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Section 3.2

$\boxed{1}$ $P_2(x)$ - ? $(0,0)$, $(\frac{\pi}{2}, 1)$, $(\pi, 0)$

ⓐ

$$\left.\begin{array}{llll} 0 & \boxed{0} & & \\ & & \frac{1-0}{\frac{\pi}{2}-0} = \boxed{\frac{2}{\pi}} & \\ \frac{\pi}{2} & 1 & & \frac{-\frac{2}{\pi}-\frac{2}{\pi}}{\pi-0} = -\frac{4}{\pi^2} \\ & & \frac{0-1}{\pi-\frac{\pi}{2}} = -\frac{2}{\pi} & \\ \pi & 0 & & \end{array}\right\} \Rightarrow$$

$$P_2(x) = 0 + \frac{2}{\pi}(x-0) - \frac{4}{\pi^2}(x-0)\left(x-\frac{\pi}{2}\right)$$

$$\boxed{P_2(x) = \frac{2}{\pi}x - \frac{4}{\pi^2}\cdot x\cdot\left(x-\frac{\pi}{2}\right)}$$

ⓑ $P_2\left(\frac{\pi}{4}\right) = \frac{2}{\pi}\cdot\frac{\pi}{4} - \frac{4}{\pi^2}\cdot\frac{\pi}{4}\left(\frac{\pi}{4}-\frac{\pi}{2}\right) = \frac{1}{2} - \frac{1}{\pi}\left(-\frac{\pi}{4}\right) = \frac{1}{2} + \frac{1}{4}$

$\boxed{P_2\left(\frac{\pi}{4}\right) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}}$ $\quad \sin\left(\frac{\pi}{4}\right) = \frac{3}{4}$

ⓒ Interpolation Error $= \frac{(x-x_1)(x-x_2)(x-x_3)\ldots(x-x_n)}{n!} f^{n}(c)$

$\Rightarrow$ Interpolation Error: $0 < c < \pi$, $|f^{n}(c)|_{max} = |\cos c|_{max} = 1$

$$\left|\sin\frac{\pi}{4} - P_2\left(\frac{\pi}{4}\right)\right| \leq \frac{\left(\frac{\pi}{4}-0\right)\cdot\left(\frac{\pi}{4}-\frac{\pi}{2}\right)\cdot\left(\frac{\pi}{4}-\pi\right)\cdot|1|}{3!} = \frac{\pi}{4}\cdot\left(-\frac{\pi}{4}\right)\left(\cdot\frac{3}{4}\pi\right) =$$

$$= \frac{3\pi^3}{6\cdot 4^3} = \frac{\pi^3}{2\cdot 64} = \boxed{\frac{\pi^3}{128}} \text{ or } \approx 0.242$$

ⓓ actual error $= \left|\sin\frac{\pi}{4} - \frac{3}{4}\right| = 0.04289 \approx \boxed{0.043}$

actual erroor $= 0.043 < \left|\sin\pi/4 - P_2\left(\frac{\pi}{4}\right)\right| = 0.242$

Answer:

(a) $P_2(x) = \frac{2}{\pi} \cdot x - \frac{4}{\pi^2} x \cdot \left(x - \frac{\pi}{2}\right)$

(b) $P_2\left(\frac{\pi}{4}\right) = \frac{3}{4}$

(c) Interpolation error $= \frac{\pi^3}{128}$ or ≈ 0.242

(d) Actual error ≈ 0.043

Actual error $\approx 0.043 <$ Interpolation error ≈ 0.242

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$\boxed{2}$ $(1,0), (2, \ln 2), (4, \ln 4)$

(a)

$$\begin{array}{lll} 1 & \boxed{0} & \frac{\ln 2 - 0}{2-1} = \boxed{\ln 2} \\ 2 & \ln 2 & \qquad \frac{\frac{\ln 2}{2} - \ln 2}{4-1} = \frac{-\frac{1}{2}\ln 2}{3} = \boxed{-\frac{1}{6}\ln 2} \\ & & \frac{\ln 4 - \ln 2}{4-2} = \frac{\ln 2}{2} \\ 4 & \ln 4 & \end{array}$$

$$P_2(x) = 0 + \ln 2(x-1) - \frac{1}{6}(x-1)(x-2)\cdot \ln 2 =$$

$$= x\cdot \ln 2 - \ln 2 - \frac{1}{6}\ln 2\,(x^2 - 3x + 2) = \ln 2\left(x - 1 - \frac{1}{6}x^2 + \frac{3x}{6} - \frac{2}{6}\right)$$

$$= \boxed{\ln 2\left(-\frac{1}{6}x^2 + \frac{3}{2}x - \frac{4}{3}\right)}$$

(b) $P(3) = \ln 2\left(-\frac{9}{6} - \frac{4}{3} + \frac{9}{2}\right) = \ln 2\left(-\frac{3}{2} + \frac{9}{2} - \frac{4}{3}\right) = \ln 2\left(3 - \frac{4}{3}\right) \Rightarrow$

$$\boxed{P(3) = \frac{5}{3}\ln 2}$$

(c) $|\ln 3 - P(3)| \le \frac{(3-1)(3-2)(3-4)}{3!} f^{(3)}(c)$, $1 < c < 4$

$f^{(1)}(x) = (\ln x)' = x^{-1}$, $f''(x) = -x^{-2}$, $f^{(3)}(x) = 2\cdot x^{-3}$

End points: $f^{(3)}(1) = 2$, $f^{(3)}(4) = 2/64$

$f^{(3)}_{max} = f^{(3)}(1) = 2 \Rightarrow f^{(3)}(c)$, $\boxed{f^{(3)}(c) \le 2}$

$$|\ln 3 - P(3)| \underset{\text{Inter.}}{\le} \left|\frac{2\cdot 1\cdot(-1)}{2\cdot 3}\right| \cdot 2 = \boxed{\frac{2}{3}} = 0.6667$$

(d) $|P(3) - \ln 3| = \left|\frac{5}{3}\ln 2 - \ln 3\right| = \ln\left(\frac{2^{5/3}}{3}\right) = \boxed{0.0566}$

error bound $= 0.6667 >$ actual error $= 0.0566$

Answer:

(a) $P_2(x) = \ln 2\left(-\frac{1}{6}x^2 - \frac{4}{3} + \frac{3}{2}x\right)$

(b) $P_2(3) = \frac{5}{3} \cdot \ln 2$

(c) Interpolation error $= \frac{2}{3}$ or 0.6667

(d) Actual error $= \ln\left(\frac{2^{5/3}}{3}\right)$ or 0.0566

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[3] $f(x) = e^{-2x}$, $x = 0, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{4}{9}, \frac{5}{9}, \frac{6}{9}, \frac{7}{9}, \frac{8}{9}, 1$

(a) Interpolation error $= \left| f(\tfrac{1}{2}) - P_9(x) \right| =$

$$= \frac{(\frac{1}{2}-0)\cdot(\frac{1}{2}-\frac{1}{9})\cdot(\frac{1}{2}-\frac{2}{9})\cdot(\frac{1}{2}-\frac{3}{9})(\frac{1}{2}-\frac{4}{9})\cdot(\frac{1}{2}-\frac{5}{9})\cdot(\frac{1}{2}-\frac{6}{9})\cdot(\frac{1}{2}-\frac{7}{9})(\frac{1}{2}-\frac{8}{9})(\frac{1}{2}-1)}{10!} \cdot f^{(10)}(c)$$

$$\left.\begin{aligned} & 0 < c < 1 \\ & f'(x) = -2^1 e^{-2x} \\ & f''(x) = 2^2 \cdot e^{-2x} \\ & f'''(x) = -2^3 \cdot e^{-2x} \\ & \vdots \\ & f^{(10)}(x) = 2^{10} \cdot e^{-2x} \end{aligned}\right\} \Rightarrow \left.\begin{aligned} & f^{(10)}(c) = 2^{10} \cdot e^{-2c} \\ & f^{(10)}(c) \text{ on end points} \\ & f^{(10)}(0) = \frac{2^{10}}{e^0} = 1024 \\ & f^{(10)}(1) = \frac{2^{10}}{e^2} = 138.58 \\ & \boxed{f^{(10)}(c)_{max} = 1024} \end{aligned}\right\} \Rightarrow$$

$$\left| f(\tfrac{1}{2}) - P_9(x) \right| = \frac{(\frac{1}{2})^2 \cdot (\frac{7}{18}) \cdot (\frac{5}{18}) \cdot (\frac{3}{18}) \cdot (\frac{1}{18}) \cdot (-\frac{1}{18}) \cdot (\frac{-3}{18}) (\frac{-5}{18}) \cdot (-\frac{7}{18})}{10!} \cdot (1024)$$

$$= \left(\frac{1}{2}\right)^2 \cdot \frac{(7 \cdot 5 \cdot 3)^2}{18^8 \cdot 10!} = 7.0579 \cdot 10^{-11} \text{ or } \approx \boxed{7.06 \cdot 10^{-11}}$$

(b) Number of decimal places guarantee to be correct if $P_9(\frac{1}{2})$ used for approximation

$\left| f(\tfrac{1}{2}) - P_9(x) \right| \leq 0.5 \cdot 10^{-p}$, where p - is number of decimal places.

$7.06 \cdot 10^{-11} \leq 0.5 \cdot 10^{-p}$

$10^{p} \leq \frac{0.5 \cdot 10^{11}}{7.06}$

$10^{p} \leq 0.071 \cdot 10^{11}$

$\log_{10} 10^{p} \leq \log_{10}(0.071 \cdot 10^{11})$

$p \leq \log_{10} 0.071 + \log_{10} 10^{11}$

$p \leq \log_{10} 0.071 + 11$

$p \leq 11 - 1.1487$

$p \leq 9.85 \Rightarrow \boxed{p = 9}$

Answer:

a) $\left|f\left(\frac{1}{2}\right) - P_9(x)\right| \simeq 7.06 \cdot 10^{-11}$

b) at least 9 decimal places

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Section 3.4

$\boxed{2}$ (a) Check the spline conditions for

$$\begin{cases} S_1(x) = 1+2x+3x^2+4x^3 & \text{on } [0,1] \\ S_2(x) = 10+20(x-1)+15(x-1)^2+4(x-1)^3 & \text{on } [1,2] \end{cases}$$

$\{(0,1),(1,10)\} \leftarrow \{(x_1 y_1),(x_2 y_2)\}$

Property 1 : $S_i(x_i) = y_i$, $S_i(x_{i+1}) = y_{i+1}$, $i=1,2,\ldots n-1$

$i=1$: $S_1(x_1) = S_1(0) = 1 = y_1 \Rightarrow$ true

$S_1(x_2) = S_1(1) = 1+2+3+4 = 10 = y_2 \Rightarrow$ true

$i=2$: $S_2(x_2) = S_2(1) = 10 = S_1(1) \Rightarrow$ true

Property 2 : $S'_{i-1}(x_i) = S'_i(x_i)$, $i=2,3,\ldots n-1$

$i=2$; $S_1'(x) = 2+6x+12\cdot x^2$, $S_2'(x) = 20+30(x-1)+12(x-1)^2$

$$\left.\begin{aligned} S_1'(x_2) &= S_1'(1) = 2+6+12 = 20 \\ S_2'(x_2) &= S_2'(1) = 20 \end{aligned}\right\} \Rightarrow S_1'(x_2) = S_2'(x_2) \Rightarrow \text{true} \quad 20=20$$

Property 3 : $S''_{i-1}(x_i) = S''_i(x_i)$, $i=2,3,\ldots n-1$

$S_1''(x) = 6+24x$, $S_2''(x) = 30+24(x-1)$

$$i=2 : \left.\begin{aligned} S_1''(x_2) &= S_1''(1) = 6+24 = 30 \\ S_2''(x_2) &= S_2''(1) = 30 \end{aligned}\right\} \Rightarrow S_1''(x_2) = S_2''(x_2) \Rightarrow \text{true} \quad 30=30$$

we can see that $S_1(x)$ and $S_2(x)$ satisfy a cubic spline all 3 properties. We can conclude ~~that~~ is a cubic spline

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Check if natural spline:

$S_1''(x_1) = S''(x_n) = 0$

$S_1''(x_1) = S_1''(0) = 6 \neq 0,\ S_2''(x_n) = S_2''(1) = 30 \neq 0$

$\Rightarrow S_1''(x_1) \neq S_2''(x_n) \neq 0 \Rightarrow$

we conclude that is not natural spline

Check if parabolically terminated:

$S_1''' = S_{n-1}''' = 0$

$S_1'''(x_2) = 24 \qquad S_2'''(x_2) = 24$

$S_1'''(x_2) = S_2'''(x_2) = 24 \neq 0 \Rightarrow$ we conclude that is not parabolicali terminated

Check if it is Not-a-knot:

$S_1'''(x_2) = S_2'''(x_2)\ ,\ S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$, or $d_1 = d_2$

$S_1'''(x_2) = S_1'''(1) = 24,\ S_2'''(x_2) = S_2'''(1) = 24$

$\Rightarrow S_1'''(x_2) = S_2'''(x_2) = 24 \Rightarrow$

we conclude that not-a-knot is satisfied

Answer: (a) 3 Continuety property of cubic speline are satisfied

(b) Natura spline → not satisfied

parabolically terminated → not satisfie

not-a-knot → satisfied.

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[3] (a) $S_1(x) = 4 - \frac{11}{4}x + \frac{3}{4}x^3$ on $[0,1]$

$S_2(x) = 2 - \frac{1}{2}(x-1) + c(x-1)^2 - \frac{3}{4}(x-1)^3$, on $[1,2]$

$(0,4), (1,2)$

Property 1:

$S_1(0) = 4 \quad S_1(1) = 2 \quad S_2(1) = 2$

Property 2:

$S_1'(x) = -\frac{11}{4} + \frac{9}{4}x^2$, $S_2'(x) = -\frac{1}{2} + 2c(x-1) - \frac{9}{4}(x-1)^2$

$S_1'(x)\big|_{x=x_2} = -\frac{11}{4} + \frac{9}{4}x^2 = S_2'(x)\big|_{x=x_2} = -\frac{1}{2} + 2c(x-1) - \frac{9}{4}(x-1)^2$

$S_1'(x_2) = S_1'(1) = -\frac{11}{4} + \frac{9}{4} = -\frac{1}{2}$

$S_2'(x_2) = S_2'(1) = -\frac{1}{2} + 2c(0) - \frac{9}{4} \cdot 0 = -\frac{1}{2}$

Property 3: $S_{i-1}''(x_i) = S_i''(x_i) \quad i = 2, \ldots n-1$

$S_1''(x) = \frac{9}{2}x$, $S_2''(x) = 2c - \frac{9}{2}(x-1)$

$S_1''(x_2) = \frac{9}{2}$, $S_2''(x_2) = S_2''(1) = 2c - 0 = 2c$

$S_1''(1) = S_2''(1) \Rightarrow \frac{9}{2} = 2c \Rightarrow \boxed{c = \frac{9}{4}}$

Check if natural spline condition satisfied

$S_1''(x_1) = S_{n-1}''(x_n) = 0$

$S_1''(x_1) = S_1''(0) = \frac{9}{2} \cdot 0 = 0$, $S_2''(x_2) = S_2''(2) = 2 \cdot \frac{9}{4} - 9/2 = 0$

=> that natural spline condition satisfied.

Check if parabollicaly terminated:

$S_1'''(x_2) = S_{n-1}'''(x_n) = 0$ or $d = d_{n-1}$

$S_1'''(x) = \frac{9}{2}$, $S_2'''(x) = -\frac{9}{2}$ =>

$\left\{\frac{9}{2} \neq -\frac{9}{2}\right\} \neq 0$ => it is not satisfied parabolicalley terminated

Check if it is not-a-knot:

$S_1'''(x_2) = S_2'''(x_2)$ and $S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$

$S_1'''(x_2) = S_1'''(1) = \frac{9}{2}$, $S_2'''(x_2) = S_2'''(1) = -\frac{9}{2}$

$S_1'''(1) = \frac{9}{2} \neq -\frac{9}{2} = S_2'''(1)$ => not-a-knot is not satisfied

Answer: $c = \frac{9}{4}$

The spline is natural

It is not parabolically terminated

It is not not-a-knot

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$\boxed{3}$ $\boxed{C}$
$$\begin{cases} S_1(x) = -2 - \frac{3}{2}x + \frac{7}{2}x^2 - x^3 \\ S_2(x) = -1 + c(x-1) + \frac{1}{2}(x-1)^2 - (x-1)^3 \\ S_3(x) = 1 + \frac{1}{2}(x-2) - \frac{5}{2}(x-2)^2 - (x-2)^3 \end{cases}$$

<u>Property 1</u> : $S_i(x_i) = y_i$, $S_i(x_{i+1}) = y_{i+1}$, $i = 1, 2 \ldots n-1$

$(0,-2), (1,-1), (2,1)$

$i=1$: $S_1(x_1) = y_1 \Rightarrow S_1(0) = -2 = y_1$

$S_1(x_2) = y_2 \Rightarrow S_1(1) = -2 - \frac{3}{2} + \frac{7}{2} - 1 = -1$

$S_2(x_2) = y_2 \Rightarrow S_2(1) = -1$

$S_2(x_3) = y_3 \Rightarrow S_2(2) = -1 + c(2-1) + \frac{1}{2}(2-1)^2 - (2-1)^3 = 1$

$\Downarrow$

$-1 + c + \frac{1}{2} - 1 = 1$

$c = 1 + 2 - \frac{1}{2} = 3 - \frac{1}{2} = \frac{5}{2}$

$\Rightarrow \boxed{c = \frac{5}{2}}$

<u>Check if natural spline property satisfied:</u>

$S_1''(x_1) = S_{n-1}''(x_n) = 0$

$S_1'(x) = -3x^2 + 7x - \frac{3}{2}$, $S_1''(x) = -6x + 7$, $S_1''(0) = 7 \neq 0$

$S_3'(x) = \frac{1}{2} - 5(x-2) - 3(x-2)^2$, $S_3''(x) = -5 - 6(x-2)$, $S_3''(2) = -5 \neq 0$

$\Rightarrow S_1''(x_1) \neq 0 \neq S_3''(x_n) \Rightarrow$ <u>natural spline</u> property is <u>not satisfy</u>

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Check if parabolically terminated satisfied:

$S_1'''(x_2) = S_{n-1}'''(x_n) = 0$, or $d_1 = 0 = d_{n-1}$

$S_1'''(x_2) = S_1'''(1) = S_3'''(1)$

$S_1'''(x) = -6 \Rightarrow S_1'''(1) = -6$

$S_2'''(x) = -6 \Rightarrow S_2'''(1) = -6$

$S_3'''(x) = -6 \Rightarrow S_3'''(1) = -6$

$\Rightarrow S_1'''(1) = S_3'''(1) \neq 0$

$\Rightarrow$ it is <u>not</u> parabolically <u>terminated</u>

<u>Check if not-a-knot</u> property satisfied

$S_1'''(x_2) = S_2'''(x_2)$ and $S_{n-2}'''(x_{n-1}) = S_{n-1}'''(x_{n-1})$

or $d_1 = d_2$, $d_{n-1} = d_{n-2}$

$S_1'''(x_2) = -6$, $S_2'''(x_2) = -6$ $\Rightarrow S_1'''(x_2) = S_2'''(x_2)$

$S_2'''(x_3) = -6$, $S_3'''(x_3) = -6$ $\Rightarrow S_2'''(x_3) = S_3'''(x_3)$

$\Rightarrow$ that not-a-knot property <u>satisfied</u>

Answer: $c = \frac{5}{2}$

<u>It is not</u> $\boxed{\text{parabolically terminated}}$

<u>It is not</u> $\boxed{\text{natural speine}}$

<u>It is</u> $\boxed{\text{not-a-knot}}$ (not-a-knot is satisfy)

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$\boxed{7}$ (b) $(-1,1), (1,1), (2,4)$

$$\begin{vmatrix} 1 & 0 & 0 \\ \delta_1 & 2(\delta_1+\delta_2) & \delta_2 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 3\left(\frac{\Delta_2}{\delta_2} - \frac{\Delta_1}{\delta_1}\right) \\ 0 \end{vmatrix} \Rightarrow$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1-(-1) & 2(2+2-1) & 2-1 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 3\left(\frac{3}{1} - \frac{0}{2}\right) = 9 \\ 0 \end{vmatrix} \Rightarrow$$

$$\begin{vmatrix} 1 & 0 & 0 \\ 2 & 6 & 1 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} c_1 \\ c_2 \\ c_3 \end{vmatrix} = \begin{vmatrix} 0 \\ 9 \\ 0 \end{vmatrix}$$

$$\left.\begin{array}{l} c_1 \cdot 1 = 0 \\ 2 \cdot c_1 + 6 \cdot c_2 + 1 \cdot c_3 = 9 \\ c_3 \cdot 1 = 0 \end{array}\right\} \Rightarrow 6c_2 = 9 \Rightarrow \boxed{\begin{array}{c} c_1 = 0 \\ c_2 = \frac{3}{2} \\ c_3 = 0 \end{array}}$$

$$d_i = \frac{c_{i+1} - c_i}{3\delta_i} \Rightarrow \boxed{d_1} = \frac{c_2 - c_1}{3\delta_1} = \frac{\frac{3}{2} - 0}{3 \cdot 2} = \frac{3}{2 \cdot 3 \cdot 2} = \boxed{\frac{1}{4}}$$

$$\boxed{d_2} = \frac{c_3 - c_2}{3\delta_1} = \frac{0 - \frac{3}{2}}{3 \cdot 1} = \frac{-3}{3 \cdot 2} = \boxed{-\frac{1}{2}}$$

$$b_i = \frac{y_{i+1} - y_i}{\delta_i} - \frac{\delta_i}{3}(2c_i + c_{i+1}) \Rightarrow \boxed{b_1} = \frac{1-1}{2} - \frac{2}{3}\left(2 \cdot 0 + \frac{3}{2}\right) = \boxed{-1}$$

$$\boxed{b_2} = \frac{3}{1} - \frac{1}{3}\left(2 \cdot \frac{3}{2} + 0\right) = 3 - 1 = \boxed{2}$$

$$\boxed{S_1(x)} = 1 - 1(x+1) + 0\cdot(x+1)^2 + \frac{1}{4}(x+1)^3 = \left| 1 - (x+1) + \frac{1}{4}(x+1)^3 \right|$$

$$\boxed{S_2(x)} = 1 - 2(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 = \left| 1 + 2(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{2}(x-1)^3 \right|$$

Answer: $\{c_1, c_2, c_3\} = \{0, \frac{3}{2}, 0\}$

$\{b_1, b_2\} = \{-1, 2\}$

$\{d_1, d_2\} = \{\frac{1}{4}, -\frac{1}{2}\}$

$S_1(x) = 1 - (x+1) + \frac{1}{4}(x+1)^3$ on $[-1, 1]$

$S_2(x) = 1 + 2(x-1) + \frac{3}{2}(x-1)^2 - \frac{1}{2}(x-1)^3$ on $[1, 2]$