DePaul University

Department of Mathematical Sciences

MAT 359/459: Simulation Models and Monte Carlo Method

Homework for Spring Quarter, 2019

Instructions:

a. *Show all your work to justify your answers.*

b. *What you submit should represent your own work. If you use other sources, cite them.*

c. *Staple and submit your work before the deadline of the homework.*

Soft deadline: Thursday, April 11, 2019:
Hard deadline: Sunday, April 14, 2019:

1. The savings account offered by Bank of Bridgeport earns 1.00% per quarter, while the saving account offered by Lake Michigan Bank earns 0.01% per day. What are the annualized rates of return assuming four quarters per year and 365 days per year for each savings account? Which account has a better return?

2. Consider a lottery with 5 tickets, and only one ticket is the winning ticket. Five people take turns to randomly choose a ticket from the lottery. Assume that you know little about probability knowledge. How can you use the Monte carlo method to estimate the chance of winning for the third person? Briefly describe.

Soft deadline: Thursday, April 18, 2019:
Hard deadline: Sunday, April 21, 2019:

3. A \$100 battery is guaranteed to last 5 years. If it fails to last 5 years, the customer received a full refund. It is found that battery life is in fact an exponential random variable, X, with mean of 30 years, i.e., X has a probability density function,

$$f(x) = \begin{cases} 0, & -\infty < x < 0 \\ \frac{1}{30} e^{-x/30}, & 0 \le x < \infty. \end{cases}$$

   (a) (**By-hand**) Compute the expected refund amount for a battery by analytic means.
   (b) (**Programming**) Assume you don't know the true answer from the above question, and you would use a Monte Carlo method to estimate the expected refund amount. What should be the simulation size with an error tolerance as 50 cents?
   (c) (**Programming**) Compute the expected refund amount by using a Monte Carlo method with $n = 10^4$ and $n = 10^6$ samples. What is the error in your method compared to the true answer computed analytically? How much smaller is the error for $n = 10^6$ than $n = 10^4$?
   (d) (**Programming/By-hand**) For your computation in the previous problem give approximate 95% confidence intervals based on your simulation data and Central Limit Theorem. Do these confidence intervals contain the true solution?

Soft deadline: Thursday, April 25, 2019:
Hard deadline: Sunday, April 28, 2019:

4. Consider the European call option for an asset price modeled by a geometric Brownian motion with initial asset price \$100, interest rate 3%, volatility 30%, and expiry date 1 year from now.

(a) According to the formula in the notes, text, or the sample computer program on d2l, compute the **exact/true** fair option price for strike prices \$75, \$100, and \$130.

(b) Using simple Monte Carlo simulation with $n = 10^5$ samples, compute the approximate value of the European call option for the parameter values above. You may use the sample program on d2l. What is the relative error for each strike price? For which strike price is the relative error largest?

Soft deadline: Thursday, May 2nd, 2019:
Hard deadline: Sunday, May 5th, 2019:

5. (Theoretical Problem) Consider the linear congruential random number generator with $M = 17$, and a possibly equal to $3, 4$, or 5, Which of these values of a would be best?

6. (Computer Problem) The Poisson random variable is a discrete random variable that models how many taxis come by in a fixed time, or how many charged particles are detected in a fixed time, or how hard drive crashes in a fixed time. The probability mass function of the Poisson random variable with mean (and variance) λ is

$$f(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!}, & x = 0, 1, ... \\ 0, & \text{otherwise.} \end{cases}$$

Write an R program to generate i.i.d. Poisson random variables with mean one, $X_1, ..., X_n$ from uniform random variables, $Z_1, ..., Z_n$ using the inverse distribution transformation method. Use your program to print out $Z_1, ..., Z_n$ and $X_1, ..., X_n$ for $n = 10$.

7. (Computer Problem) Consider a distribution with CDF

$$F(y) = \frac{1}{1 + e^{-y}}, \quad y \in \mathbb{R}$$

Use the inverse distribution transformation method to generate random variables with above CDF from $U[0, 1]$ random variables. Generate 1000 such random numbers.

Soft deadline: Thursday, May 16th, 2019:
Hard deadline: Sunday, May 19th, 2019:

8. (Computer Problem) Consider the situation where the maximum temperature in degrees Farenheit for the seven successive days in a certain week is the vector random variable, $(T_1, ..., T_7)$, where

$$T_1 \sim \mathcal{U}(70, 80),$$

$$T_{j+1} = 14 + 0.8T_j + 3X_j, \ j = 1, ...6,$$

where $X_1, ..., X_6$ i.i.d. $\mathcal{N}(0, 1)$. A weather derivative pays \$100 if there are two or more days with maximum temperatures below 70 degrees. Using Monte Carlo simulation compute the fair price of this derivative with relative error of no more than 1%.

Soft deadline: Thursday, May 23rd, 2019:
Hard deadline: Sunday, May 26th, 2019:

9. (Theoretical & Computer Problem) Consider a distribution with density function as

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1 - x)^{\beta-1}, \quad 0 \le x \le 1$$

Derive a rejection-acceptance method for generating random variables with above pdf with $\alpha = 2, \beta = 2$ from $U[0, 1]$ random variables. Generate 1000 such random numbers.

10. (Computer problem) Consider the problem of pricing *lookback* options for a stock modeled by a geometric Brownian motion with an initial price of \$100, a volatility of 40%, and zero interest rate. Let the expiry time be 12 weeks in the future (consider 52 weeks a year), and let the monitoring frequency be weekly.

    a) Find the fair price of both the put and call options.

    b) Does the put or the call have a higher price? What is a possible intuitive explanation?