Linear algebra II
Linear functions and matrix products

Topics we'll cover

$U^T \cdot V = U \cdot V$

1. Linear functions
2. Matrix-vector products
3. Matrix-matrix products

Linear and quadratic functions

In one dimension:

- Linear: $f(x) = 3x + 2$ — one variable
- Quadratic: $f(x) = 4x^2 - 2x + 6$

In higher dimension, e.g. $x = (x_1, x_2, x_3)$:

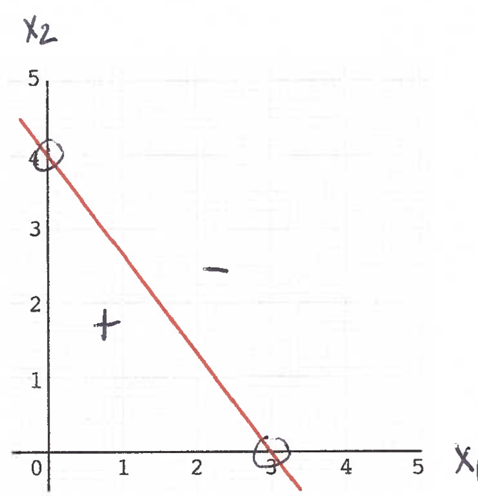
- Linear: $3x_1 - 2x_2 + x_3 + 4$ — several variables but first degree
- Quadratic: $x_1^2 - \underbrace{2x_1x_3}_{\text{pairwise product}} + 6x_2^2 + 7x_1 + 9$ — second degree including pairwise products

Linear functions and dot products

Linear separator $4x_1 + 3x_2 = 12$:

$(3,0)$: $4(3) + 3(0) = 12$, $12 = 12$

$(0,4)$: $4(0) + 3(4) = 12$, $12 = 12$



For $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, linear separators are of the form:

$$w_1x_1 + w_2x_2 + \cdots + w_dx_d = c.$$

Can write as $w \cdot x = c$, for $w = (w_1, \ldots, w_d)$.

$4x_1 + 3x_2 = 12$ can be written as $\begin{pmatrix} 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 12$

More general linear functions

no x_2 so coef is 0 and so is x_4

A linear function from $\mathbb{R}^4$ to $\mathbb{R}$: $f(x_1, x_2, x_3, x_4) = 3x_1 - 2x_3$

$$f(x_1, x_2, x_3, x_4) = (3, 0, -2, 0) \cdot (x_1, x_2, x_3, x_4)$$

A linear function from $\mathbb{R}^4$ to $\mathbb{R}^3$: $f(x_1, x_2, x_3, x_4) = (4x_1 - x_2, x_3, -x_1 + 6x_4)$

$(0,0,1,0)$ $\quad$ $(-1, 0, 0, 6)$

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}}_{x} \to \begin{pmatrix} (4,-1,0,0) \cdot x \\ (0,0,1,0) \cdot x \\ (-1,0,0,6) \cdot x \end{pmatrix} =$$

$$= \begin{bmatrix} 4 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Matrix-vector product

Product of matrix $M \in \mathbb{R}^{r \times d}$ and vector $x \in \mathbb{R}^d$: $\quad\rightarrow$ series of dot products

$$\underbrace{\overset{M}{\begin{bmatrix} - & M_1 & - \\ - & M_2 & - \\ & \vdots & \\ - & M_r & - \end{bmatrix}}}_{\substack{\longleftarrow d \longrightarrow \\ r \times d}} \underbrace{\begin{bmatrix} | \\ x \\ | \end{bmatrix}}_{d \times 1} = \underbrace{\begin{bmatrix} M_1 \cdot x \\ M_2 \cdot x \\ \vdots \\ M_r \cdot x \end{bmatrix}}_{r \times 1}$$

The identity matrix

square matrix

The $d \times d$ **identity matrix** I_d sends each $x \in \mathbb{R}^d$ to itself.

$$I_d = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$Ix = x$

Matrix-matrix product

Product of matrix $A \in \mathbb{R}^{r \times k}$ and matrix $B \in \mathbb{R}^{k \times p}$:

$$\underbrace{\overbrace{\begin{bmatrix} — & A_1 & — \\ — & A_2 & — \\ & \vdots & \\ — & A_r & — \end{bmatrix}}^{A}}_{r \times k} \underbrace{\overbrace{\begin{bmatrix} | & & | \\ B^{(1)} & \cdots & B^{(p)} \\ | & & | \end{bmatrix}}^{B}}_{k \times p} = \underbrace{\overbrace{\begin{array}{c|c} & j \\ \hline i & \begin{bmatrix} A_i \cdot B^{(j)} \end{bmatrix} \end{array}}^{AB}}_{r \times p}$$

inner dimension shall agree

ij entry of AB is simply the dot product of $A_i \cdot B^{(j)}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 1+2\cdot 0+3\cdot(-1) & 1\cdot 0+2\cdot 2+3\cdot 1 \\ 4\cdot 1+5\cdot 0+6(-1) & 4\cdot 0+5\cdot 2+6\cdot 1 \\ \underbrace{4\cdot 1+5\cdot 0+(-6)}_{4-6} & \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ -2 & 16 \end{bmatrix}$$

2×3 $\quad$ 3×2

Matrix products

If $A \in \mathbb{R}^{r\times k}$ and $B \in \mathbb{R}^{k\times p}$, then AB is an $r \times p$ matrix with (i,j) entry

$$(AB)_{ij} = (\text{dot product of } i\text{th row of } A \text{ and } j\text{th column of } B) = \sum_{\ell=1}^{k} A_{i\ell}B_{\ell j}$$

$(AB)^T$ $\quad$ $B^T \quad A^T$

$p\times r$ $\quad$ $\underbrace{p\times k \quad k\times r}_{p\times r}$

- $I_k B = B$ and $A I_k = A$
- Can check: $(AB)^T = B^T A^T$
- For two vectors $u, v \in \mathbb{R}^d$, what is $u^T v$?

$$\begin{pmatrix} u_1 \\ \vdots \\ u_d \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} = u^T v = (u_1 \dots u_d) \begin{pmatrix} v_1 \\ \vdots \\ v_d \end{pmatrix} = u \cdot v$$

Dot product $= u^T v$ between/of u & v

Some special cases

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$

For vector $x \in \mathbb{R}^d$, what are $x^T x$ and xx^T?

single number ← $x^T x = x \cdot x = \|x\|^2 \Longrightarrow x^T x = x \cdot x = \|x\|^2$

you get full $d \times d$ matrix ← $xx^T = \underset{d\times 1}{\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}} \underset{1 \times d}{(x_1 \dots x_d)} = \underset{d \times d}{\begin{pmatrix} x_i x_j \end{pmatrix}}$

cannot change order of matrix multiplication

Associative but not commutative

- Multiplying matrices is **not commutative**: in general, $AB \neq BA$

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$$

$AB \neq BA$

- But it is **associative**: $ABCD = (AB)(CD) = (A(BC))D$, etc.

You can parenthesize the expression as long as you maintain the order of the matrices

Example: if $x \in \mathbb{R}^d$ has length 2, what is $\underbrace{x^T x}_{\|x\|^2} \underbrace{x^T x}_{\|x\|^2} \underbrace{x^T x}_{\|x\|^2} \underbrace{x^T x}_{\|x\|^2}$?

$\|x\|^8 = 2^8$

length of x is 2 so 2^8