Linear algebra III
Square matrices as quadratic functions

Topics we'll cover

1. Square matrices as quadratic functions
2. Special cases of square matrices: symmetric and diagonal
3. Determinant
4. Inverse

A special case

Recall: For vector $x \in \mathbb{R}^d$, we have $x^T x = \|x\|^2$. $= x^T \underbrace{I x}_{x}$

What about $x^T M x$, for arbitrary $d \times d$ matrix M?

$$(x_1 \ldots x_d) \begin{bmatrix} & & \\ & M & \\ & & \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix}$$

$$1 \times d \qquad d \times d \qquad d \times 1$$

$$= \sum_{i,j} M_{ij} x_i x_j = \sum_{i,j} M_{ij} x_i x_j$$

What is $x^T M x$ for $M = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$?

M_{11}, M_{12}, M_{21}, M_{22}

$$(x_1\ x_2)\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = (x_1\ x_2)\begin{pmatrix} x_1 + 2x_2 \\ 3x_2 \end{pmatrix} = x_1^2 + 2x_1x_2 + 3x_2^2$$

Using formula:

$$\sum_{i,j} M_{ij} x_i x_j = M_{11} x_1 x_1 + M_{12} x_1 x_2 + M_{21} x_2 x_1 + M_{22} x_2 x_2$$
$$= x_1^2 + 2x_1x_2 + 3x_2^2$$

Square matrices are quadratic forms.

Quadratic functions

Let M be any $d \times d$ (**square**) matrix.
For $x \in \mathbb{R}^d$, the mapping $x \mapsto x^T M x$ is a **quadratic function** from $\mathbb{R}^d$ to $\mathbb{R}$:

$$\underset{\mathbb{R}^d}{x} \rightarrow x^T M x = \sum_{i,j=1}^{d} M_{ij} x_i x_j.$$

What is the quadratic function associated with $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 3 & 4 & 5 \end{pmatrix}$?

$$\underset{x}{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}} \rightarrow 1x_1^2 + 2x_2^2 + 3x_3x_1 + 4x_3x_2 + 5x_3^2$$

Write the quadratic function $f(x_1, x_2) = \overset{1}{x_1^2} + \underbrace{2}_{a+b=2} x_1 x_2 + \overset{3}{3} x_2^2$ using matrices and vectors.

$$\begin{bmatrix} \underline{1} & \underline{a} \\ \underline{b} & \underline{3} \end{bmatrix}$$

also $a+b=2$
Possible matrices

$$\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$$

Special cases of square matrices

- **Symmetric**: $M = M^T$

X this is not symmetric

$$\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}}_{\text{symmetric}}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 3 & 4 & 6 \end{pmatrix}$$

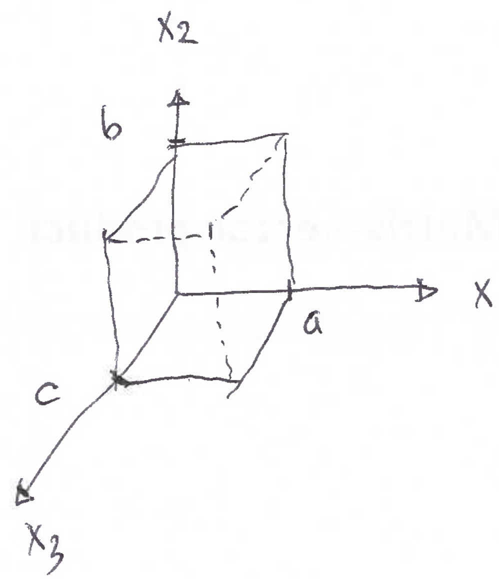
- **Diagonal**: $M = \mathrm{diag}(m_1, m_2, \ldots, m_d)$

diagonal

$$\mathrm{diag}(1, 4, 7) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$B = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

$|B| = abc$

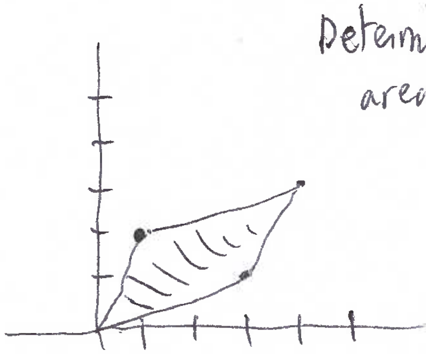


Determinant of a square matrix

Determinant of $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $|A| = ad - bc$.

Example: $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$

$|A| = 3 \times 2 - 1 \times 1 = 5$



Determinant is the area of parallelogram extended

In higher dimension, parallelepiped

Inverse of a square matrix

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$.
Notation: A^{-1}.

Example: if $A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$ then $A^{-1} = \begin{pmatrix} 0 & -1/2 \\ 1/2 & 1/4 \end{pmatrix}$. Check!

$$\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1/2 \\ 1/2 & 1/4 \end{pmatrix} = \begin{pmatrix} \overbrace{1\cdot 0 + 2\cdot 1/2}^{1} & \overbrace{1(-\frac{1}{2}) + 2\cdot\frac{1}{4}}^{0} \\ \underbrace{-2\cdot 0 + 0\cdot 1/2}_{0} & \underbrace{-2\cdot(-\frac{1}{2}) + 0\cdot\frac{1}{4}}_{+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\text{Identity Matrix}}$$

Inverse of a square matrix, cont'd

The **inverse** of a $d \times d$ matrix A is a $d \times d$ matrix B for which $AB = BA = I_d$.
Notation: A^{-1}.

Singular $\equiv$ not invertible

- Not all square matrices have an inverse
- Square matrix A is invertible if and only if $|A| \neq 0$
- What is the inverse of $A = \mathrm{diag}(a_1, \ldots, a_d)$?

$$\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{pmatrix}$$

$d \times d$

$\det = a_1 \cdot a_2 \cdots a_d$

Need $a_i \neq 0$

Inverse $\begin{pmatrix} 1/a_1 & & 0 \\ & \ddots & \\ 0 & & 1/a_d \end{pmatrix}$