Problem Set 6

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Problem 6-2: Reductions and algorithms

Solution 6.2a

No, it is not possible to determine the existence of a vertex cover of size $|V| - 1000$ in polynomial time.

Reduction

In an independent set of size, $|V|/\log^3|V|$ has the vertex cover of size $|V| - |V|/\log^3|V|$. If $|V|/\log^3|V| \leq 1000$, then we remove $1000 - |V|/\log^3|V|$ edges, thus we have a vertex cover of size $|V| - 1000$.

Let Graph $G = (V, E)$ have $n = |V|$ vertices $v_1, v_2, \ldots v_n$. We create a graph $G' = (V', E')$ by removing edges such that the number of edges moved is $1000 - |V|/\log^3|V|$. Thus, G' has a vertex cover of size $|V| - 1000$. This reduction from G to G' can be done in polynomial time as it involved removing at most 1000 edges in the adjacency list matrix.

Lemma 0.1. *Vertex Cover of size ($|V| - 1000$) $\leq_p$ Independent Set*

Proof. If we have a black box to solve Independent Set, then we can decide whether G' has a vertex cover of size ($|V| - 1000$) by asking the black box whether G has an independent set of size at least $|V|/\log^3|V|$. Thus, this completes the proof.

□

Lemma 0.2. *Independent Set $\leq_p$ Vertex Cover of size ($|V| - 1000$)*

Proof. If we have a black box to solve Vertex Cover of size ($|V| - 1000$), then we can decide whether G has an independent set of size at least $|V|/\log^3|V|$ by asking the black box whether G' has a vertex cover of size at most $|V| - 1000$.

Thus, this completes the proof.

□

Solution 6.2b

We will first prove that a clique and independent set have the same complexity (that both are NP).

A clique is a subset of vertices of a graph such that there is an edge between any two vertices in a clique.

Thus, if $G = (V, E)$ is a graph, a clique is a subset S of V such that for every (u, v) in S there is an edge (u, v) in G.

We define the complement of G as $G*$. $G*$ has the /same set of vertices as G. For every edge (u, v) in G, there is no edge (u, v) in $G*$. For every edge (u, v) not in G, there is an edge (u, v) in $G*$.

Thus, the problem of finding a clique of size k in G is equivalent to finding an independent set of size k in $G*$. Assume S is such an independent set in $G*$. Thus by definition of independent set for every node u, v in S there is not edge (u, v) in $G*$. By definition of construction of $G*$, it implies that for every node u, v in S, there is an edge in G. Thus, the set S is a clique in G.

We can prove that

Lemma 0.3. *Clique* $\leq_p$ *Independent Set*

Proof. If we have a black box to solve Independent Set, then we can decide whether G has a clique of size at least k by asking the black box whether G* has an independent set of size at least k. Thus, this completes the proof.

□

Lemma 0.4. *Independent Set* $\leq_p$ *Clique*

Proof. If we have a black box to solve Clique , then we can decide whether G has an independent set of size at least k by asking the black box whether G* has a clique of size at least k. Thus, this completes the proof.

□

No, it is not possible to determine the existence of a clique of size $|V|/\log^3|V|+1000$ in polynomial time.

Reduction:

Let Graph G has an independent set of size $|V|/\log^3|V|$. Thus, the complement graph $G*$ will have a clique of size $|V|/\log^3|V|$.
Now we reduce graph G by selecting 1000 vertices $v_1, \ldots v_1 000$ in a set S such that there exists an edge between any two vertices in the set S. Thus, the complement graph $G*$ will have a clique of size $|V|/\log^3|V| + 1000$. This concludes the reduction.