Probability review I: Probability spaces, events, and conditioning

Topics we'll cover

1. How to define the **probability space** for an experiment in which outcomes are random.
2. How to formulate an **event** of interest.
3. The probability that two events both occur.
4. The **conditional probability** that an event occurs, given that some other event has occurred.
5. **Bayes' rule**.

Probability spaces

You roll two dice.

What is the probability they add to 10?

The **probability space** has two components:

1. **Sample space** (space of outcomes).

$(4,2)$
$(1,6)$

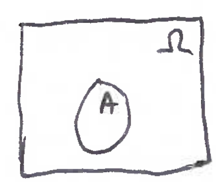
$$\Omega = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots,(6,6)\}$$
$$= \{1,2,\ldots,6\} \times \{1,2,\ldots,6\} = \{1,2,3,4,5,6\}^2$$

2. **Probabilities of outcomes**, summing to 1.

Each outcome has probability $1/36$

Probability $= \frac{1}{\Omega} = \frac{1}{6^2} = \frac{1}{36}$

Events



Probability space:

- Outcomes: $\Omega = \{\text{all possible pairs of dice rolls}\}$
- Every pair $z = (z_1, z_2) \in \Omega$ has probability $1/36$.

Event of interest: the two dice add up to 10.

$A \subseteq \Omega$

$$A = \{(z_1, z_2) : z_1 + z_2 = 10\}$$
$$= \{(4,6),(5,5),(6,4)\}$$

$$P(A) = 3 \times \frac{1}{36} = \frac{1}{12}$$

To add up to 10, z_1 must at least be 4

1 dice $= \{1,2,3,4,5,6\}$

Multiple events

You have ten coins. Nine are fair, but one is a bad coin that always comes up tails.

- You close your eyes and pick a coin at random.
- You toss it four times, and it comes up tails every time.

What is the probability you picked the bad coin? $\longrightarrow$ Selected / desired event

(coin #, toss1, toss2, toss3, toss4)

$$\Omega = \{1,2,3,\dots,10\} \times \{H,T\} \times \{H,T\} \times \{H,T\} \times \{H,T\}$$
$$= \{1,2,\dots,10\} \times \{H,T\}^4$$

- Ten coins: nine are fair, one is a bad coin that always comes up tails.
- You pick a coin at random, toss it four times, and it's tails every time.

$A \doteq$ picked the bad coin
$= \{(10,-,-,-,-)\}$

But we get to observe
$B =$ all coins are tails
$= \{(\_,T,T,T,T)\}$

$P(A \text{ and } B) = P(A \cap B)$
$= P(\text{bad coin})\, P(\text{all tails} \mid \text{bad coin})$
$= \frac{1}{10} \times 1 = \frac{1}{10}$ $\Rightarrow P(A \cap B) = \frac{1}{10}$

out of 10 coins

Probability that all tosses = tails But all were tails so 1

Conditioning

For two events A, B, **conditional probability**

$$\underbrace{\Pr(B|A)}_{\text{"given"}} = \text{probability that } B \text{ occurs, given that } A \text{ occurs}$$

Conditioning formula: $\boxed{\Pr(A \cap B) = \Pr(A)\Pr(B|A)}$

In our example:

- A: the bad coin is chosen
- B: all four tosses are tails

Want $\Pr(A|B)$

Rearranging: $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{1/10}{P(B)} = \dfrac{1/10}{5/32} = 0.64$

- Ten coins: nine are fair, one is a bad coin that always comes up tails.
- You pick a coin at random, toss it four times, and it's tails every time.

Event A: the bad coin is chosen. Event B: all tails

$P(\text{all tails}) = P(\text{bad coin, all tails}) + P(\text{not bad coin, all tails})$

$$= \underbrace{\frac{1}{10}}_{\text{bad coin}}\overbrace{(1)}^{\text{all tails}} + \underbrace{\frac{9}{10}}_{\text{not bad coin} \to \text{fair coin}} \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) \times \left(\frac{1}{2}\right) =$$

$$P(B) = \frac{1}{10} + \frac{9}{10} \times \left(\frac{1}{16}\right) = \frac{25}{160} = 5/32$$

Bayes' rule

Two events A, B

- We are interested in A
- We can observe B

If we find out B occurred, how does it alter the probability of A?

$$\boxed{\text{Bayes' rule: } \Pr(A|B) = \underbrace{\Pr(A)}_{P(A)} \times \underbrace{\frac{\Pr(B|A)}{\Pr(B)}}_{\text{multiplicative correction factor (mcf)}}}$$

it changed $P(A)$ by mcf

$$P(A \cap B) = P(A)\, P(B|A)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)\, P(B|A)}{P(B)}$$

BAYES' RULE: Fundamental formula for inference in machine learning.