Sample problems [1]

1. The following questions are related to basic concepts in graph theory. Recall that a directed graph is a graph with directed edges. A directed edge is an edge (u, v), $u, v \in V$ going from u to v but not from v to u.
   (a) Give a simple alternative characterization of the class of all undirected graphs that have exactly one spanning tree.
   (b) What is a "topological ordering" of a directed graph?
2. The following questions are related to strongly connected components (SCCs). Recall that a strongly connected component in a directed graph is a maximal subset of the vertices of the graph such that every pair of vertices in this graph has a path between them in both directions.
   (a) Prove or disprove: There exists a directed graph $G = (V, E)$ and a subset $\{u, v, w\}$ of V such that u and v belong to the same SCC, u and w belong to the same SCC, but v and w do not belong to the same SCC.
   (b) Give an algorithm to identify if a given directed graph is strongly connected and analyze its runtime and complexity.
   (c) Give an example of a directed graph that has five SCCs, and such that the number of SCCs can be reduced to one by adding a single directed edge between two existing vertices.
3. We have seen that Dijkstra's algorithm can be used to solve the single-source shortest paths (SSSP) problem in directed graphs with nonnegative edge weights, but that its running time is slightly super-linear. Give a linear-time algorithm to solve the special case of the SSSP problem in which all of the edges in the input directed graph have the same positive weight Δ.
4. Let $G = (V, E)$ be a connected undirected graph where each edge has weight 10 or 11. The weight of a minimum spanning tree is the sum of the weights of all the edges of the minimum spanning tree. let weight of the minimum spanning tree of G be be W^*, and let T be a spanning tree of G with weight W where $W > W^*$. Prove or disprove: The graph G has a spanning tree with weight equal to $W - 1$.
5. Let $G = (V, E)$ be an edge-weighted, connected, undirected graph. Let V' be a subset of V and let E' denote the set of all edges in E with one endpoint in V' and one endpoint in $V - V'$. Let e denote an edge of minimum weight in E', and assume that no other edge in E' has the same weight as e (i.e., e is the unique minimum-weight edge in E'). Prove or disprove: Every minimum spanning tree of G includes edge e.
6. Recall that when mergesort is invoked on an array of size $n = 2^k$, where k is some positive integer, the "top-level" invocation makes two "second-level" recursive invocations on arrays of size $n/2$, and performs $O(n)$ additional comparisons.

   Joe claims to have discovered a new recursive comparison-based sorting algorithm, similar in structure to mergesort, but with the following specific characteristics. Joe's algorithm only operates on arrays with size equal to a power of 8. For an input array of size $8^0 = 1$, Joe's algorithm uses zero comparisons. For an input array of size $n = 8^k$, where k is some positive integer, the top-level invocation of Joe's recursive sorting routine makes at most ℓ second-level recursive invocations on arrays of size $\frac{n}{8}$, for some positive integer ℓ that is independent of n. The top-level invocation of Joe's recursive sorting routine performs at most $100n$ additional comparisons beyond those performed within the ℓ second-level invocations.

[1] Questions provided by Prof. Greg Plaxton

(a) Let $C(n)$ denote the worst-case number of comparisons performed by Joe's algorithm on any input of size n, where n is a power of 8. Write a recurrence for establishing an upper bound on $C(n)$.

(b) Bearing in mind that the performance of mergesort is known to be asymptotically optimal, what is the least integer ℓ for which Joe's claim is plausible? Justify your answer.

7. Let A be a collection of n points in the plane such that the Euclidean distance between any two points in A is at least 1, i.e., for all pairs of points (x, y) and (x', y') in A, we have

$$\sqrt{(x - x')^2 + (y - y')^2} \geq 1.$$

Let B denote the set of all points (x, y) in A such that $|x| \leq 10$ and $|y| \leq 10$. Prove that there exists a positive integer c, independent of n, such that $|B| \leq c$.