Measures of embedding for interval-valued fuzzy sets

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Abstract

Interval-valued fuzzy sets are a generalisation of classical fuzzy sets where the membership values are intervals. The epistemic interpretation of interval-valued fuzzy sets assumes that there is one real-valued membership degree of an element within the membership interval of possible membership degrees. Considering this epistemic interpretation, we propose a new measure, called IV-embedding, to compare the precision of two interval-valued fuzzy sets. An axiomatic definition for this concept as well as a construction method are provided. The construction method is based on aggregation operators and the concept of interval embedding, which is also introduced and deeply studied.

Keywords: Interval-valued fuzzy sets; embedding; inclusion.

1. Introduction

Fuzzy sets were originally introduced by Lotfi A. Zadeh as an extension of classical set theory [36], where the Boolean characteristic function of a set is replaced with a function into the real unit interval $[0, 1]$. They represent the core of soft computing, since they allow more flexible information processing systems (see, for instance [28]). They have been applied in many different areas, such as artificial intelligence, machine learning, information processing, statistics and data analysis, control system, decision sciences, economics, medicine or engineering. A complete revision about these applications has been shown in [14, 15]. Since there are many real problems where it may not be possible to narrow the membership information down to an exact number, more sophisticated extensions of fuzzy sets have been proposed where the membership value itself is made less precise by turning it into a collection of possible values rather than a single real number in $[0, 1]$. This idea leads to a more general formulation of fuzzy sets where membership values are intervals (interval-valued fuzzy sets) or functions (type-2 fuzzy sets) [24]. Interval-valued fuzzy sets are one of the most important extensions that were independently introduced by Zadeh [37], Grattan-Guiness [19], Jahn [22] and Sambuc [33] in the seventies. They are able to jointly work with vagueness and uncertainty.

For interval-valued fuzzy sets, we can consider the epistemic or the ontic interpretation [12]. In our study the first one will be the chosen. Thus, we assume that there is one actual, real-valued membership degree of an element within the membership interval of possible membership degrees.

Since this generalisation of fuzzy sets was introduced, many different properties and relations between interval-fuzzy sets have been studied. Concepts such as clouds, distance, similarity, inclusion or entropy measures have been studied. Neumanier [29] proposed the concept of a cloud, using interval-value fuzzy sets, as an efficient method to represent a family of probabilities. Dubois and Prade [13] studied the connection between clouds, interval-valued fuzzy sets and possibility theory. Grzegorzewski [20] studied the concept of distance between interval-valued fuzzy sets based on the Hausdorff metric. Bustince and Burillo [7] and Szmidt and Kacprzyk [35] proposed different notions of entropy of interval valued fuzzy sets from different point of views. Bustince *et al.* in [8] or Zeng and Guo [38] introduced axiomatic definitions of inclusion measures for interval-valued fuzzy sets. More recently, Bustince *et al.* in [10] study a new class of similarity measures between interval-valued fuzzy sets based that considers the width of intervals so that the uncertainty of the output is strongly related to the uncertainty of the input. The degree of inclusion and similarity between interval-valued fuzzy sets is constructed based on the precedence relation, aggregation and uncertainty assessment in [31]. This problem is important in many different frameworks such as medical diagnosis [11, 32] where the need of some kind of comparison between IVFS is present.

Thus, the study of how to compare and to establish relationships between two interval-valued fuzzy set is a challenging topic. In particular, the precision of the membership of an interval-valued fuzzy set is also important. In this work embedding measures will be introduced as a tool to compare wether one interval-valued fuzzy set is a less precise description than another interval-valued fuzzy set or not. The rest of this paper is organized as follows: Section 2 provides some basic concepts about interval-valued fuzzy sets. Section 3 gives the definitions of embedding measure for intervals, as well as some properties and construction methods. Section 4 proposes a coherent axiomatic definition for the concept of embedding measure for interval-valued fuzzy sets, studying in detail this concept. Finally, Section 6 shows a case study and Section 7 draws conclusions.

2. Basic concepts

Let X denote the universe of discourse. An interval-valued fuzzy set of X is a mapping $A : X \to L([0, 1])$ such that $A(x) = [\underline{A}(x), \overline{A}(x)]$, where $L([0, 1])$ denotes the family of closed intervals included in the unit interval $[0, 1]$. Thus, an interval-valued fuzzy set A is totally characterized by two mappings, $\underline{A}$ and $\overline{A}$, from X into $[0, 1]$ such that $\underline{A}(x) \leq \overline{A}(x)$ for any $x \in X$. These maps represent the lower and upper bound for the membership function. Thus, A is a classical fuzzy set if $\underline{A}(x) = \overline{A}(x)\, \forall x \in X$. Obviously, fuzzy sets are particular cases of interval-valued fuzzy sets. The collection of all the interval-valued fuzzy sets in X is denoted by $IVFS(X)$.

There are two essential relations in $IVFS(X)$ in which this paper is focused: the inclusion and the embedding. Both are based on orders on $L([0,1])$, but their meaning is totally different. While the inclusion represents that an interval-valued fuzzy set is totally contained in another one, the embedding happens when an interval-valued fuzzy set has a more precise information about the real membership-function than another one. Thus, we will consider two partial orders on $L([0,1])$, defined for any $a, b \in L([0,1])$ with $a = [\underline{a}, \overline{a}]$ and $b = [\underline{b}, \overline{b}]$ as follows:

- Contained ($a \subseteq b$): it is said that a is included in b if and only if $\underline{b} \leq \underline{a} \leq \overline{a} \leq \overline{b}$.
- Lattice order ($a \leq b$) [18]: it is said that a is lower than or equal to b if and only if $\underline{a} \leq \underline{b}$ and $\overline{a} \leq \overline{b}$.

From them, we can obtain also two partial orders on $IVFS(X)$ defined, for any $A, B \in IVFS(X)$ as follows:

- Embedded: it is said that A is embedded in B, and it is denoted by $A \sqsubseteq B$, if and only if $A(x) \subseteq B(x), \forall x \in X$.
- Included: it is said that A is included in B, and it is denoted by $A \subseteq B$, if and only if $A(x) \leq B(x), \forall x \in X$.

Example 2.1. *Consider $A, B, C, D \in IVFS(X)$ as in Figure 1. Note that A is embedded in B as the uncertainty about the real membership value is lower for A than for B. However, $A \not\subseteq B$. On the other hand, we have that $C \subseteq D$, but $C \not\sqsubseteq D$.*

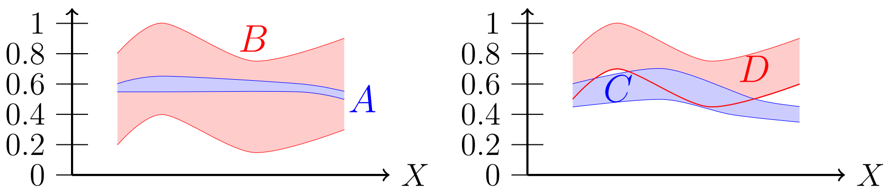

Figure 1: Embedding versus inclusion.

Thus, with a simple example it is easy to see that embedding and inclusion are totally different concepts.

Since the membership value of an interval-valued fuzzy set is an interval, in order to study their embedding measures, we will start by studying the analogous concept for intervals. This will be a first step for defining embeddings for interval-valued fuzzy sets. Moreover, we will use the families of embedding for intervals to obtain embeddings in $IVFS(X)$.

3. Embedding measures for intervals

In this section the axioms that characterize an embedding for intervals are introduced. More precisely, we will try to define a measure of the degree of embedding for one element in $L([0,1])$ into another one.

The embedding degree between two intervals differs axiomatically and philosophically from the inclusion degree as consequence of the conditions (axioms) imposed in order to achieve a good representation of the embedding. The first requirement is that an interval is totally embedded in another one, that is, all the points of the first interval are also points of the second one, just in the case the degree of embedding is the maximum. Moreover, if two intervals do not have elements in common, none of the two is embedded into the other one. The third requirement is that the embedding is increasing in the second variable. This means that if the interval b is embedded in the interval c, it is logical that the embedding of any other interval a in c is always greater than or equal to the embedding in b.

The three conditions above express the essence of the notion of embedding for intervals and represent the minimal requirements for a function to be a embedding. Thus,

Definition 3.1. *The function $E : L([0,1]) \times L([0,1]) \to [0,1]$ is an embedding on $L([0,1])$ if the following properties are hold:*

A1: $E(a,b) = 1$ if and only if $a \subseteq b$.

A2: If $a \cap b = \emptyset$, then $E(a,b) = E(b,a) = 0$.

A3: If $b \subseteq c$, then $E(a,b) \leq E(a,c)$.

for any $a,b,c \in L([0,1])$.

Some interesting properties for embeddings on $L([0,1])$ are:

Proposition 3.2. *For any $a,b,c \in L(]0,1])$ we have that:*

(1) $E(a \cap b, a) = E(a \cap b, b) = E(a, a \cup b) = E(b, a \cup b) = 1$.
(2) $E(a, b \cap c) \leq \min\{E(a,b), E(a,c)\}$.

Proof. (1) Since $a \cap b \subseteq a \subseteq a \cup b$, $a \cap b \subseteq b \subseteq a \cup b$, this is immediate from Axiom A1.
(2) Since $b \cap c \subseteq b$ and $b \cap c \subseteq c$, we obtain from Axiom A3 that $E(a, b \cap c) \leq E(a,b)$ and $E(a, b \cap c) \leq E(a,c)$.

□

Once the axioms that define the embedding for intervals as well as some important properties are introduced, some construction methods are studied. The first one is based on the width of the intervals and the second one on implications.

3.1. Construction of a embedding based on interval width

In this subsection, a construction method for interval embedding based on measuring and comparing the width of the related intervals is proposed. Apart from obtaining a family of embedding measures, we will use it to analyse the converse implications for Axioms A2 and A3 in Definition 3.1. Since it is based on the width of the intervals, we first recall this well-known concept and fix the related notation.

Definition 3.3. *Let a be an interval in $L([0,1])$ with $a = [\underline{a}, \overline{a}]$. The width of the interval a is given by [3]:*

$$w(a) = \overline{a} - \underline{a}$$

Thus, we can use the comparison of the interval width to measure the embedding degree.

Proposition 3.4. *Let $\phi : \Delta \to [0,1]$ be a map with $\Delta = \{(x,y,z) \in [0,1] \times (0,1] \times [0,1] \,|\, x \leq y, x \leq z\}$ such that*

- $\phi(0, y, z) = 0$,
- $\phi(x, y, z) = 1 \Leftrightarrow x = y$,
- *ϕ is increasing in the first and third component,*

the map E_ϕ defined by

$$E_\phi(a,b) = \begin{cases} 1 & \text{if } w(a) = 0,\ a \in b \\ 0 & \text{if } w(a) = 0,\ a \notin b \\ \phi(w(a \cap b), w(a), w(b)) & \text{otherwise} \end{cases}$$

is an embedding for intervals.

Proof. It is clear that E_ϕ is well-defined since its value between 0 and 1. Thus, let's prove that E_ϕ is an embedding for intervals.

- Axiom A1: If a is just one point, $E_\phi(a,b) = 1$ iff $a \in b$, so $a \subseteq b$. If a is different from a point, $E_\phi(a,b) = \phi(w(a \cap b), w(a), w(b)) = 1$ iff $w(a \cap b) = w(a)$ by the second property of ϕ. As $a \cap b \subseteq a$, the only possibility to have the same width is that $a \cap b = a$ and this is equivalent to say that $a \subseteq b$.

- Axiom A2: If $a \cap b = \emptyset$, we will consider again two cases. If a is just one point, $E_\phi(a,b) = 0$ iff $a \notin b$, which is equivalent to say that $a \cap b = \emptyset$. If a is different from a point, then $E_\phi(a,b) = \phi(0, w(a), w(b)) = 0$ by the first property of ϕ. Moreover, if $a \cap b = \emptyset$, then $b \cap a = \emptyset$ and therefore $E_\phi(b,a) = 0$.

- Axiom A3: If $b \subseteq c$, then we will consider again the two cases. If a is just a point and $E_\phi(a,b) = 0$, then the proof of this axiom is trivial. Otherwise $E_\phi(a,b) = 1$ and $a \in b$. Thus, $a \in c$ and therefore $E_\phi(a,c) = 1$. If a is not a point, as ϕ is increasing in the first and third component and we have that $w(a \cap b) \leq w(a \cap c)$ and $w(b) \leq w(c)$, then $E_\phi(a,b) = \phi(w(a \cap b), w(a), w(b)) \leq \phi(w(a \cap c), w(a), w(c)) = E_\phi(a,c)$

□

Corolary 3.5. *If we consider the function $E_w : L([0,1]) \times L([0,1]) \to [0,1]$ defined by:*

$$E_w(a,b) = \begin{cases} 1 & \textit{if } w(a) = 0,\ a \cap b \neq \emptyset \\ 0 & \textit{if } w(a) = 0,\ a \cap b = \emptyset \\ \frac{w(a\cap b)}{w(a)} & \textit{if } w(a) \neq 0 \end{cases}$$

then E_w is an embedding for intervals.

Proof. It is an immediate consequence of Proposition 3.4, since $E_w = E_\phi$ with $\phi(x,y,z) = x/y$. □

Example 3.6. *Let's study if the interval $[0.2, 0.5]$ is embedded in the intervals $[0.6, 0.7]$, $[0.4, 0.7]$ or $[0.1, 0.7]$. Using the function $E_w(a,b)$ it is obtained that*

$$E_w([0.2,0.5],[0.6,0.7]) = \frac{0}{0.3} = 0$$
$$E_w([0.2,0.5],[0.4,0.7]) = \frac{0.1}{0.3} = 1/3$$
$$E_w([0.2,0.5],[0.1,0.7]) = \frac{0.3}{0.3} = 1$$

Note that the results make sense since it seems clear that $[0.2, 0.5]$ is not embedded in $[0.6, 0.7]$, it is embedded in some degree in $[0.4, 0.7]$ and it is totally embedded in $[0.1, 0.7]$.

In addition, note that Axiom A2 is fulfilled in only one direction. If we consider $a = [0.2, 0.5]$ and $b = [0.5, 0.6]$, we have that $a \cap b \neq \emptyset$, but $E_w(a,b) = 0$.

The same happens for Axiom A3. Thus, if we consider $a = [0.1, 0.6]$, $b = [0.4, 0.7]$ and $c = [0.2, 0.9]$, we have that $b \subseteq c$, but $E_w(b,a) = \frac{2}{3}$ and $E_w(c,a) = \frac{4}{7}$, that is, $E_w(b,a) > E_w(c,a)$.

Another particular family of embeddings for intervals that can be obtained from the construction proposed in Proposition 3.4 is introduced in the next proposition.

Proposition 3.7. *The function $E_\lambda : L([0,1]) \times L([0,1]) \to [0,1]$ defined by:*

$$E_\lambda(a,b) = \begin{cases} 1 & \textit{if } a \subseteq b \\ 0 & \textit{if } a \cap b = \emptyset \\ \lambda & \textit{otherwise} \end{cases}$$

with $\lambda \in [0,1)$ is an embedding for intervals.

Proof. It is again a consequence of Proposition 3.4. In this case the map ϕ is given by:

$$\phi(x,y,z) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = y \\ \lambda & \text{otherwise} \end{cases}$$

□

Although λ cannot be equal to 1 (in that case, it does not fulfil Axiom A1), we have considered that it could be equal to zero. In fact, in this case we obtain a lower bound for the family of interval embedding measures as the following proposition states.

Proposition 3.8. *For any embedding E we have that $E_0 \leq E$, that is, the previous mapping for the case $\lambda = 0$ is the minimum embedding function.*

Proof. If $E_0(a, b) = 0$, the proof is immediate. Otherwise, we have that $a \subseteq b$ and $E_0(a, b) = 1$. But as E is an embedding function we know that $E(a, b) = 1$ by Axiom A1. Thus, in that case, $E_0(a, b) = E(a, b)$. □

3.2. Construction of embedding based on implications

In this section other construction method for embeddings using implications is introduced. Firstly, we define a new kind of functions called F-functions and then we extend this construction method for implications.

Definition 3.9. *The function $F : [0, 1]^2 \to [0, 1]$ is called a F-function if the following properties hold:*

F1. F is decreasing in the first variable,

F2. F is increasing in the second variable,

F3. $F(x, y) = 1 \Leftrightarrow x \leq y$.

Proposition 3.10. *Let $F : [0, 1]^2 \to [0, 1]$ be a F-function, the map $E_F : L([0, 1]) \times L([0, 1]) \to [0, 1]$ defined by:*

$$E_F(a, b) = \begin{cases} 0 & \text{if } a \cap b = \emptyset \\ min(F(\underline{b}, \underline{a}), F(\overline{a}, \overline{b})) & \text{otherwise} \end{cases}$$

is an embedding for intervals.

Proof. E_F is well-defined since the values of F are in the interval $[0, 1]$. Let's check the three axioms defining an embedding:

- $E_F(a, b) = 1$ iff $F(\underline{b}, \underline{a}) = 1$ and $F(\overline{a}, \overline{b}) = 1$ iff $\underline{b} \leq \underline{a}$ and $\overline{a} \leq \overline{b}$ iff $a \subseteq b$.
- If $a \cap b = \emptyset$ then trivially $E_F(a, b) = E_F(b, a) = 0$.
- If $b \subseteq c$ and $a \cap c = \emptyset$ then $b \cap c = \emptyset$ and therefore $E_F(a, b) = E_F(a, c) = 0$. If $b \subseteq c$ and $a \cap c \neq \emptyset$. In the case $a \cap b = \emptyset$, then $E_F(a, b) = 0 \leq E_F(a, c)$. In the case $a \cap b \neq \emptyset$, as $\underline{c} \leq \underline{b}$ and $\overline{b} \leq \overline{c}$ and F is decreasing in the first variable and increasing in the second variable, we have $F(\underline{b}, \underline{a}) \leq F(\underline{c}, \underline{a})$ and $F(\overline{a}, \overline{b}) \leq F(\overline{a}, \overline{c})$, then $E_F(a, b) \leq E_F(a, c)$.

□

Thus, we can use F-functions to obtain embedding measures for intervals. We are mainly interested in a particular case of F-functions, the ones based on fuzzy implications [26], that is, a generalization of classical implications to fuzzy logic case. We use the following definition, which is equivalent to the one introduced by Fodor and Roubens [17].

Definition 3.11. *An implication function is a function $I : [0,1]^2 \to [0,1]$ verifying:*

I1. I is decreasing in the first variable,

I2. I is increasing in the second variable,

I3. $I(0,0) = I(1,1) = 1$ and $I(1,0) = 0$.

Note that sometimes property (I3) is replaced by the conditions $I(0,x) = 1$ and $I(x,1) = 1$ for every $x \in [0,1]$, yielding an equivalent definition [1]. Some extra properties could be required to implication functions. In particular we are going to deal with implication functions fulfilling:

I4. $I(x,y) = 1 \Leftrightarrow x \leq y$.

This is usually called the ordering property (see, for instance, [1, 2, 34]) and it is fulfilled by a lot of examples of basic implication functions.

It is immediate that any implication function fulfilling property $I4$ is a particular case of a F-function. Thus, as a consequence of Proposition 3.10, a way to define a embedding from an implication function is defined.

Corolary 3.12. *Let I be an implication function fulfilling Property $I4$, the map $E_I : L([0,1]) \times L([0,1]) \to [0,1]$ defined by:*

$$E_I(a,b) = \begin{cases} 0 & \text{if } a \cap b = \emptyset \\ min(I(\underline{b},\underline{a}), I(\overline{a},\overline{b})) & \text{otherwise} \end{cases}$$

is an embedding for intervals.

A family of implication functions that satisfies properties I1, I2, I3 and I4 can be obtained as follows.

Proposition 3.13. *Consider $\Delta = \{(x,y) \in [0,1]^2 \mid x > y\}$. Let $f : \Delta \to [0,1)$ be a function such that*

- *f is decreasing in the first variable,*
- *f is increasing in the second variable,*
- *$f(1,0) = 0$,*

then the function $I_f : [0,1]^2 \to [0,1]$ defined as

$$I_f(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ f(x,y) & \text{otherwise} \end{cases}$$

is an implication function fulfilling Property $I4$.

Proof. It is immediate that I_f fulfils Axioms I1 and I2 and Property I4. For Axiom I3 we have that $I_f(0,0) = I_f(1,1) = 1$ and $I_f(1,0) = f(1,0) = 0$ by definition. $\square$

It is possible to establish an order among the implication functions introduced in Proposition 3.13 by taking into account the order among the functions used to generated them.

Corolary 3.14. *Consider f_1 and f_2 such that $f_1 \leq f_2$, i.e. $f_1(x,y) \leq f_2(x,y)$ for all $(x,y) \in \Delta$, then $I_{f_1} \leq I_{f_2}$ and $E_{I_{f_1}} \leq E_{I_{f_2}}$.*

Proof. It is immediate that from $f_1 \leq f_2$ we have that $I_{f_1} \leq I_{f_2}$. For the associated embeddings, we have that:

If $a \cap b = \emptyset$ then $E_{I_{f_1}}(a,b) = E_{I_{f_2}}(a,b) = 0$ by definition.

In the case $a \cap b \neq \emptyset$, as $I_{f_1}(\underline{b}, \underline{a}) \leq I_{f_2}(\underline{b}, \underline{a})$ and $I_{f_1}(\overline{a}, \overline{b}) \leq I_{f_2}(\overline{a}, \overline{b})$, then $E_{I_{f_1}}(a,b) \leq E_{I_{f_2}}(a,b)$. $\square$

As a consequence of Proposition 3.13, some well-known implication functions fulfilling Property I4 can be obtained as well as their associated embeddings. We have considered the notation introduced in [2] for naming the implication functions.

- For $f_{LK}(x,y) = 1 - x + y$, we obtain the Lukasiewicz implication function:

$$I_{LK}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 1 - x + y & \text{otherwise} \end{cases}$$

  The corresponding Lukasiewicz embedding for intervals is:

$$E_{LK}(a,b) = \begin{cases} 0 & \text{if } a \cap b = \emptyset \\ \min(1 - \underline{b} + \underline{a}, 1 - \overline{a} + \overline{b}, 1) & \text{otherwise} \end{cases}$$

  since

$$\min(I_{LK}(\underline{b}, \underline{a}), I_{LK}(\overline{a}, \overline{b})) = \min(1 - \underline{b} + \underline{a}, 1 - \overline{a} + \overline{b}, 1).$$

- For $f_{FD}(x,y) = max(1 - x, y)$, we obtain the Fodor implication function:

$$I_{FD}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ max(1 - x, y) & \text{otherwise} \end{cases}$$

  Thus, the Fodor embedding for intervals is:

$$E_{FD}(a,b) = \begin{cases} 0 & \text{if } a \cap b = \emptyset \\ 1 & \text{if } a \subseteq b \\ max(1 - \overline{a}, \overline{b}) & \text{if } \underline{b} \leq \underline{a} \leq \overline{b} < \overline{a} \\ max(1 - \underline{b}, \underline{a}) & \text{if } \underline{a} < \underline{b} \leq \overline{a} \leq \overline{b} \\ \min\{max(1 - \overline{a}, \overline{b}), max(1 - \underline{b}, \underline{a})\} & \text{if } b \subset a \end{cases}$$

since $\min(I_{FD}(\underline{b},\underline{a}), I_{FD}(\overline{a},\overline{b})) = 1$ if $I_{FD}(\underline{b},\underline{a}) = I_{FD}(\overline{a},\overline{b}) = 1$. That is possible if $a \subseteq b$. On the other hand, $\min(I_{FD}(\underline{b},\underline{a}), I_{FD}(\overline{a},\overline{b})) = max(1-\overline{a},\overline{b})$ if $I_{FD}(\underline{b},\underline{a}) = 1$ and $I_{FD}(\overline{a},\overline{b}) = max(1-\overline{a},\overline{b})$. Thus in this case, according to definition of I_{FD}, $\underline{b} \leq \underline{a}$ and $\overline{b} < \overline{a}$. The proof of the other two cases is analogous.

- For $f_{GD}(x,y) = y$, we obtain the Gödel implication function:

$$I_{GD}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$$

As $\min(I_{GD}(\underline{b},\underline{a}), I_{GD}(\overline{a},\overline{b})) = 1$ if and only if $I_{GD}(\underline{b},\underline{a}) = I_{GD}(\overline{a},\overline{b}) = 1$. That is possible if $a \subseteq b$. On the other hand, $\min(I_{GD}(\underline{b},\underline{a}), I_{GD}(\overline{a},\overline{b})) = \underline{a}$ if and only if $I_{GD}(\underline{b},\underline{a}) = \underline{a} \leq I_{GD}(\overline{a},\overline{b})$ and then according to definition of I_{GD}, $\underline{a} < \underline{b}$. In a similar way, $\min(I_{GD}(\underline{b},\underline{a}), I_{GD}(\overline{a},\overline{b})) = \overline{b} \leq I_{GD}(\underline{b},\underline{a})$. Thus, $\underline{b} \leq \underline{a}$ and $\overline{b} < \overline{a}$.

Thus, we obtain that the Gödel embedding for intervals is:

$$E_{GD}(a,b) = \begin{cases} 0 & \text{if } a \cap b = \emptyset \\ 1 & \text{if } a \subseteq b \\ \overline{b} & \text{if } \underline{b} \leq \underline{a} \leq \overline{b} < \overline{a} \\ \underline{a} & \text{otherwise} \end{cases}$$

- For $f_{GG}(x,y) = y/x$, we obtain the Goguen implication function [18]:

$$I_{GG}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$$

Thus, the Goguen embedding:

$$E_{GG}(a,b) = \begin{cases} 0 & \text{if } a \cap b = \emptyset \\ 1 & \text{if } a \subseteq b \\ \min\{\overline{b}/\overline{a}, \underline{a}/\underline{b}\} & \text{otherwise} \end{cases}$$

In this case, $\min(I_{GG}(\underline{b},\underline{a}), I_{GG}(\overline{a},\overline{b})) = 1$ if and only if $I_{GG}(\underline{b},\underline{a}) = I_{GG}(\overline{a},\overline{b}) = 1$. That is possible if $a \subseteq b$.

- For $f_{RS}(x,y) = 0$, we obtain the Rescher implication function:

$$I_{RS}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

and therefore, the Rescher embedding is:

$$E_{RS}(a,b) = \begin{cases} 1 & \text{if } a \subseteq b \\ 0 & \text{otherwise} \end{cases}$$

Remark 3.15. *Note that it is possible to establish an order among the different afore introduced embeddings as a consequence of Corollary 3.14. In particular:*

$$E_{RS} \leq E_{GD} \leq E_{FD} \leq E_{LK}$$

and

$$E_{RS} \leq E_{GD} \leq E_{GG} \leq E_{LK}.$$

This is a consequence of the relationship among the functions used to generate this embeddings. Thus, for any $(x, y) \in \Delta$*, it is clear that:*

$$f_{RS}(x, y) = 0 \leq y = f_{GD}(x, y),$$

$$f_{GD}(x, y)) = y \leq \max(1 - x, y) = f_{FD}(x, y)$$

and

$$f_{FD}(x, y) = \max(1 - x, y) \leq 1 - x + y = f_{LK}(x, y).$$

Apart from that, we have that

$$f_{GD}(x, y) = y \leq y/x = f_{GG}(x, y),$$

since $0 \leq y < x \leq 1$ *and so* $1/x \geq 1$*. On the other hand, we know that* $x > y$ *and* $1 - x \geq 0$*. Thus,* $y(1 - x) \leq x(1 - x)$ *and then* $y \leq x(1 - x + y)$ *which is equivalent to say that* $y/x \leq 1 - x + y$*, that is,*

$$f_{GG}(x, y) \leq f_{LK}(x, y), \forall (x, y) \in \Delta.$$

Moreover, we know that there is not an order relation between E_{GG} *and* E_{FD} *since, for instance,*

$$\begin{aligned} E_{GG}([0.1, 1], [0.5, 1]) &= \min(1, 0.1/0.5) = 0.2 \\ E_{FD}([0.1, 1], [0.5, 1]) &= \min(\max(0, 1), \max(0.5, 0.1)) = 0.5 \end{aligned}$$

but

$$\begin{aligned} E_{GG}([0.3, 1], [0.5, 1]) &= \min(1, 0.3/0.5) = 0.6 \\ E_{FD}([0.3, 1], [0.5, 1]) &= \min(\max(0, 1), \max(0.5, 0.3)) = 0.5. \end{aligned}$$

Graphically, this was represented in Figure 2

The following example shows these relationships.

Example 3.16. *Let us consider several combinations for intervals* a *and* b *to show the relations existing among the different embeddings obtained from implication functions.*

In the first example, we consider different intervals $b = [0.2, 0.6]$*,* $b = [0.2, 0.8]$*,* $b = [0.4, 0.6]$*,* $b = [0.4, 0.8]$ *. Then we obtain the value of the embedding for the preceding* b *with respect to the interval* $a_\alpha = [\alpha, \alpha + 0.1]$ *for* $\alpha \in [0, 0.9]$ *(Figure 3).*

Then, with the same intervals b *we consider different intervals* $a_\beta = [\beta, \beta + 0.5]$ *for* $\beta \in [0, 0.5]$ *(Figure 4).*

We can see in both examples that although the different examples of embedding measures give us different values for the embedding degree, it is the same in all the cases when we are sure that one set is embedded in the other or when one set is not embedding at all in the other. They also allow us to see the general order relationship among these embedding measures obtained from implication functions established in Figure 2.

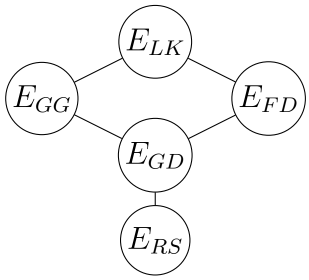

Figure 2: Order relationships among the different examples of embedding measures based on implications.

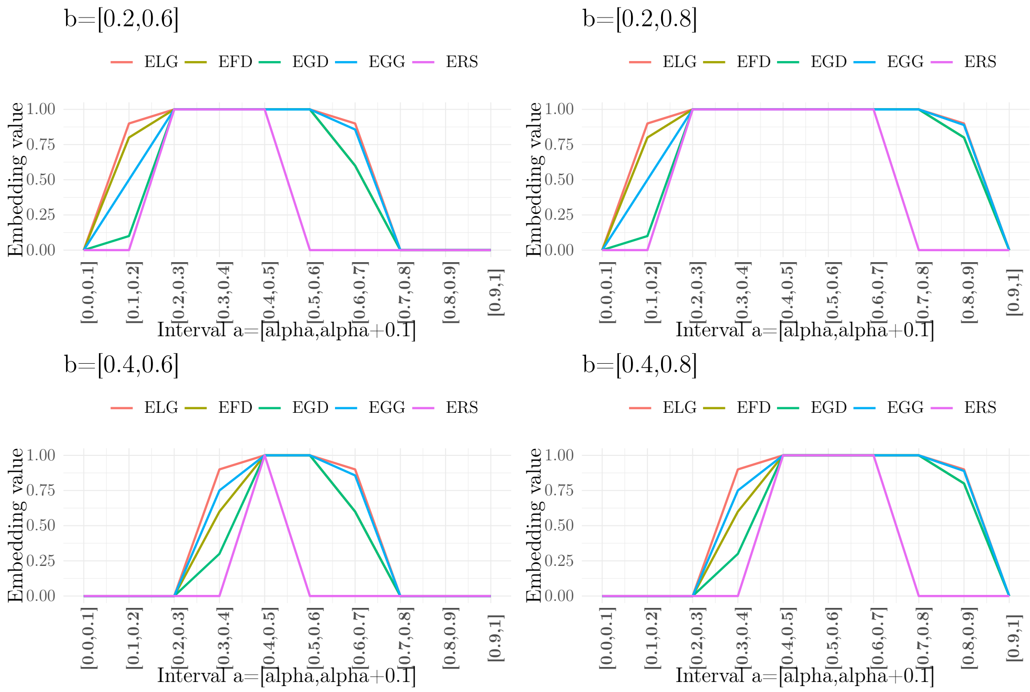

Figure 3: $a_\alpha = [\alpha, \alpha + 0.1]$ for $\alpha \in [0, 0.9]$

4. Embedding measures for Interval-valued fuzzy sets

At the previous section we study a measure of the degree in which an interval is embedded in another one. However, as we commented, that was just a first step in order to compare two interval-valued fuzzy sets, that is, to measure the degree of embedding of an interval-valued fuzzy set in another one. Taking into account the ideas considered for Definition 3.1, we can propose the following definition.

Definition 4.1. *The mapping:*

$$\mathcal{E} : IVFS(X) \times IVFS(X) \to [0, 1]$$

is an interval-valued embedding, in short IV-embedding, if the following properties hold:

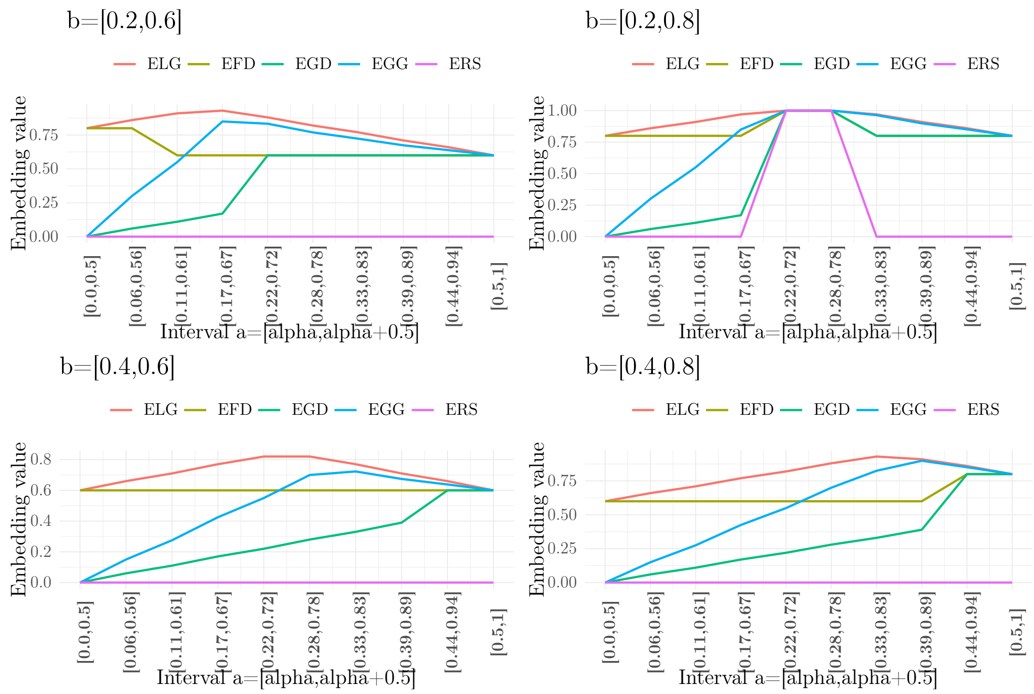

Figure 4: $a_\alpha = [\alpha, \alpha + 0.5]$ for $\alpha \in [0, 0.5]$

P1: $\mathcal{E}(A, B) = 1$ *iff* $A \sqsubseteq B$.
P2: If $A(x) \cap B(x) = \emptyset$ *for all* $x \in X$, *then* $\mathcal{E}(A, B) = 0$.
P3: If $\mathcal{E}(B, C) = 1$ *then* $\mathcal{E}(A, B) \leq \mathcal{E}(A, C)$ *for every* $A \in IVFS(X)$.

A general method to obtain IV-embedding measures based on aggregating the degree of embedding at any point of the referential set is introduced. As the notion of aggregation function is a cornerstone of this approach, the concept is briefly defined as follows.

Definition 4.2. *[5, 6, 27] A function* $\mathcal{M} : [0, 1]^n \to [0, 1]$ *is an aggregation function whenever it is increasing in each argument and it satisfies the boundary conditions:* $\mathcal{M}(0, \ldots, 0) = 0$ *and* $\mathcal{M}(1, \ldots, 1) = 1$.

Note that an embedding measure is equal to one only when the first set is totally embedded in the second one. Then, it is necessary to consider not just any aggregation function, but the ones satisfying the strict property ([1]). That means that $\mathcal{M}(x_1, x_2, \ldots, x_n) = 1$ only if $x_i = 1$ for all $i = 1, \ldots, n$.

Examples of strict aggregation functions are the arithmetic mean, the geometric mean and the harmonic mean or any t-norm and any t-conorm (see, for instance, [23]) generated by their aggregation functions. Thus, any of these functions could be considered to obtain embedding measures for interval-valued fuzzy sets as follows.

Proposition 4.3. *Let X be a finite referential. If $\mathcal{M} : [0,1]^n \to [0,1]$ is an strict aggregation function and $E : L([0,1])^2 \to [0,1]$ is an interval embedding, then the function $\mathcal{E}_{\mathcal{M}} : IVFS(X) \times IVFS(X) \to [0,1]$ given by:*

$$\mathcal{E}_{\mathcal{M}}^{E}(A,B) = \underset{x\in X}{\mathcal{M}} E(A(x), B(x))$$

is an IV-embedding.

Proof. P1: From its definition, $\mathcal{E}_{\mathcal{M}}^{E}(A,B) = 1$ iff $\underset{x\in X}{\mathcal{M}} E(A(x), B(x)) = 1$. Since $\mathcal{M}$ has the one strict property, this is equivalent to say that $E(A(x), B(x)) = 1$ for all $x \in X$. By Axiom A1 in Definition 3.1, this is equivalent to $A(x) \subseteq B(x), \forall x \in X$ which means that $A \sqsubseteq B$.

P2: If $A(x) \cap B(x) = \emptyset$ for all $x \in X$, then $E(A(x_i), B(x_i)) = 0$ by Axiom A2 in Definition 3.1. Thus, $\underset{x\in X}{\mathcal{M}} E(A(x), B(x)) = 0$ and as consequence $\mathcal{E}_{\mathcal{M}}^{E}(A,B) = 0$.

P3: Let $A \in IVFS(X)$. If $\mathcal{E}_{\mathcal{M}}^{E}(B,C) = 1$, then $B \sqsubseteq C$ as we just proved in P1. Thus, $B(x) \subseteq C(x)$ for all $x \in X$. Therefore, $E(A(x), B(x)) \leq E(A(x(x), C(x))$ for all $x \in X$. Then,

$$\underset{x\in X}{\mathcal{M}} E(A(x), B(x)) \leq \underset{x\in X}{\mathcal{M}} E(A(x), C(x))$$

by the monotonicity of $\mathcal{M}$ and therefore $\mathcal{E}_{\mathcal{M}}^{E}(A,B) \leq \mathcal{E}_{\mathcal{M}}^{E}(A,C)$.

□

Remark 4.4. *Proposition 4.3 allows to construct different IV-embeddings from the interval embeddings described in Subsections 3.1 and 3.2 .*

The lower bound for the family of IV-embedding measures on $IVFS(X)$ is obtained if E_0 (see Proposition 3.7) is considered. For this particular measure and the aggregation function $\mathcal{M}^0(x_1, x_2, \ldots, x_n) = 0$ if there exists at least one element x_i such that $x_i \neq 1$, the following IV-embedding is obtained:

$$\mathcal{E}_{\mathcal{M}^0}^{E_0}(A,B) = \begin{cases} 1 & \text{if } A \sqsubseteq B \\ 0 & \text{otherwise} \end{cases}$$

This is the smallest IV-embedding on $IVFS(X)$ as it follows from the following proposition.

Proposition 4.5. $\mathcal{E}_{\mathcal{M}^0}^{E_0} \leq \mathcal{E}$ *for any IV-embedding* $\mathcal{E}$.

Proof. If $A \sqsubseteq B$, then $\mathcal{E}_{\mathcal{M}^0}^{E_0}(A,B) = 1$, but as $\mathcal{E}$ is an IV-embedding function we know that $\mathcal{E}(A,B) = 1$ by property 1. Thus, in that case, $\mathcal{E}_{\mathcal{M}^0}^{E_0}(A,B) = \mathcal{E}(A,B)$.

On the other hand, if $A \not\sqsubseteq B$, then $\mathcal{E}_{\mathcal{M}^0}^{E_0}(A,B) = 0$ and therefore it is immediate that $\mathcal{E}_{\mathcal{M}^0}^{E_0}(A,B) \leq \mathcal{E}(A,B)$. □

It is clear that $\mathcal{E}_{\mathcal{M}^0}^{E_0}$ is a very drastic embedding measure. Some richer examples are proposed below.

Example 4.6. *Let us consider the embedding for intervals E_w which was introduced in Corollary 3.5 and let us consider the embedding measures for interval-valued fuzzy sets obtained from it by the minimum t-norm, the product t-norm, the arithmetic mean, the geometric mean and the harmonic mean, which will be denoted by $\mathcal{E}^w_M$, $\mathcal{E}^w_P$, $\mathcal{E}^w_{AM}$, $\mathcal{E}^w_{GM}$ and $\mathcal{E}^w_{HM}$, respectively.*

Let X be the referential defined as $X = \{x/10 : x \in \mathbb{N}, x \leq 100\}$. In order to see the influence of the aggregation function, we consider different possibilities for one set to obtain the degree of embedding in another fix set. Thus, let A_ϵ the family of interval-valued fuzzy sets defined by

$$A_\epsilon(x) = \left[0.25 - \frac{(x-5)^2}{100}, \frac{(x-5)^2}{100} + \epsilon\right],$$

$\forall x \in X, \forall \epsilon \in [0.25, 0.75]$ and let B be the interval-valued fuzzy set defined by

$$B(x) = \left[0, \frac{x^3 - 15x^2 + 50x}{400} + 0.87\right], \quad \forall x \in X$$

Some of the sets A_ϵ are represented by circles in Figure 5 and B is represented by triangles.

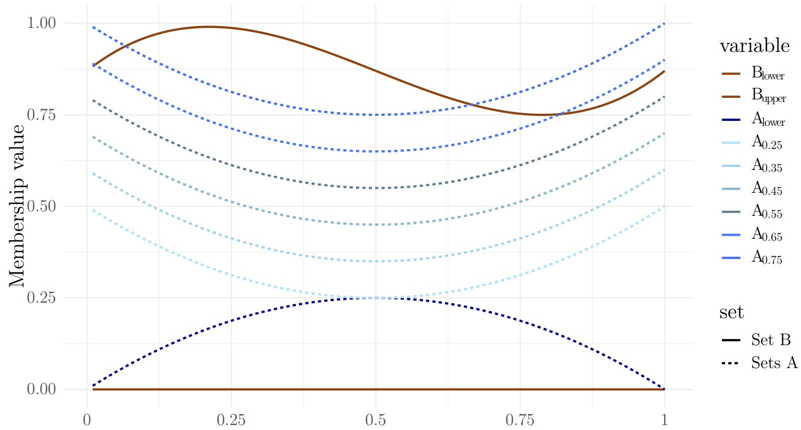

Figure 5: Membership functions for A_ϵ and B.

In Figure 6 we have represented the embedding degree obtained by $\mathcal{E}^w_M$, $\mathcal{E}^w_P$, $\mathcal{E}^w_{AM}$, $\mathcal{E}^w_{GM}$ and $\mathcal{E}^w_{HM}$ for different values of ϵ.

We can see at this simple example the influence of the chosen aggregation function. From any $\epsilon \in [0.6141, 0.75]$, A_ϵ is not totally embedded in B and the values for the embedding degree are totally different if we consider the product t-norm or the arithmetic mean to aggregate the degrees at any point of X. For the different means, we can see the results are in this case similar, but not equal, as we can see on the right part of Figure 6. More in detail, we can see some values of the embedding measures for the cases $\epsilon \in \{0.53, 0.63, 0.73\}$ in Table 1. Note for example that $A_{0.53} \sqsubseteq B$, but $A_{0.63} \not\sqsubseteq B$ and we are even more secure that $A_{0.73} \not\sqsubseteq B$.

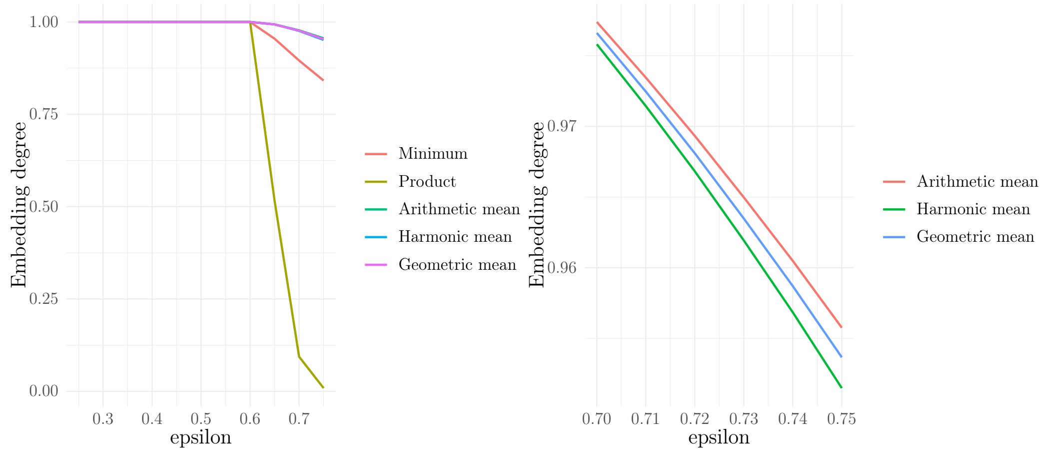

Figure 6: Behaviour of different embedding measures.

ϵ	$\mathcal{E}_M^w$	$\mathcal{E}_P^w$	$\mathcal{E}_{AM}^w$	$\mathcal{E}_{GM}^w$	$\mathcal{E}_{HM}^w$
0.53	1	1	1	1	1
0.63	0.9796	0.79925	0.9978	0.9978	0.9978
0.73	0.8625	0.0215	0.9642	0.9627	0.9611

Table 1: Values of different embedding measures for measuring the degree in which $A_{0.73}$ is embedded in B.

5. IV-embedding degree for the main operations in $IVFS(X)$

In this section it is studied how the above defined IV-embedding behaves when the intersection, union and complement of $IVFSs(X)$ are considered. Thus, these concepts are first introduced. Trying to be as general as possible, we will define the intersection by means of t-norms on $L([0,1])$ and the union by means of t-conorms on $L([0,1])$. Bedregal et al. [4, 3] proposed a generalization of the concept of t-norm for interval values based on the definition of a t-norm on $[0,1]$. It was given as follows:

Definition 5.1. *The mapping $T : L([0,1]) \times L([0,1]) \to L([0,1])$ is an interval-valued t-norm, IV t-norm for short, if it satisfies the following properties:*

- *T1. Symmetry: $T(a,b) = T(b,a)$ for each $a,b \in L([0,1])$;*
- *T2. Associativity: $T(a,T(b,c)) = T(T(a,b),c)$ for each $a,b,c \in L([0,1])$;*
- *T3. $\leq$-Monotonicity: $T(a_1,b_1) \leq T(a_2,b_2)$ for each $a_1,a_2,b_1,b_2 \in L([0,1])$ such that $a_1 \leq a_2$ and $b_1 \leq b_2$;*
- *T4. $\subseteq$-Monotonicity: $T(a_1,b_1) \subseteq T(a_2,b_2)$ for each $a_1,a_2,b_1,b_2 \in L([0,1])$ such that $a_1 \subseteq a_2$ and $b_1 \subseteq b_2$;*
- *T5. 1-Identity: $T([1,1],a) = a$ for each $a \in L([0,1])$.*

Thus, in general, we could use the IV t-norms to define the intersection between two interval-valued fuzzy sets.

Definition 5.2. *Let A and B be two elements in $IVFS(X)$. For any IV t-norm T we can define its T-intersection as the interval-valued fuzzy set given by*

$$A \cap_T B(x) = T(A(x), B(x))$$

Since this concept should keep the original idea of intersection for classical sets, the intersection should be included in both intersected sets.

Proposition 5.3. *Let $A, B \in IVFS(X)$ and T an IV t-norm. Then $A \cap_T B \subseteq A$ and $A \cap_T B \subseteq B$.*

Proof. For any $A, B \in IVFS(X)$, it is clear that $B(x) \leq [1, 1]$ for any $x \in X$. Thus,

$$A \cap_T B(x) = T(A(x), B(x)) \leq T(A(x), [1, 1]) = A(x)$$

by applying Axioms T3 and T5 in Definition 5.1. Therefore, $A \cap_T B \subseteq A$. Analogously we could prove that $A \cap_T B \subseteq B$. □

Bedregal considered a particular case of IV t-norms, called representable. Thus, they proved (see Theorem 4.1 in [4]) that if we consider two t-norms $t, t' : [0, 1]^2 \to [0, 1]$ with $t(x, y) \leq t'(x, y)$ for every $x, y \in [0, 1]$ then

$$T(a, b) = [t(\underline{a}, \underline{b}), t'(\overline{a}, \overline{b})]$$

is an IV t-norm and T was said to be derived from t and t'.

Given any representable IV t-norm t' it is clear that $t \leq t'$ if t is the minimum t-norm. Thus, since t is the greatest t-norm, then we obtain the representable IV t-norm:

$$T_M(a, b) = [\min(\underline{a}, \underline{b}), \min(\overline{a}, \overline{b})]$$

for any $a, b \in L([0, 1])$. It is then straightforward that

$$A \cap_{T_M} B(x) = [\min(\underline{A(x)}, \underline{B(x)}), \min(\overline{A(x)}, \overline{B(x)})], \quad \forall x \in X$$

which is, in fact, the usual definition for the intersection of two interval-valued fuzzy sets (see [9]). As we will focus on this IV t-norm, we will use the notation $\cap$ for $\cap_{T_M}$ for sake of simplicity.

Note that for this particular case of representable IV t-norm, the intersection is not only included in both sets (see Proposition 5.3) but also the "greatest" set fulfilling this property.

Proposition 5.4. *Consider $A, B \in IVFS(X)$. For any $C \in IVFS(X)$ such that $C \subseteq A$ and $C \subseteq B$, we have that $C \subseteq A \cap B$.*

Proof. If $C \subseteq A$, then $C(x) \leq A(x)$ for any $x \in X$. Analogously we obtain that $C(x) \leq B(x)$ for any $x \in X$. Thus, for any $x \in X$ we have that

$$C(x) = T_M(C(x), C(x)) \leq T_M(A(x), B(x)) = A \cap B(x)$$

by applying Axiom T3 in Definition 5.1. Thus, $C \subseteq A \cap B$. $\square$

Proposition 5.3 shows that the intersection is included in any of the sets. However it is not embedded. This is again a consequence of the different ideas behind inclusion and embedding in $IVFS(X)$.

Example 5.5. *Consider $A, B, C \in IVFS(X)$,*

$$A \cap B(x) = [\min(\underline{A(x)}, \underline{B(x)}), \min(\overline{A(x)}, \overline{B(x)})], \quad \forall x \in X$$

we have that

X	x	y
A	$[0.2, 0.8]$	$[0.3, 0.4]$
B	$[0.3, 0.7]$	$[0.1, 0.5]$
C	$[0.4, 0.5]$	$[0.2, 0.4]$
$A \cap B$	$[0.2, 0.7]$	$[0.1, 0.4]$
$B \cap C$	$[0.3, 0.5]$	$[0.1, 0.4]$

then $A \cap B \not\sqsubseteq A$ and $A \cap B \not\sqsubseteq B$ but $B \cap C \sqsubseteq B$ and $B \cap C \not\sqsubseteq C$.

Thus, it is not possible to obtain properties in general for the embedding of the intersection of two $IVFS(X)$.

However, consider the interval-valued fuzzy set defined by the collection of fuzzy sets that could be the real membership value in accordance to both sets, defined as follows

$$A \otimes B(x) = A(x) \cap B(x), \forall x \in X$$

Since we require $A \otimes B$ to belong to $IVFS(X)$ this set can be only considered if $A(x) \cap B(x) \neq \emptyset, \forall x \in X$. An example of this operation is shown in Figure 7.

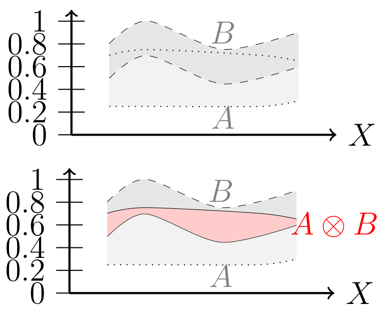

Figure 7: Membership values for $A \otimes B$.

Thus $A \otimes B$ represents the greatest Interval-valued fuzzy sets included in A and B. For these operation we can prove the following properties.

Proposition 5.6. *Let $\mathcal{E}$ be an IV-embedding. For any $A, B, C \in IVFS(X)$ we have that:*

(1) *If $A(x) \cap B(x) \neq \emptyset, \forall x \in X$, then $\mathcal{E}(A \otimes B, A) = \mathcal{E}(A \otimes B, B) = 1$.*
(2) *If $A(x) \cap B(x) \neq \emptyset, \forall x \in X$, then $\mathcal{E}(C, A \otimes B) \leq \min(\mathcal{E}(C, A), \mathcal{E}(C, B))$.*
(3) *If $A \sqsubseteq B$, then $\mathcal{E}(A, A \cap B) = \mathcal{E}(A \cap B, B) = 1$.*
(4) *If $A \sqsubseteq B$, then $\mathcal{E}(C, A) \leq \mathcal{E}(C, A \cap B) \leq \mathcal{E}(C, B)$.*

Proof. (1) If $A(x) \cap B(X) \neq \emptyset \, \forall x \in X$, $A \otimes B$ is well defined and it is immediate from its definition that $A \otimes B \sqsubseteq A$ and $A \otimes B \sqsubseteq B$. Thus, $\mathcal{E}(A \otimes B, A) = \mathcal{E}(A \otimes B, B) = 1$ from Axiom P1.

(2) Since $A \otimes B \sqsubseteq A$ and $A \otimes B \sqsubseteq B$, from Axiom P3 we have that $\mathcal{E}(C, A \otimes B) \leq \mathcal{E}(C, A)$ and $\mathcal{E}(C, A \otimes B) \leq \mathcal{E}(C, B))$.

(3) If $A \sqsubseteq B$, then for any $x \in X$, we have that

$$\min(\underline{A(x)}, \underline{B(x)}) \leq \underline{A(x)} \leq \overline{A(x)} = \min(\overline{A(x)}, \overline{B(x)})$$

and

$$\underline{B(x)} = \min(\underline{A(x)}, \underline{B(x)}) \leq \min(\overline{A(x)}, \overline{B(x)}) \leq \overline{B(x)}$$

This means that $A \sqsubseteq A \cap B$ and $A \cap B \sqsubseteq B$. Then, from Axiom P1, we have that $\mathcal{E}(A, A \cap B) = \mathcal{E}(A \cap B, B) = 1$.

(4) As it was proved before, $A \sqsubseteq A \cap B$ and $A \cap B \sqsubseteq B$ and then this property is a direct consequence of Axiom P3.

□

Now, we are going to do a parallel study for the union. In a similar way, the definition of t-conorm is necessary for introducing the concept of union between two interval-valued fuzzy sets. A t-conorm is a symmetric, associative, monotonic map from $[0, 1]^2$ into $[0, 1]$ with zero as neutral element. Taking into account its strong relationship with t-norms and Definition 5.1, we can consider the following definition for a t-conorm on $L([0, 1])$.

Definition 5.7. *The mapping $S : L([0, 1]) \times L([0, 1]) \to L([0, 1])$ is an interval-valued t-conorm, IV t-conorm for short, if it satisfies Axioms T1, T2, T3 and T4 and the axiom*

S5. 0*-Identity: $S([0, 0], a) = a$ for each $a \in L([0, 1])$.*

IV t-conorms allow us to generalize the definition of union of interval-valued fuzzy sets as follows.

Definition 5.8. *Let A and B be two elements in $IVFS(X)$. For any IV t-conorm S we can define its S-union as the interval-valued fuzzy set given by*

$$A \cup_S B(x) = S(A(x), B(x)), \forall x \in X$$

As well as it happened for the intersection, the union fulfils the following natural requirement.

Proposition 5.9. *Let $A, B \in IVFS(X)$ and S an IV t-conorm. Then $A \subseteq A \cup_S B$ and $B \subseteq A \cup_S B$.*

Proof. For any $A, B \in IVFS(X)$, it is clear that $[0, 0] \leq B(x)$ for any $x \in X$. Thus,

$$A(x) = S(A(x), [0, 0]) \leq S(A(x), B(x)) = A \cup_S B(x)$$

by applying the $\leq$-monotonicity and the 0-identity of S. Therefore, $A \subseteq A \cup_S B$. The proof of $B \subseteq A \cup_S B$ is analogous. □

Now, we are going to characterize the representable IV t-conorms. It is easy to prove that for any two t-conorms s and s' in $[0, 1]$ with $s \leq s'$, the map

$$S(a, b) = [s(\underline{a}, \underline{b}), s'(\overline{a}, \overline{b})]$$

is an IV t-conorm. In the particular case that s' is the maximum t-conorm, the union of two interval-valued fuzzy sets is given by:

$$A \cup_{S_M} B(x) = [\max(\underline{A(x)}, \underline{B(x)}), \max(\overline{A(x)}, \overline{B(x)})]$$

since the maximum is the lowest t-conorm. This is the usual union for interval-valued fuzzy sets considered in the literature (see again [9]) and then it is just denoted as $\cup$. For this union, we can assure that the union is the lowest interval-valued fuzzy set including the two joined sets.

Proposition 5.10. *Let $A, B \in IVFS(X)$. For any $C \in IVFS(X)$ such that $A \subseteq C$ and $B \subseteq C$, we have that $A \cup B \subseteq C$.*

Proof. Since $A(x) \leq C(x)$ and $B(x) \leq C(x)$ for any $x \in X$, we have that

$$A \cup B(x) = S_M(A(x), B(x)) \leq S_M(C(x), C(x)) = C(x)$$

by the $\leq$-monotonicity of S_M and the idempotency of the maximum t-conorm. □

If we consider again the sets considered in Example 5.5, we have that $A \not\sqsubseteq A \cup B$, $B \not\sqsubseteq A \cup B$ and $B \not\sqsubseteq B \cup C$, but $C \not\sqsubseteq B \cup C$. However, we can consider the interval-valued fuzzy set representing the fuzzy sets which are in A or B, that is,

$$A \oplus B(x) = A(x) \cup B(x), \forall x \in X.$$

Since $A \oplus B$ must belong to $IVFS(X)$, $A(x) \cap B(x) \neq \emptyset$ for any $x \in X$. It is immediate to prove that $A \sqsubseteq A \oplus B$ and $B \sqsubseteq A \oplus B$.

Proposition 5.11. *Let $\mathcal{E}$ be an IV-embedding. For any $A, B, C \in IVFS(X)$ we have that:*

(1) *If $A(x) \cap B(x) \neq \emptyset, \forall x \in X$, then $\mathcal{E}(A, A \otimes B) = \mathcal{E}(B, A \otimes B) = 1$.*
(2) *If $A(x) \cap B(x) \neq \emptyset, \forall x \in X$, then $\max(\mathcal{E}(C, A), \mathcal{E}(C, B)) \leq \mathcal{E}(C, A \otimes B)$.*

(3) *If* $A \sqsubseteq B$*, then* $\mathcal{E}(A, A \cup B) = \mathcal{E}(A \cup B, B) = 1$.
(4) *If* $A \sqsubseteq B$*, then* $\mathcal{E}(C, A) \leq \mathcal{E}(C, A \cap B) \leq \mathcal{E}(C, B)$.

Proof. (1) It is immediate from Axiom P1.
(2) It is immediate from Axiom P3.
(3) If $A \sqsubseteq B$, then for any $x \in X$, we have that

$$\max(\underline{A(x)}, \underline{B(x)}) = \underline{A(x)} \leq \overline{A(x)} \leq \max(\overline{A(x)}, \overline{B(x)})$$

that is, $A \sqsubseteq A \cup B$ and

$$\underline{B(x)} \leq \max(\underline{A(x)}, \underline{B(x)}) \leq \max(\overline{A(x)}, \overline{B(x)}) = \overline{B(x)}$$

which means that $A \cup B \sqsubseteq B$. Then, from Axiom P1, we have that $\mathcal{E}(A, A \cup B) = 1$ and $\mathcal{E}(A \cup B, B) = 1$.
(4) As we showed at the previous property, we have that $A \sqsubseteq A \cap B$ and $A \cap B \sqsubseteq B$ and then this property is a direct consequence of Axiom P3.

□

In order to define the complement of any IVFS, we need to introduce the negations on $L([0, 1])$ based on the classical fuzzy negations (see, among others, [16, 21, 25, 30]). Thus, Bedregal et al. introduced the following negation for intervals.

Definition 5.12. *[3] An IV negation is a mapping* $N : L([0, 1]) \to L([0, 1])$ *which is decreasing and such that* $N([0, 0]) = [1, 1]$ *and* $N([1, 1]) = [0, 0]$*. An IV negation* N *is strong if* $N(N(a)) = a$ *for every* $a \in L([0, 1])$*.*

Note that if $n : [0, 1] \to [0, 1]$ is a negation, then $N(a) = [n(\overline{a}), n(\underline{a})]$ is an IV negation.

IV negation allow us to generalize the definition of complement of interval-valued fuzzy sets as follows:

$$A^c(x) = N(A(x)), \forall x \in X$$

for any $A \in IVFS(X)$.

Henceforth, we will consider n as the standard fuzzy negation, that is,

$$A^c_N(x) = [1 - \overline{A(x)}, 1 - \underline{A(x)}], \forall x \in X$$

From this definition, it is clear that, in general, A is neither embedded in A^c nor A^c is embedded in A. Therefore, we can obtain properties for these embeddings just in some specific cases.

Proposition 5.13. *Let* $\mathcal{E}$ *be an IV-embedding. For any* $A \in IVFS(X)$*, we have that:*

(1) *If* $0.5 \notin A(x) \, \forall x \in X$*, then* $\mathcal{E}(A, A^c) = \mathcal{E}(A^c, A) = 0$.
(2) *If* $\underline{A(x)} + \overline{A(x)} = 1 \, \forall x \in X$*, then* $\mathcal{E}(A, A^c) = \mathcal{E}(A^c, A) = 1$.

Proof. (1) If $0.5 \notin A(x) \, \forall x \in X$, then $A(x) \cap A^c(x) = \emptyset$ for any $x \in X$, it is straightforward from Axiom P2.
(2) If $\underline{A(x)} + \overline{A(x)} = 1 \, \forall x \in X$, then $A = A^{`c}$ and therefore $\mathcal{E}(A, A^c) = \mathcal{E}(A^c, A) = 1$ by Axiom P1.

□

6. Case study

The aim of this case study is to test the effectiveness of a medical treatment for a diabetic patient. This treatment would be based on an "intelligent system" that continuously and automatically measures glucose levels in blood (in milligrams per decilitre) and injects the patient with the appropriate dose of medication. A "normality band" for glucose levels has been previously established throughout the day, particularly in relation to meal time. These values are scaled in $[0,1]$ by an affine application $f : [40, 440] \to [0,1]$, $f(t) = \frac{t-40}{400}$. In the same line, the 24 hours interval time is scaled also to $[0,1]$. The "normality band" can be modeled as a $[\underline{B(x)}, \overline{B(x)}]$ set where X is the variable representing the different moments of the day where glucose levels are measured. Glucose level at a moment x is "appropriate" if it is within $\underline{B(x)}$ and $\overline{B(x)}$. Suppose that during several days the glucose level is measured at different moments of the day, being x_{i_j} the glucose level at moment i of day j. Consider the IVFS $[\underline{A(x)}, \overline{A(x)}]$ such that $\underline{A(x_i)} = \min_j\{x_{i_j}\}$ and $\overline{A(x_i)} = \max_j\{x_{i_j}\}$ for every time i. The embedding degree of A in B could be useful to measure the effectiveness of the medication proposed. Consider that glucose level is measure every 15 minutes starting at 00:00. Thus X takes 96 different values (it is worth noting that if the reference set X is uniformly distributed and its cardinal increases, the embedding function, E, approaches a continuous function on $[0,1]$ and the value of the embedding tends to be obtained by means of an integral). Note that it is also necessary to select an aggregation function. Such aggregation should give more weight to values that are farther away from the normality values. This "distancing" should be measured in relative terms: if normality levels at $8h$ are to be between 60 and 100, it is much more alarming to have 50 than to have 120.

Let us consider that normality value at 00:00 is within 80 and 120 milligrams per decilitre (between 0.1 and 0.2 in $[0,1]$ scale). The $[\underline{B(x)}, \overline{B(x)}]$ associate to normality values during a day is represented in Figure 8.

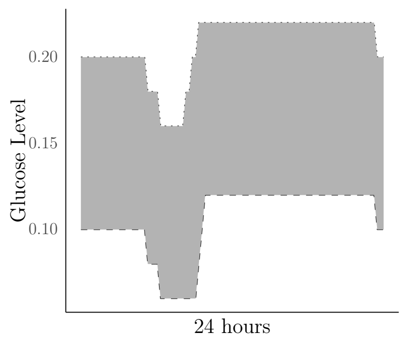

Figure 8: Normality band of glucose levels in 24 hours

$[\underline{A(x)}, \overline{A(x)}]$ is represented in Figure 9 together with the IVFS(X) $[\underline{B(x)}, \overline{B(x)}]$ representing the normality level. Noting x_0 as the measurement at $00:00$ hours; for each i

in 1 y 95, $i = 4t + s$ with $s \in \{0, 1, 2, 3\}$, then x_i is the measurement at t hours and $15s$ minutes.

Note that $\overline{A(x_i)}$ is between 0.12 and 0.15 for the 40 first measurements in a day. $\overline{A(x_{41})} = 0.11$, $\overline{A(x_{42})} = 0.10$, $\overline{A(x_{43})} = 0.09$, $\overline{A(x_{44})} = 0.08$, $\overline{A(x_{45})} = 0.07$, $\overline{A(x_{46})} = 0.07$, $\overline{A(x_{47})} = 0.06$, $\overline{A(x_{48})} = 0.05$, $\overline{A(x_{49})} = 0.08$, $\overline{A(x_{50})} = 0.1$; $\overline{A(x_i)}$ is within normality levels for all the other x_i.

On the other hand, $\overline{A(x_i)} = 0.25$ for $i \in \{0, 1, \ldots, 32\}$; $\overline{A(x_{33})} = 0.26$, $\overline{A(x_{34})} = 0.28$, $\overline{A(x_{35})} = 0.30$, $\overline{A(x_{36})} = 0.30$, $\overline{A(x_{37})} = 0.28$, $\overline{A(x_{38})} = 0.26$, $\overline{A(x_{39})} = 0.24$, $\overline{A(x_{40})} = 0.22$; Finally $\overline{A(x)}$ is within normality levels until 23 : 00 hours; Finally, $\overline{A(x_{93})} = 0.24$, $\overline{A(x_{94})} = 0.24$ and $\overline{A(x_{95})} = 0.25$.

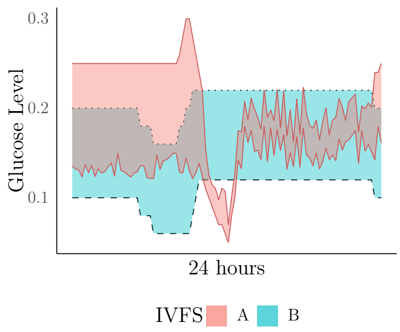

Figure 9: Embedding of A in B

For each $x_i \in X$ the embedding degree of $A(x)$ in $B(x)$ is defined as

$$E\big(A(x), B(x)\big) = 1 - \left(Max\left\{ \frac{\overline{A(x)} - \overline{B(x)}}{1 - \overline{B(x)}}, 0 \right\} + Max\left\{ \frac{\underline{B(x)} - \underline{A(x)}}{\underline{B(x)}}, 0 \right\} \right)$$

Using different aggregation functions, different embedding values are obtained, showing the ability of embeddings to indicate glucose behavior.

(1) When the aggregation function is mean, $\mathcal{M}_a$, then :

$$\mathcal{E}^E_{\mathcal{M}_a}(A, B) = \frac{\sum_{x \in X} E\big(A(x), B(x)\big)}{96} \approx 0.935$$

(2) If we consider the mean $\mathcal{M}_b$ considering only the set X_b composed by the 51 x_i with $E\big(A(x_1), B(x_i)\big) \neq 1$:

$$\mathcal{E}^E_{\mathcal{M}_b}(A, B) = \frac{\sum_{x \in X_b} E\big(A(x), B(x)\big)}{51} \approx 0.877$$

(3) If the aggregation function is the mean $\mathcal{M}_c$ considering only the set $X_c = \{x_i | i = 31, 42, 43, 44, 45, 46, 47, 48, 49, 50\}$ associated to the 10 highest values of $E\big(A(x_i), B(x_i)\big)$, then:

$$\mathcal{E}^E_{\mathcal{M}_c}(A, B) = \frac{\sum_{x \in X_c} E\big(A(x), B(x)\big)}{10} \approx 0.674$$

Thus, the notion of embedding could be useful to model such situations.

7. Conclusions

In this work we have introduced the concept of embedding for interval valued fuzzy sets as a functions that takes values on $[0, 1]$. In addition it is also provided a construction method to obtain IV-embedding measures based on aggregating the degree of embedding at any point of the referential set. Thus, an axiomatic definition to characterize interval embeddings is also presented, highlighting its difference with regard to the concept of inclusion. Some construction methods for interval embeddings are also presented mainly based on the interval width as well as on implication operators. The proposed IV-embeddings are an efficient tool to compare the precision of two different interval-valued fuzzy sets that is a relevant topic in many applications such as image processing for example. In the future we plan to extend the definition of $\mathcal{E}$ to provide IV-embeddings measured in terms of intervals instead of a number in $[0, 1]$ and to apply them to solve real problems. In addition, we plan to study these measures to any interval $[a, b] \in \mathcal{R}$.

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