Mathematical Modeling

Task 2

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I. For each of the following discrete dynamical systems:

(a) Xn+1 = 1 - 0.5 Xn

(b) xn+1 = 2 Xn - 3 Xn^2

(c) Xn+1 = 2 Xn - (1 / 3)* Xn^3

(a) **(i) Find the fixed points (steady states) and test for stability using the first-derivative test.**

Xn+1 = 1 - 0.5 Xn

X* = 1 - 0.5 X*

X*(1 + 0.5) = 1

X* = 2 / 3

Fixed point X* = 2 / 3

First derivative test:

f'(x) = -0.5

f'(X* = 2 / 3) = -0.5

| f'(X* = 2 / 3) | = 0.5 < 1 , => that fixed point X* = 2 / 3 is stable

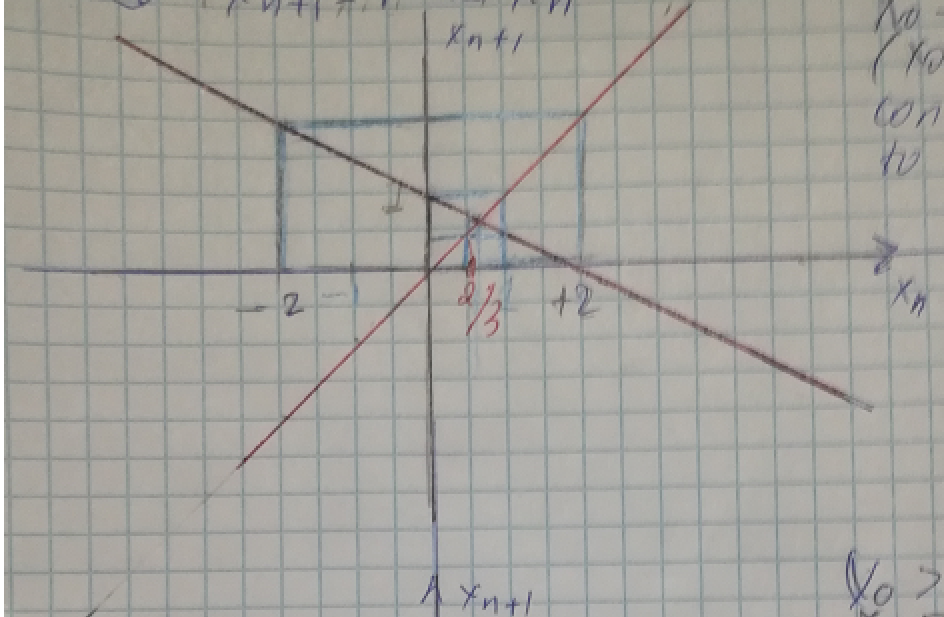
(ii) Draw a cobweb diagram (by hand) and try to describe the long-term behavior of theiterates for various initial conditions x0. Check your work using the Java applet or the Maple `cobweb.mw' le (posted on D2L Week 2 folder

Problem 1

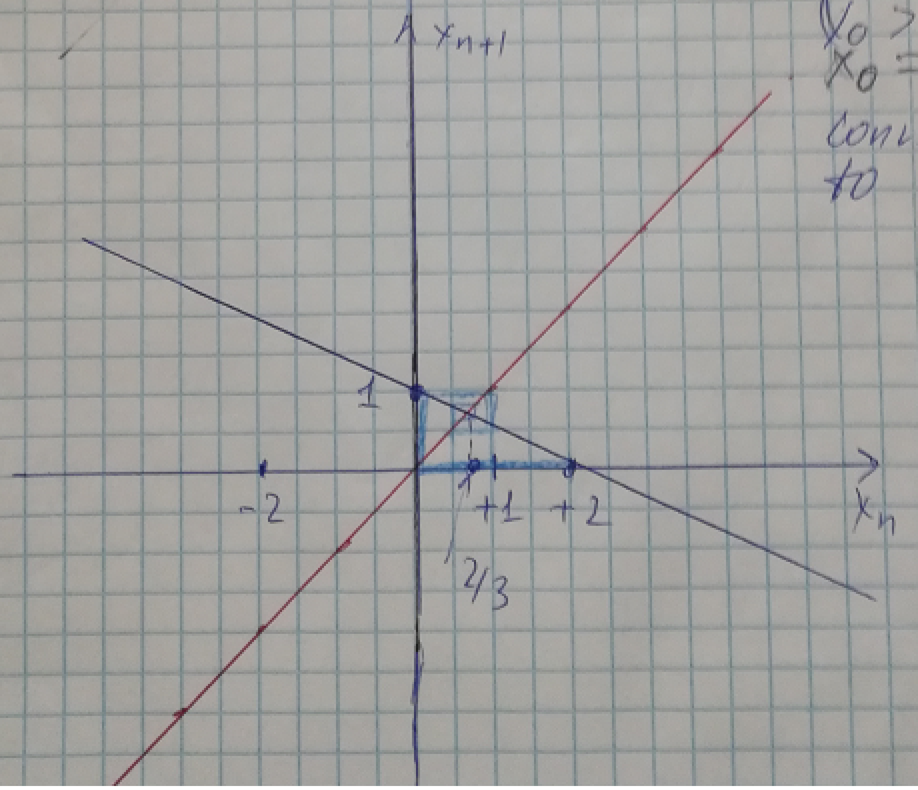
Draw cobweb Diagrams

(a) $x_{n+1} = 1 - 0.5\,x_n$

$x_0 = -2.0$
($x_0 < 0$)
converge to $2/3$



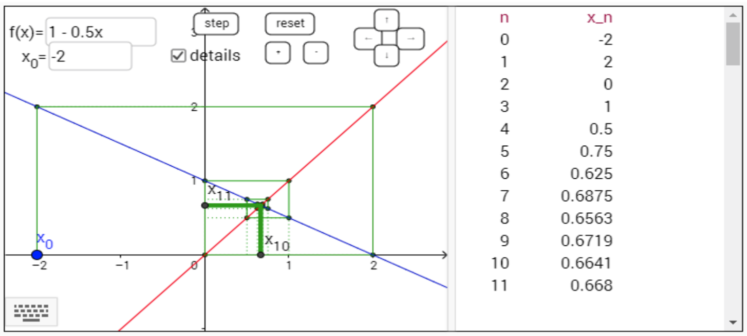
($x_0 > 0$)
$x_0 = 2.0$
converge to $2/3$



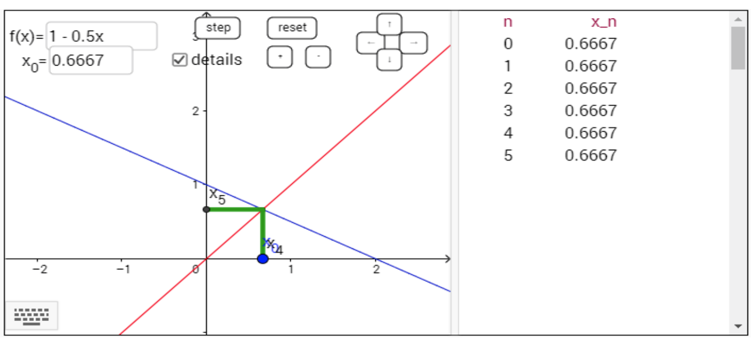
(iii)For each stable fixed point (if there are any), use the cobwebbing technique to estimate its basin of attraction (the set of all initial values that get attracted by the stable fixed point).

Basin of attraction for $X^* = 2/3$ $\forall$ X0 $\in (-\infty, +\infty)$
Xn+1 converge to $X^* = 2/3$

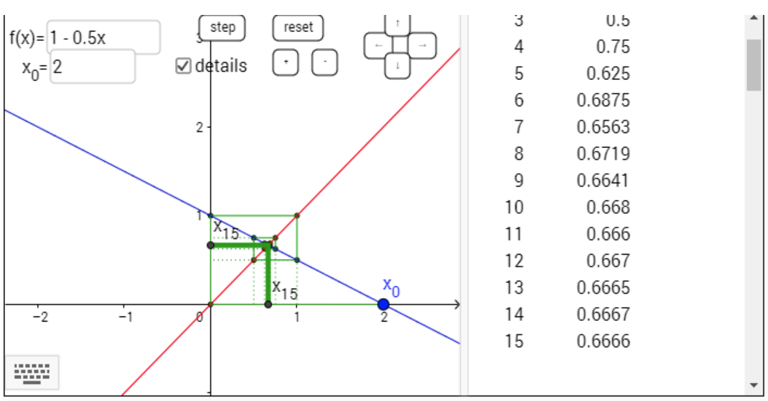
Cobweb
X0 = -2, Converge to 2 / 3



X0 = 2 / 3, Converge to 2 / 3



X0 = 2, Converge to 2 / 3



(b) (i) Find the fixed points (steady states) and test for stability using the

$X_{n+1} = 2X_n - 3X_n^2$

$X^* = 2X^* - 3X^{*2}$

$X^*(1 - 3X^*) = 0$

$X^*_1 = 0$, $X^*_2 = 1/3$

Fixed points: $X^*_1 = 0$, $X^*_2 = 1/3$

First derivative test:

$f'(x) = 2 - 6X$

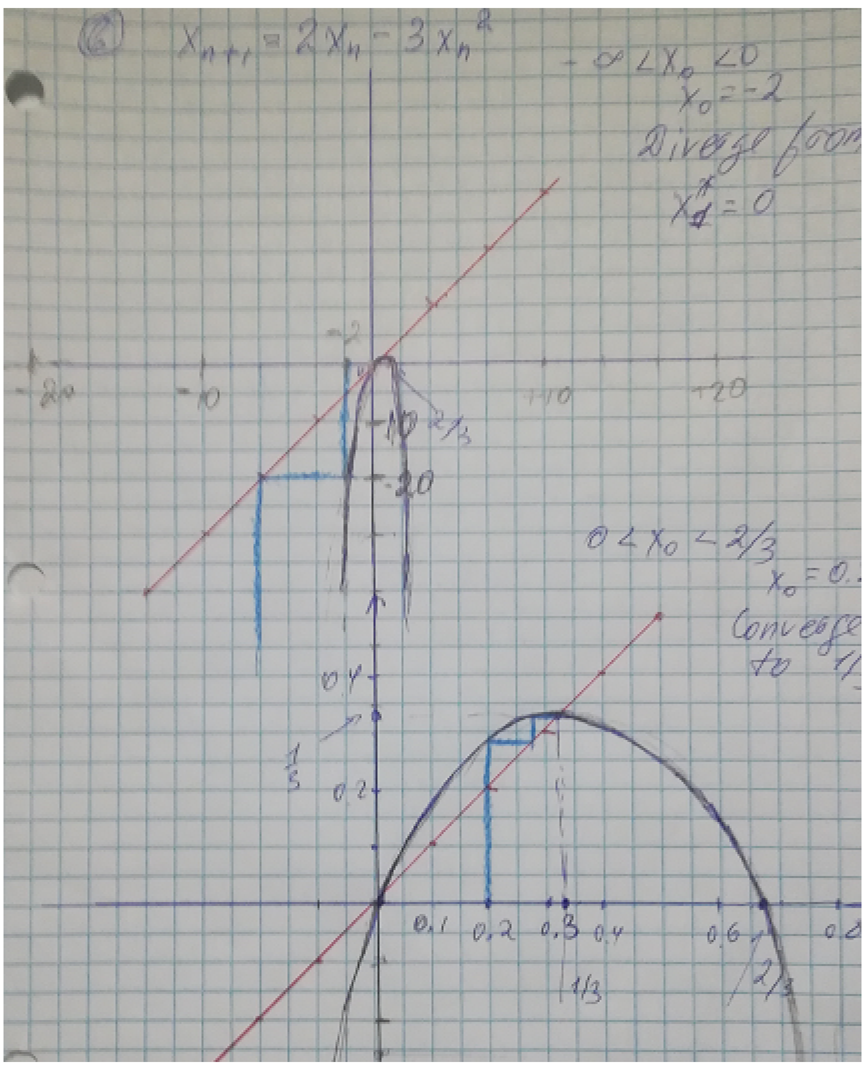
$f'(X^*_1 = 0) = 2 - 6 * 0 = 2$

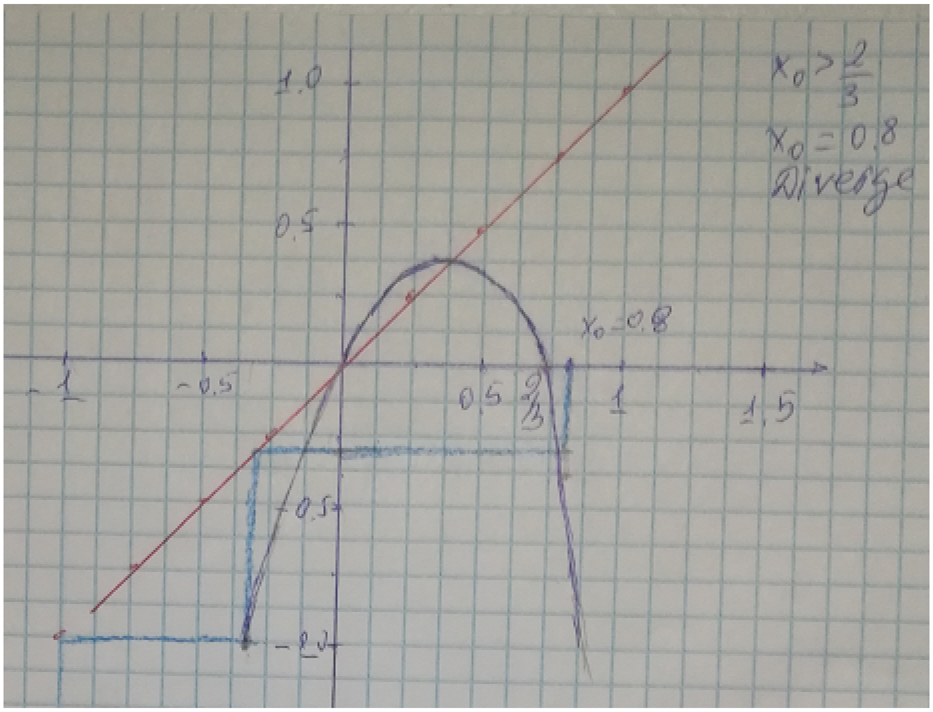
$f'(X^*_2 = 1/3) = 2 - 6 * 1/3 = 0$

$| f'(X^*_1 = 0) | = 2 > 1$, => that fixed point $X^*_1 = 0$ is unstable and X_{n+1} will diverge from $X^*_1 = 0$

$| f'(X^*_2 = 1/3) | = 0 < 1$, => $X^*_2 = 1/3$ is stable and X_{n+1} will converge to $X^*_2 = 1/3$

(ii) Draw a cobweb diagram (by hand)





(iii) For each stable fixed point (if there are any), use the cobwebbing technique to estimate its basin of attraction (the set of all initial values that get attracted by the stable fixed point).

Roots are:

2X- 3X^2 = 0

X(2 - 3X) = 0, => X1 = 0, X2 = 2 / 3

Basin of attraction for X*2 = 1 / 3 :

∀ X0 ∈] 0 , 2 / 3 [

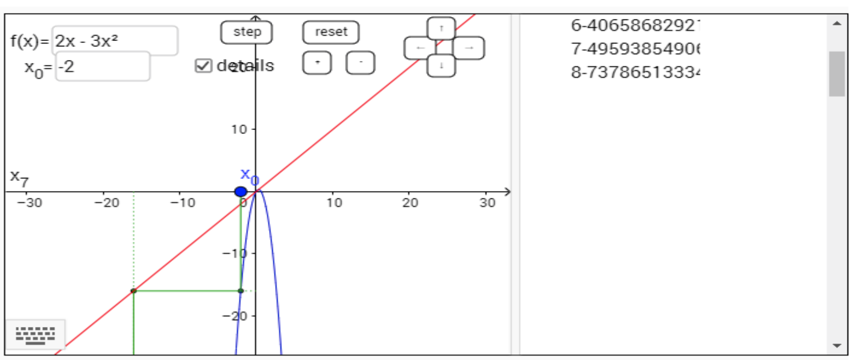
Xn+1 will converge to X*2 = 1 / 3

For X0 = 2 / 3 and X0 = 0 Xn+1 = 0

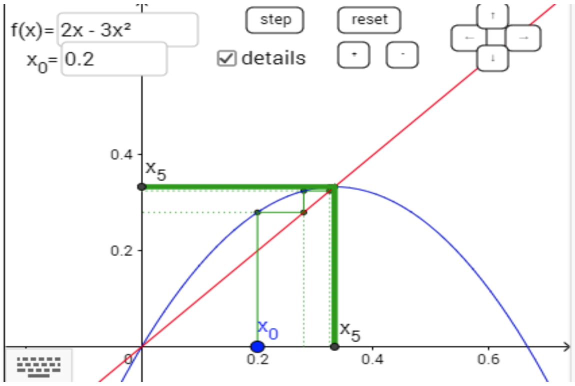
∀ X0 ∈ (**-**-∞ , 0 [U] 2 / 3 , +∞**)** Xn+1 diverge from X*2 = 1 / 3

Cobweb

X0 = -2 , Diverge

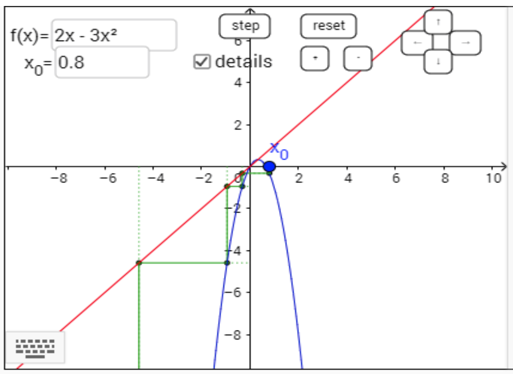


X0 = 0.2, Converge to X*2 = 1 / 3



n	x_n
0	0.2
1	0.28
2	0.3248
3	0.3331
4	0.3333
5	0.3333

X0 = 0.8 > 2 / 3, Diverge from X*2 = 1 / 3



n	x_n
0	0.8
1	-0.32
2	-0.9472
3	-4.586
4	-72.2651
5	-15811.2691
6	-750020317.8

(c) (i) Find the fixed points (steady states) and test for stability using the Xn+1 = 2Xn - (1 / 3)Xn^3

X* = 2X* - (1 / 3)X*^3

2X* - (1 / 3)X*^3 - X* = 0

X* - (1 / 3)X*^3 = 0

X* (1 - (1 / 3) X*^2} = 0 = >

1 - (1 / 3)X*^2 = 0 => X*^2 = 3 , X*2 = √3, X*3 = -√3

Fixed points: X*1 = 0, X*2 = √3, X*3 = -√3

First derivative test:

f'(x) = 2 - X^2

f'(X*1 = 0) = 2 - 0^2 = 2

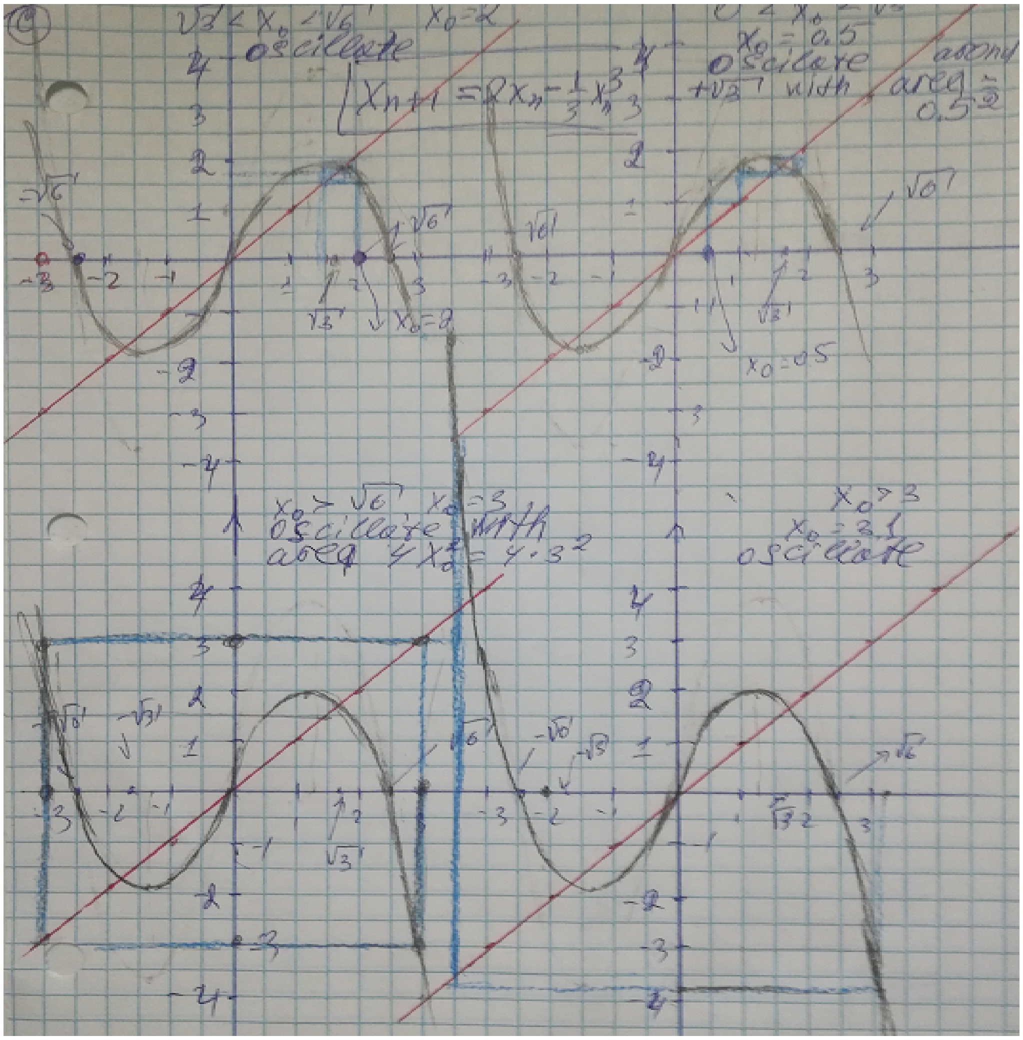
f'(X*2 = √3) = 2 - (√3)^2 = -1

f'(X*2 = -√3) = 2 - (-√3)^2 = -1

| f'(X*1 = 0) | = 2 > 1 , => that fixed point X*1 = 0 is unstable and Xn+1 will diverge from X*1= 0

| f'(X*2 =√3) | = |-1| = 1 ,
=> That fixed point X*2 = √3 is inconclusive
| f'(X*2 =-√3) | = |-1| = 1 ,
=> That fixed point X*2 = -√3 is inconclusive

(ii) Draw a cobweb diagram (by hand)



(iii) For each stable fixed point (if there are any), use the cobwebbing technique to estimate its basin of attraction (the set of all initial values that get attracted by the stable fixed point).

Roots are:
2X - (1 / 3)X^3 = 0
X(2 - (1 / 3)X^2) = 0
(1 / 3)X^2 = 2,   X^2 = 6, =>

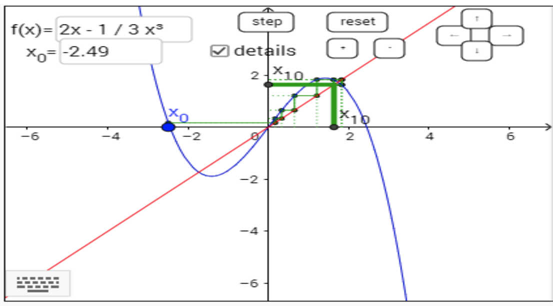
$X1 = 0$, $X2 = \sqrt{6}$, $X3 = -\sqrt{6}$

For $X0 = \sqrt{3}$, $Xn+1 = \sqrt{3}$
For $x0 = -\sqrt{3}$, $Xn+1 = -\sqrt{3}$
For $X0 = 0$, $X0 = \pm\sqrt{6}$, $Xn+1 = 0$
For $X0 \in$ **(** $-\infty$, $-\sqrt{6}[\ \cup \ ] -\sqrt{6}, -\sqrt{3} [\ \cup \ ] -\sqrt{3} , 0 [\ \cup \ ]0, \sqrt{3}[\ \cup \ ]\sqrt{3} , \sqrt{6} [\ \cup \ ]\sqrt{6} , +\infty$**)**
Xn+1 oscillate

Both fixed points do not have the Basin of attraction

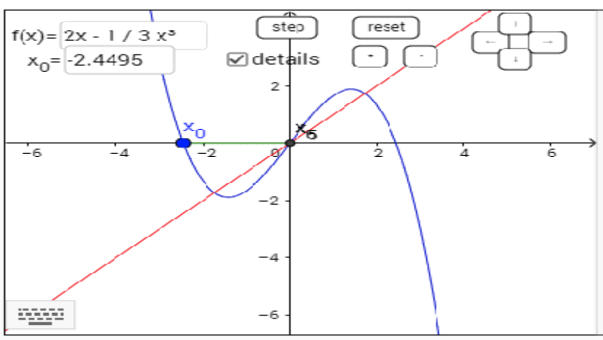
Cobweb

X0 = -2.49 < -$\sqrt{6}$, Xn+1 oscillate



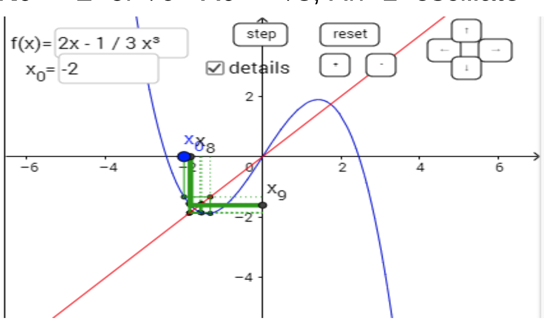
1	0.1661
2	0.3306
3	0.6492
4	1.2072
5	1.828
6	1.6199
7	1.8229
8	1.6267
9	1.8186
10	1.6323

X0 = -$\sqrt{6}$, Xn+1 = 0



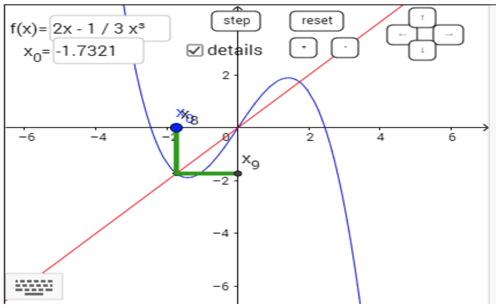
1	0
2	0
3	0
4	0
5	0
6	0

X0 = -2 or $\sqrt{6} < X0 < -\sqrt{3}$, Xn+1 oscillate



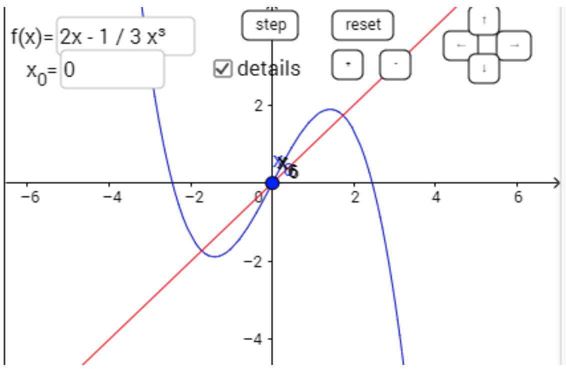
1	-1.3333
2	-1.8765
3	-1.5504
4	-1.8586
5	-1.5772
6	-1.8466
7	-1.5942
8	-1.8378
9	-1.6065

X0 = -√3, Xn+1 = -√3



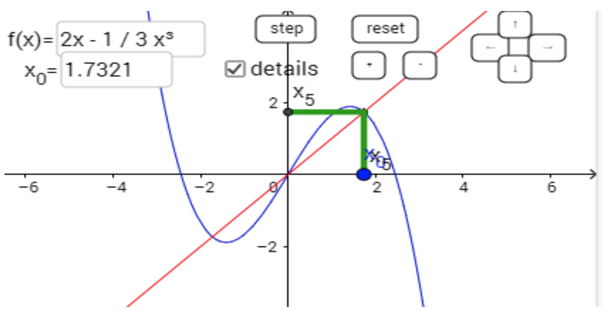
1	-1.7321
2	-1.7321
3	-1.7321
4	-1.7321
5	-1.7321
6	-1.7321
7	-1.7321
8	-1.7321
9	-1.7321

X0 = 0, Xn+1 = 0



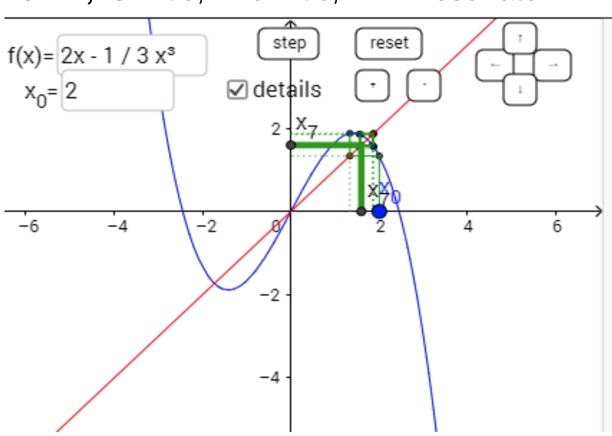
1	0
2	0
3	0
4	0
5	0
6	0

X0 = 0.5, or   0< X0 < √3, Xn+ 1 = √3



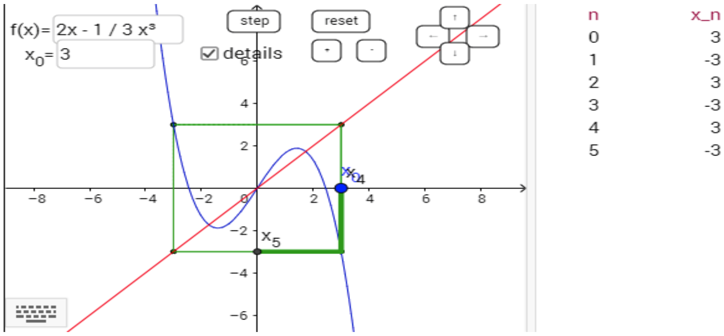
1	1.7321
2	1.7321
3	1.7321
4	1.7321
5	1.7321

X0 = 2, or   √3,< X0 < √6, Xn+1 oscillate

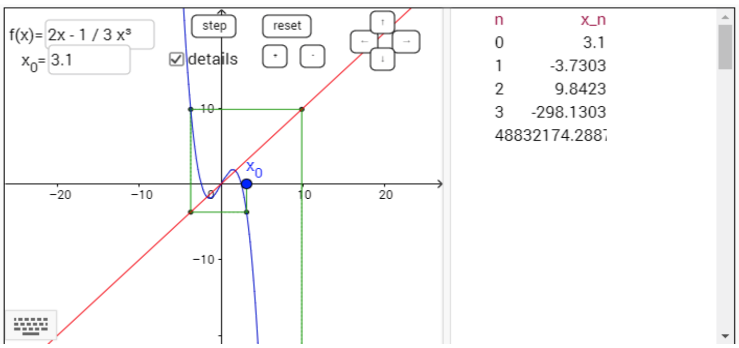


1	1.3333
2	1.8765
3	1.5504
4	1.8586
5	1.5772
6	1.8466
7	1.5942

X0 = 3, or X0 > √6,, Xn+1 oscillate



X0 = 3.1, X0>3 , Xn+1 oscillate



II. Here is a simple discrete model for the spread of a contagious disease: Let Xn denote the percentage of the student population at time n (in days) in a college dorm with a severe case of the flu. If all are susceptible of the disease, then
(1 - Xn) represents the percentage of those susceptible but not yet infected at time n. If those infected remain contagious, then the percentage of newly infected individuals at time n+1, xn+1-xn, is assumed to be proportional to the number of all possible interactions between those infected and those not infected at time
n. In other words: Xn+1 - Xn = k* Xn *(1 - xn); for some fraction k, 0 < k < 1.

Question: Analyze the long-term behavior of the percentage of infected students xn. In other words, and the xed points, analyze their stability, draw cobweb diagrams.
Hint: The difference equation can be rewritten as Xn+1 = Xn + k * Xn - k* X^2 =
(k+1)Xn-k*Xn^2, so the dynamic rule is given by the function
f(x) = (k + 1)*X - kX^2, with 0 <= x <= 1

Xn+1 = Xn + k * Xn (1-Xn) =Xn + kXn - kXn^2
Xn+1 = Xn(1 + k) - kXn^2, 0 < k < 1, 0 <= x <=1
X* = X*(1 + k) - kX*^2
X* + kX* - kX*^2 - X* = 0
kX*(1-X*) = 0
X*1 = 0, X*2 = 1
Fixed point are: X*1 = 0, X*2 = 1
First Derivative Test:
f'(x) = (X(1+k) - kX^2)' = k + 1 - 2kX = 1 + k(1-2X)
f'(X*1=0) = 1 + k(1 - 2 * 0) = 1 + k
f'(X*2=1) = 1 + k(1 - 2 * 1) = 1 - k

| f'(X*1=0) | = | 1 + k |
Because <0<k<1 then | 1 + k | is always > 1 , therefore fixed point X*1 = 0
Is unstable and Xn+1 diverge from X*1 = 0

| f'(X*2=1) | = | 1 - k | < 1   there the following system of inequalities has to have a solution for fixed point to be stable
1 - k < 1   k > 0   k > 0
-(1-k) < 1 =>   -1 + k < 1   =>   k < 2   => solution   0 < k < 2
Because we have a condition 0< k < 1
And fixed point X*2=1 is stable and converge to X*2=1

Find Basin of attraction for fixed point X*2=1, 0 <= x <= 1
Roots are :
X(1 + k) - kX^2 = 0, => X1 = 0, X2 = 1 + 1/k

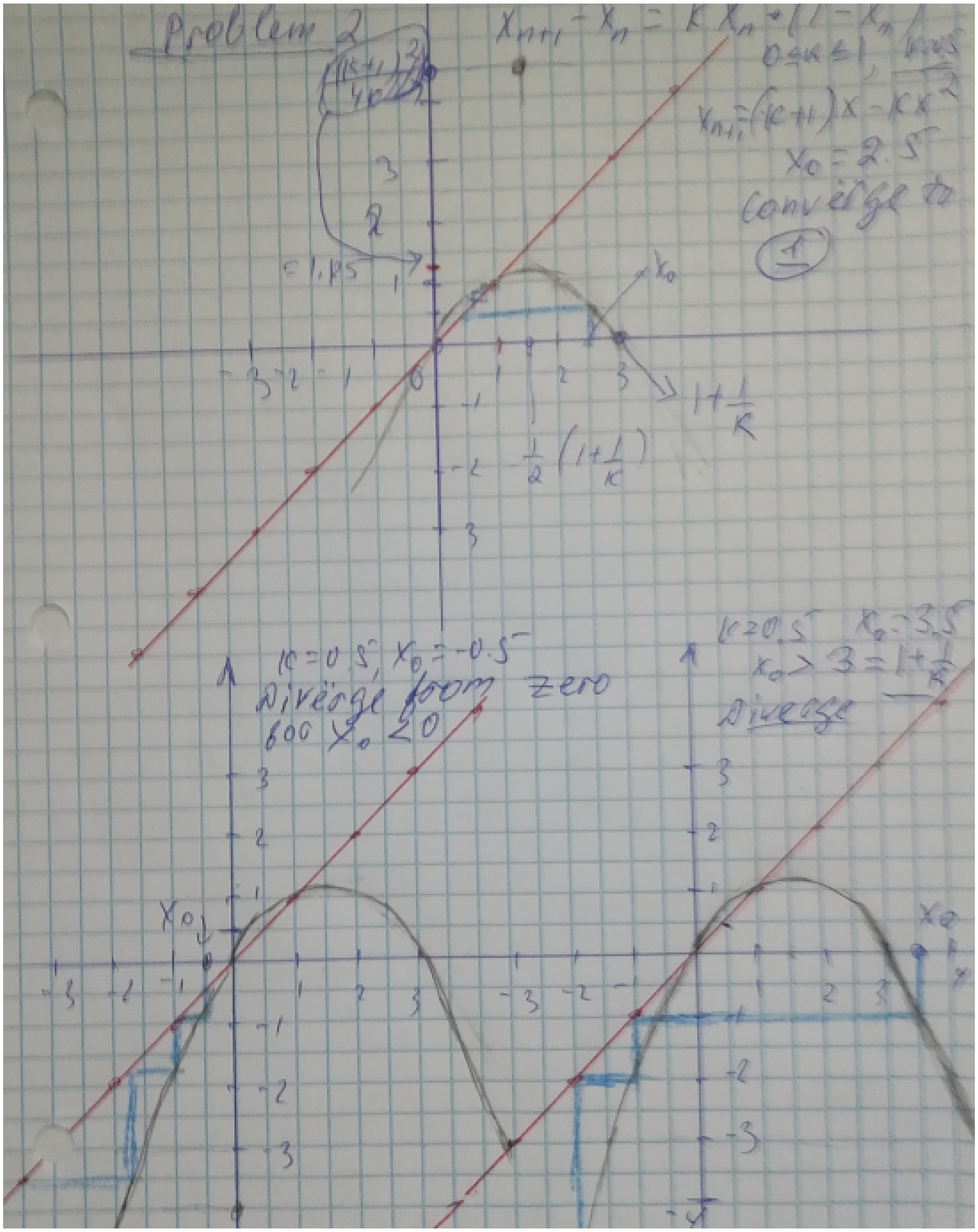
Basin of attraction for a fixed point X*2=1 is:

∀ X0 ∈] 0 , 1 + 1/k [Xn+1 will converge to X*2=1

∀ X0 ∈] 1, + ∞) Xn+1 will diverge

For X0 = 0 , Xn+1 = 0

For X0 = 1 , Xn+1 = 1



III. Consider a slightly modified version of the logistic model given by the recursive relation:
$x_{n+1} = r * x_n * (1 - x_n)$
where r is a positive growth parameter, $r > 0$. The goal is to understand how the dynamics of this model changes with respect to parameter r.
(a) Show that the model admits two fixed points: $x=0$ and $x = (r - 1)=r$.

$X^* = r * X^* \ (1 - X^*)$
$X^* (r - 1 - r^*X^*) = 0$ ->

$X^*1 = 0$,
$r^*X^* = r - 1$ ->

$X^*1 = 0$, $X^*2 = (r - 1) / r$

(b) Investigate the stability of each xed point (using the rst derivative test): show that
$x = 0$ is stable if $0 < r < 1$, and $x = (r - 1)=r$ is stable if $1 < r < 3$.

$f'(X) = (rX - rX^2)' = r - 2rx$
$f'(X^*1=0) = r(1 - 2 * 0) = r$
$f'(X^*2=(r-1)/r) = r(1 - 2 * (r-1)/r) = r - 2(r-1) = 2-r$
$| f'(X^*1=0) | = |r| < 1$ for fixed point to be stable, -> system of inequalities

$r < 1$ $r< 1$
$-r < 1$ -> $r > 0$ or $r < 1$ U $r > 0$
$0< r < 1$ is a condition for fixed point $X^*1 = 0$ to be stable

$| f'(X^*2=(r-1)/r) | = | 2 - r| < 1$ for fixed point to be stable, -> system of inequalities
$2 - r < 1$ $r > 1$ $r > 1$
$-(2 - r) < 1$ -> $-2 + r < 1$ -> $r < 3$ -> $r > 1$ U $r < 3$
$1 < r < 3$ is the condition for fixed point $X^*2 = (r - 1)/r$ to be stable

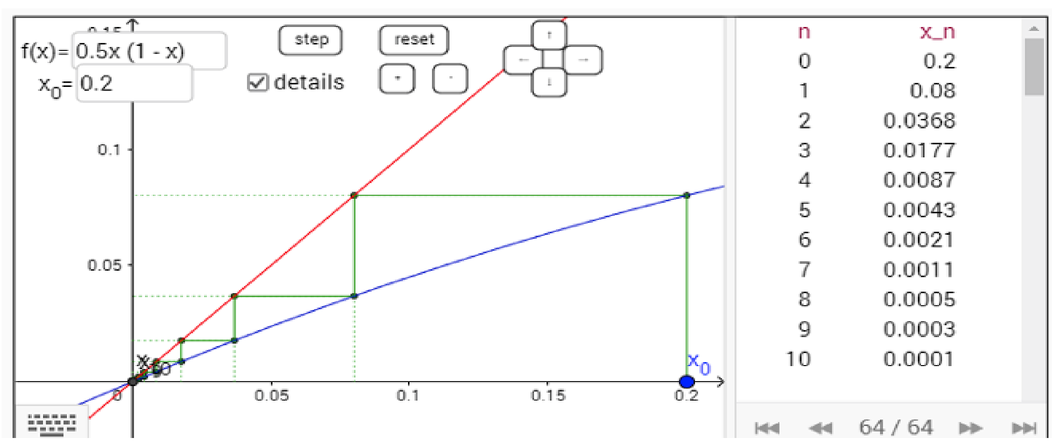
(c) For each stable fixed point, use the cobwebbing technique to estimate its basin of attraction: the set of all initial values x0 > 0 that get attracted by the fixed point.

Answer for Fixed point X*1 = 0, X0 > 0 AND 0< r <1
Basin of attraction for X*1 =0 : { X0 ∈ (0,1) U x0 ∈ (1, 1/r [} ->
Xn+1 will converge to X*1 = 0

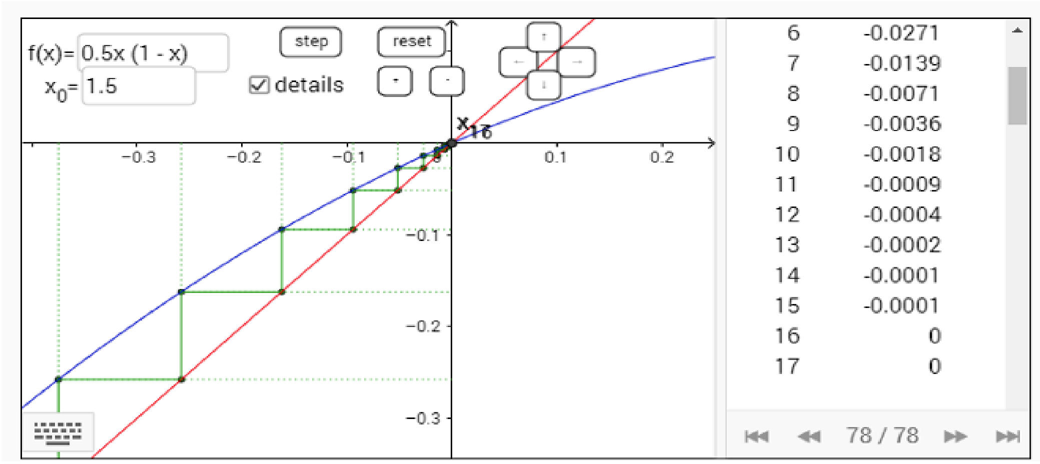
For x0 ∈ (-∞, 0[U] 1/r , +∞) Xn+1 will diverge
For x0 = 1/r , Xn+1 will oscillate between X0 and -(X0-1)

Cobweb for Fixed point X* = 0

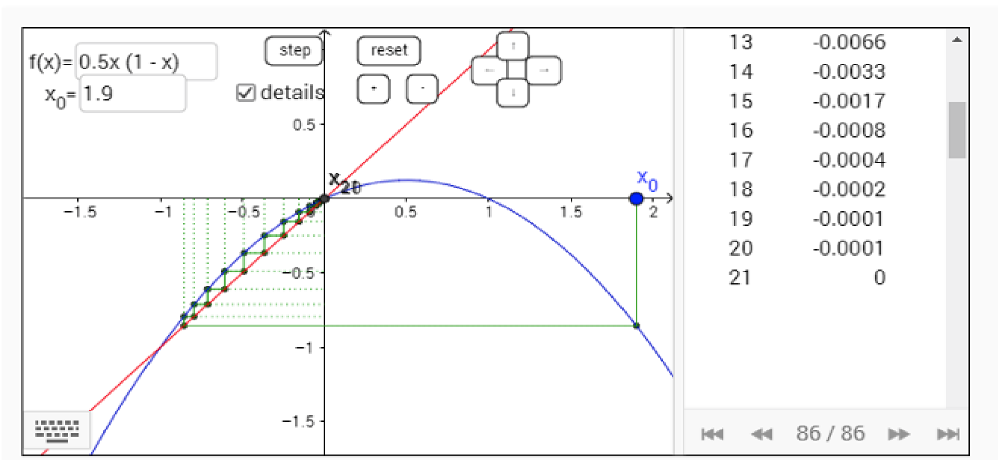
r = 0.5, X0 = 0.2, 0<X0<1 And X0 != 1/r
Converge to 0



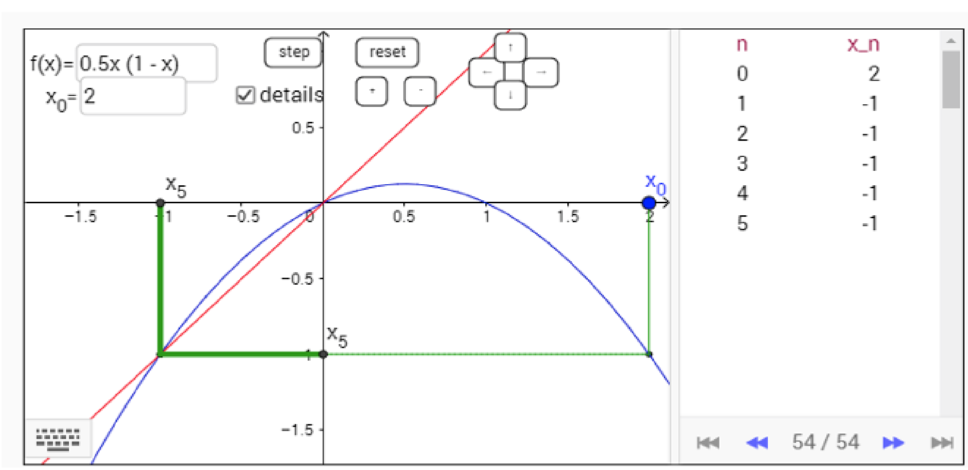
r = 0.5 , X0 =1.5, X0 > 1 And X0 != 1/r
Converge to 0



r = 0.5, X0 = 1.9 , X0 > 1 And X0 < 1/r
Converge to 0

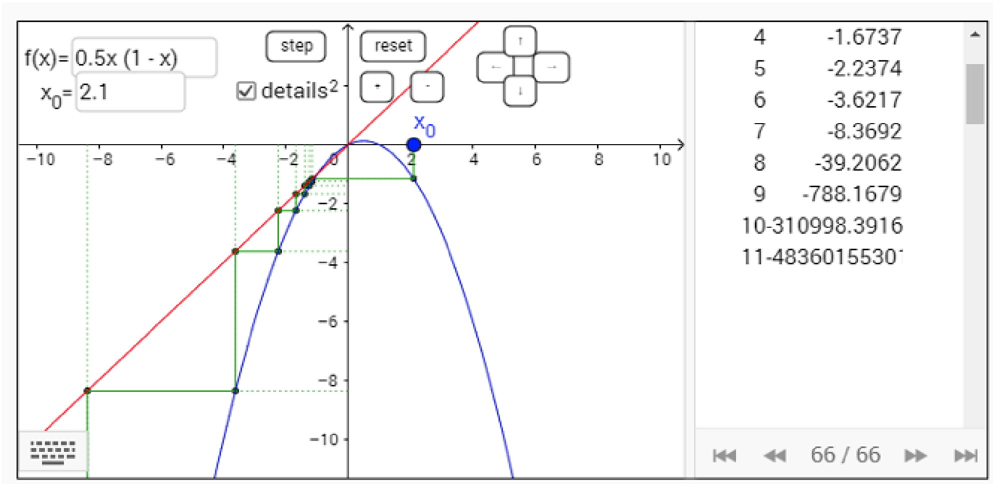


r = 0.5, X0 = 2, X0 > 1 And X0 = 1/r
Oscillates between X0 and -1 or between 2 and -1



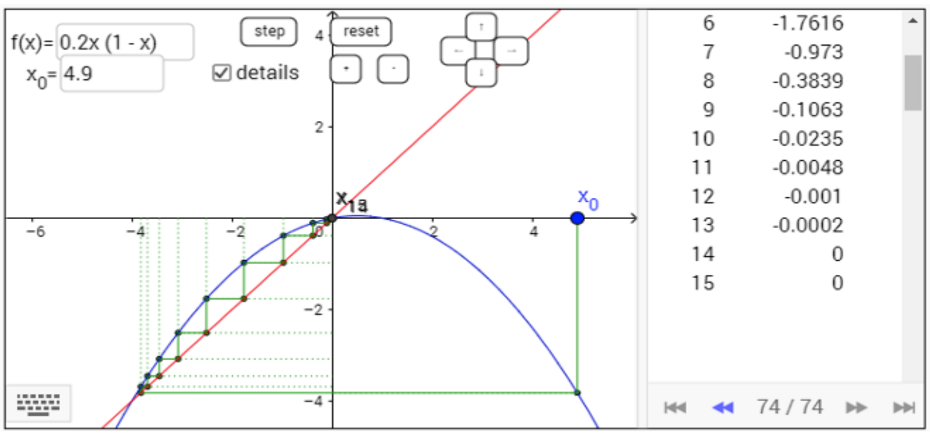
r = 0.5, X0 = 2.1 , X0> 1 And X0 > 1/r = 2

Diverge



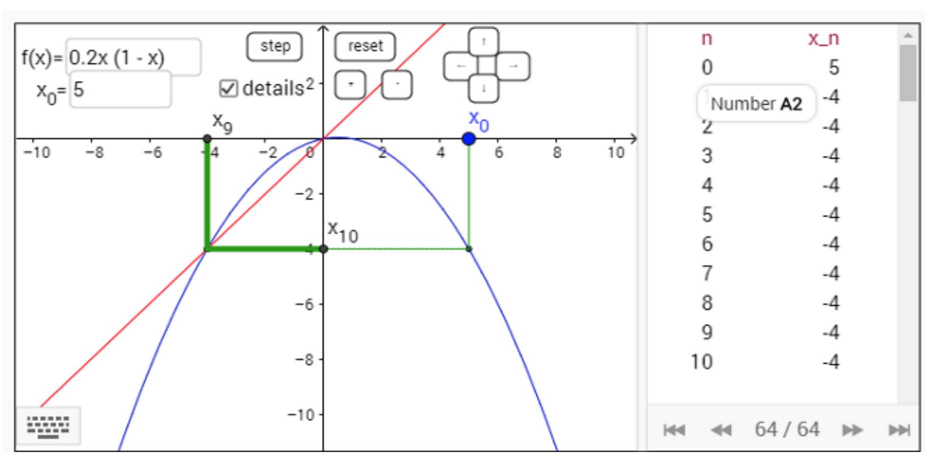
r = 0.2, X0 = 4.9 X0> 1 And X0<1/r=5

converge to 0



r = 0.2, X0=5.0, X0 > 1 And X0 = 1/r =5

Oscillates between X0 and -4 or between 5 and -4



Answer for Fixed point X*2 = (r - 1) / r, X0 > 0 AND 1< r <3

Basin of attraction for X*2 = (r - 1) / r : { X0 ∈] 0,1[U x0 ∈ (1, 1/r [} ->

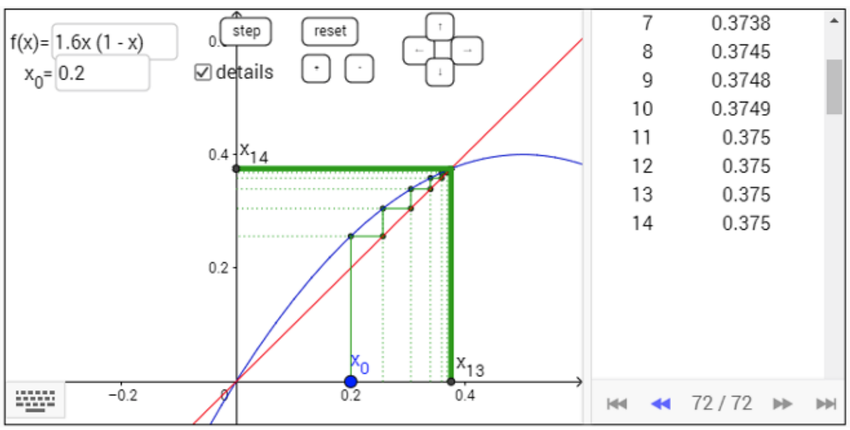
Xn+1 will converge to X*2 = (r - 1) / r

For x0 ∈ (-∞, 0[U] 1/r , +∞) Xn+1 will diverge
For x0 = 1 Xn+1 = 0
For x0 = 1/r , Xn+1 will oscillate between X0 and -(X0-1)

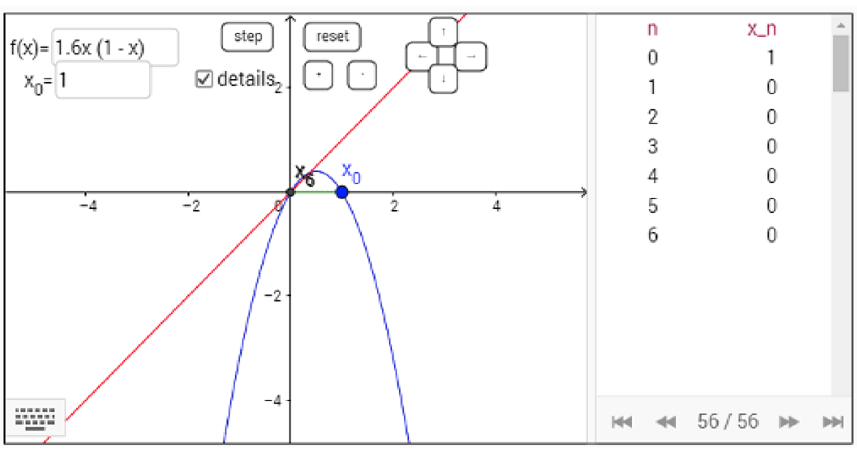
Answer for Fixed point X*2 = (r - 1) / r, X0 > 0   AND   r >= 3
Basin of attraction for X*2 = (r - 1) / r : { X0 = 1/r } ->
Xn+1 will converge to X*2 = (r - 1) / r

For x0 ∈] 0 , 1/r [U] 1/r+∞) Xn+1 will oscillate between 0.01 to 0.99
For x0 = 1 Xn+1 = 0

Cobweb for Fixed point X* = (r - 1) / r, 1 < r < 3
r =1.6, X0 = 0.2,   0 < X0 < 1
Converge to (r-1)/r = (1.6 -1)/1.6 = 0.3745

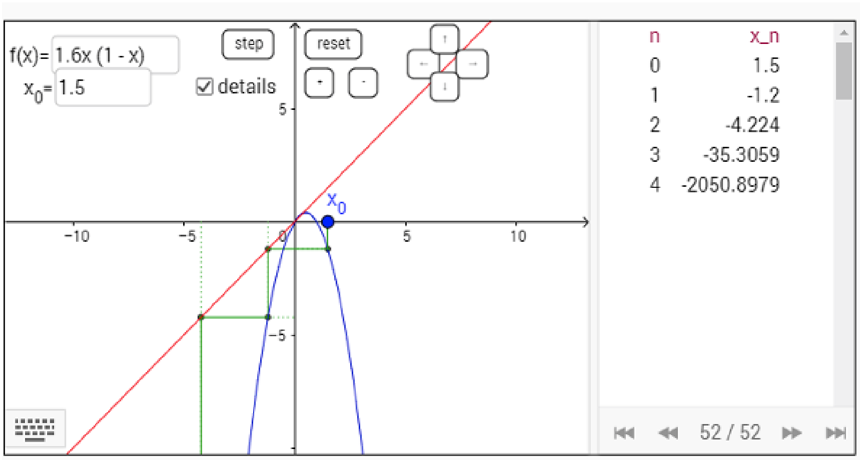


r =1.6, X0 = 1.0
Always = 0



r =1.6, X0 = 1.5 , X0 > 1

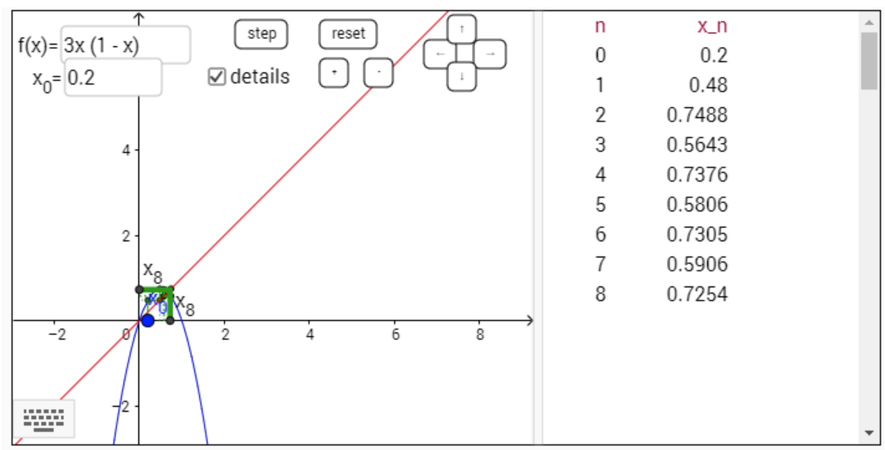
Diverge



Cobweb for Fixed point X* = (r - 1) / r, r >= 3

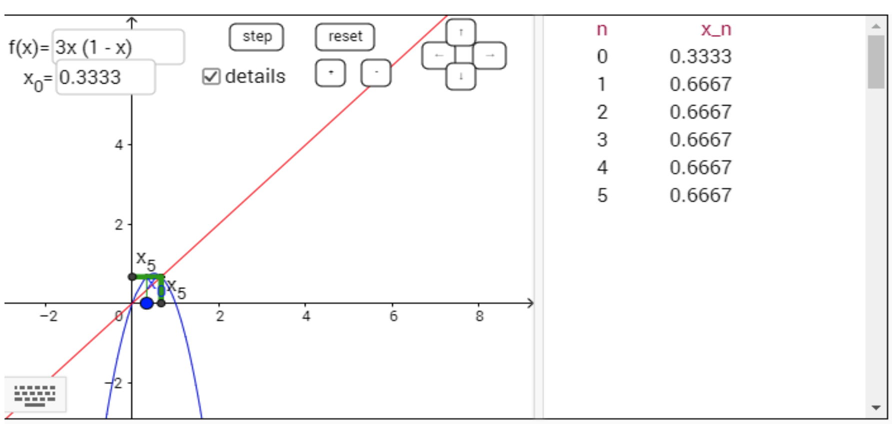
r = 3, X0 = 0.2, 0 < X0 < 1/r= 1/3

Xn+1 will oscillate between 0.01 to 0.99



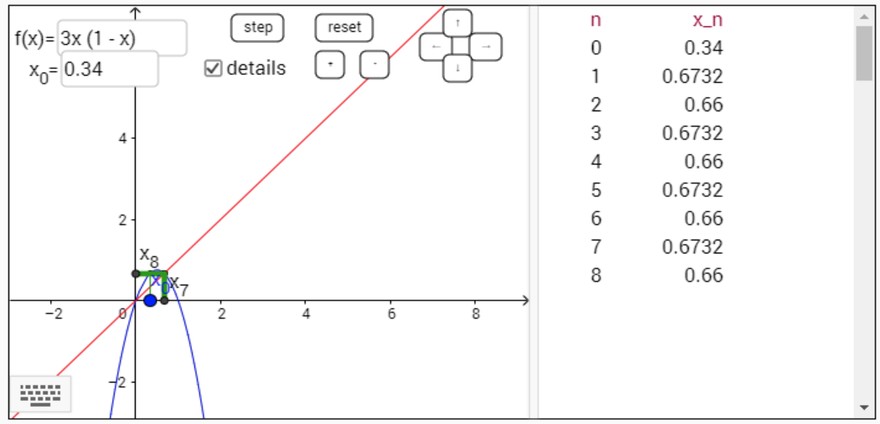
r = 3, X0 = 1/r = 1/3

Xn+1 will Xn+1 will converge to X*2 = (r - 1) / r = (3-1)/3 = ⅔



r = 3, X0 = 0.34,   1/3 = 1/r < X0 < +∞

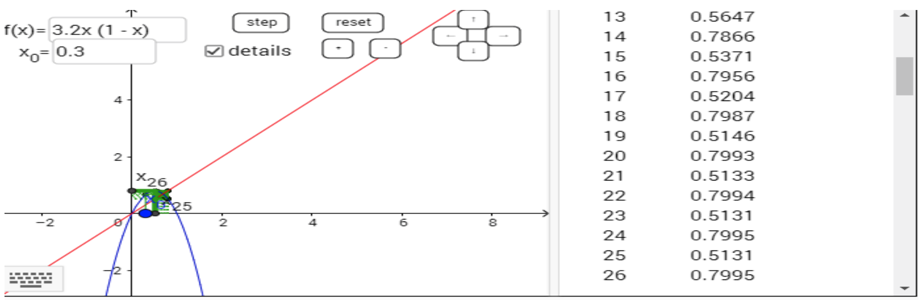
Xn+1 will oscillate between 0.01 to 0.99



(d) Use the cobweb applet or the 'cobweb' Maple le to investigate what happens with the values of xn for various values of r > 3: r = 3:2, r = 3:5, r = 3:65, r = 3:8293, r = 4.

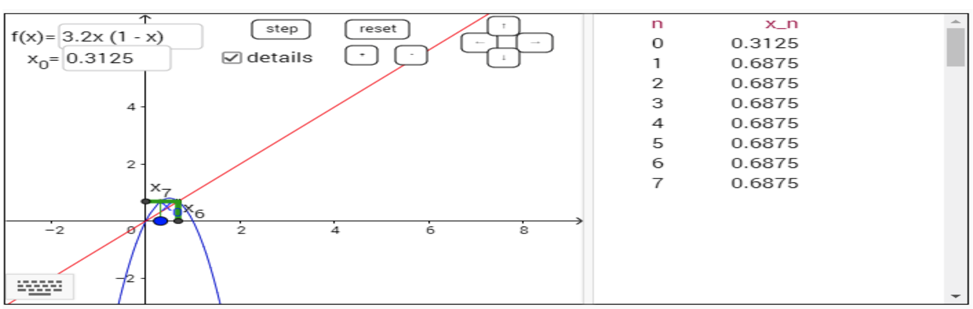
r = 3.2, X0 = 0.3, 0 < X0 < 1/r = (1 / 3.2) = 0.3125

Xn+1 will oscillate between 0.5 to 0.8 in this case



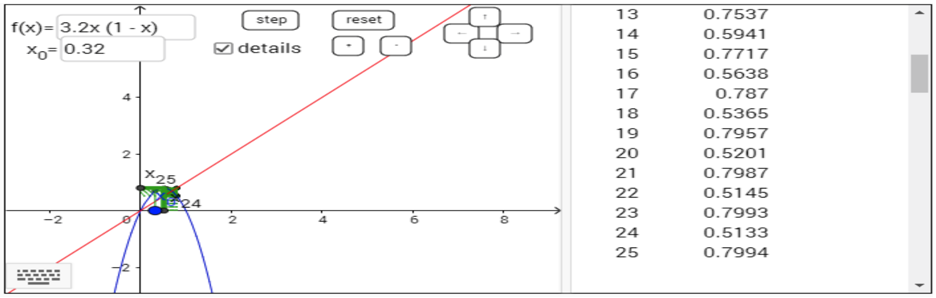
r = 3.2, X0 = 1/r = 1 / 3.2 = 0.3125

Xn+1 will converge to (r-1)/r = (3.2-1)/3.2 = 2.2 / 3.2 = 0.6875



r = 3.2, X0 = 0.32, X0 > 1/r = (1 / 3.2) = 0.3125

Xn+1 will oscillate between 0.5 to 0.8 in this case

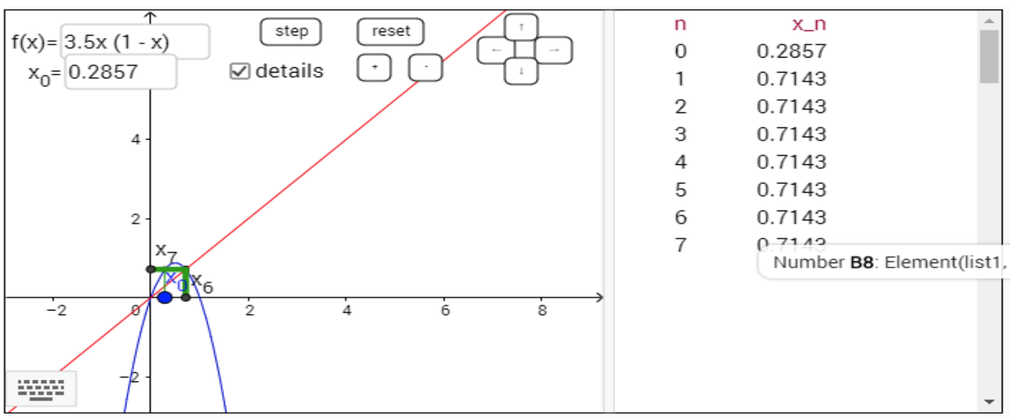


r = 3.5, X0 = 0.2, 0 < X0 < 1/r = (1 / 3.5) = 0.2857

Xn+1 will oscillate between 0.3 to 0.9 in this case

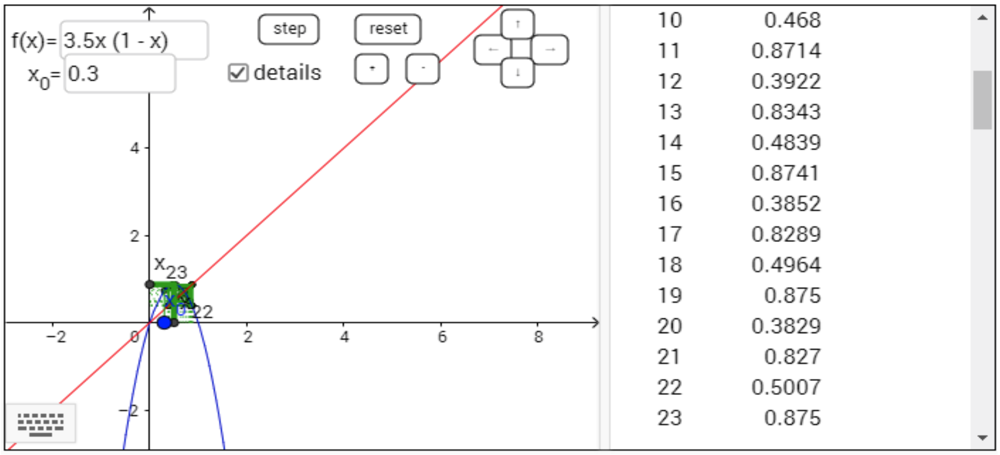
r = 3.5, X0 = 1 / 3.5 , X0 = 1/r = (1 / 3.5) = 0.2857

Xn+1 will converge to (r-1)/r = (3.5-1)/3.5 = 2.5 / 3.5 = 0.7143

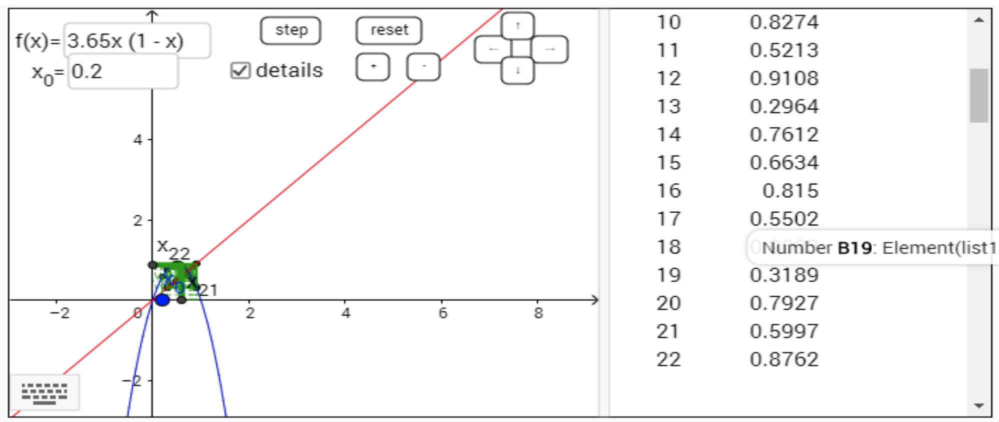


r = 3.5, X0 = 0.30, X0 > 1/r = (1 / 3.5) = 0.2857

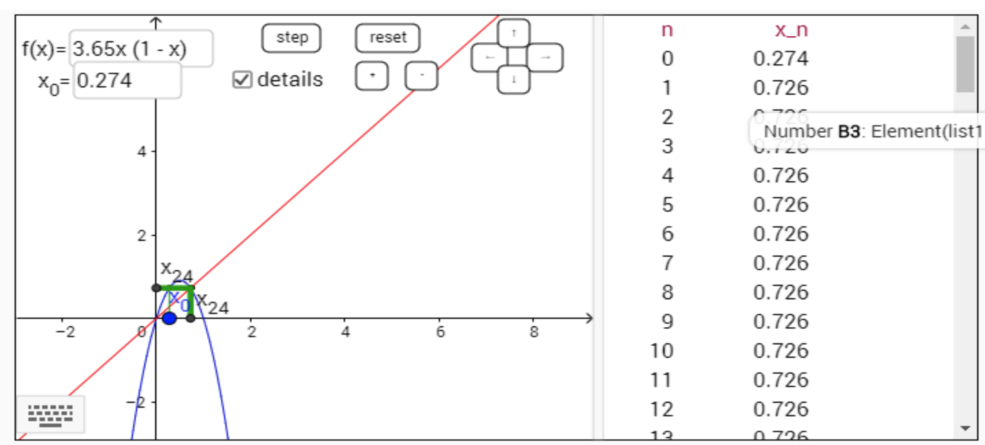
Xn+1 will oscillate between 0.3 to 0.9 in this case



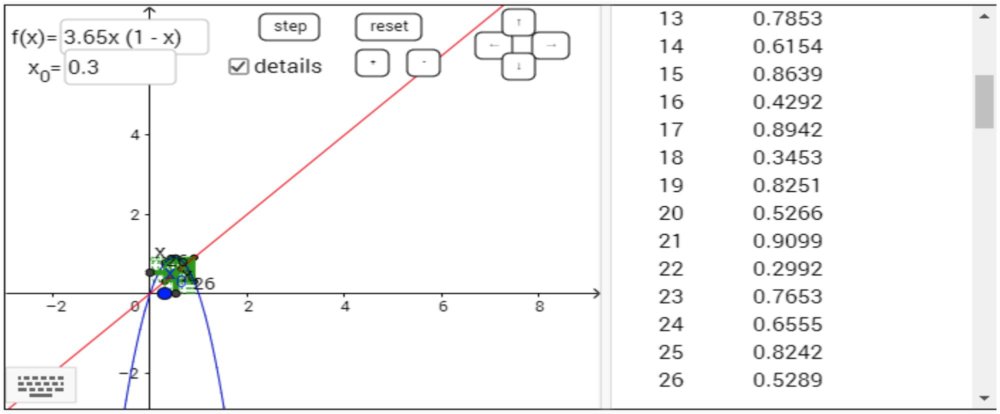
r = 3.65, X0 = 0.2, 0 < X0 < 1/r = (1 / 3.65) = 0.2740
Xn+1 will oscillate between 0.2 to 0.99 in this case



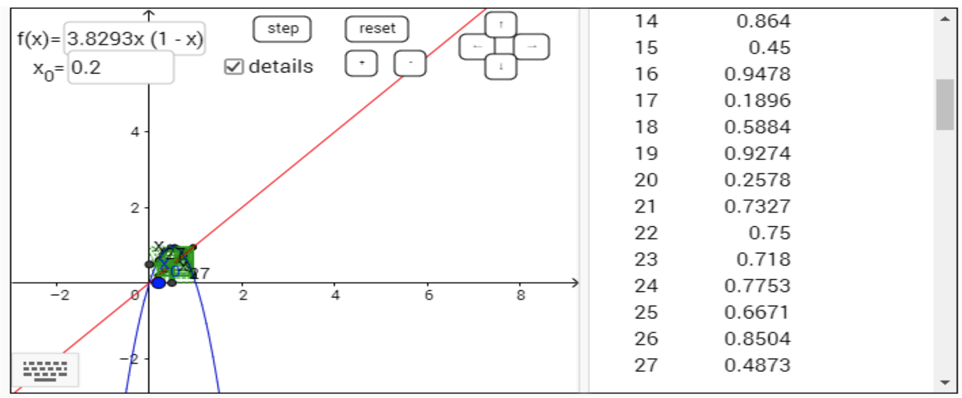
r = 3.65, X0 = 1 / 3.65 , X0 = 1/r = (1 / 3.65) = 0.2740
Xn+1 will converge to (r-1)/r = (3.65-1)/3.65 = 2.65 / 3.65 = 0.7260



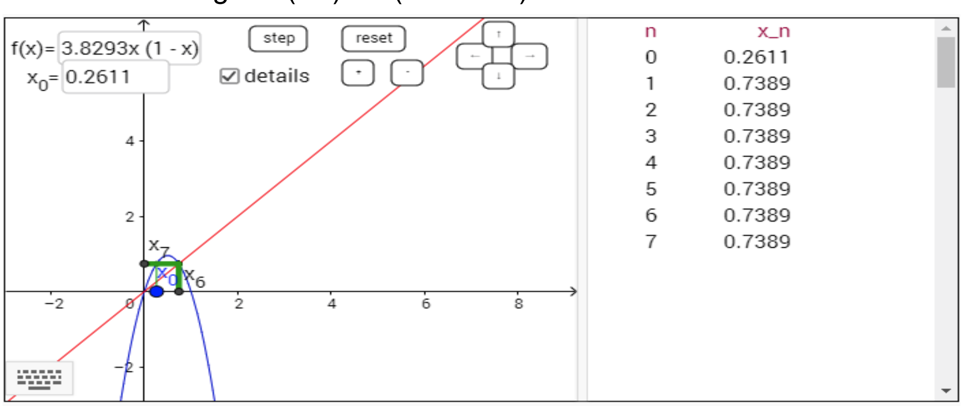
r = 3.65, X0 = 0.3, X0 > 1/r = (1 / 3.65) = 0.2740
Xn+1 will oscillate between 0.2 to 0.99 in this case



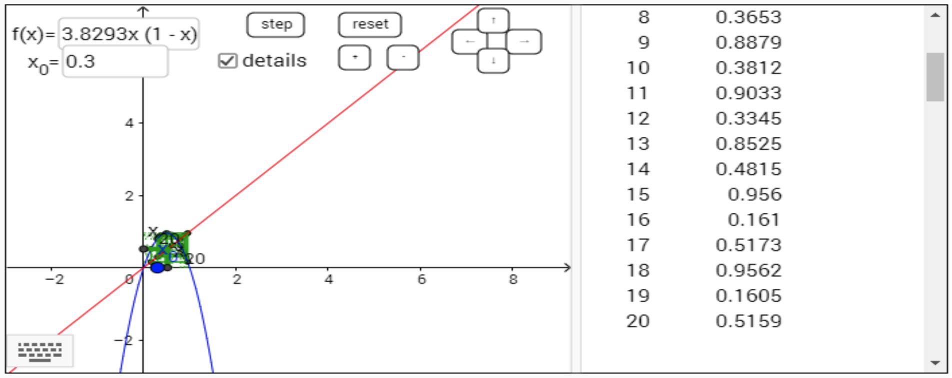
r = 3.8293, X0 = 0.2, 0 < X0 < 1/r = (1 / 3.8293) = 0.2611
Xn+1 will oscillate between 0.1 to 0.99 in this case



r = 3.8293, X0 = 1 / 3.8293 , X0 = 1/r = (1 / 3.8293) = 0.2611
Xn+1 will converge to (r-1)/r = (3.8293-1)/3.8293 = 2.8293 / 3.8293 = 0.7389

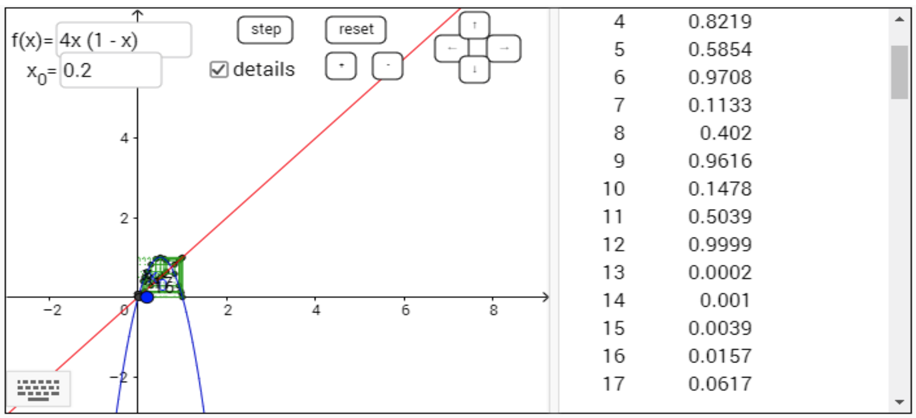


r = 3.8293, X0 = 0.3, X0 > 1/r = (1 / 3.8293) = 0.2611
Xn+1 will oscillate between 0.1 to 0.99 in this case



r = 4.0, X0 = 0.2, 0 < X0 < 1/r = (1 / 4.0) = 0.25

Xn+1 will oscillate between 0. to 0.99 in this case

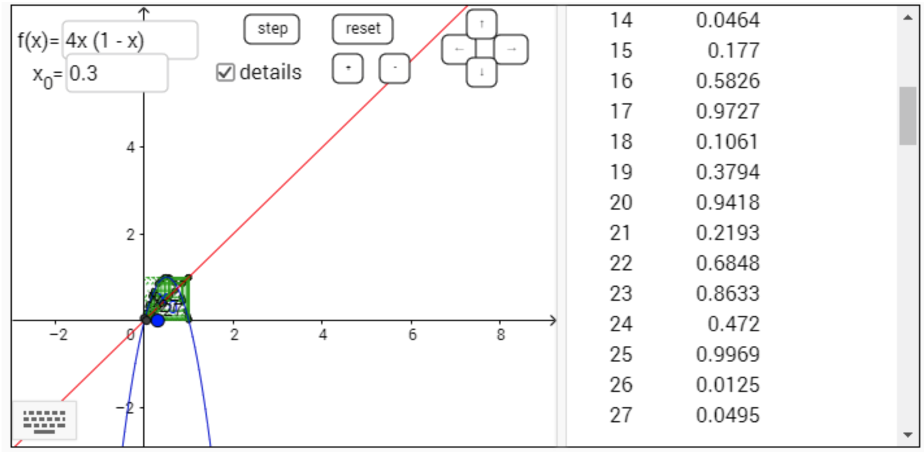


r = 4.0, X0 = 1 / 4.0 = 0.25 , X0 = 1/r = (1 / 4.0) = 0.25

Xn+1 will converge to (r-1)/r = (4.0-1)/4.0 = 3 / 4 = 0.75

r = 4.0, X0 = 0.3, X0 > 1/r = (1 / 4.0) = 0.25

Xn+1 will oscillate between 0 to 0.99 in this case



Summary . We can see that with r > 3, r = 3.2, 3.5, 3.65, 3.8293, 4 area of oscillation is increasing
And area is in between]0 , 1[

4. The table below shows the US population between 1900-2010 (in thousands).
1900 1910 1920 1930 1940 1950 1960 1970 1980 1990 2000 2010
75,995 91,972 105,711 122,755 131,669 150,697 179,323 203,212 226,505 248,710 281,416 308,746

We want to t a discrete logistic model xn+1 - xn = Rxn(1 - xn=K) for this data. In other words, we need to nd the best parameters R and K.
This can be done by trying to t a line for the per-capita growth rate [xn+1 □ xn]=xn vs. xn (since this trend line should look like R(1 - xn=K) = R - Rxn=K.
a) Use the Excel le posted on D2L (or create your own) to plot [xn+1□xn]=xn vs xn, and nd the trend line. Use the equation of the trend line to estimate parameters R and K. Comment on the carrying capacity K. (Here is a bit of help with the trend line, Excel should give you y = -0:0000003391x + 0:1926214683; use it to nd R and K.)
b) Run a projection for the US population from 1900-2050 using the logistic model (from the initial population at 1900). Compare with the actual values.

(Xn+1 - Xn) /Xn = -RXn/K + R
Slope and Intercept found from Excel. (excel document attached electronically and Printed below.)

Slope	-0.000000339 0621884	
Intercept	0.1926214683	

System of equations

Slope = -R/K -> K = - R/Slope -> K = - 0.1926214683 / Slope
Intercept = R R = Intercept R = 0.1926214683

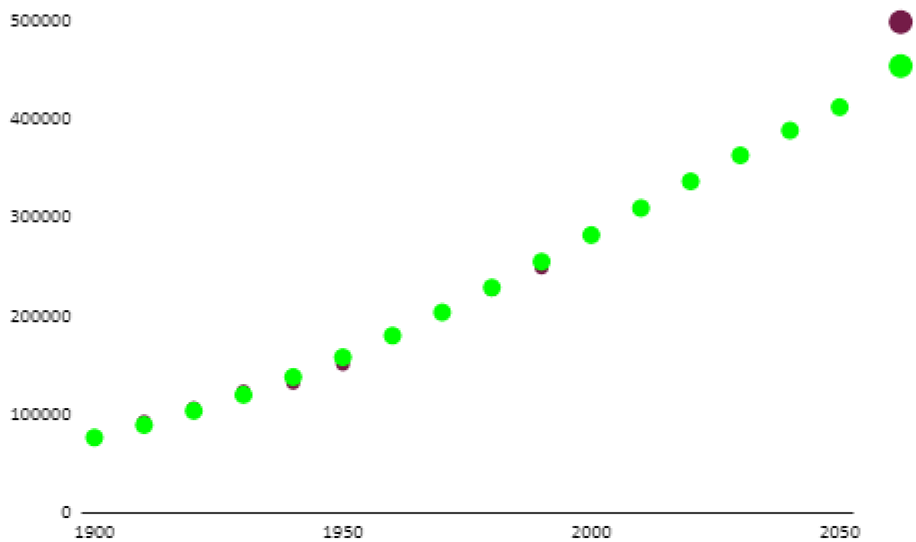
K = - 0.1926214683 /(-0.0000003390621884) = 568100.7050165
R = 0.1926214683

Xn+1 = Xn + R * Xn * (1 - Xn / K) ,
Xn+1 = (-R/K) * Xn^2 + R*Xn + Xn = (-R/K) * Xn^2 + R*(Xn + 1)

Xn+1 = (-R/K) * Xn^2 + R*(Xn + 1)
Dynamic System then is :
Xn+1 = (-0.0000003390621884) * Xn^2 + 0.1926214683(Xn + 1), X0 = 75995
Output of Excel is shown below for this Logistic Model Fit of data with X0 = **75995**
Excel file also attached as separate file.

b) Run a projection for the US population from 1900-2050 using the logistic model (from the initial population at 1900). Compare with the actual values.

year	population	ratio (x(n+1)-x(n)) /x(n)	Logistic Model Fit
1900	75995	0.2102375156	75995
1910	91972	0.1493824207	88675.10296
1920	105711	0.1612320383	103089.6926
1930	122755	0.07261618671	119343.6024
1940	131669	0.1445138947	137502.515
1950	150697	0.1899573316	157577.8223
1960	179323	0.1332177133	179511.5186
1970	203212	0.1146241364	203163.2203
1980	226505	0.098033156	228301.9265
1990	248710	0.1315025532	254605.2575
2000	281416	0.0971160133	281668.384
2010	308746		309023.5582
2020			336169.1963
2030			362605.2746
2040			387870.062
2050			411572.5673
Slope	-0.000000339 0621884		
Intercept	0.1926214683		
R	0.1926214683		
K	568100.705		



The above model makes a good fit for a modeling data