Def. flow network

A flow network is a digraph $G = (V, E)$ with source $s \in V$, sink $t \in V$ and non-negative integer capacity $c(e)$ for each $e \in E$.

- no parallel edges
- no edge enters s
- no edge leaves t

Def. s-t cut:

A s-t cut is a partition (A, B) of the vertices with $s \in A$ and $t \in B$.

Def capacity of a cut:

The capacity of an s-t-cut (A, B) sum of the capacities of the edges from A to B.

$$cap(A, B) = \sum_{e \text{ out of } A} c(e)$$

Def s-t flow:

An s-t flow is a function from the edges to the integers satisfying the following

- $\forall e \in E \quad : \quad 0 \leq f(e) \leq c(e)$
- $\forall v \in V - \{s,t\} : \sum_{e \text{ into } v} f(e) = \sum_{e \text{ out of } v} f(e)$

Def value of a flow

The value of a flow f is

$$val(f) = \sum_{e \text{ out of } s} f(e)$$

BIPARTITE MATCHING

Def: Given an undirected graph G = (V, E of a subset of edges $M \subseteq E$ is a matching if each node appears in at most one edge in M.

Max matching: Given a graph, find a max cardinality matching.

Def. A graph G is bipartite if the node can be partitioned into two subsets L and R s.t. every edge connects a node in L to a node in R

Bipartite matching: Given a bipartite graph $G = (L \cup R, E)$ find a max-matching. (by reducing to max-flow.)

Soln:

① Consider a flow network G' s.t

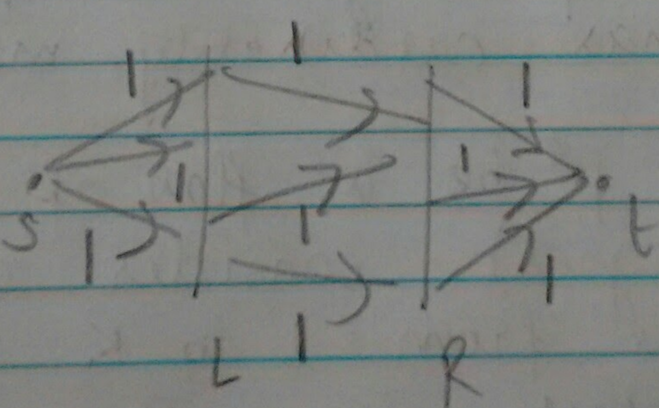
$$G' = (L \cup R \cup \{s,t\}, E')$$

where ⓐ all edges are directed from L to R and match the edges in G.

ⓑ There are edges from s to every vertex in L

ⓒ There are edges to t from every vertex in R

ⓓ $c(e) = 1 \quad \forall e \in E'$



Thm: Max cardinality matching
= value of integer max flow

Pf

① max cardinality matching
$\leq$ value of integer max flow

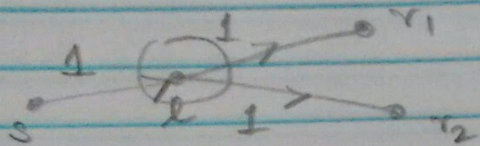
Pf: Let M^* be a max. cardinality matching of size m^*.
consider the flow f that sends one unit of flow along each of the matched edges.
This creates a flow of value m^*

$\therefore \exists$ a flow f which has value m^*

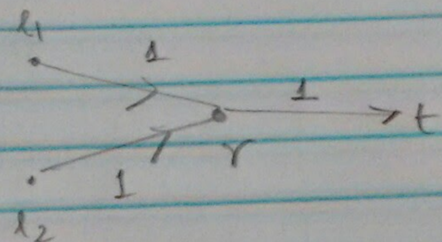
$\therefore$ max card. matching
= value of f
$\leq$ max value of flow

② value of integer max flow
$\leq$ max cardinality matching

Pf. let f^* be a flow of max value (integral). consider all edges from L to R involved in this flow. These form a matching. Suppose they did not, that would mean that there was a vertex that was a part of two edges. This is not possible because of the definition of a flow and the fact that all the capacities in this network are 1.



not possible, because $\sum$ flow in $=$ $\sum$ flow out



not possible for the same reason.

$\therefore$ The flow gives you a matching on the bipartite graph of size $=$ value of the flow

$\therefore$ consider f^* the flow of maximum value; v^*
$\exists$ a matching M s.t. size of $M = v^*$

$\therefore$ max size of matching $\geq$ size of $M = v^* =$ max value of fl

① & ② give us what we want