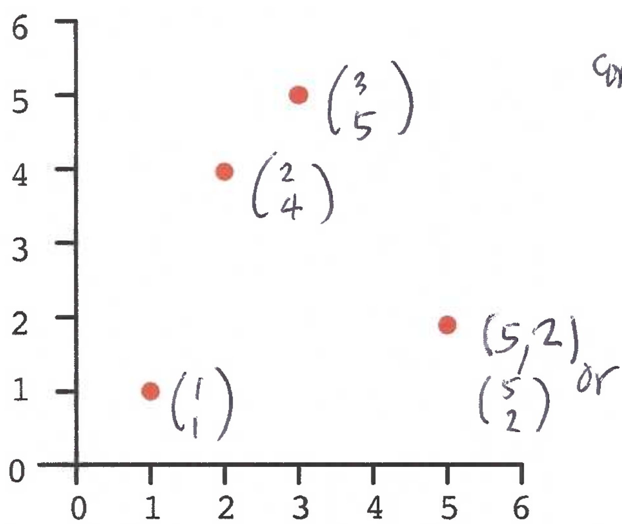
WEEK 3

Linear algebra I
Basic vector-matrix notation, and dot products

Topics we'll cover

1. Representing data using vectors and matrices
2. Vector and matrix notation
3. Taking the transpose
4. Dot products, angles, and orthogonality

Data as vectors and matrices



Data Matrix

Adopt this convention $\longrightarrow$ $\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 5 \\ 5 & 2 \end{bmatrix}$ 4×2 put data as rows

$$\begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 4 & 5 & 2 \end{bmatrix}$$

2×4

Matrix-vector notation

Vector $x \in \mathbb{R}^d$:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_d \end{pmatrix}$$

$d \times 1$

Matrix $M \in \mathbb{R}^{r \times d}$:

$$M = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1d} \\ M_{21} & M_{22} & \cdots & M_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ M_{r1} & M_{r2} & \cdots & M_{rd} \end{pmatrix} \quad \updownarrow r$$

$\longleftarrow d \longrightarrow$

M_{ij} = entry at row i, column j

$r \times d$ matrix

Transpose of vectors and matrices

→ switch rows & columns

$$x = \begin{pmatrix} 1 \\ 6 \\ 3 \\ 0 \end{pmatrix} \text{ has } \textbf{transpose} \quad x^T = \begin{pmatrix} 1 & 6 & 3 & 0 \end{pmatrix}$$

4×1 $\qquad 1 \times 4$

$$M = \begin{pmatrix} 1 & 2 & 0 & 4 \\ 3 & 9 & 1 & 6 \\ 8 & 7 & 0 & 2 \end{pmatrix} \text{ has } \textbf{transpose} \quad M^T = \begin{bmatrix} 1 & 3 & 8 \\ 2 & 9 & 7 \\ 0 & 1 & 0 \\ 4 & 6 & 2 \end{bmatrix}$$

3×4 $\qquad 4 \times 3$

- $(A^T)_{ij} = A_{ji}$
- $(A^T)^T = A$ ←

Adding and subtracting vectors and matrices

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 7 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2+1 & 1+2 \\ 0+3 & 2+4 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 6 \end{pmatrix}$$

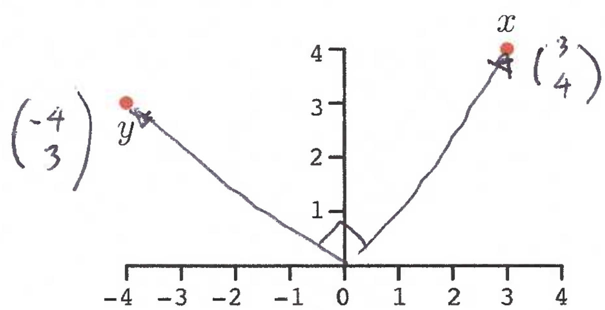
Dot product of two vectors

Dot product of vectors $x, y \in \mathbb{R}^d$:

$$x \cdot y \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d.$$

What is the dot product between these two vectors?

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 3 \end{pmatrix} = 3 \times (-4) + 4 \times 3 = 0$$



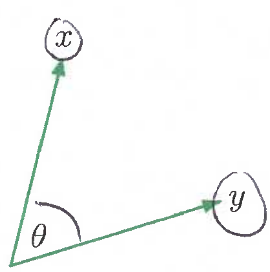
Dot products and angles

What is $x \cdot x$?

$x \cdot x = x_1^2 + x_2^2 + \cdots + x_d^2 = \|x\|^2$

Dot product of vectors $x, y \in \mathbb{R}^d$: $x \cdot y \;=\; x_1 y_1 + x_2 y_2 + \cdots + x_d y_d$.

Tells us the angle between x and y:



$$\cos\theta \;=\; \frac{x \cdot y}{\|x\| \, \|y\|}.$$

$x \cdot y = \emptyset \Rightarrow \cos\theta = \emptyset \Rightarrow \theta = 90^\circ$

x is **orthogonal** (at right angles) to y if and only if $x \cdot y = 0$

When x, y are **unit vectors** (length 1): $\cos\theta = x \cdot y$

What is $x \cdot x$?

Using Angle formula, angle between x and itself is zero $\cos\theta = 1 \Rightarrow \theta = 0^\circ$

so $\cos\emptyset = 1$, $1 = \cos\emptyset = \frac{x \cdot x}{\|x\| \, \|x\|} \Rightarrow x \cdot x = \|x\|^2$

if interested only in direction, we normalize, results in unit vector

normalize vector to unit one.

length of vector doesn't matter