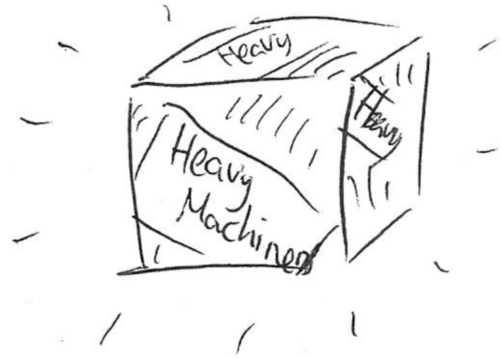
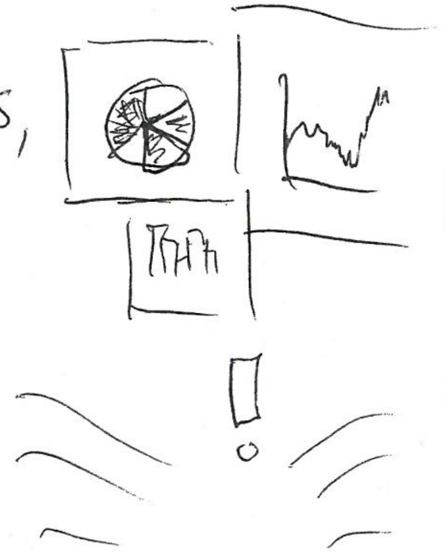
Linear Programming



The Campaign Consultant

Two policies, two demographics, have estimates on the number of votes per dollar advertising in support of each policy.



Want to win majority in each demographic, by spending as little as possible.

	age 20-30	30-40
gun control	2	3
legalizing marijuana	3	-1
population in millions	12	10

← made up figures!!

Let x = money spent on first policy

y = money spent on second policy.

Can formulate the problem as a "linear program"

$$\min \quad x+y$$

subject to

$$2x + 3y \geq 6 \quad \longleftrightarrow \quad y \geq 2 - \tfrac{2}{3}x$$

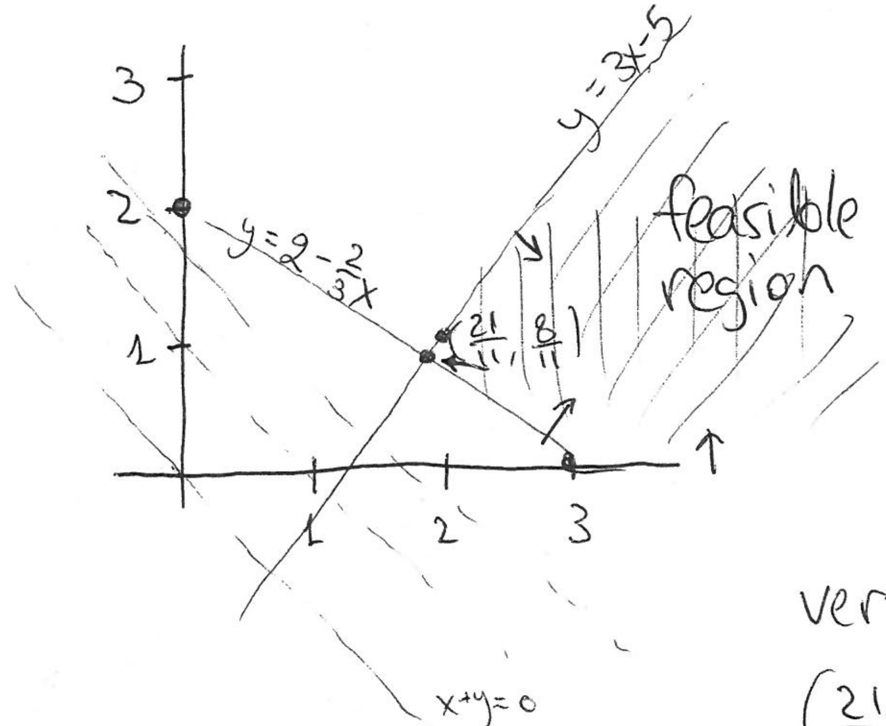
$$3x - y \geq 5 \quad \longleftrightarrow \quad y \leq 3x-5$$

$$x, y \geq 0$$

In general: min/max a linear objective subject to linear constraints.

$$\min \sum c_i x_i \qquad\qquad \max \sum c_i x_i$$

$$\text{s.t. } Ax \geq b \qquad\qquad \text{s.t. } Ax \leq b$$



vertices → value

$\left(\frac{21}{11}, \frac{8}{11}\right) \rightarrow \frac{29}{11}$

$(3, 0) \rightarrow 3$

Important Observations

- The feasible region is a convex polytope.
- The optimum is obtained in a vertex of the polytope.

Linear Programs Can be Solved Efficiently!

<u>Known algorithms</u>

- <u>Simplex</u> walks from vertex to vertex in direction $\vec{c}$.
    - Very useful in practice.
    - Takes exponential time in worst-case.
    - There are poly-time versions from the 2000's.
- <u>Ellipsoid</u> guarantee that opt is in ellipsoid, keep shrinking ellipsoid.
    - First poly-time algo, not used in practice.
- <u>Interior Point Method</u> random walk in polytope guided by $\vec{c}$.
    - Poly time & active area of research.

<u>However</u>, adding the restriction that the solution is over integers or $\{0,1\}$ makes the problem NP-hard!!

<u>Still</u>, LP generalizes essentially every problem we saw!

<u>Max Flow</u>

$$\max \sum_{v} f(s,v)$$

$$\text{s.t.} \quad f(u,v) = -f(v,u) \quad \forall u,v \in V$$

$$\sum_{u} f(u,v) = 0 \quad \forall v \neq s,t \in V$$

$$f(u,v) \leq c(u,v) \quad \forall u,v \in V$$

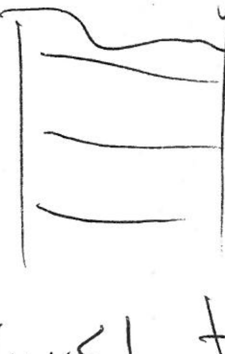
<u>Max Perfect Matching</u>

The vertices of this polytope are all integral!

$$\max \sum_{u,v} w_{u,v} x_{u,v}$$

$$\sum_{u:(u,v)\in E} x_{u,v} = 1 \quad \forall v \in V$$

$$\sum_{v:(u,v)\in E} x_{u,v} = 1 \quad \forall u \in V$$



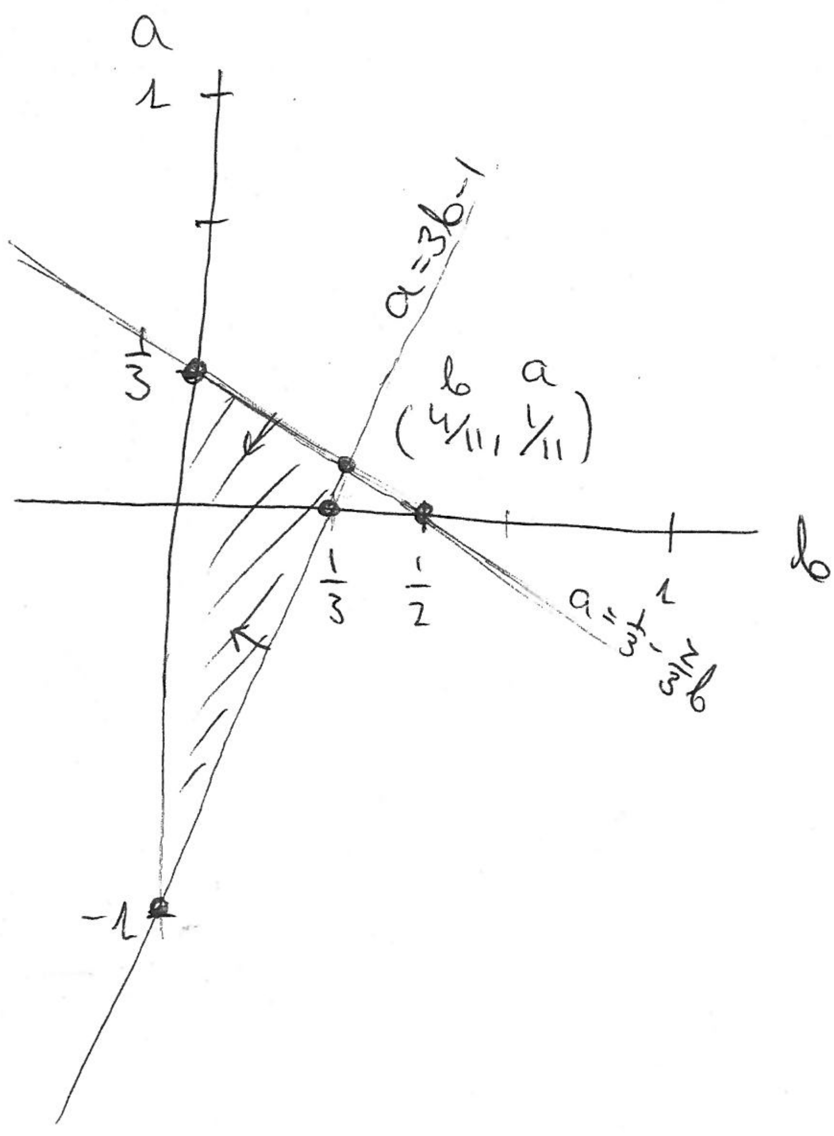
$$0 \leq x_{u,v} \leq 1 \quad \forall u,v$$

This problem is also a linear program denoted P^*:

$$\max \; 5a + 6b$$

$$\text{s.t.} \quad 3a + 2b \leq 1 \iff a \leq \frac{1}{3} - \frac{2}{3}b$$

$$3b - a \leq 1 \iff a \geq 3b - 1$$



$$6 \cdot \frac{4}{11} + 5 \cdot \frac{1}{11} = \frac{29}{11}$$

$$\|$$

$$\frac{1}{11}(3x - y) + \frac{4}{11}(2x + 3y)$$

$$\|$$

$$x + y$$

In general

primal	dual
min cx	max by
s.t. $Ax \geq b$	s.t $A^T y \leq c$

Weak Duality

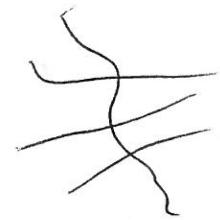
value of feasible solution to primal $\geq$ value of feasible solution to dual

Strong Duality

opt of primal (if well-defined) $=$ opt of dual

<u>Duality</u> Suppose that someone gives you a solution of an LP and claims it's optimal. Can you certify?

E.g. for flow can show a cut that reached capacity.



Recall LP from before:

$$P: \quad \min \; x+y$$
$$\text{s.t.} \quad 3x-y \geq 5$$
$$2x+3y \geq 6$$
$$x,y \geq 0$$

I can prove that opt ≥ 2:

$$x+y \geq \frac{1}{3}(2x+3y) \geq 2$$

Can we do better? Want a,b s.t.

$$x+y \geq b(2x+3y) + a(3x-y)$$

& $5a+6b$ is as large as possible.