Divide and Conquer

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1 Paradigm

1. Break problem into sub-problems on smaller input size.
2. Solve each sub-problem recursively.
3. Combine solutions to overall solution.

2 Example: Merge Sort

$\boxed{7}\ \boxed{8}\ \boxed{1}\ \boxed{5}\ \boxed{12}\ \boxed{3}\ \boxed{6}\ \boxed{9}$

1. Divide the array into two halves
2. Sort each half
3. Merge the two halves

$\boxed{1}\ \boxed{5}\ \boxed{7}\ \boxed{8}\quad \boxed{3}\ \boxed{6}\ \boxed{9}\ \boxed{12}$

$\boxed{1}\ \boxed{3}\ \boxed{5}\ \boxed{6}\ \boxed{7}\ \boxed{8}\ \boxed{9}\ \boxed{12}$

3 Recurrence

$T(n)$ = run-time on n numbers.
Recurrence: $T(n) = 2 \times T(\frac{n}{2}) + \theta(n)$

3.1 How to solve recurrences?

Two methods for figuring out the solution.
One method for proving the solution once you already know it.

Example:
$T(n) = 2T(\frac{n}{2}) + \theta(n)$
$T(1) = \theta(1)$

First: Work with explicit constants
$T(n) = 2T(\frac{n}{2}) + cn$
$T(1) = c$

3.2 First Method:

Recursion Tree

$T(n) \to cn$

$T(\frac{n}{2}) \quad T(\frac{n}{2}) \quad \to 2\,c\,\frac{n}{2} = c\,n$

$T(\frac{n}{4}) \quad T(\frac{n}{4}) \quad T(\frac{n}{4}) \quad T(\frac{n}{4}) \quad \to 4\,c\,\frac{n}{4} = c\,n$

Overall depth of the tree $= \log n$

Note 3.1. every layer is $c\,n$

Note 3.2. Overall: $c\,n \log n = \theta(n \log n)$ where c is a constant.

Why is it so important to work with explicit constants?

Example

$T(n) = 2\,T(n-1) + \theta(1)$

$T(1) = \theta(1)$

$T(n) \to \theta(1) \to c$

$T(n-1) \quad T(n-1) \to 2\,\theta(1) = \theta(1) \to 2\,c$

$T(n-2) \quad T(n-2) \quad T(n-2) \quad T(n-2) \to 4\,\theta(1) = \theta(1) \to 4\,c$

Assuming the total is $n\theta(1) = \theta(n)$ is WRONG.
$T(n) \geq 2^{(n-1)}$.
we have this issue because we didn't use explicit constants!

3.3 Iteration Method

Unroll the recurrence
$T(n) = 2\,T(n-1) + c$

$T(1) = c$

$T(n) = 2\,T(n-1) + c$
$= 2(2\,T(n-2) + c) + c$
$= 2(2(2\,T(n-3) + c) + c) + c$
$= 2(2(2 \ldots 2(T(1+c) + c) + c) + \ldots) + c$

Suppose that $T(1) = c$
$\therefore T(n) = 2^{n-1}\,c + 2^{n-2}\,c + 2^{n-3}\,c + \ldots + c$

* this is very subtle and delicate. you must be very careful *
$\therefore c(2^{n-1} + 2^{n-2} + 2^{n-3} + \ldots + 1) = c(2^n - 1)$

$\therefore c\,(2^n - 1) = \theta(2^n)$

Note 3.3. Geometric Sum: $a + a\,q + a\,q^2 + a\,q^2 + \ldots + a\,q^{n-1} = a\,\frac{q^n - 1}{q-1}$

3.3.1 Example

$T(n) = 2\,T(\frac{n}{2}) + c\,n$
$T(1) = c$

$T(n) = 2\,T(\frac{n}{2}) + c\,n = 2(2\,T(\frac{n}{4}) + c\,\frac{n}{2}) + c\,n$
$= 2^{\log n}T(1) + 2^{\log n - 1}c\,n/2^{\log n - 1} + \ldots + 2\,c\,\frac{n}{2} + c\,n$
$= n\,c\,(\log n + 1) = \theta(n \log n)$

3.4 Substitution method: Proof by Induction

3.4.1 Example 1

$T(n) = 2\,T(\frac{n}{2}) + c\,n$
$T(1) = c$

Claim: $T(n) = c\,n\,(\log n + 1) \quad \forall n$

Proof. By induction on n
Base: $n = 1 \quad T(1) = c = c \cdot (\log 1 + 1) = c$
Hypothesis: True for $n - 1$

Induction or Step n:
$T(n) = 2\,T(\frac{n}{2}) + c\,n$
$= 2\,c\,\frac{n}{2}(\log\frac{n}{2}) + c\,n$
$= c\,n\,(\log n - 1 + 1) + c\,n$
$= c\,n\,\log n + c\,n$ $\square$

Note 3.4.
$\log\frac{a}{b} = \log a - \log b$
$\log 2 = 1$

3.4.2 Example 2

$T(n) = 2\,T(n-1) + 1$
$T(1) = 1$

Claim: $T(n) = 2^n - 1 \quad \forall n$

Proof. By induction on n
Base: $n = 1 \quad T(1) = 1 = 2^1 - 1 = c$
Hypothesis: True for $n - 1$
Induction or Step n:
$T(n) = 2\,T(n-1) + 1$
$= 2(2^{n-1} - 1) + 1$
$= 2^n - 2 + 1$
$= 2^n - 1$ $\square$

4 Master Theorem

$T(1) = \theta(1) \qquad T(n) = a\,T(\frac{n}{b}) + \theta(n^k)$
$a \geq 1 \quad b \geq 1 \qquad a, b$ are constants. k is a non-negative constant

Then

1. $k < \log_b a \quad \Rightarrow \quad T(n) = \theta(n^{\log_b a})$
2. $k = \log_b a \quad \Rightarrow \quad T(n) = \theta(n^{\log_b a}\log n)$
3. $k > \log_b a \quad \Rightarrow \quad T(n) = \theta(n^k)$

4.1 Example

$T(n) = aT(\frac{n}{b}) + \theta(n) \quad , a < b$
$T(n) = \theta(n)$