Problem Set 1

This problem set is due at **10:00 am** on **Tuesday, January 31st**.

Problem 1- 1: Growth Sort the following functions so f appears before g if $f = O(g)$:

$$n^{0.99},\ \log_{1.1} n,\ 10^{1249},\ (\log_2 n)^2,\ 2^{(\ln\ln n)^2},\ 10^n,\ \ln\ln n,\ 2^{n^2},\ (\log_{10} n)^n,\ 1000n + 10^{10}.$$

Provide a one line explanation for each pair of consecutive functions in the sorted list.
Solution: 10^{1249}, $\ln\ln n$, $\log_{1.1} n$, $(\log_2 n)^2$, $2^{(\ln\ln n)^2}$, $n^{0.99}$, $1000n{+}10^{10}$, 10^n, $(\log_{10} n)^n$, 2^{n^2}
Explanation:

$$\begin{aligned} 10^{1249} &= \ln\ln(e^{e^{10^{1249}}}) \\ &\le \ln\ln n, \text{ when } n \ge e^{e^{10^{1249}}} \end{aligned}$$

$$\begin{aligned} \ln\ln n &\le \ln n, \text{ when } n \ge 1 \\ &< \frac{\ln n}{\ln 1.1},\ \ln 1.1 < 1 \\ &= \log_{1.1} n \end{aligned}$$

$$\begin{aligned} \log_{1.1} n &= \frac{\log_2 n}{\log_2 1.1} \\ &\le C \cdot \log_2 n \cdot \log_2 n, \text{ when } C \ge \frac{1}{\log_2 1.1} \text{ and } n \ge 2 \end{aligned}$$

$$\begin{aligned} (\log_2 n)^2 &= 2^{\log_2((\log_2 n)^2)} \\ &= 2^{2\log_2(\log_2 n)} \\ &= 2^{2\frac{\ln(\frac{\ln n}{\ln 2})}{\ln 2}} \\ &= 2^{2\frac{\ln\ln n - \ln\ln 2}{\ln 2}} \\ &\le 2^{\frac{2}{\ln 2}\ln\ln n} \\ &\le 2^{(\ln\ln n)^2}, \text{ when } \ln\ln n \ge \frac{2}{\ln 2} \text{ or } n \ge e^{e^{\frac{2}{\ln 2}}} \end{aligned}$$

$$
\begin{aligned}
2^{(\ln \ln n)^2} &= 2^{(\ln t)^2}, \text{ let t = } \ln n \\
&\leq 2^{\sqrt{t}^2}, \forall t > 1 \text{because } \sqrt{t} > \ln(t) \text{ for all } t > 1 \\
&= 2^t \\
&\leq 2^{\frac{t \cdot 0.99}{\ln 2}}, \frac{0.99}{\ln 2} \approx 1.428 \\
&= 2^{\frac{\ln n \cdot 0.99}{\ln 2}} \text{ Plug } t \text{ back in.} \\
&= 2^{\log_2 n \cdot 0.99} \\
&= (2^{\log_2 n})^{0.99} \\
&= n^{0.99}
\end{aligned}
$$

$$
\begin{aligned}
n^{0.99} &\leq n \\
&< 1000n + 10^{10}
\end{aligned}
$$

$$
\begin{aligned}
1000n + 10^{10} &= 10^3 \cdot n + 10^{10} \\
&\leq 10^{10} \cdot n + 10^{10} \\
&= 10^{10}(n+1) \\
&< C \cdot 10^n, \text{ when } C = 10^{10} \text{ and } n > 1
\end{aligned}
$$

$$
\begin{aligned}
10^n &= (\log_{10}(10^{10}))^n \\
&\leq (\log_{10} n)^n, \text{ when } n \geq 10^{10}
\end{aligned}
$$

$$
\begin{aligned}
(\log_{10} n)^n &= 2^{\log_2((\log_{10} n)^n)} \\
&= 2^{n \cdot \log_2(\log_{10} n)} \\
&< 2^{n \cdot n}
\end{aligned}
$$

Problem 1- 2: A New Order Let G be an undirected graph on N vertices where each vertex has degree at most 2.

(a) Suppose that we perform a BFS of G. Let $v_1, \ldots, v_N$ be the vertices of G in the order they are visited in the search. Prove or disprove: every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i.
Answer: Yes.
Proof:
For each connected component G' in G, let V' be the set of all vertices in G' and $v_i \in V'$ be the ith vertex visited relative to the first vertex visited in G'.

Lemma 1 *Let M_k be the set of vertices marked after visiting v_k. $|M_k| \leq 2$ for all k.*

Proof. Proof by induction on k.

If $k = 1$, then v_1 is the first vertex visited in G' so no vertices in G' can be marked immediately prior to visiting v_1. Additionally, because BFS visits one connected component at a time, no vertices in $G \setminus G'$ can be marked immediately prior to visting v_1. Therefore, no vertices are marked immediately prior to visiting v_1. After visiting v_1, only two vertices can be marked, because v_1 has at most 2 neighbors. Therefore, $|M_k| \leq 2$.

If $k > 1$, $|M_{k-1}|$ is the number of vertices immediately prior to visiting v_k, and $v_k \in M_{k-1}$. Let N_k be the neighbors of v_k that are so far unvisited. Because $v_k \neq v_1$, v_k must have one neighbor already visited, and v_k has at most two neighbors, so $|N_k| \leq 1$. In the process of visiting v_k, v_k is unmarked and become visited, and all unvisited neighbors of v_k become marked. Thus $M_k = M_{k-1} \setminus [v_k] \cup N_k$. By the inductive hypothesis, $|M_{k-1}| \leq 2$, so $|M_k| \leq 2 - 1 + |N_k| \leq 2 - 1 + 1 = 2$.

Let (v_i, v_j) be any edge in G' such that $i < j$. After v_i is visited, $v_j \in M_i$. Because $|M_i| \leq 2$ and BFS visits vertices in the order they are marked, at most one other vertex can be visted between v_i and v_j, so $j - i \leq 2$.

(b) Suppose that we perform a DFS of G. Let $v_1, \ldots, v_N$ be the vertices of G in the order they are visited in the search. Prove or disprove: every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i.

Answer: No. A counterexample is a cycle of size 100. Observe that the vertex visited in the end is adjacent to the vertex that you start with. This gives you the edge (v_1, v_{100}).