Problem Set 3

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1 Problem 3-1: Updating MST after edge addition.

Input: $G(V, E)$ weighted graph w/non-negative weights. $T = MST(G)\,, T = (V, E')\,, E' \subseteq E$. $e \in V \times V - E$. $e = (u, v)$
Output: MST of Graph $G' = (V, E \bigcup \{e\})$
Explanation of Algorithm: For this algorithm, we must use the cycle lemma. Let $e = (u, v)$ where $u, v \in V$. There exists a path from u to v in T. Inclusion of edge e results in a cycle in T. So the algorithm finds a cycle in T and removes the highest weighted edge in the cycle.

1.1 Algorithm

Please refer Algorithm 1 MST Edge addition.

1.2 Complexity

$m = |E'|$ (# of edges)
$n = |V|$ (# of nodes)
$m \leq n - 1$
Line 7 in Algorithm 1 loops m times. Within the loop, Find-Set is called, which has a complexity of $\log n$. Thus, the overall complexity is $O(n \log n)$.

Algorithm 1 MST Edge addition

procedure EDGE ADDITION(G, T, e) ▷ $G(V, E)$ weighted graph w/non-negative weights. $T = MST(G), T = (V, E'), E' \subseteq E$. $e \in V \times V - E$. $e = (u, v)$

    Make-Set(u)

    Make-Set(v)

    $U \leftarrow$ Union(u, v) ▷ U is the representative of the set after the Union operation

    $e' \leftarrow e$

    $w' \leftarrow w(e)$ ▷ Initialize w' to the weight of edge $e = (u, v)$

    for all edges $f = (s, t) \in E'$ such that Find(s) or Find$(t) = U$ **do**

        if Find-Set$(s) \notin U$ **then**

            $U \leftarrow$Union$(U$,$s)$ ▷ Update representative of the set

        else

            $U \leftarrow$Union$(U$,$t)$ ▷ Update representative of the set

        end if

        if $w' < w((s, t))$ **then** ▷ Found a heaver edge

            $e' \leftarrow (s, t)$

            $w' \leftarrow w((s, t))$

        end if

    end for

    if $e' \neq e$ **then** ▷ An existing edge in the graph G is heavier than e

        $E'' \leftarrow E' \bigcup e - e'$

        $T' \leftarrow (V, E'')$

    else

        $T' \leftarrow T$

    end if

    return T'

end procedure

1.3 Correctness

Lemma 1.1. *The addition of the edge e to T results in a cycle. The edge e is added to the MST if there exists another edge e' in the cycle whose weight is greater than the weight of e.*

Proof. In T, which is an MST of G, there is a path from vertex u to v. Therefore, the addition of the edge e to T will result in a cycle since there are multiple paths from vertex u to v.
Assume an edge e' such that the weight of e' is greater than the weight of e and e' is part of the cycle.
Consider $T' \triangleq T - \{e'\} \bigcup \{e\}$
Claim 1: Weight of T' only smaller or equal to weight of T because e' is the heaviest edge.
Claim 2: T' spans all vertices.
Thus, the edge e' can be removed and replaced by e, and we get a Tree T that has a smaller total sum of weights. If no such e' exists, e is the heaviest edge in the cycle and is not added to T. □