WEIGHTED SET COVER

INPUT: Set U s.t. $|U| = n$

$S_1, S_2, \ldots, S_m \subseteq U$

weights $w_1, \ldots, w_m$.

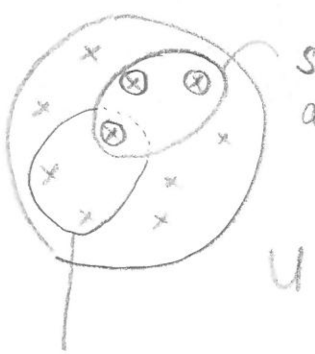
OUTPUT: Find a minimum weight Set cover.

Key lemma:

$\forall i$, total charge on elements of $S_i \leq w_i \left(\sum_{i=1}^{|S_i|} \frac{1}{i} \right)$

$$\leq w_i \left(\sum_{i=1}^{n} \frac{1}{i} \right)$$

— ∘ —



S_i covers 3 elements and charges each '⊗' $\frac{w_i}{3}$

S_j covers $\underline{\underline{2}}$ new elements and charges each element $w_j/2$.

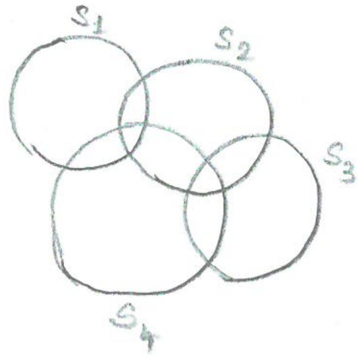
— ∘ —

Recall that the <u>total of all charges</u> = <u>weight of greedy set cover</u>.

we would like to relate the weight of the greedy set cover to $w^* =$ weight of an optimal set cover.

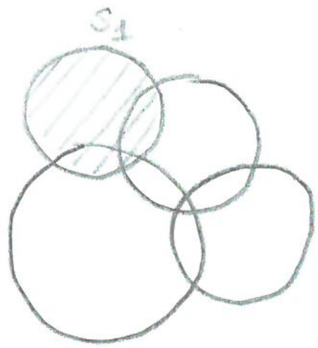
We would like to show :

$$\text{weight of greedy set cover} \leq \omega^* \underbrace{\left(\sum_{i=1}^{n} \frac{1}{i}\right)}_{H_n} \approx \omega^* \log(n).$$



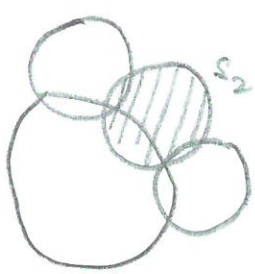
Suppose $U = S_1 \cup S_2 \cup S_3 \cup S_4$ and $S_1, \ldots, S_4$ is the optimal set cover.

step 1



Total charge we pick up on elements of $S_1 \leq \omega_1 H_n$

step 2



Total charge we pick up on elements of $S_2 \leq \omega_2 H_n$

so on

Adding up our inequalities, we get

$$\sum \text{charge}(S_i) \leq \omega^* H_n$$

Last time, we showed that there is an example on which our algorithm achieves this $\log(n)$ approximation ratio.

WEIGHTED VERTEX COVER (SECTIONS 11.4, 11.6)

VERTEX COVER IS A SPECIAL CASE OF THE SET COVER PROBLEM.

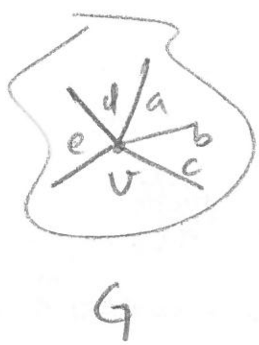
INPUT : $G = (V, E, \omega : E \longrightarrow \mathbb{R}_{\geq 0})$

OUTPUT : A vertex cover C s.t.

$$\sum_{v \in C} \omega_v$$

is minimized.

How to view this as a set cover problem — view the vertices as sets!



$\dashrightarrow \quad v = \{a, b, c, d, e\}$

easy to see (via reducing to the weighted set cover problem) we can get an approximation ratio of size H_d where d is max degree.

$\sim \log d. \leq \log(|V|-1)$. $(\max \deg \leq |V|-1)$.

We can get a 2-approx for weighted vertex cover.

Algorithm presented in section 11.4 is again based on 'charges'.

- We will maintain a non-negative charge c_e for each edge e.
- Initialize $c_e := 0 \quad \forall e \in E$
- Maintain invariant that $\forall u \in V \sum_{(u,v)\in E} c_{(u,v)} \leq w_u$. $(\star)$
- At any stage, the charges induce a coloring of the vertices. If inequality for a vertex is tight (i.e. an equality), then color u red else color u blue

inequality tight $\longleftrightarrow$ RED

not tight $\longleftrightarrow$ BLUE

Algorithm:

while $\exists (u,v) \in E$ such that u, v are BLUE

increase $c_{(u,v)}$ until at least one of u, v becomes RED

Note $(\star)$ is preserved! work this out.

Return the set of RED vertices.

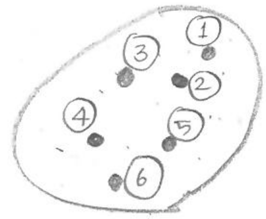
Let U^* be an optimal vertex cover with $w(U^*) = w^*$.

Let U be the output of our algorithm.

want to show $w(U) \leq 2w^*$.

<u>claim 1</u> $w(U^*) \geq \sum_{e \in E} c_e$ = SUM OF ALL EDGE CHARGES

<u>Proof idea</u> :



G

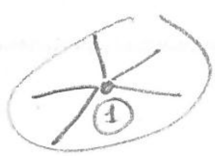
$\boxed{\cdot}$ ← vertex

$\boxed{\bullet}$ ← elements of U^*

Maintain $w_{①} + w_{②} + w_{③} + \dots$

At the same time, pick up charges

This happens because the invariant is maintained for the charges



$w_{①}$ $\geq$ TOTAL CHARGE ON EDGES ADJACENT TO ①

$\therefore$ $w_{ⓘ} \geq$ total remaining charge on edges adjacent to ⓘ

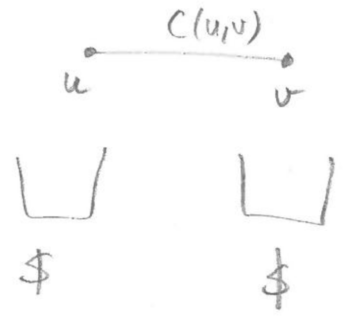
$$\begin{array}{ccc} \sum w_{ⓘ} & \geq & \sum \text{total} \dots \\ \| & & \| \\ w(U^*) & \geq & \sum_{e \in E} c_e \end{array}$$

claim 2 : $w(U) \leqslant 2 \sum_{e \in E} c_e$

$= 2 \times \{\text{TOTAL OF ALL CHARGES}\}$

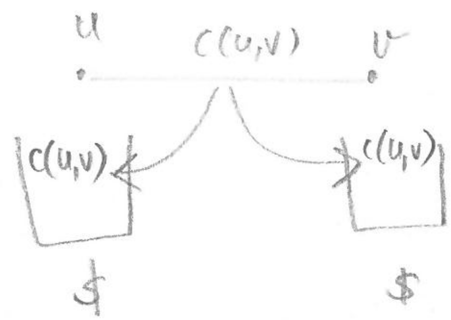
Set of RED vertices

Proof idea :



Bag of \$ on each vertex

at the end of our algorithm



put $c(u,v)$ \$ into each bag at u & at v

observe that total \$ in bag at u $= w_u$ if $u \in U$.

[this is because at the red vertices, the inequality is tight.]

∴ Total \$ distributed to vertices in U $= \sum_{u \in U} w_u = W(U)$

$\therefore$ Also 2 × TOTAL OF ALL CHARGES

$\geq$ TOTAL \$ DISTRIBUTED

$$2 \sum_{e \in E} c_e \geq \text{total \$ distributed to vertices in } U = \sum_{u \in U} w_u = W(U)$$

The result follows from claim 1 & claim 2.

11.6 Another 2-approx for weighted vertex cover.

"LP-rounding" technique.

(1) WRITE WEIGHTED. V.C. PROBLEM AS AN INTEGER LINEAR PROGRAM

$$\min \sum x_u w_u$$

w_u ← GIVEN

x_u ← VARIABLE

s.t.

$$x_u \in \{0,1\}$$

edge-many inequalities $\left\{ \; x_u + x_v \geq 1 \quad \forall (u,v) \in E \right.$

(2) "RELAX" INTEGRALITY CONSTRAINTS TO GET AN LP

replace $x_u \in \{0,1\}$ with $x_u \geq 0 \quad \forall u \in U$

After you relax the ILP you get an LP that can be solved in polynomial time.
solve to get an optimal fractional solution.

3 "ROUND" Your fractional solution to get a solution in your integral feasible space.

i.e. x_i's are fractional

"round" to x_i's 0,1.

if $x_u \geq 1/2$ —— add u to the vertex cover

$x_u < 1/2$ —— do not add u back.

When you do this, the objective fn. goes up by at most 2×.
because you replaced all $x_u \geq 1/2$ by 1.
and just got rid of all $x_u < 1/2$.