a)
    (i) Distinguish uniform circular motion from non- uniform circular motion
    (ii) A racing car (figure 1) goes around a circular curve as fast as it can without skidding. The radius of the curve is 50m and the road is banked at 20° to allow faster speed. If the coefficient of static friction between the road and the tyre is 0.80 resolve forces into horizontal and vertical component and the equation. Apply Newton's law of motion and the equation for maximum frictional force to determine maximum speed of the car.



12. NECTA 1989/P2/1
    (a) When is a corner said to be correctly banked? Explain briefly why a banked corner can be negotiated at a greater speed than the corner on a flat road?
    (b) On end of a light inelastic string of length L is attached to a fixed point and the other end supports a small bob of mass M. The bob set into oscillation so that it describes a horizontal circle with a constant angular velocity , ω if the string makes and angle θ with the vertical show that:-
        (i) The tension in the string $T_o = M\omega^2 L$
        (ii) $\theta = \cos^{-1}\left[\frac{g}{L\omega^2}\right]$
        (iii) The period of revolution of the bob $T = 2\pi\sqrt{\frac{L\cos\theta}{g}}$
    (c) An empty cylindrical tin of internal diameter 0.10m is placed on the centre of a turn – table so that the axis of symmetry of the tin coincided with the axis of rotation of turn – table is set into rotation it is found that a small object placed on the inner vertical surface of the tin remain impenged there if the speed of rotation of turn table is 28rads^{-1} (or more).
        (i) Draw a diagram to show all the forces which act on the object and
        (ii) Calculate the coefficient of friction between the object and the inner surface of the tin.

13. NECTA 1988/P1/2
    (a) Stone tied to an inextensible string is whirled in a vertical circle at a constant speed , V. if the string snaps as a result of extension.
        (i) At what position of motion of store is this likely to occur?

(ii) Sketch the path followed by the string snaps to the time it hits the ground.

(b) A body of weight , W slides without friction from the rest at the top of a sphere of radius, r if the body falls through a vertical distance, h (h < r) deduce an expression in terms of w , r and h for the normal reaction s of the sphere.

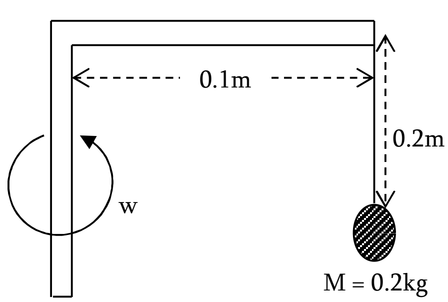
Ans: $S = \left(\frac{r - 3h}{r}\right)W$

14. (a) Why doesn't water contained in a bucket fall off when the bucket is whirled in a vertical plane?
(b) A gramophone record rotates at 33½ r.p.m and has a radius of 0.15m. determine:-
(i) Its angular velocity
(ii) The speed of a point on its circumference.
(iii) The speed of a point mid – way between its centre and its circumference.

15. A pendulum bob of mass 2kg is attached to one end of the string of length 1.2m. the bob moves in horizontal circle such away that the string is inclined at 30o to the vertical. Calculate :-
(a) The tension in the string.
(b) Its velocity.
(c) The period of the motion.

16. (a) How many revolution per second must apparatus figure below rotates about the vertical axis in order that the cord shall make and angle of 45^{o} to the vertical and hence find the tension on the string?
(b) Find the angle θ which makes with the vertical so that the system is rotate at $1.5 rads^{-1}$.

Ans: (a) 1.01Hz , T = 2.77N.



17. (a) What is banking of roads? Why is it necessary?
(b) (i) Obtain an expression for angle of banking of a curved road. Show that it is independent of the mass of vehicle?
(ii) On what factors angle of banking depends.

18. (a) Explain centripetal force and centrifugal force. Why centrifugal force is called a pseudo force?
(b) Derive the expression for the maximum speed of a vehicle on the banked road state the factors on which the optimum speeds depends.

19. A store of mass 0.4g is tied to a string and rotated in a vertical circle of radius 1.2m. Calculate the speed of the stone for which the tension in the string is zero at the highest point of the circle. What is the tension at the lowest point in this case?

Ans: 3.43m/s , 23.52N.

20. A bucket of water tied to one end of a rope of length 3m is rotated in a vertical circle about the other end in such a way that water in it does not spill. Calculate the minimum velocity of the bucket at which this happens.

Ans: 1.823rad/s.