Lecture Notes #1
Economics 120B
Econometrics

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Brief Overview of the Course

Economics suggests important relationships, often with policy implications, but virtually never suggests quantitative magnitudes of causal effects.

- What is the *quantitative* effect of reducing class size on student achievement?
- How does another year of education change earnings?
- What is the price elasticity of cigarettes?
- What is the effect on output growth of a 1 percentage point increase in interest rates by the Fed?
- What is the effect on housing prices of environmental improvements?

This course is about using data to measure causal effects.

- Ideally, we would like an experiment
  - what would be an experiment to estimate the effect of class size on standardized test scores?
- But almost always we only have observational (nonexperimental) data.
  - returns to education
  - cigarette prices
  - monetary policy
- Most of the course deals with difficulties arising from using observational to estimate causal effects
  - confounding effects (omitted factors)
  - simultaneous causality
  - "correlation does not imply causation"

In this course you will:

- Learn methods for estimating causal effects using observational data;
- Learn some tools that can be used for other purposes, for example forecasting using time series data;
- Focus on applications as well as provide some theory to understand the methods;
- Learn to evaluate the regression analysis of others – this means you will be able to read/understand empirical economics papers in other econ courses;
- Get some hands-on experience with regression analysis in your problem sets.

Review of Probability and Statistics

(SW Chapters 2, 3)

Empirical problem: Class size and educational output

- Policy question: What is the effect on test scores (or some other outcome measure) of reducing class size by one student per class? By 8 students/class?
- We must use data to find out (is there any way to answer this *without* data?)

The California Test Score Data Set

All K-6 and K-8 California school districts (n = 420)

Variables:

- 5^{th} grade test scores (Stanford-9 achievement test, combined math and reading), district average
- Student-teacher ratio (STR) = no. of students in the district divided by no. full-time equivalent teachers

Initial look at the data:

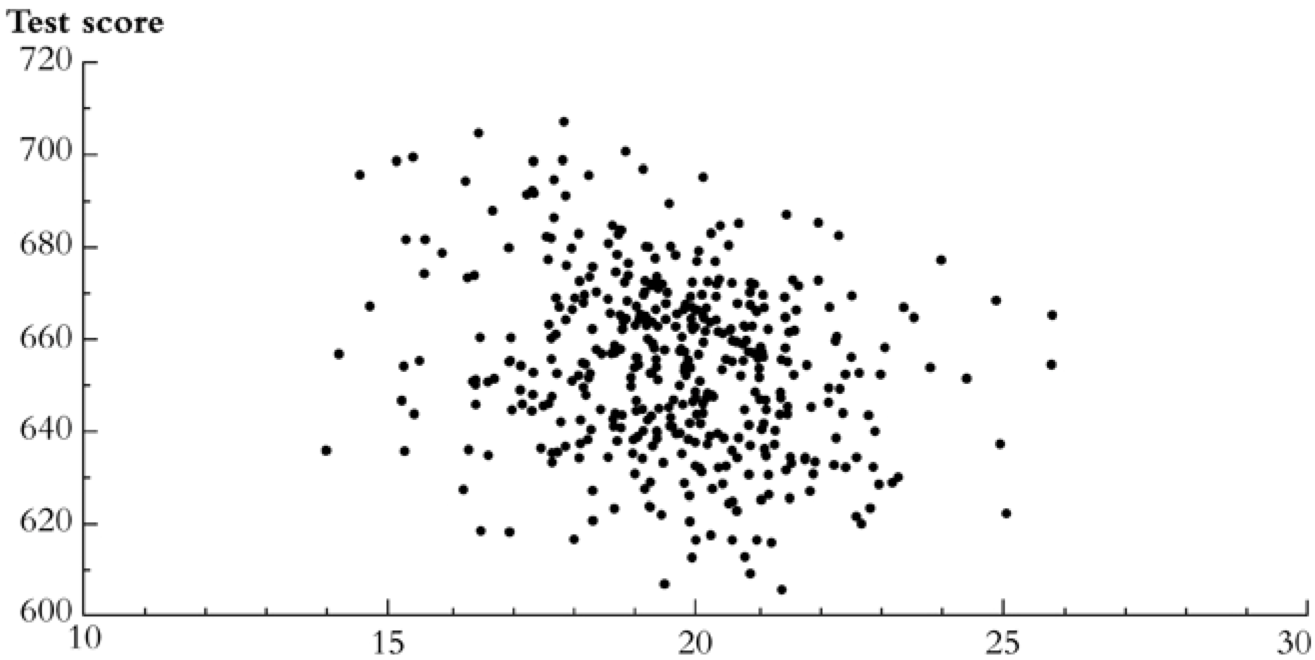
(You should already know how to interpret this table)

TABLE 4.1 Summary of the Distribution of Student–Teacher Ratios and Fifth-Grade Test Scores for 420 K–8 Districts in California in 1998

			Percentile						
	Average	Standard Deviation	10%	25%	40%	50% (median)	60%	75%	90%
Student–teacher ratio	19.6	1.9	17.3	18.6	19.3	19.7	20.1	20.9	21.9
Test score	665.2	19.1	630.4	640.0	649.1	654.5	659.4	666.7	679.1

- This table doesn't tell us anything about the relationship between test scores and the *STR*.

Do districts with smaller classes have higher test scores?
Scatterplot of test score v. student-teacher ratio



What does this figure show?

We need to get some numerical evidence on whether districts with low STRs have higher test scores – but how?

1. Compare average test scores in districts with low STRs to those with high STRs ("***estimation***")

2. Test the "null" hypothesis that the mean test scores in the two types of districts are the same, against the "alternative" hypothesis that they differ ("***hypothesis testing***")

3. Estimate an interval for the difference in the mean test scores, high v. low STR districts ("***confidence interval***")

Initial data analysis: Compare districts with "small" (STR < 20) and "large" (STR ≥ 20) class sizes:

Class Size	Average score ($\bar{Y}$)	Standard deviation (s_Y)	n
Small	657.4	19.4	238
Large	650.0	17.9	182

1. ***Estimation*** of Δ = difference between group means
2. ***Test the hypothesis*** that $\Delta = 0$
3. Construct a ***confidence interval*** for Δ

1. Estimation

$$\bar{Y}_{\text{small}} - \bar{Y}_{\text{large}} = \frac{1}{n_{\text{small}}}\sum_{i=1}^{n_{\text{small}}} Y_i - \frac{1}{n_{\text{large}}}\sum_{i=1}^{n_{\text{large}}} Y_i$$

$$= 657.4 - 650.0$$

$$= 7.4$$

Is this a large difference in a real-world sense?

- Standard deviation across districts = 19.1
- Difference between 60^{th} and 75^{th} percentiles of test score distribution is 666.7 – 659.4 = 7.3
- This is a big enough difference to be important for school reform discussions, for parents, or for a school committee?

2. Hypothesis testing

Difference-in-means test: compute the *t*-statistic,

$$t = \frac{\bar{Y}_s - \bar{Y}_l}{\sqrt{\frac{s_s^2}{n_s} + \frac{s_l^2}{n_l}}} = \frac{\bar{Y}_s - \bar{Y}_l}{SE(\bar{Y}_s - \bar{Y}_l)} \qquad \text{(remember this?)}$$

where $SE(\bar{Y}_s - \bar{Y}_l)$ is the "standard error" of $\bar{Y}_s - \bar{Y}_l$, the subscripts *s* and *l* refer to "small" and "large" STR districts, and

$$s_s^2 = \frac{1}{n_s - 1}\sum_{i=1}^{n_s}(Y_i - \bar{Y}_s)^2 \text{ (etc.)}$$

Compute the difference-of-means *t*-statistic:

Size	$\bar{Y}$	$s_{Y.}$	n
small	657.4	19.4	238
large	650.0	17.9	182

$$t = \frac{\bar{Y}_s - \bar{Y}_l}{\sqrt{\frac{s_s^2}{n_s} + \frac{s_l^2}{n_l}}} = \frac{657.4 - 650.0}{\sqrt{\frac{19.4^2}{238} + \frac{17.9^2}{182}}} = \frac{7.4}{1.83} = 4.05$$

|t| > 1.96, so reject (at the 5% significance level) the null hypothesis that the two means are the same.

3. Confidence interval

A 95% confidence interval for the difference between the means is,

$$(\overline{Y}_s - \overline{Y}_l) \pm 1.96 \times SE(\overline{Y}_s - \overline{Y}_l)$$

$$= 7.4 \pm 1.96 \times 1.83 = (3.8, 11.0)$$

Two equivalent statements:

1. The 95% confidence interval for Δ doesn't include 0;
2. The hypothesis that $\Delta = 0$ is rejected at the 5% level.

What comes next…

- The mechanics of estimation, hypothesis testing, and confidence intervals should be familiar
- These concepts extend directly to regression and its variants
- Before turning to regression, however, we will review some of the underlying theory of estimation, hypothesis testing, and confidence intervals:
  - Why do these procedures work, and why use these rather than others?
  - So we will review the intellectual foundations of statistics and econometrics

Review of Statistical Theory

1. **The probability framework for statistical inference**
2. Estimation
3. Testing
4. Confidence Intervals

The probability framework for statistical inference

(a) Population, random variable, and distribution
(b) Moments of a distribution (mean, variance, standard deviation, covariance, correlation)
(c) Conditional distributions and conditional means
(d) Distribution of a sample of data drawn randomly from a population: $Y_1, \ldots, Y_n$

(a) Population, random variable, and distribution

Population

- The group or collection of all possible entities of interest (school districts)
- Often, we will think of populations as very large or infinite

Random variable Y

- Numerical summary of a random outcome (district average test score, district STR)

Population distribution of Y

- The probabilities of different values of Y that occur in the population, for ex. $\Pr[Y = 650]$ (when Y is discrete)
- or: The probabilities of sets of these values, for ex.

  $\Pr[640 \leq Y \leq 660]$ (when Y is continuous).

(b) Moments of a population distribution: mean, variance, standard deviation, covariance, correlation

mean = expected value (expectation) of Y

$= E(Y)$

$= \mu_Y$

= long-run average value of Y over repeated realizations of Y

variance $= E(Y - \mu_Y)^2$

$= \sigma_Y^2$

= measure of the squared spread of the distribution

standard deviation $= \sqrt{\text{variance}} = \sigma_Y$

Moments, ctd.

$$\textbf{\textit{skewness}} = \frac{E\left[\left(Y-\mu_Y\right)^3\right]}{\sigma_Y^3}$$

$=$ measure of asymmetry of a distribution

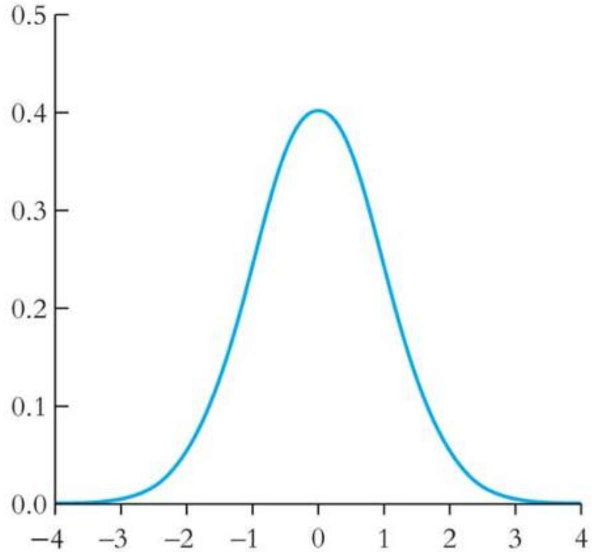
- *skewness* $= 0$: distribution is symmetric
- *skewness* $>$ ($<$) 0: distribution has long right (left) tail

$$\textbf{\textit{kurtosis}} = \frac{E\left[\left(Y-\mu_Y\right)^4\right]}{\sigma_Y^4}$$

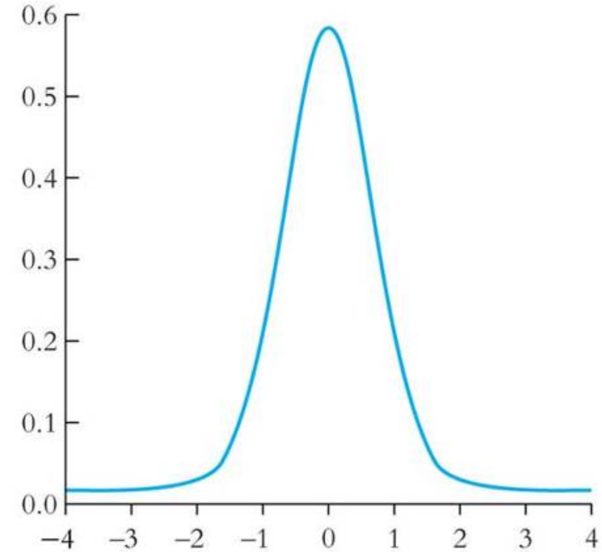
$=$ measure of mass in tails

$=$ measure of probability of large values

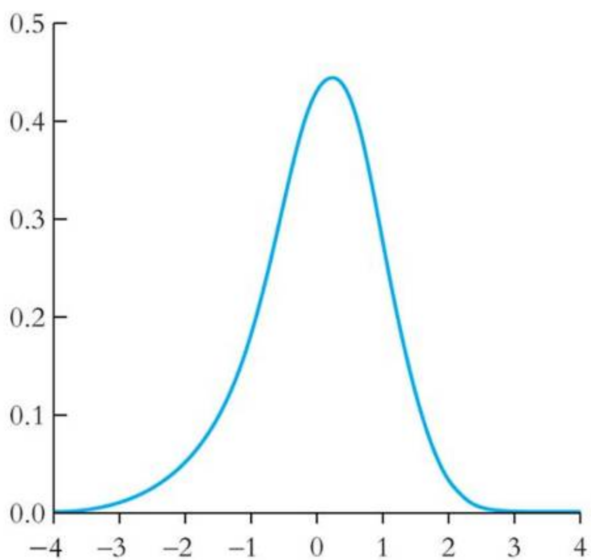
- *kurtosis* $= 3$: normal distribution
- *skewness* > 3: heavy tails (“***leptokurtotic***”)



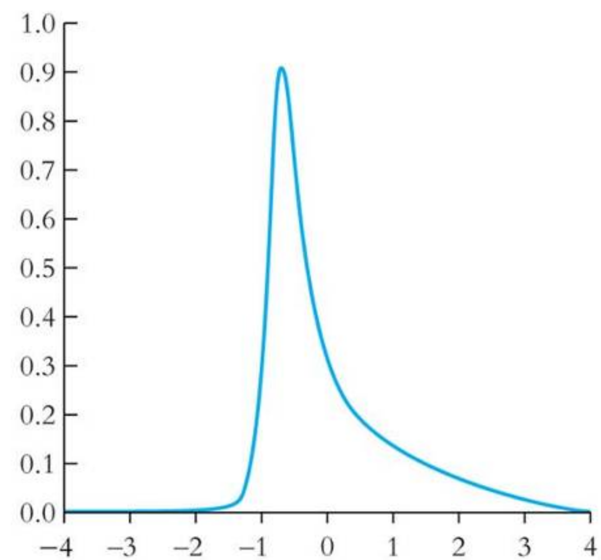
(a) Skewness = 0, kurtosis = 3



(b) Skewness = 0, kurtosis = 20



(c) Skewness = −0.1, kurtosis = 5



(d) Skewness = 0.6, kurtosis = 5

2 random variables: joint distributions and covariance

- Random variables X and Z have a ***joint distribution***
- The ***covariance*** between X and Z is

$$\mathrm{cov}(X,Z) = E[(X - \mu_X)(Z - \mu_Z)] = \sigma_{XZ}.$$

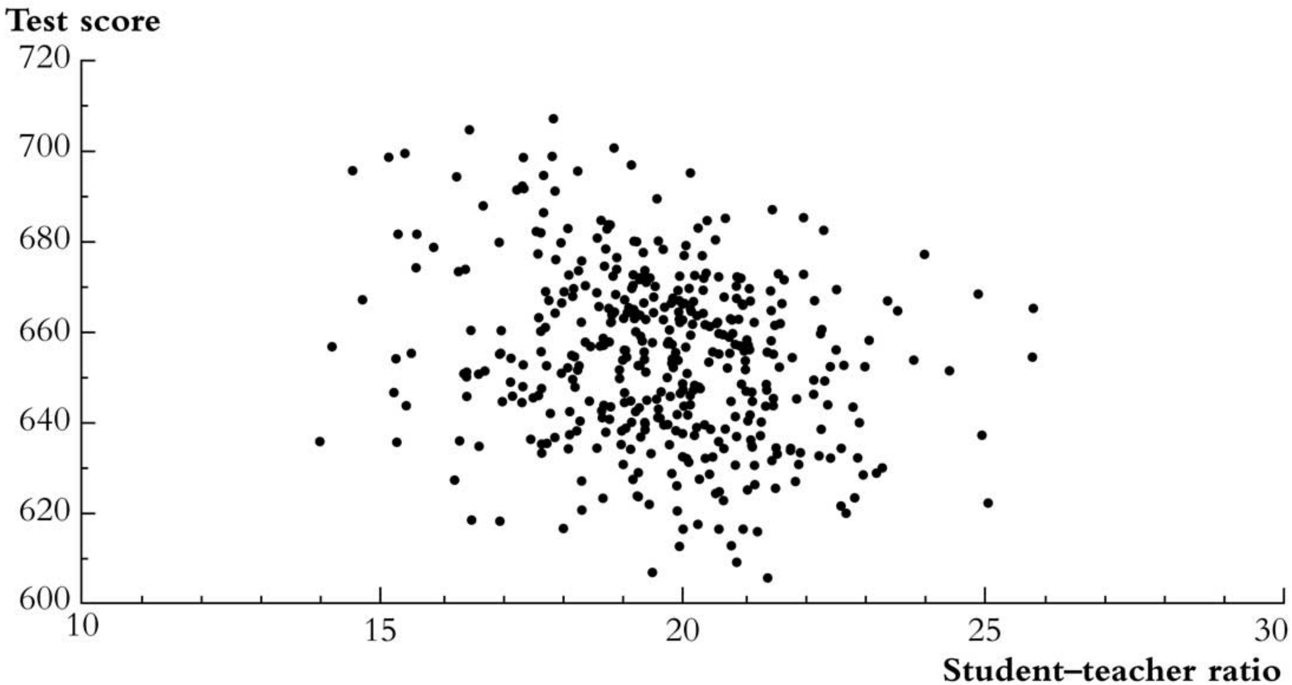
- The covariance is a measure of the linear association between X and Z; its units are units of $X \times$ units of Z
- $\mathrm{cov}(X,Z) > 0$ means a positive relation between X and Z
- If X and Z are independently distributed, then $\mathrm{cov}(X,Z) = 0$ (but not vice versa!!)
- The covariance of a r.v. with itself is its variance:

$$\mathrm{cov}(X,X) = E[(X - \mu_X)(X - \mu_X)] = E[(X - \mu_X)^2] = \sigma_X^2$$

The covariance between Test Score and STR is negative:

FIGURE 4.2 Scatterplot of Test Score vs. Student–Teacher Ratio (California School District Data)

Data from 420 California school districts. There is a weak negative relationship between the student–teacher ratio and test scores: The sample correlation is −0.23.



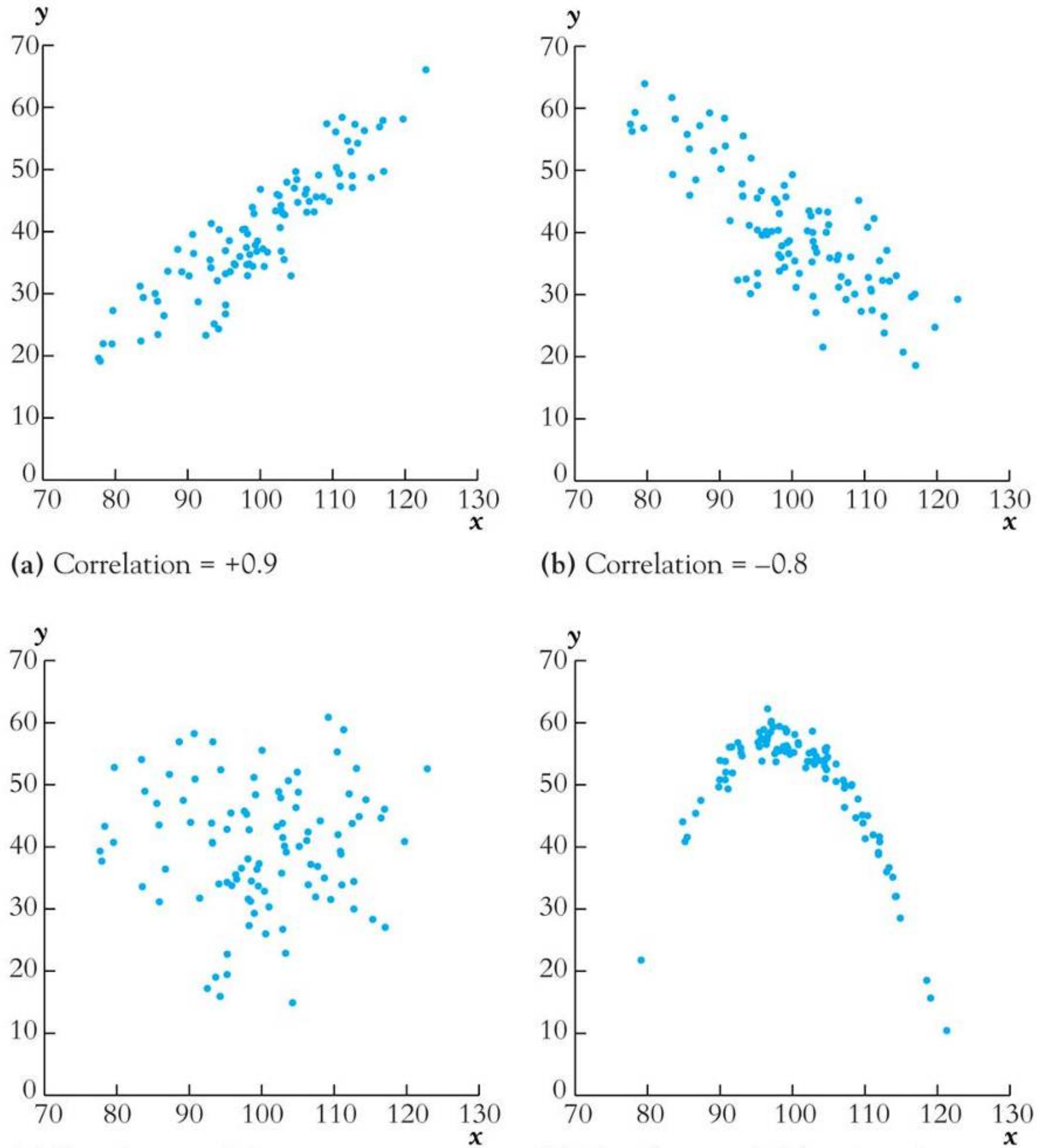
so is the ***correlation***…

The *correlation coefficient* is defined in terms of the covariance:

$$\operatorname{corr}(X,Z) = \frac{\operatorname{cov}(X,Z)}{\sqrt{\operatorname{var}(X)\operatorname{var}(Z)}} = \frac{\sigma_{XZ}}{\sigma_X \sigma_Z} = r_{XZ}$$

- $-1 \leq \operatorname{corr}(X,Z) \leq 1$
- $\operatorname{corr}(X,Z) = 1$ mean perfect positive linear association
- $\operatorname{corr}(X,Z) = -1$ means perfect negative linear association
- $\operatorname{corr}(X,Z) = 0$ means no linear association

The correlation coefficient measures linear association



(a) Correlation = +0.9

(b) Correlation = –0.8

(c) Correlation = 0.0

(d) Correlation = 0.0 (quadratic)

(c) Conditional distributions and conditional means

Conditional distributions

- The distribution of Y, given value(s) of some other random variable, X
- Ex: the distribution of test scores, given that STR < 20

Conditional expectations and conditional moments

- *conditional mean* = mean of conditional distribution
  $= E(Y|X = x)$ (***important concept and notation***)
- *conditional variance* = variance of conditional distribution
- *Example*: $E(Test\ scores|STR < 20)$ = the mean of test scores among districts with small class sizes

The difference in means is the difference between the means of two conditional distributions:

Conditional mean, ctd.

$\Delta = E(Test\ scores|STR < 20) - E(Test\ scores|STR \geq 20)$

Other examples of conditional means:

- Wages of all female workers (Y = wages, X = gender)
- Mortality rate of those given an experimental treatment (Y = live/die; X = treated/not treated)
- If $E(X|Z)$ = const, then corr(X,Z) = 0 (not necessarily vice versa however)

The conditional mean is a (possibly new) term for the familiar idea of the group mean

(d) Distribution of a sample of data drawn randomly from a population: $Y_1,\ldots, Y_n$

We will assume simple random sampling

- Choose and individual (district, entity) at random from the population

Randomness and data

- Prior to sample selection, the value of Y is random because the individual selected is random
- Once the individual is selected and the value of Y is observed, then Y is just a number – not random
- The data set is $(Y_1, Y_2,\ldots, Y_n)$, where Y_i = value of Y for the i^{th} individual (district, entity) sampled

Distribution of $Y_1,\ldots, Y_n$ under simple random sampling

- Because individuals #1 and #2 are selected at random, the value of Y_1 has no information content for Y_2. Thus:
  - Y_1 and Y_2 are ***independently distributed***
  - Y_1 and Y_2 come from the same distribution, that is, Y_1, Y_2 are ***identically distributed***
  - That is, under simple random sampling, Y_1 and Y_2 are independently and identically distributed (***i.i.d.***).
  - More generally, under simple random sampling, $\{Y_i\}$, $i = 1,\ldots, n$, are i.i.d.

This framework allows rigorous statistical inferences about moments of population distributions using a sample of data from that population …

1. The probability framework for statistical inference
2. **Estimation**
3. Testing
4. Confidence Intervals

Estimation

$\bar{Y}$ is the natural estimator of the mean. But:

(a) What are the properties of $\bar{Y}$?

(b) Why should we use $\bar{Y}$ rather than some other estimator?

- Y_1 (the first observation)
- maybe unequal weights – not simple average
- median($Y_1, \ldots, Y_n$)

The starting point is the sampling distribution of $\bar{Y}$ …

(a) The sampling distribution of $\bar{Y}$

$\bar{Y}$ is a random variable, and its properties are determined by the ***sampling distribution*** of $\bar{Y}$

- The individuals in the sample are drawn at random.
- Thus the values of $(Y_1,\ldots, Y_n)$ are random
- Thus functions of $(Y_1,\ldots, Y_n)$, such as $\bar{Y}$, are random: had a different sample been drawn, they would have taken on a different value
- The distribution of $\bar{Y}$ over different possible samples of size n is called the ***sampling distribution*** of $\bar{Y}$.
- The mean and variance of $\bar{Y}$ are the mean and variance of its sampling distribution, $E(\bar{Y})$ and $\text{var}(\bar{Y})$.
- The concept of the sampling distribution underpins all of econometrics.

The sampling distribution of $\bar{Y}$, ctd.

Example: Suppose Y takes on 0 or 1 (a ***Bernoulli*** random variable) with the probability distribution,

$$\Pr[Y = 0] = .22, \Pr(Y = 1) = .78$$

Then

$$E(Y) = p\times 1 + (1 - p)\times 0 = p = .78$$

$$\sigma_Y^2 = E[Y - E(Y)]^2 = p(1 - p) \quad \textit{[remember this?]}$$

$$= .78\times (1 - .78) = 0.1716$$

The sampling distribution of $\bar{Y}$ depends on n.

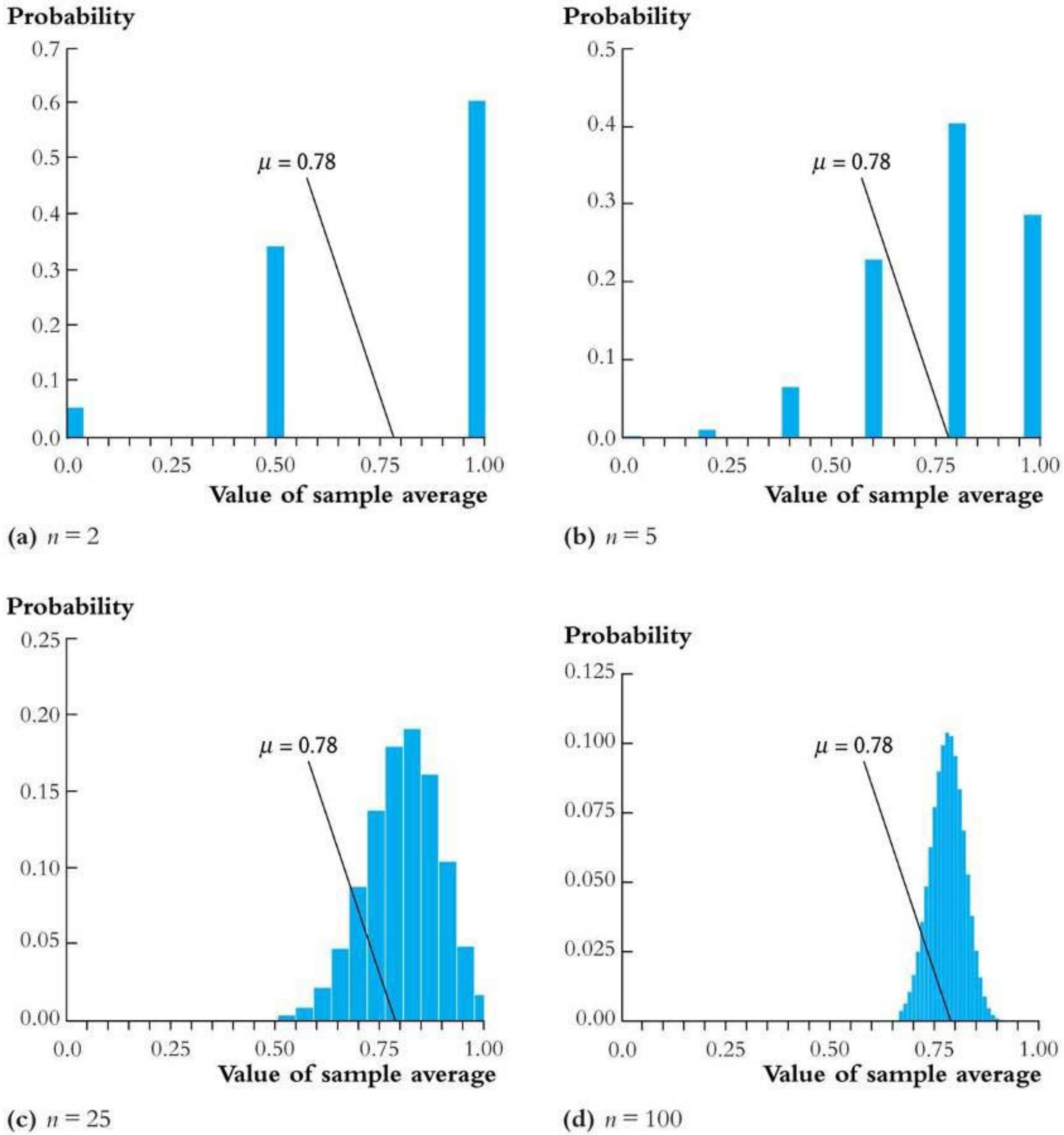
Consider $n = 2$. The sampling distribution of $\bar{Y}$ is,

$$\Pr(\bar{Y} = 0) = .22^2 = .0484$$

$$\Pr(\bar{Y} = ½) = 2\times .22\times .78 = .3432$$

$$\Pr(\bar{Y} = 1) = .78^2 = .6084$$

The sampling distribution of $\bar{Y}$ when Y is Bernoulli (p = .78):



Things we want to know about the sampling distribution:

- What is the mean of $\bar{Y}$?
  - If $E(\bar{Y})$ = true $\mu = .78$, then $\bar{Y}$ is an ***unbiased*** estimator of μ
- What is the variance of $\bar{Y}$?
  - How does var($\bar{Y}$) depend on n (famous $1/n$ formula)
- Does $\bar{Y}$ become close to μ when n is large?
  - Law of large numbers: $\bar{Y}$ is a ***consistent*** estimator of μ
- $\bar{Y} - \mu$ appears bell shaped for n large…is this generally true?
  - In fact, $\bar{Y} - \mu$ is approximately normally distributed for n large (Central Limit Theorem)

The mean and variance of the sampling distribution of $\bar{Y}$

General case – that is, for Y_i i.i.d. from any distribution, not just Bernoulli:

mean: $E(\bar{Y}) = E(\frac{1}{n}\sum_{i=1}^{n} Y_i) = \frac{1}{n}\sum_{i=1}^{n} E(Y_i) = \frac{1}{n}\sum_{i=1}^{n}\mu_Y = \mu_Y$

Variance:

$$\begin{aligned}
\text{var}(\bar{Y}) &= E[\bar{Y} - E(\bar{Y})]^2 \\
&= E[\bar{Y} - \mu_Y]^2 \\
&= E\left[\left(\frac{1}{n}\sum_{i=1}^{n} Y_i\right) - \mu_Y\right]^2 \\
&= E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_Y)\right]^2
\end{aligned}$$

so $$\mathrm{var}(\bar{Y}) = E\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_Y)\right]^2$$

$$= E\left\{\left[\frac{1}{n}\sum_{i=1}^{n}(Y_i - \mu_Y)\right] \times \left[\frac{1}{n}\sum_{j=1}^{n}(Y_j - \mu_Y)\right]\right\}$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E\left[(Y_i - \mu_Y)(Y_j - \mu_Y)\right]$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{cov}(Y_i, Y_j)$$

$$= \frac{1}{n^2}\sum_{i=1}^{n}\sigma_Y^2$$

$$= \frac{\sigma_Y^2}{n}$$

Mean and variance of sampling distribution of $\bar{Y}$, ctd.

$$\boxed{E(\bar{Y}) = \mu_Y}$$

$$\boxed{\operatorname{var}(\bar{Y}) = \frac{\sigma_Y^2}{n}}$$

Implications:

1. $\bar{Y}$ is an *unbiased* estimator of μ_Y (that is, $E(\bar{Y}) = \mu_Y$)
2. $\operatorname{var}(\bar{Y})$ is inversely proportional to n
   - the spread of the sampling distribution is proportional to $1/\sqrt{n}$
   - Thus the sampling uncertainty associated with $\bar{Y}$ is proportional to $1/\sqrt{n}$ (larger samples, less uncertainty, but square-root law)

The sampling distribution of $\bar{Y}$ when n is large

For small sample sizes, the distribution of $\bar{Y}$ is complicated, but if n is large, the sampling distribution is simple!

1. As n increases, the distribution of $\bar{Y}$ becomes more tightly centered around μ_Y (the *Law of Large Numbers*)
2. Moreover, the distribution of $\bar{Y} - \mu_Y$ becomes normal (the *Central Limit Theorem*)

The *Law of Large Numbers*:

An estimator is ***consistent*** if the probability that its falls within an interval of the true population value tends to one as the sample size increases.

If $(Y_1,\ldots,Y_n)$ are i.i.d. and $\sigma_Y^2 < \infty$, then $\bar{Y}$ is a consistent estimator of μ_Y, that is,

$$\Pr[|\bar{Y} - \mu_Y| < \varepsilon] \to 1 \text{ as } n \to \infty$$

which can be written, $\bar{Y} \xrightarrow{p} \mu_Y$

("$\bar{Y} \xrightarrow{p} \mu_Y$" means "$\bar{Y}$ converges in probability to μ_Y").

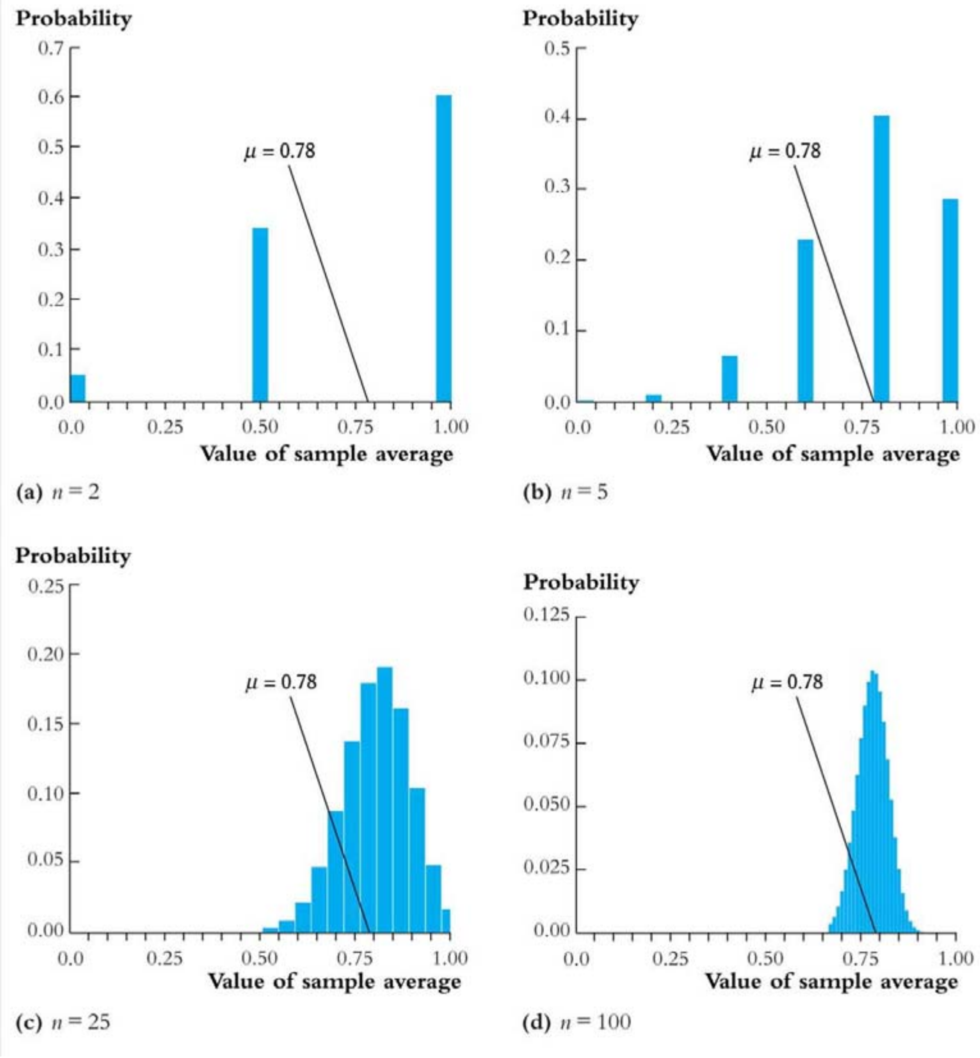
(*the math*: as $n \to \infty$, $\text{var}(\bar{Y}) = \dfrac{\sigma_Y^2}{n} \to 0$, which implies that $\Pr[|\bar{Y} - \mu_Y| < \varepsilon] \to 1$.)

The *Central Limit Theorem* (CLT):

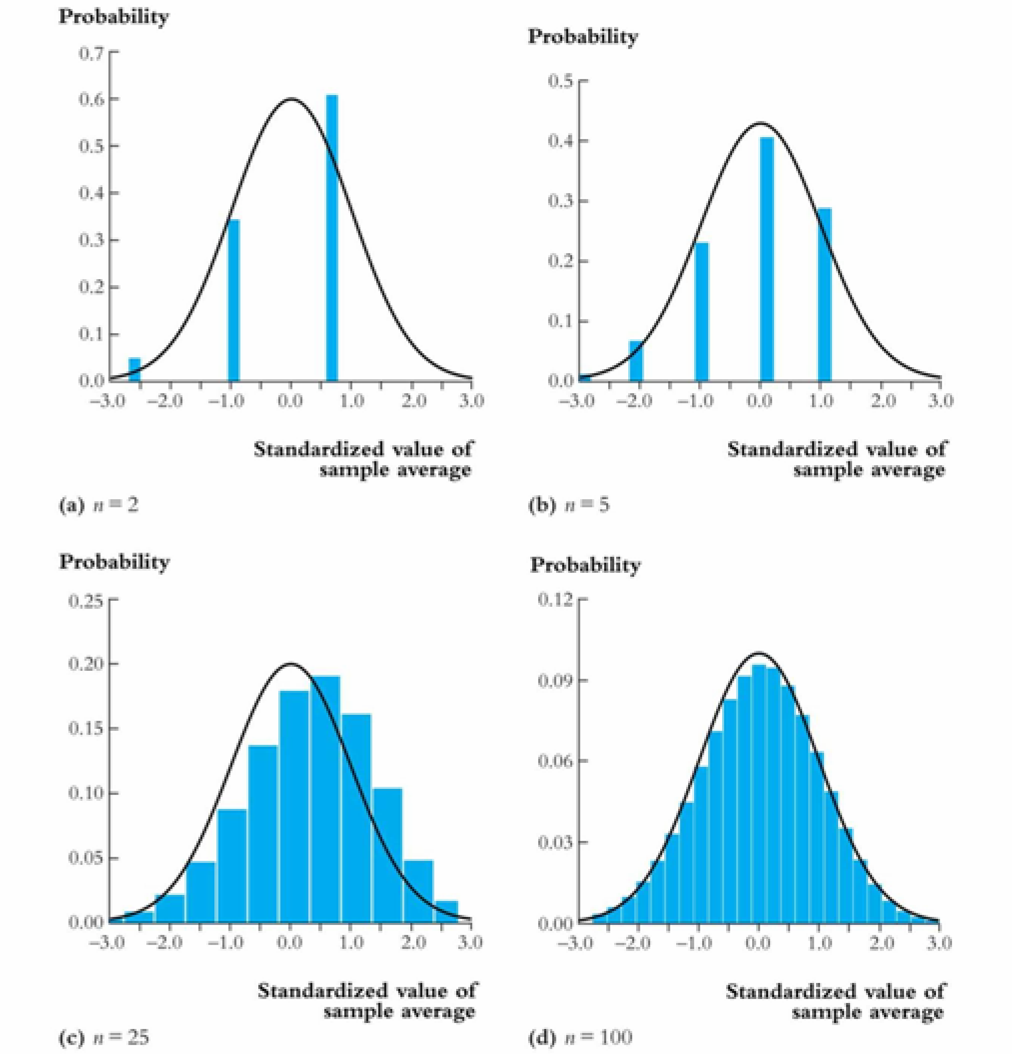
If $(Y_1,\ldots,Y_n)$ are i.i.d. and $0 < \sigma_Y^2 < \infty$, then when n is large the distribution of $\bar{Y}$ is well approximated by a normal distribution.

- $\bar{Y}$ is approximately distributed $N(\mu_Y, \frac{\sigma_Y^2}{n})$ ("normal distribution with mean μ_Y and variance σ_Y^2/n")
- $\sqrt{n}(\bar{Y} - \mu_Y)/\sigma_Y$ is approximately distributed $N(0,1)$ (standard normal)
- **That is, "standardized" $\bar{Y}$ = $\frac{\bar{Y} - E(\bar{Y})}{\sqrt{\text{var}(\bar{Y})}}$ = $\frac{\bar{Y} - \mu_Y}{\sigma_Y/\sqrt{n}}$ is approximately distributed as $N(0,1)$**
- **The larger is n, the better is the approximation.**

Sampling distribution of $\bar{Y}$ when Y is Bernoulli, p = 0.78:



Same example: sampling distribution of $\dfrac{\bar{Y} - E(\bar{Y})}{\sqrt{\mathrm{var}(\bar{Y})}}$:



Summary: The Sampling Distribution of $\bar{Y}$

For $Y_1,\ldots,Y_n$ i.i.d. with $0 < \sigma_Y^2 < \infty$,

- The exact (finite sample) sampling distribution of $\bar{Y}$ has mean μ_Y ("$\bar{Y}$ is an unbiased estimator of μ_Y") and variance σ_Y^2/n
- Other than its mean and variance, the exact distribution of $\bar{Y}$ is complicated and depends on the distribution of Y (the population distribution)
- When n is large, the sampling distribution simplifies:
  - $\boxed{\bar{Y} \xrightarrow{p} \mu_Y}$ (Law of large numbers)
  - $\boxed{\dfrac{\bar{Y} - E(\bar{Y})}{\sqrt{\text{var}(\bar{Y})}} \text{ is approximately } N(0,1)}$ (CLT)

(b) Why Use $\bar{Y}$ To Estimate μ_Y?

- $\bar{Y}$ is unbiased: $E(\bar{Y}) = \mu_Y$
- $\bar{Y}$ is consistent: $\bar{Y} \xrightarrow{p} \mu_Y$
- $\bar{Y}$ is the "least squares" estimator of μ_Y; $\bar{Y}$ solves,

$$\min_m \sum_{i=1}^{n} (Y_i - m)^2$$

so, $\bar{Y}$ minimizes the sum of squared "residuals"

optional derivation (also see App. 3.2)

$$\frac{d}{dm}\sum_{i=1}^{n}(Y_i - m)^2 = \sum_{i=1}^{n}\frac{d}{dm}(Y_i - m)^2 = -2\sum_{i=1}^{n}(Y_i - m)$$

Set derivative to zero and denote optimal value of m by $\hat{m}$:

$$\sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \hat{m} = n\hat{m} \text{ or } \hat{m} = \frac{1}{n}\sum_{i=1}^{n} Y_i = \bar{Y}$$

Why Use $\bar{Y}$ To Estimate μ_Y?, ctd.

- $\bar{Y}$ has a smaller variance than all other *linear unbiased* estimators: consider the estimator, $\hat{\mu}_Y = \frac{1}{n}\sum_{i=1}^{n} a_i Y_i$, where $\{a_i\}$ are such that $\hat{\mu}_Y$ is unbiased; then $\text{var}(\bar{Y}) \leq \text{var}(\hat{\mu}_Y)$ (proof: SW, Ch. 17)
- $\bar{Y}$ isn't the only estimator of μ_Y – can you think of a time you might want to use the median instead?

1. The probability framework for statistical inference
2. Estimation
3. **Hypothesis Testing**
4. Confidence intervals

Hypothesis Testing

The ***hypothesis testing*** problem (for the mean): make a provisional decision, based on the evidence at hand, whether a null hypothesis is true, or instead that some alternative hypothesis is true. That is, test

H_0: $E(Y) = \mu_{Y,0}$ vs. H_1: $E(Y) > \mu_{Y,0}$ (1-sided, >)

H_0: $E(Y) = \mu_{Y,0}$ vs. H_1: $E(Y) < \mu_{Y,0}$ (1-sided, <)

H_0: $E(Y) = \mu_{Y,0}$ vs. H_1: $E(Y) \neq \mu_{Y,0}$ (2-sided)

Some terminology for testing statistical hypotheses:

p-value = probability of drawing a statistic (e.g. $\bar{Y}$) at least as adverse to the null as the value actually computed with your data, assuming that the null hypothesis is true.

The ***significance level*** of a test is a pre-specified probability of incorrectly rejecting the null, when the null is true.

Calculating the p-value based on $\bar{Y}$:

$$\boxed{p\text{-value} = \Pr_{H_0}[|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|]}$$

where $\bar{Y}^{act}$ is the value of $\bar{Y}$ actually observed (nonrandom)

Calculating the p-value, ctd.

- To compute the p-value, you need the to know the sampling distribution of $\bar{Y}$, which is complicated if n is small.
- If n is large, you can use the normal approximation (CLT):

$$p\text{-value} = \Pr_{H_0}[|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|],$$

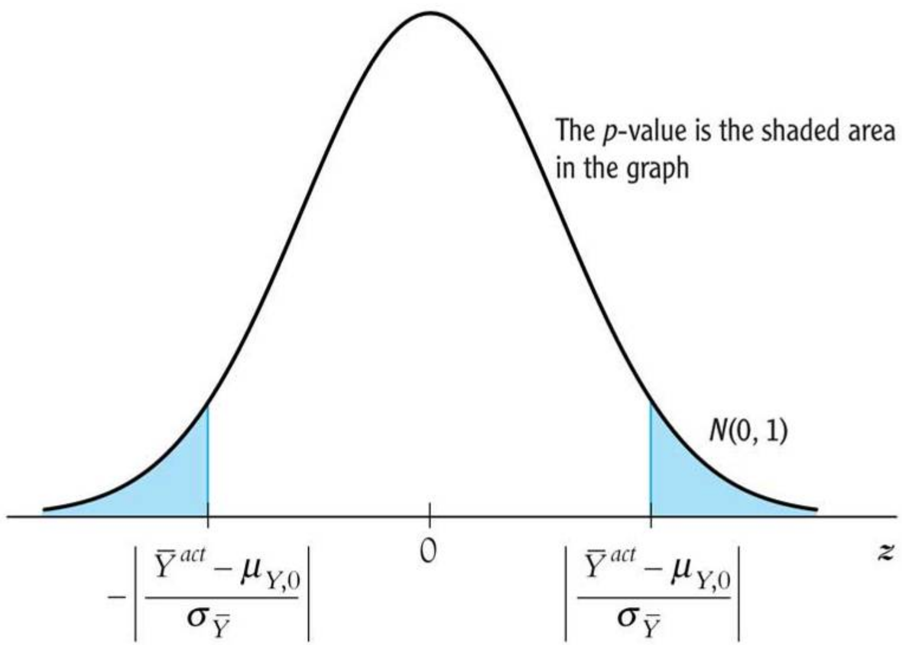
$$= \Pr_{H_0}\left[\left|\frac{\bar{Y} - \mu_{Y,0}}{\sigma_Y/\sqrt{n}}\right| > \left|\frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_Y/\sqrt{n}}\right|\right]$$

$$= \Pr_{H_0}\left[\left|\frac{\bar{Y} - \mu_{Y,0}}{\sigma_{\bar{Y}}}\right| > \left|\frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_{\bar{Y}}}\right|\right]$$

$$\cong \text{ probability under left+right } N(0,1) \text{ tails}$$

where $\sigma_{\bar{Y}}$ = std. dev. of the distribution of $\bar{Y}$ = $\sigma_Y/\sqrt{n}$.

Calculating the p-value with σ_Y known:



- For large n, p-value = the probability that a $N(0,1)$ random variable falls outside $|(\bar{Y}^{act} - \mu_{Y,0})/\sigma_{\bar{Y}}|$
- In practice, $\sigma_{\bar{Y}}$ is unknown – it must be estimated

Estimator of the variance of Y:

$$s_Y^2 = \frac{1}{n-1}\sum_{i=1}^{n}(Y_i - \bar{Y})^2 = \text{“sample variance of } Y\text{”}$$

Fact:

If $(Y_1,\ldots,Y_n)$ are i.i.d. and $E(Y^4) < \infty$, then $s_Y^2 \xrightarrow{p} \sigma_Y^2$

Why does the law of large numbers apply?

- Because s_Y^2 is a sample average; see Appendix 3.3
- Technical note: we assume $E(Y^4) < \infty$ because here the average is not of Y_i, but of its square; see App. 3.3

Computing the p-value with σ_Y^2 estimated:

$$
\begin{aligned}
p\text{-value} &= \Pr_{H_0}[|\bar{Y} - \mu_{Y,0}| > |\bar{Y}^{act} - \mu_{Y,0}|], \\
&= \Pr_{H_0}\left[\left|\frac{\bar{Y} - \mu_{Y,0}}{\sigma_Y / \sqrt{n}}\right| > \left|\frac{\bar{Y}^{act} - \mu_{Y,0}}{\sigma_Y / \sqrt{n}}\right|\right] \\
&\cong \Pr_{H_0}\left[\left|\frac{\bar{Y} - \mu_{Y,0}}{s_Y / \sqrt{n}}\right| > \left|\frac{\bar{Y}^{act} - \mu_{Y,0}}{s_Y / \sqrt{n}}\right|\right] \text{ (large } n\text{)}
\end{aligned}
$$

so

$$\boxed{p\text{-value} = \Pr_{H_0}[|t| > |t^{act}|]} \quad (\sigma_Y^2 \textbf{ estimated})$$

$$\cong \text{ probability under normal tails outside } |t^{act}|$$

where $t = \dfrac{\bar{Y} - \mu_{Y,0}}{s_Y / \sqrt{n}}$ (the usual t-statistic)

What is the link between the *p*-value and the significance level?

The significance level is prespecified. For example, if the prespecified significance level is 5%,

- you reject the null hypothesis if $|t| \geq 1.96$
- equivalently, you reject if $p \leq 0.05$.
- The *p*-value is sometimes called the ***marginal significance level***.
- Often, it is better to communicate the *p*-value than simply whether a test rejects or not – the *p*-value contains more information than the "yes/no" statement about whether the test rejects.

At this point, you might be wondering,...

What happened to the *t*-table and the degrees of freedom?

<u>Digression: the Student *t* distribution</u>

If Y_i, $i = 1,\ldots, n$ is i.i.d. $N(\mu_Y, \sigma_Y^2)$, then the t-statistic has the Student t-distribution with $n - 1$ degrees of freedom.

The critical values of the Student t-distribution is tabulated in the back of all statistics books. Remember the recipe?

1. Compute the t-statistic
2. Compute the degrees of freedom, which is $n - 1$
3. Look up the 5% critical value
4. If the t-statistic exceeds (in absolute value) this critical value, reject the null hypothesis.

Comments on this recipe and the Student *t*-distribution

1. The theory of the *t*-distribution was one of the early triumphs of mathematical statistics. It is astounding, really: if *Y* is i.i.d. normal, then you can know the *exact*, *finite-sample* distribution of the *t*-statistic – it is the Student *t*. So, you can construct confidence intervals (using the Student *t* critical value) that have *exactly* the right coverage rate, no matter what the sample size. This result was really useful in times when "computer" was a job title, data collection was expensive, and the number of observations was perhaps a dozen. It is also a conceptually beautiful result, and the math is beautiful too – which is probably why stats profs love to teach the *t*-distribution. But….

Comments on Student t distribution, ctd.

2. If the sample size is moderate (several dozen) or large (hundreds or more), the difference between the *t*-distribution and N(0,1) critical values are negligible. Here are some 5% critical values for 2-sided tests:

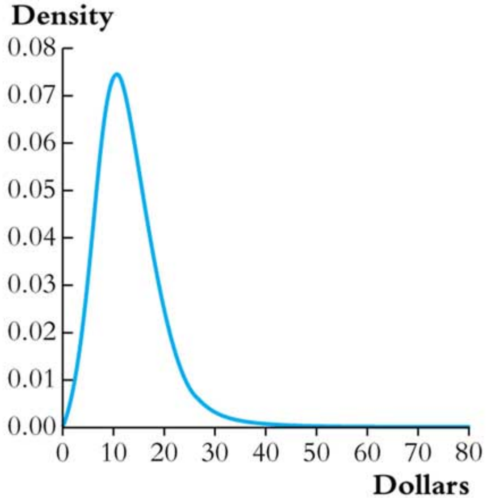
degrees of freedom ($n-1$)	5% *t*-distribution critical value
10	2.23
20	2.09
30	2.04
60	2.00
∞	1.96

Comments on Student t distribution, ctd.

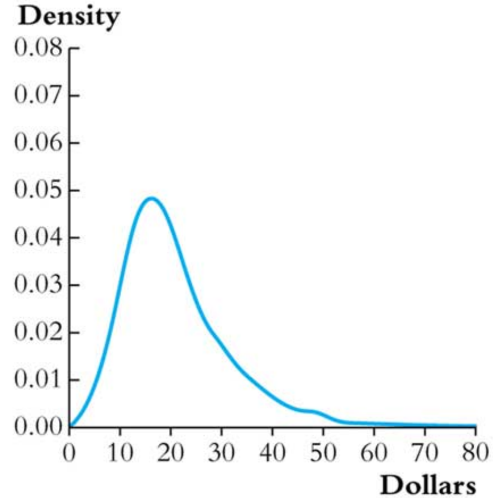
3. So, the Student-t distribution is only relevant when the sample size is very small; but in that case, for it to be correct, you must be sure that the population distribution of Y is normal. In economic data, the normality assumption is rarely credible. Here are the distributions of some economic data.
   - Do you think earnings are normally distributed?
   - Suppose you have a sample of $n = 10$ observations from one of these distributions – would you feel comfortable using the Student t distribution?

FIGURE 2.4 **Conditional Distribution of Average Hourly Earnings of U.S. Full-Time Workers in 2004, Given Education Level and Gender**

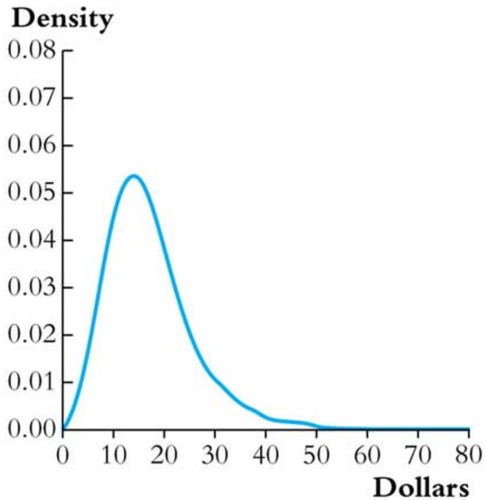
The four distributions of earnings are for women and men, for those with only a high school diploma (a and c) and those whose highest degree is from a four-year college (b and d).



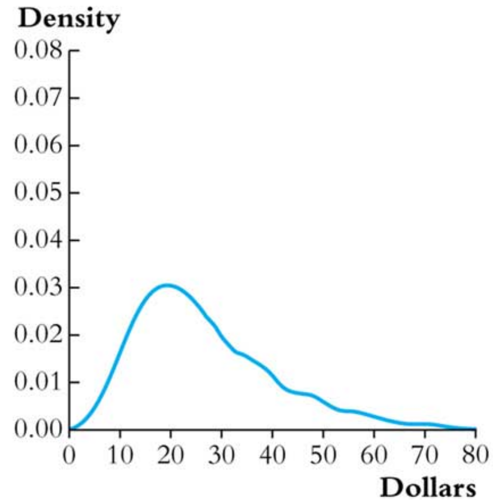
(a) Women with a high school diploma



(b) Women with a college degree



(c) Men with a high school diploma



(d) Men with a college degree

The Student-t distribution – summary

- The assumption that Y is distributed $N(\mu_Y,\sigma_Y^2)$ is rarely plausible in practice (income? number of children?)
- For $n > 30$, the t-distribution and $N(0,1)$ are very close (as n grows large, the t_{n-1} distribution converges to $N(0,1)$)
- The t-distribution is an artifact from days when sample sizes were small and "computers" were people
- For historical reasons, statistical software typically uses the t-distribution to compute p-values – but this is irrelevant when the sample size is moderate or large.
- For these reasons, in this class we will focus on the large-n approximation given by the CLT

1. The probability framework for statistical inference
2. Estimation
3. Testing
4. **Confidence intervals**

Confidence Intervals

A 95% ***confidence interval*** for μ_Y is an interval that contains the true value of μ_Y in 95% of repeated samples.

Digression: What is random here? The values of $Y_1,\ldots,Y_n$ and thus any functions of them – including the confidence interval. The confidence interval it will differ from one sample to the next. The population parameter, μ_Y, is not random, we just don't know it.

Confidence intervals, ctd.

A 95% confidence interval can always be constructed as the set of values of μ_Y not rejected by a hypothesis test with a 5% significance level.

$$\{\mu_Y: \left|\frac{\bar{Y}-\mu_Y}{s_Y/\sqrt{n}}\right| \leq 1.96\} = \{\mu_Y: -1.96 \leq \frac{\bar{Y}-\mu_Y}{s_Y/\sqrt{n}} \leq 1.96\}$$

$$= \{\mu_Y: -1.96\frac{s_Y}{\sqrt{n}} \leq \bar{Y} - \mu_Y \leq 1.96\frac{s_Y}{\sqrt{n}}\}$$

$$= \{\mu_Y \in (\bar{Y} - 1.96\frac{s_Y}{\sqrt{n}}, \bar{Y} + 1.96\frac{s_Y}{\sqrt{n}})\}$$

This confidence interval relies on the large-n results that $\bar{Y}$ *is approximately normally distributed and* $s_Y^2 \xrightarrow{p} \sigma_Y^2$.

Summary:

From the two assumptions of:

(1) simple random sampling of a population, that is, $\{Y_i, i = 1,\ldots,n\}$ are i.i.d.

(2) $0 < E(Y^4) < \infty$

we developed, for large samples (large n):

- Theory of estimation (sampling distribution of $\bar{Y}$)
- Theory of hypothesis testing (large-n distribution of t-statistic and computation of the p-value)
- Theory of confidence intervals (constructed by inverting test statistic)

Are assumptions (1) & (2) plausible in practice? **Yes**

Let's go back to the original policy question:

What is the effect on test scores of reducing STR by one student/class?

Have we answered this question?

FIGURE 4.2 Scatterplot of Test Score vs. Student–Teacher Ratio (California School District Data)

Data from 420 California school districts. There is a weak negative relationship between the student–teacher ratio and test scores: The sample correlation is −0.23.