Class Notes 2. Bond Pricing with Uncertainty

Prof. Aaron Tornell

UCLA

Rate of Return on a Bond

- In order to buy one bond at time t, you need to pay p_t.
- The bond will pay at $t+1$, a coupon c_{t+1}
- And you will be able to sell it at a price, p_{t+1}
- Thus, your return will be: c_{t+1} plus your capital gain, $p_{t+1} - p_t$.
- To obtain the "rate of return" (RoR), we divide by the original cost of the bond:

$$R_{t+1} = \underbrace{\frac{c_{t+1}}{p_t}}_{\text{current yield}} + \underbrace{\frac{p_{t+1}-p_t}{p_t}}_{\text{capital gains rate}}$$

How to Determine the Price of a Bond?

- To answer this question we will make two assumptions:
  1. Suppose that investors are **risk-neutral**.
  2. Consider a **riskless one-period bond** that costs \$1 at time t and pays $1 + r$ at $t + 1$.
- NOTE: To avoid confusion, we will denote the risk-free one-period interest rate by i^f rather than i_t.
- Now consider a one-period bond that "promises" to pay next period a coupon c_{t+1} and the face value F_{t+1}.

How to Determine the Price of a Bond?

- An investor with \$100, has two available investments
  1. Buy 100 riskless one-period bonds and pay \$100 at time t. She will receive at $t+1$
  $$\$100 \cdot [1 + i^f] \tag{1}$$
  2. Use the \$100 to buy $\$100/p_t$ bonds at time t. She will receive at $t+1$
  $$\frac{\$100}{p_t} \cdot [c_{t+1} + F_{t+1}] \tag{2}$$

How to Determine the Price of a Bond?

- Because all investors are risk-neutral, in equilibrium the returns in (1) and (2) must be the same.

$$\begin{aligned}
\$100 \cdot [1 + i^f] - \$100 &= \frac{\$100}{p_t} \cdot [c_{t+1} + F_{t+1}] - \$100 \\
[1 + i^f] &= \frac{1}{p_t} \cdot [c_{t+1} + F_{t+1}] \\
p_t &= \frac{c_{t+1} + F_{t+1}}{1 + i^f} \qquad (3)
\end{aligned}$$

How to Determine the Price of a Bond?

- What would be the price of the bond if it has a longer maturity?
- At $t+1$, the investor will sell the bond for p_{t+1}.
  Thus she will receive at $t+1$,

$$\frac{\$100}{p_t} \cdot [c_{t+1} + p_{t+1}] \tag{4}$$

- Instead of (3), we have that the bond price is

$$p_t = \frac{c_{t+1} + p_{t+1}}{1 + i^f} \tag{5}$$

How to Determine the Price of a Bond?

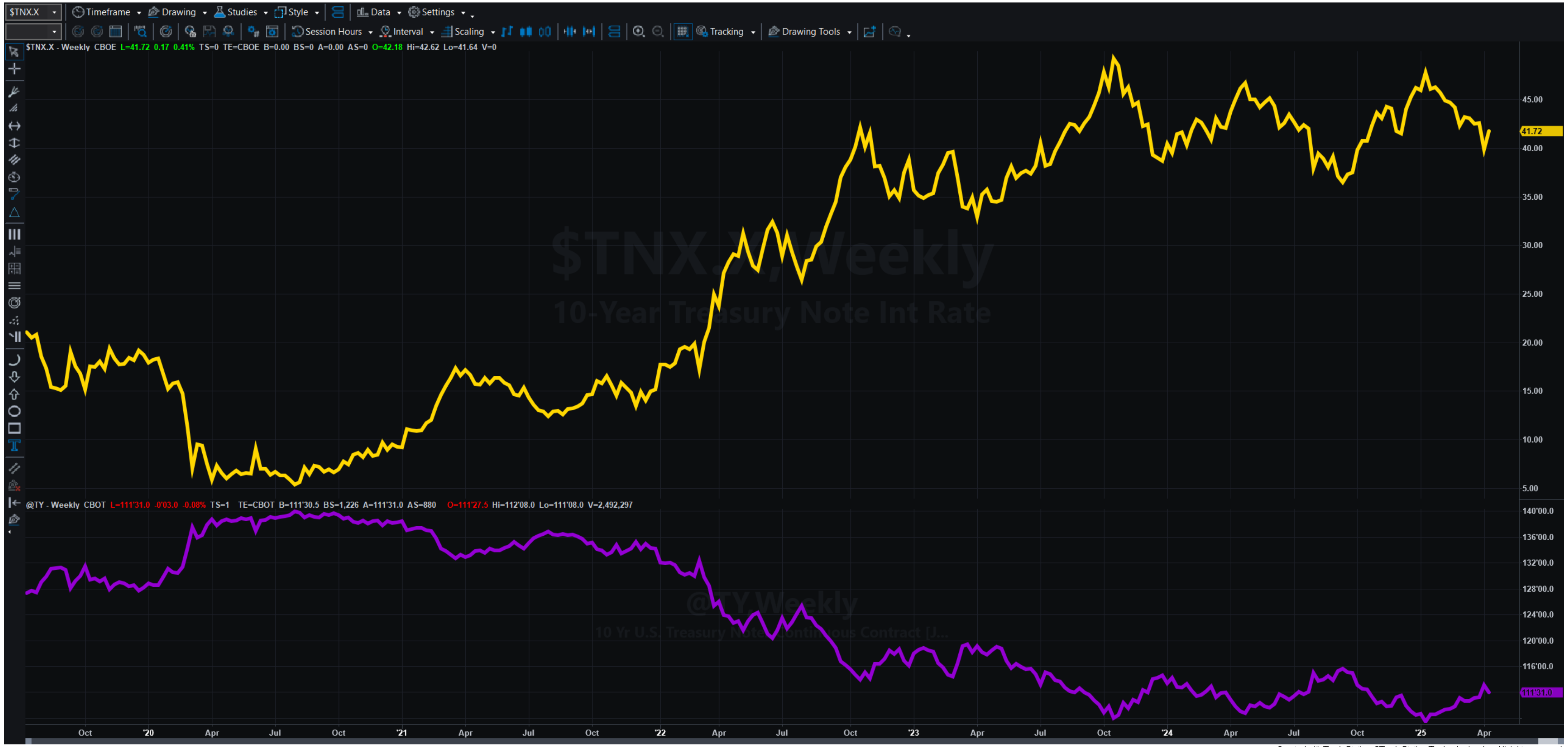
- What is the price of a one-period bond if investors expect a "haircut"? That is, they expect to get $E_t(F_{t+1})$ instead of the promised face value F_{t+1}.

  Thus the bond price is

$$p_t = \frac{c_{t+1} + E_t(F_{t+1})}{1 + i^f} \tag{6}$$

  where $E_t(F_{t+1})$ is the expected value of F_{t+1}.

Ten-year Treasuries Interest Rate (Yellow)
Ten-Year Treasuries Price (Purple)
From 2020 to 4/8/2025



How to Determine the Price of a Bond?

- To make things simple suppose that there are only two possibilities: at $t+1$ the bond will either repay the full face value or default and repay zero.
  If the probability of repayment is u we get
  $F_{t+1} = \begin{cases} F & \text{with probability} \;\; u \\ 0 & \text{with probability} \;\; 1-u \end{cases}$
- Therefore the expected value of F_{t+1} is
  $E_t(F_{t+1}) = u \cdot F + [1-u] \cdot 0 = u \cdot F$
- If we replace in (6), it follows that the price of a one-period bond with zero coupon if investors expect a "haircut" is given by

$$p_t = \frac{0 + u \cdot F}{1 + i^f} = \frac{u \cdot F}{1 + i^f} \tag{7}$$

How to Determine the Price of a Bond?

- What is the price of a multi-period bond if investors are uncertain about the future price of the bond?
- Combining (5) and (6) we get

$$p_t = \frac{c_{t+1} + E_t(p_{t+1})}{1 + i^f} \tag{8}$$

Spread High Yield (BB or Lower) minus Treasuries: 4.6% on 4/7/2025