Fitting Social Network Models to Data

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Statistical Analysis of Networks

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Fitting a General Model to Data

- Suppose we have network data and want to see how the models fit
- Initially focus on models where the Y_{ij} are independent (given covariates X), so that

$$\text{logit}[P(Y_{ij} = 1 | X = x, \beta)] = \beta_1 x_{1,ij} + \beta_2 x_{2,ij} \ldots + \beta_K x_{K,ij}$$

  for some choice of covariates $x_{1,ij}, x_{2,ij}, \ldots, x_{K,ij}$

Estimation based on the Likelihood Function

- Develop a framework based on the likelihood function
- What is maximum likelihood estimation?
  Find β that makes the observed network most likely under the model with β as the parameter.

The log of the *likelihood* (as a function of β) is

$$\ell(\beta|y,\ x) \equiv \log[P(Y = y|X = x,\ \beta)] \quad \beta \in \mathrm{R}^K, \quad \text{given } y \in \mathcal{Y}$$

Example: Homogeneous Bernoulli model Rényi and Erdős model

- Y_{ij} are independent and equally likely

$$P(Y_{ij} = 1 | X = x, \beta) = \frac{\exp(\beta)}{1 + \exp(\beta)} \qquad \forall \;\; i, j = 1, \ldots, n$$

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- Equivalently

$$\log \text{odds}(Y_{ij} = 1 | X = x, \beta) = \beta \qquad \forall \; i, j = 1, \ldots, n$$

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More abstractly:

$$P(Y = y|X = x,\ \beta) = \frac{e^{\beta g(y)}}{\kappa(\beta)} \qquad y \in \mathcal{Y}$$

$\kappa(\beta) = [1 + \exp(\beta)]^N$

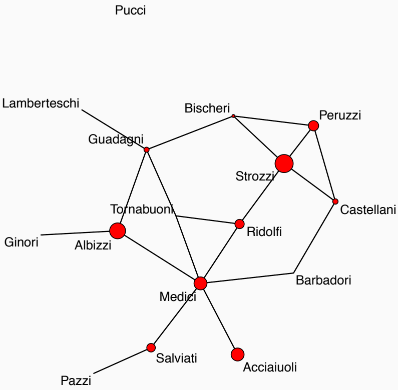
$$g(y) = \sum_{i,j} y_{ij} \quad \text{the number of ties}$$

The log of the *likelihood* (as a function of β) is

$$\ell(\beta|y,\, x) \equiv \log[P(Y = y|X = x,\ \beta)] \ = \ \beta g(y) - N \log[1 + \exp(\beta)] \quad -\infty$$

Example: Effect on the number of Priorates on centrality

Let X be the number of priorates (seats on the civic council) held by the family between 1282- 1344.



Joint specification

Recall

$$P(Y = y|X = x,\ \beta) = \frac{e^{\beta_1 g_1(y)+\beta_2 g_2(y)\ldots+\beta_K g_K(y)}}{c(\beta)} \qquad\qquad y \in \mathcal{Y}$$

where

$$g_k(y) = \sum_{i,j} x_{1,ij} y_{ij}, \qquad k = 1, \ldots, K$$

for the covariates $x_{1,ij}, x_{2,ij}, \ldots, x_{K,ij}$

The *log-likelihood* is

$$\begin{aligned} & \ell(\beta; y,\ x) \equiv \log\left[P(Y = y|X = x,\ \beta)\right] \\ = \ & \beta_1 g_1(y) + \beta_2 g_2(y) \ldots + \beta_K g_K(y) \\ - \ & \sum_{i,j} \log[1 + \exp(\sum_{k=1}^{K} \beta_k x_{k,ij})] \end{aligned}$$

Maximum Likelihood Estimation

Our goal is to find the

Maximum likelihood estimator (MLE):
The maximizer $\hat{\beta}$ of the loglikelihood function, $\ell(\beta|y,\ x)$

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where

$$I(\beta) = \mathbb{E}_Y\left[-\frac{\partial^2 \ell(\beta; y,\ x)}{\partial \beta^T \partial \beta}(\beta)\right] \tag{1}$$

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- We can compute $p-$values and confidence intervals

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- In particular $\text{deviance}(\beta = 0) - \text{deviance}(\beta = \hat{\beta})$

Example: Effect on the number of Priorates on centrality

For $k = 1, ..., n$:

$$x_{ij}^{priorates} = \text{Number of priorates for } i + \text{Number of priorates for } j$$

$$x_{ij}^{density} = 1/N$$

Example: Effect on the number of Priorates on centrality

```
Formula:   flomarriage ~ density + nodecov("priorates")

                    Estimate Std. Error p-value
density             -307.82    64.181     <1e-04 ***
nodecov.priorates    0.01643  0.0073    0.0276 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05

    Null  Deviance: 166.355  on 120  degrees of freedom
 Residual Deviance: 102.921  on 118  degrees of freedom
          Deviance:  63.435  on   2  degrees of freedom
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It is approximately χ^2 on K degrees of freedom.

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Does the model fit perfectly?

Compare the *residual deviance* to a χ^2 on $N - K$ degrees of freedom.

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