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1 Problem 1-2: A New Order

Let G be an undirected graph on N vertices where each vertex has degree at most 2.

1.1 (a)

Suppose that we perform a BFS of G . Let v_1, \dots, v_N be the vertices of G in the order they are visited in the search. Prove or disprove: every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i .

This holds true by the following proofs. We first prove that in this BFS search, every layer has at most 2 vertices. Secondly, we prove that the labeling of each edge is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i .

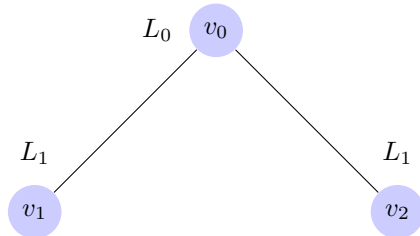
Lemma 1.1. *If G be an undirected graph on N vertices where each vertex has degree at most 2, then every layer L_i in a BFS has at most 2 vertices.*

Proof. Proof By Induction: On i , $L_i = \{\text{number of vertices in layer } L_i \text{ at distance } i\}$

Base Case: $i = 0$
 $i = 0$. $L_0 = \{v_0\}$, v_0 only vertex in layer L_0 . This base case holds because there is only one vertex in L_0

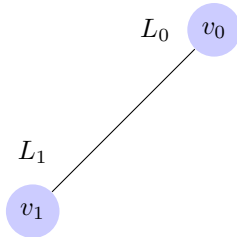
Base Case: $i = 1$

Case 1:



This case holds true because v_0 has a degree of 2 and layer L_1 has 2 vertices.

Case 2:



This case holds true because v_0 has a degree of 1 and layer L_1 has 1 vertices.

Induction Hypotheses:

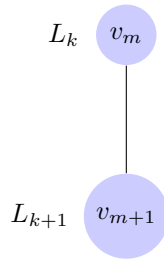
$\forall i \leq k \quad L_i = \{\text{Has at most 2 vertices}\}$

Induction Step:

For $k + 1$ we want to show $L_{k+1} = \{\text{Has at most 2 vertices}\}$

Case 1:

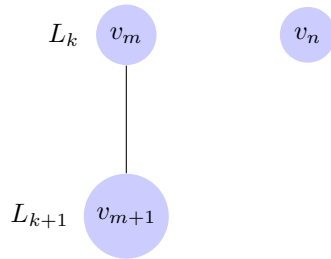
Layer L_k has 1 vertex and it has 1 child.



Layer L_{k+1} has one vertex, and thus Case 1 holds true.

Case 2:

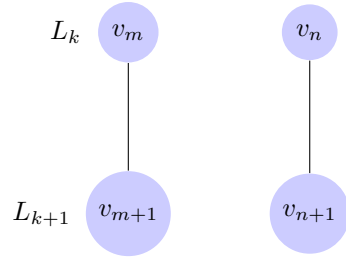
Layer L_k has 2 vertices and only one vertex has 1 child.



Layer L_{k+1} has one vertex, and thus Case 2 holds true.

Case 3:

Layer L_k has 2 vertices and each vertex has 1 child.



Layer L_{k+1} has 2 vertices, and thus Case 3 holds true.

Case 4:

Layer L_k has 2 vertices and each vertex has 0 children.

A BFS search will never reach L_{k+1} because there are no additional vertices to search.

□

Lemma 1.2. *If a BFS is performed on G , let v_1, \dots, v_N be the vertices of G in the order they are visited in the search. Every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i .*

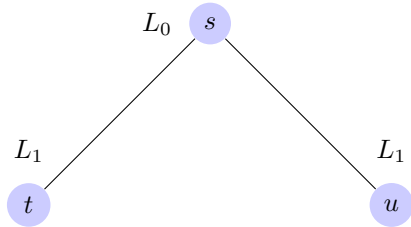
Proof. Proof By Induction: On i , every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) .

Base Case: $i = 0$

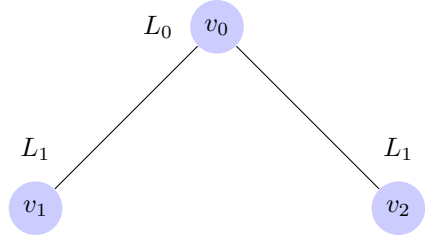
$i = 0$. $L_0 = \{s\}$, s only vertex in layer L_0 . This base case holds because there is only one vertex in L_0 and thus s would be labeled as v_0 .

Base Case: $i = 1$

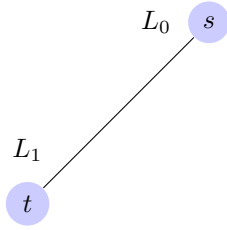
Case 1:



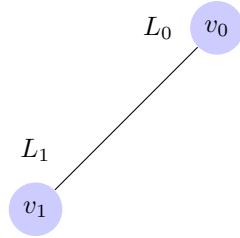
This case holds true because s would be labeled as v_0 , t will be labeled as v_1 , and u will be labeled as v_2 . The edges are (v_0, v_1) and (v_0, v_2) . This is shown below.



Case 2:



This case holds true because s would be labeled as v_0 and t will be labeled as v_1 . The edge is (v_0, v_1) . This is shown below.



Induction Hypotheses:

$\forall i \leq k$ Every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i .

Induction Step:

For $k + 1$ we want to show that every edge in G is of the form (v_k, v_{k+1}) or (v_k, v_{k+2})

By Lemma 1.1, each Layer can have at most 2 vertices. Thus, we will only consider up to 2 vertices for this proof by induction.

Case 1:

There is one vertex labeled as v_k and it has 1 child.

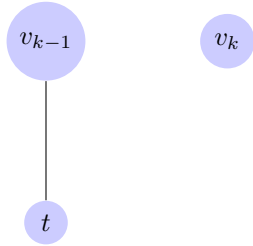


The vertex t would be labeled as v_{k+1} since it is the next vertex to be searched. The edge is (v_k, v_{k+1}) . Thus Case 1 holds true.

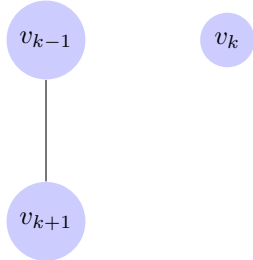


Case 2:

At some layer, there are 2 vertices v_{k-1} and v_k and only 1 vertex has 1 child.

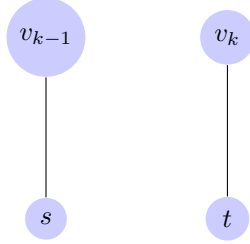


The vertex t would be labeled as v_{k+1} since it is the next vertex to be searched in BFS. The edge is (v_{k-1}, v_{k+1}) . Thus Case 2 holds true.

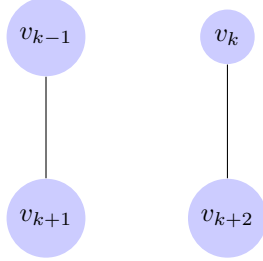


Case 3:

At some layer, there are 2 vertices v_{k-1} and v_k and each vertex has 1 child.

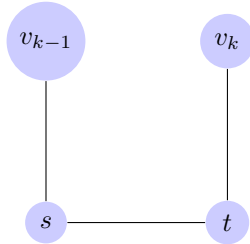


The vertex s would be searched before vertex t because it was identified before t because the parent of s was identified prior to the parent of t . The vertices s and t would be labeled as v_{k+1} and v_{k+2} respectively since they are the next vertices to be searched in BFS. The edges are (v_{k-1}, v_{k+1}) and (v_k, v_{k+2}) . Thus Case 3 holds true.

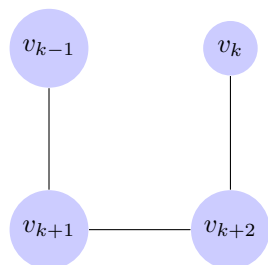


Case 4:

At some layer, there are 2 vertices v_{k-1} and v_k and each vertex has 1 child, and there is an edge between the children.



The vertex s would be searched before vertex t because it was identified before t because the parent of s was identified prior to the parent of t . The vertices s and t would be labeled as v_{k+1} and v_{k+2} respectively since they are the next vertices to be searched in BFS. The fact that there is an edge between s and t doesn't matter. The edges are (v_{k-1}, v_{k+1}) , (v_k, v_{k+2}) , and (v_{k+1}, v_{k+2}) . Thus Case 4 holds true.



Case 5:

At some layer, there are 2 vertices and each vertex has 0 children.

A BFS search will conclude at this layer because there are no further children to search.

□

1.2 (b)

Suppose that we perform a DFS of G . Let v_1, \dots, v_N be the vertices of G in the order they are visited in the search. Prove or disprove: every edge in G is of the form (v_i, v_{i+1}) or (v_i, v_{i+2}) for some i .

This claim does not prove true and I will disprove it using a proof by contradiction.

Proof by Contradiction:

Proof. Based on lemma 1.1, the only vertex that can have two children is the start node. Assume a start node with two children and that the left child has k descendants, where $k > 1$. Additionally, assume that each of these k descendants does not have an edge to a descendent of the right child of the start node.

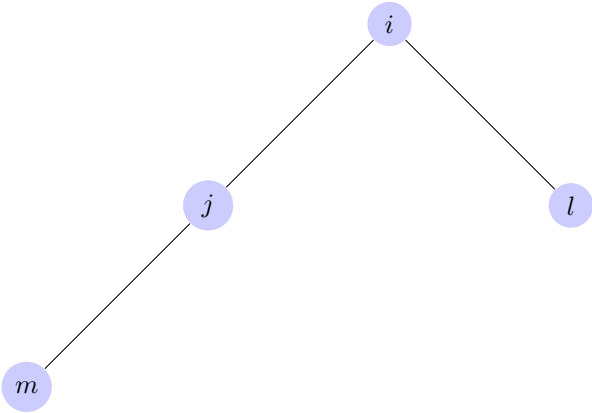
The start node will be labeled as v_0 . The left child will be labeled as v_1 since it is the first child to be searched. Following the DFS, all of v_1 's descendants will be searched. Thus, the last descendant to be searched from the left child will be labeled as v_k . Thus the left subtree will be labeled v_1, \dots, v_k .

The next child to be searched will be the right child of the start node. This vertex will be labeled v_{k+1} . Thus, there is an edge in G of the form (v_0, v_{k+1}) where $k > 1$. This concludes the proof by contradiction.

□

Example:

Before DFS Search:



After DFS Search:

