Problem Set 9

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collaborators: none

Problem 9-2: Load Balancing

Part A

Lemma 0.1. *k is at most $2m$.*

Proof. Proof by Contradiction: Let us assume case for when $k > 2m$. We consider the case for where we are about to assign task $i = 2m+1$ on a machine. The load on each machine $L > \frac{2L*}{3}$.
When you add the i'th job, you're adding a job with weight $w_i > \frac{L*}{3}$.
Thus, when the job is assigned on a machine, the total load on machine $L > \frac{2L*}{3} + \frac{L*}{3} > L*$. L* is the maximum load on a machine. Since the maximum load cannot be exceeded, we have a contradiction. This completes the proof. □

Part B

The SORTEDBALANCE algorithm sorts the tasks based on the weights in decreasing order. For each task, it is allocated to the least loaded machine. We consider two cases.
Case 1: $0 \leq k \leq m$ Since $k \leq m$, the algorithm allocates a task to each machine. Since we allocate based on sorted order, $L** = w_1$. For any task i where $i \leq k$.

$$w_i \leq L** \tag{1}$$

$$L** \leq L \tag{2}$$

$$L** > \frac{L*}{3} \tag{3}$$

Case 2: $m < k \leq 2m$ When $k > m$, the algorithm allocates more than one task to at least one machine. Additionally, the maximum amount of tasks one can allocate to any machine is 2 tasks. If we allocate more than two tasks, the total weight allocated on the machine exceeds $L*$. Thus we can claim the following:

$$L** > \frac{2L*}{3} \tag{4}$$

Part C

From the class lectures, we know that the following two claims are true:

$$w \leq L* \tag{5}$$

$$L \leq L* \tag{6}$$

When $i \leq k$, we can claim

$$w_i > \frac{L*}{3} \tag{7}$$

For $i > k$ we can claim

$$w_i \leq \frac{L*}{3} \tag{8}$$

Let's assume that all tasks up to k have already been allocated to the machines. We are allocating task i where $i > k$ to a machine.
Case 1: $L = 0$

$$L + w_i = w \tag{9}$$

By equation 8

$$w_i \leq \frac{L*}{3} \tag{10}$$

Case 2: $L > 0$
By equations 6 and 10

$$L + w_i \leq L* + \frac{L*}{3} \tag{11}$$

$$L* + \frac{L*}{3} = \frac{4L*}{3} \tag{12}$$

Thus,

$$L \leq \frac{4L*}{3} \tag{13}$$