Introduction to Project Topics

Kaicong Sun

Organizational Issues

- Projects can be done in groups of two or three.
- Submit the codes and report to Kaicong.Sun@ipvs.uni-stuttgart.de
  - Template in ILias
  - Workflow
  - Codes with comments
  - If possible, compare with CPU implemetation
- Written report: 6-12 pages
- Submission deadline: 31.03.2019

Topics

- Implementation of the Norm for Modulation Transfer Function (MTF) measurement: ASTM-E 1695-95
- Implementation of (Alternating Direction Method of Multipliers) ADMM optimizor for given energy function using Newton's methd to solve nonconvex subproblem
- Implementation of (Alternating Direction Method of Multipliers) ADMM optimizor for given energy function using Limited-memory BFGS (L-BFGS) method to solve nonconvex subproblem
- Implementation of (Alternating Direction Method of Multipliers) ADMM optimizor for given energy function using ADAM method to solve nonconvex subproblem
- 2D Fourier Transform
- Canny edge detector

Topics

Modulation Transfer Function (MTF) Based on ASTM-E1695-95

Modulation transfer function (MTF) is widely used as a metric for spatial resolution assessment. This project is aimed to implement the MTF measurement of the computed tomography (CT) system based on the norm ASTM-E 1695-95 [1].
CT images of the test object, i.e., a phantom disk made of Aluminium with diameter 20mm, are given. Specifically,

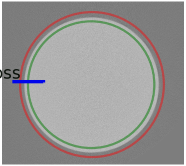
- Three test CT images are given with Input4, Input7, Input10.
- In the file Readme you can find the pixelsize of each CT image, which will be needed when you calculate MTF.
- Compare your MTF curves with the corresponding given MTF curves, noticing the setup parameters: binsize, search distance and fit point count.

Modulation Transfer Function (MTF) Based on ASTM-E1695-95

- Calculate the centre of the phantom in the CT slice.
- Choose inner and outer radii with respect to the centre of circle that bracket the edge.
- Segregate the region between inner and outer radii with bins sized to a small fraction of one pixel.
- Averaging the value of bins according to the distance to the centre.
- Smoothing the averaged curve crossing the edge and do a piece-wise, least-squares cubic fit (ERF).
- Calculate the first derivative of the curve ERF to get PSF.
- Calculate the Fourier Transform of the PSF and normalize the maxima to one (MTF).

Modulation Transfer Function (MTF) Based on ASTM-E1695-95

Profile across
Inner and
outer radii



1. Split 1 pixel to subpixels (according to table in ASTM)
2. Calculate the center and radius of the circle
3. Define the region with outer and inner radii
4. Average the all the profiles in the region
5. Piece-weise least-squares cubic fit
6. Got the white curve beneath

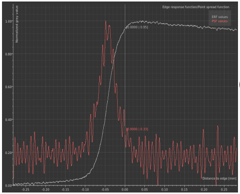
Edge Spread Function (ESF)

First derivative

Point Spread Function (PSF)

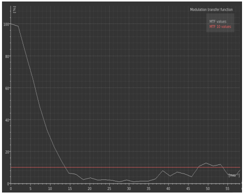
1D Fourier Transform

Modulation Transfer Function (MTF)



Source from tool: VGStudio

1D FFT
(padding)



Source from tool: VGStudio

Alternating Direction Method of Multipliers (ADMM)

- An optimization problem solover with good robustness of method of multipliers
- Support decomposition

ADMM (Alternating Direction Method of Multipliers) deals with the following problem [2].

$$\begin{aligned} \textit{minimize} \quad & f(x) + g(y) \\ \textit{subject to} \quad & Ax + By = c \end{aligned} \tag{1}$$

Alternating Direction Method of Multipliers (ADMM)

Here, f, g are assumed convex. An auxiliary variable (Lagrange multiplier) z is introduced to form an function $L_\rho(x, y, z)$

$$L_\rho(x, y, z) = f(x) + g(y) + z^T(Ax + By - c) + \frac{\rho}{2}\|Ax + By - c\|_2^2 \quad (2)$$

where ρ is a tunning parameter. Then, we can iteratively solve for x, y, z in three seperate steps (subproblems):

$$\begin{aligned} x^{k+1} &= \arg\min_x L_\rho(x, y^k, z^k) \\ y^{k+1} &= \arg\min_y L_\rho(x^{k+1}, y, z^k) \\ z^{k+1} &= z^k + \rho(Ax^{k+1} + By^{k+1} - c) \end{aligned} \quad (3)$$

Alternating Direction Method of Multipliers (ADMM)

Proximal operator of a function $f(x)$ is defined as:

$$prox_f(v) = \arg\min_x (f(x) + \frac{1}{2}\|x - v\|_2^2) \tag{4}$$

- Proximal operator is defined in a certain format [3]
- Proximal operator can be solved analytically which accerlates the computation speed

The above mentioned three steps of ADMM can benefit from the proximal operator in terms of computation complexity if a reasonable decomposition of your energy function can be determined so that any step of the three could match the format of proximal operator.

Energy Function

$$J = \min_{x} \frac{1}{2}(\sum_{i=1}^{M} ||y_i - A_i I_0 \exp(-x)||^2_{W_i} + < \log(B_i I_0 \exp(-x) + \sigma_i^2), 1 >)$$
$$+ \beta \sum_{p=-w}^{w} \sum_{q=-w}^{w} \gamma(p,q) \parallel x - S_x^p S_y^q x \parallel_1 + \chi_C(x), \quad (5)$$

where $< \cdot >$ indicates a pointwise multiplication of two vectors.
I_0 is a constant. A_i and B_i are constant matrices. σ_i is constant vector.
S_x, S_y are shift operators along x- and y-axis.
w is a constant for the window size and β is constant weight.
y_i are the input images. x is the expected output image.
M expresses the number of inputs y_i, we make $M = 4$.
W_i is a diagonal weight matrix W_i and can be expressed as

$$W_i = \text{diag}\{\frac{1}{B_{ik} I_0 \exp(-x) + \sigma_{ik}^2}\}.$$

Energy Function

$$J = \min_x \frac{1}{2}\left(\sum_{i=1}^{M} ||y_i - A_i I_0 \exp(-x)||^2_{W_i} + < \log(B_i I_0 \exp(-x) + \sigma_i^2), 1 >\right)$$

$$+ \beta \sum_{p=-w}^{w} \sum_{q=-w}^{w} \gamma(p,q) \parallel X - S_x^p S_y^q X \parallel_1 + \chi_C(x), \quad (6)$$

Here, $\| \cdot \|_1$ indicates the Euclidean l-1 norm.
We can define $\gamma(p,q) = \alpha^{|p|+|q|}$ where α is a constant.
$\mathcal{X}_B(X)$ is the indicator function of the convex set B which constrains the nonnegativity of the reconstructed X with
$C = \{x : x_K \geq 0, \forall K \in \{1, \ldots, N\}\}$ and

$$\mathcal{X}_{\mathcal{C}}(x) = \begin{cases} 0, & x \subseteq C \\ +\infty, & x \subsetneq C. \end{cases} \quad (7)$$

Energy Function in ADMM

The energy function J can be decomposed into multiple f_i as follows:

$$
\begin{aligned}
&f_i(x_i) := \frac{1}{2}||y_i - A_i I_0 \exp(-x_i)||^2_{W_i} + < \log(B_i I_0 \exp(-x_i) + \sigma_i^2), 1 >) \\
&\text{for } i = 1, \ldots, M \\
&f_i(x_i) := \beta\gamma(p,q)||x_i - S_x^p S_y^q x_i||_1 \text{ for } i = M+1, \ldots, M + w^2 \\
&f_i(x_i) := \chi_C(x_i) \text{ for } i = M + w^2 + 1
\end{aligned}
\tag{8}
$$

Note that

$$< \log(B_i I_0 \exp(-x_i) + \sigma_i^2), 1 > = \sum_{k=1}^{n} \log(I_0 < [B_i]_k, \exp(-x_i) > + [\sigma_i]_k^2).$$

Energy Function in ADMM

The augmented Lagrangian for the seperated energy function is then

$$\mathcal{L}(x, z, p) = \sum_{i=1}^{M+w^2+1} f_i(x_i) + \sum_{i=1}^{M+w^2+1} < p_i, x_i - z > + \frac{1}{2} \sum_{i=1}^{M+w^2+1} ||x_i - z||^2$$

with the concensus constrain $x_i = z$ for $i \in \{1, \cdots, M + w^2 + 1\}$. The energy function can then be solved seperately by ADMM in the standard way:

$$\begin{aligned} x_i^{k+1} &= \arg\min_{x_i} f_i(x_i) + < p_i^k, x_i - z^k > + \frac{\rho_i}{2} ||x_i - z^k||_2^2 \\ z^{k+1} &= \arg\min_{z} - \sum_{i=1}^{M+w^2+1} < p_i^k, z > + \frac{1}{2} \sum_{i=1}^{M+w^2+1} \rho_i ||x_i^{k+1} - z||_2^2 \\ p_i^{k+1} &= p_i^k + \rho_i (x_i^{k+1} - z^{k+1}). \end{aligned} \tag{9}$$

Energy Function in ADMM

$$\begin{aligned}
x_i^{k+1} &= \arg\min_{x_i} f_i(x_i) + < p_i^k, x_i - z^k > + \frac{\rho_i}{2} ||x_i - z^k||_2^2 \\
z^{k+1} &= \arg\min_{z} - \sum_{i=1}^{M+w^2+1} < p_i^k, z > + \frac{1}{2} \sum_{i=1}^{M+w^2+1} \rho_i ||x_i^{k+1} - z||_2^2 \\
p_i^{k+1} &= p_i^k + \rho_i (x_i^{k+1} - z^{k+1}).
\end{aligned} \tag{10}$$

Admm for a global consensus can be simplified further. We have thus

$$z^{k+1} := \frac{1}{\rho} \sum_{i=1}^{M+w^2+1} \rho_i x_i^{k+1} \text{ where } \rho := \sum_{i=1}^{M+w^2+1} \rho_i.$$

Hence, we simply to:

$$\begin{aligned}
x_i^{k+1} &= \arg\min_{x_i} f_i(x_i) + < p_i^k, x_i - z^k > + \frac{\rho_i}{2} ||x_i - z^k||_2^2 \\
p_i^{k+1} &= p_i^k + \rho_i (x_i^{k+1} - \frac{1}{\rho} \sum_{i=1}^{M} \rho_i x_i^{k+1}).
\end{aligned} \tag{11}$$

Energy Function in ADMM

For some nonconvex $f_i(x_i)$, i.e., $f_{1,\cdots,M}$, one needs to use, e.g., Newton's method, quasi-Newton's method or adaptive moment estimation (ADAM) to solve it.
We compute the partial derivative of $g_i(x_i)$ and $h_i(x_i)$:

$$g_i(x_i) := \frac{1}{2}||y_i - A_i I_0 \exp(-x_i)||^2_{W(x_i)} = \frac{1}{2}\sum_{k=1}^{n} \frac{(y_{ik} - A_{ik} I_0 \exp(-x_i))^2}{B_{ik} I_0 exp(-x_i) + \sigma_{ik}^2}$$

$$\begin{aligned} \frac{\partial g_i}{\partial x_{ij}}(x_i) = \sum_{k=1}^{n} \Bigg[& \frac{(y_{ik} - A_{ik} I_0 \exp(-x_i))A_{ikj} I_0 \exp(-x_{ij})}{(B_{ik} I_0 \exp(-x_i) + \sigma_{ik}^2)} \\ & + \frac{1}{2} \frac{B_{ikj} I_0 exp(-x_{ij}) \left(y_{ik} - A_{ik} I_0 \exp(-x_{ij})\right)^2}{(B_{ik} I_0 exp(-x_i) + \sigma_{ik}^2)^2} \Bigg] \end{aligned} \tag{12}$$

Here j means the jth element in the vectorized x_i and A_{ik} indicates the $k$the row of the matrx A_i.

Energy Function in ADMM

If we formulate in matrixwise, we have:

$$\frac{\partial g_i}{\partial x_i}(x_i) = \frac{A_i^T\left(y_i - A_i I_0 \exp(-x_i)\right) I_0 \exp(-x_i)}{\left(B_i I_0 \exp(-x_i) + \sigma_i^2\right)} + \frac{1}{2}\frac{B_i I_0 exp(-x_i)\left(y_i - A_i I_0 \exp(-x_i)\right)^2}{(B_i I_0 exp(-x_i) + \sigma_i^2)^2}. \tag{13}$$

$$h_i(x_i) := < \log(B_i I_0 \exp(-x_i) + \sigma_i^2), 1 >$$

$$\frac{\partial h_i}{\partial x_{ij}}(x_i) = \sum_{k=1}^{n}\left[-\frac{1}{B_{ik} I_0 \exp(-x_i) + \sigma_{ik}^2} \cdot B_{ikj} I_0 exp(-x_{ij})\right] \tag{14}$$

In matrixwise, we have:

$$\frac{\partial h_i}{\partial x_i}(x_i) = -\frac{1}{B_i I_0 \exp(-x_i) + \sigma_i^2} \cdot B_i^T I_0 exp(-x_i) \tag{15}$$

For the other f_i, i.e., $f_{M+1,\cdots,M+w^2+1}$, one can benefit from proximal operator and soft threshold [8].

ADMM with Newton's Method

Although ADMM is originally designated for convex problem. It is robust enough even for some nonconvex problems.

For the nonconvex subproblems of ADMM, one could use Newton's method to solve them [4]. For the other subproblems, one could take advantage of proximal operator.

For Newton's method, you need to compute the gradient and inverse Hessian of the energy functon on X.

ADMM with L-BFGS

For the nonconvex subproblems of ADMM, one could use a quasi-Newton's method L-BFGS to solve them [5][6].

For the other subproblems, one could take advantage of proximal operator.

For L-BFGS, an estimation for the inverse Hessian is calculated to reduce the computation load.

ADMM with ADAM

For the nonconvex subproblems, one could use adaptive moment estimation (ADAM)to solve them [7].

- A method for stochastic optimation which combines the advantages of two stochastic gradient descent methods AdaGrad and RMSProp.
- An algorithm for first-order gradient-based optimization of stochastic objective functions, based on adaptive estimates of lower-order moments. Specifically, it updates the stepsize not only based on the average first moment (the mean) as in RMSProp, but also making use of the average of the second moments of the gradients (the uncentered variance).

2D Fast Fourier Transform

Implement a Fast Fourier Transform on the GPU. Support for non-power-of-two input sizes is optional.

Canny Edge Detector

Implement the Canny edge detector on the GPU. The program should include graphical output (e.g. using OpenGL). There also should be an option to output the various intermediate stages.

Sources

1 Standard Test Method for Measurement of Computed Tomography (CT) System Performance.

2 Alternating Direction Method of Multipliers.

3 Proximal Algorithms.

4 Newton's Method for Unconstrained Optimization.

5 Quasi-Newton methods.

6 Optimization methods.

7 ADAM: A METHOD FOR STOCHASTIC OPTIMIZATION.

8 Statistical Image Reconstruction Using Mixed Poisson-Gaussian Noise Model for X-Ray CT.