Network Flow

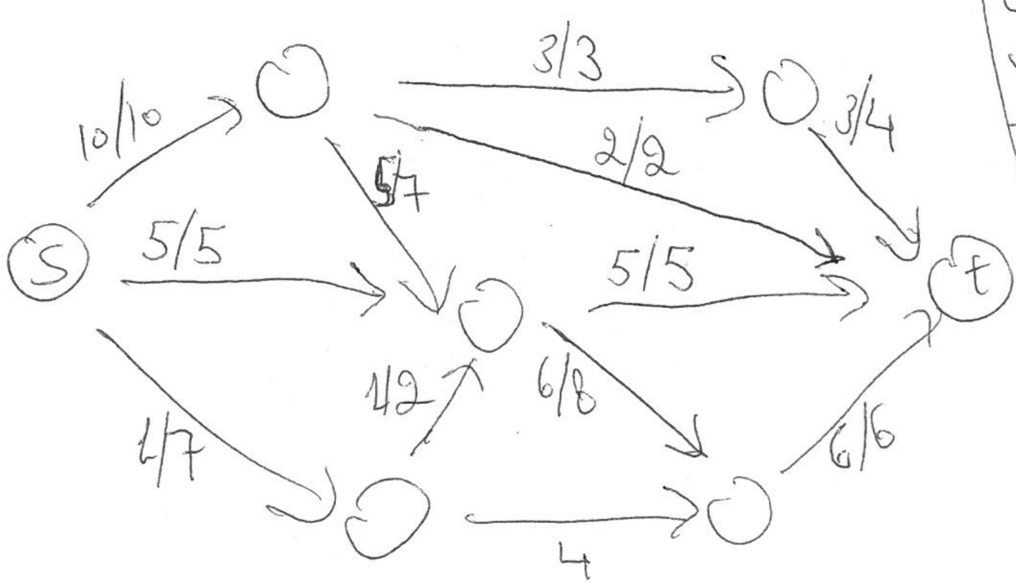
Input: Directed graph $G=(V,E)$
with capacities $c: E \to \mathbb{R}^+$
source $s \in V$
target $t \in V$

Output: Max flow $f: \cancel{V \times V} \to \mathbb{R}^+$

* $0 \le f(e) \le c(e) \quad \forall e \in E$ — Capacity
  $f(u,v)=0 \quad (u,v) \notin E$
* $\sum'_{u \xrightarrow[e]{} v} f(e) = \sum'_{v \xrightarrow{e} u} f(e) \quad \forall v \in V - \{s,t\}$ — Conservation

Maximize $val(f) = \sum'_{s \xrightarrow[e]{}} f(e)$

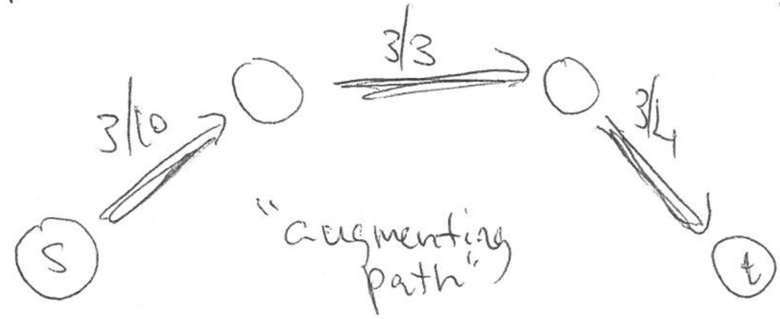
generalizes: assignment, matching, transportation problems



Ford-Fulkerson Algorithm

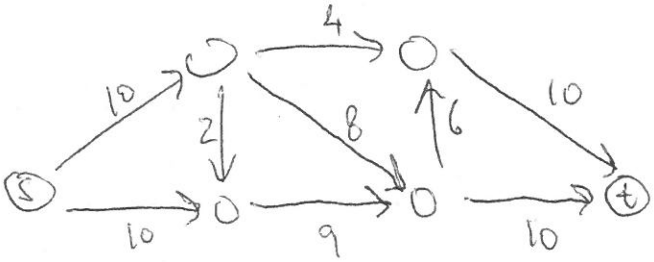
Repeat:

* Pick a path from s to t, flow on it as much as possible.
* Update the network accordingly.

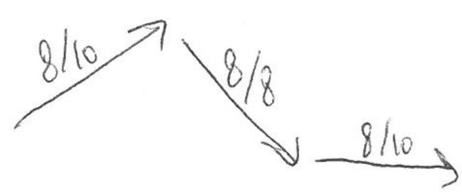


at most 3 units can flow on this path.

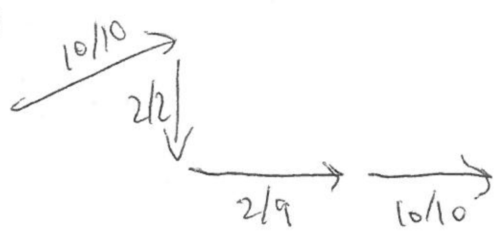
Does it work?



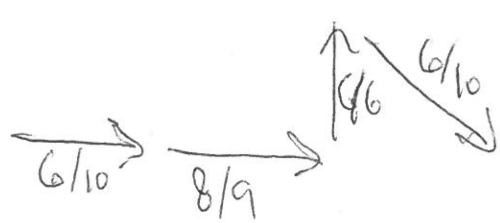
I



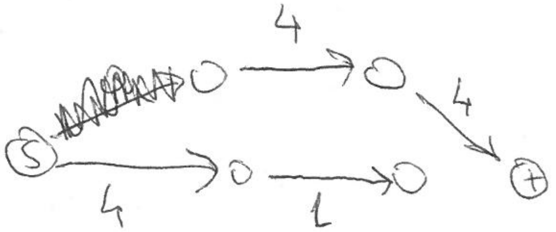
II



III

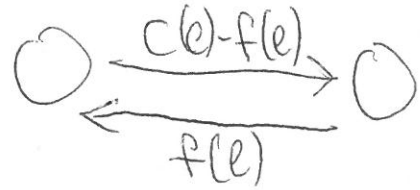


Remaining:



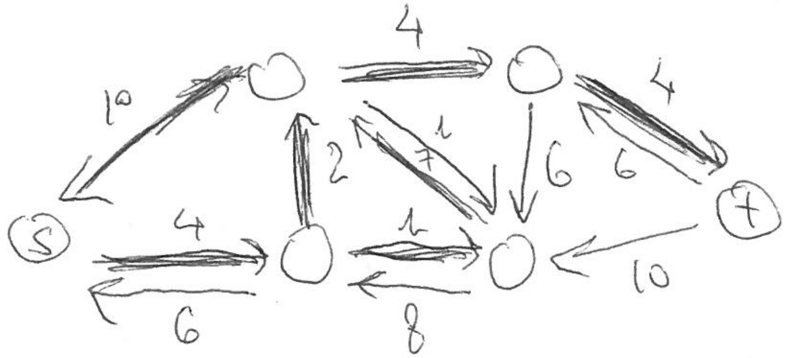
can't continue but only flew 16

Therefore, when we flow f(e) on edge e
we update the network



"residual network"

In the example with flow 16, the residual
network will be:



So it's possible to flow 3 more!

Every iteration takes linear time in the size of
the graph.

How many iteration can there be?

If $|f^*|$ is the size of max flow, then at most $|f^*|$.

- Each time the value increases by at least 1.

Maximality of flow

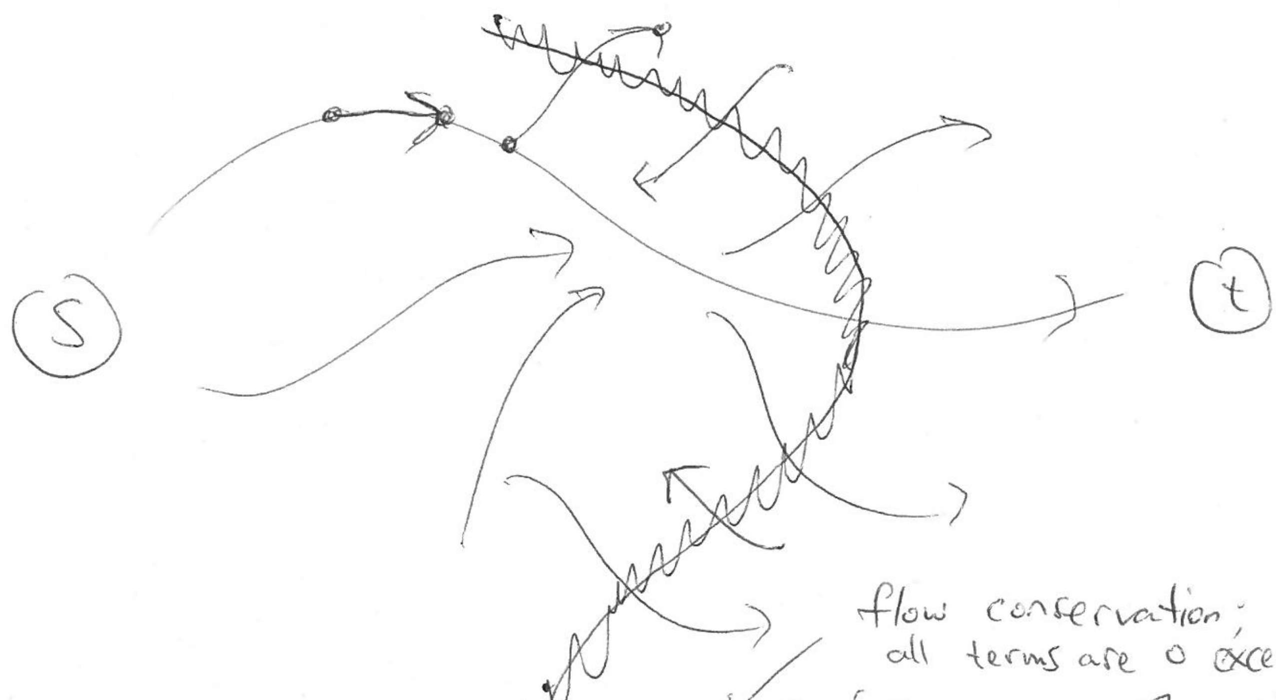
Why does getting stuck (i.e. no more augmenting paths in residual network) mean we found max flow?

Flow value lemma

Let f be a flow

$(C, V-C)$ be an s-t cut, $s \in C$, $t \in V-C$.

Then, net flow across $(C, V-C)$, i.e., $\sum\limits_{\substack{e=(u,v)\\ u\in C\\ v\in V-C}} f(e) - \sum\limits_{\substack{e=(v,u)\\ v\in V-C\\ u\in C}} f(e)$, equals the value of f.



flow conservation; all terms are 0 except for $u=s$.

Pf value of $f = \sum\limits_{e=(s,v)} f(e) = \sum\limits_{u\in C}\left(\sum\limits_{e=(u,v)} f(e) - \sum\limits_{e=(v,u)} f(e)\right)$

if $u \in C$ & $v \in C$ we get a cancelation

$= \sum\limits_{\substack{e=(u,v)\\ u\in C\\ v\in V-C}} f(e) - \sum\limits_{\substack{e=(v,u)\\ u\in C\\ v\in V-C}} f(e)$

$\square$

Weak Duality

Let f be a flow.

$(C, V-C)$ an s-t cut, $s \in C$, $t \in V-C$.

Then, value of f is at most the capacity of $(C, V-C)$, i.e. $\sum_{\substack{e=(u,v) \\ u \in C \\ v \in V-C}} c(e)$.

Pf value of f $\overset{\text{flow value lem}}{=}$ $\sum_{\substack{e=(u,v) \\ u \in C \\ v \in V-C}} f(e) - \sum_{\substack{e=(v,u) \\ u \in C \\ v \in V-C}} f(e)$

$$\leq \sum_{\substack{e=(u,v) \\ u \in C \\ v \in V-C}} f(e) \underset{\text{capacity}}{\leq} \sum_{\substack{e=(u,v) \\ u \in C \\ v \in V-C}} c(e) = \text{capacity of } (C, V-C).$$

$\blacksquare$

Min Cut - Max Flow Thm

Value of max flow = Capacity of Min cut.

In fact: The following are equivalent:

(1) There is an s-t cut $(C, V-C)$ whose capacity is the value of f.

(2) f is max flow.

(3) There are no augmenting paths with respect to f.

Pf

$\boxed{(1) \Rightarrow (2)}$ Suppose $(C, V-C)$ is an s-t cut of capacity value of f.

For any flow f', its value is at most this capacity (weak duality) which is the value of f.

$\Rightarrow$ f is max flow.

$\boxed{(2) \Rightarrow (3)}$ Assume there is an augmenting path wrt f. Then f can be improved and is not max

$\boxed{(3) \Rightarrow (1)}$

Let f be a flow with no augmenting paths.
Let C = nodes reachable from s in residual network. Note that $s \in C$, $t \in V-C$.

$$\text{value of } f \underset{\substack{\text{flow}\\ \text{val}\\ \text{lem}}}{=} \sum_{\substack{e=(u,v)\\ u \in C\\ v \in V-C}} f(e) - \sum_{\substack{e=(v,u)\\ u \in C\\ v \in V-C}} f(e) = \sum_{\substack{e=(u,v)\\ u \in C\\ v \in V-C}} c(e) = \text{capacity of } (C, V-C).$$