Mathematical Modeling

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##

Problem 1
Dynamic Systems

> $a := 0.175$

$$a := 0.175 \tag{1}$$

> $f := 0.4$

$$f := 0.4 \tag{2}$$

> $rbc := rsolve(\{c(n) = (1-f)\cdot c(n-1) + a\cdot f\cdot c(n-2), c(0) = 30, c(1) = 25, \}, c(n))$

$$rbc := -5\left(-\frac{1}{10}\right)^n + 35\left(\frac{7}{10}\right)^n \tag{3}$$

> $seq(evalf(rbc), n = 0..10)$

30., 25., 17.10000000, 12.01000000, 8.403000000, 5.882500000, 4.117710000, 2.882401000, 2.017680300, 1.412376250, 0.9886633710 **(4)**

>

> $a := 1$

$$a := 1 \tag{5}$$

> $f := 0.0001$

$$f := 0.0001 \tag{6}$$

> $rbc := rsolve(\{c(n) = (1-f)\cdot c(n-1) + a\cdot f\cdot c(n-2), c(0) = 30, c(1) = 25, \}, c(n))$

$$rbc := \frac{250030}{10001} + \frac{50000\left(-\frac{1}{10000}\right)^n}{10001} \tag{7}$$

> $seq(evalf(rbc), n = 0..10)$

30., 25., 25.00050000, 25.00049995, 25.00049995, 25.00049995, 25.00049995, 25.00049995, 25.00049995, 25.00049995, 25.00049995 **(8)**

> $f := 0.1$

$$f := 0.1 \tag{9}$$

> $rbc := rsolve(\{c(n) = (1-f)\cdot c(n-1) + a\cdot f\cdot c(n-2), c(0) = 30, c(1) = 25, \}, c(n))$

$$rbc := \frac{280}{11} + \frac{50\left(-\frac{1}{10}\right)^n}{11} \tag{10}$$

> $seq(evalf(rbc), n = 0..10)$

30., 25., 25.50000000, 25.45000000, 25.45500000, 25.45450000, 25.45455000, 25.45454500, 25.45454550, 25.45454545, 25.45454546 **(11)**

> $f := 0.5$

$$f := 0.5 \tag{12}$$

> $rbc := rsolve(\{c(n) = (1-f)\cdot c(n-1) + a\cdot f\cdot c(n-2), c(0) = 30, c(1) = 25, \}, c(n))$

$$rbc := \frac{10\left(-\frac{1}{2}\right)^n}{3} + \frac{80}{3} \quad \mathbf{(13)}$$

> $seq(evalf(rbc), n = 0..10)$

30., 25., 27.50000000, 26.25000000, 26.87500000, 26.56250000, 26.71875000, 26.64062500, 26.67968750, 26.66015625, 26.66992188 **(14)**

> $f := 0.7$

$$f := 0.7 \quad \mathbf{(15)}$$

> $rbc := rsolve(\{c(n) = (1 - f) \cdot c(n-1) + a \cdot f \cdot c(n-2), c(0) = 30, c(1) = 25, \}, c(n))$

$$rbc := \frac{50\left(-\frac{7}{10}\right)^n}{17} + \frac{460}{17} \quad \mathbf{(16)}$$

> $seq(evalf(rbc), n = 0..50)$

30., 25., 28.50000000, 26.05000000, 27.76500000, 26.56450000, 27.40485000, 26.81660500, 27.22837650, 26.94013645, 27.14190448, 27.00066686, 27.09953320, 27.03032676, 27.07877127, 27.04486011, 27.06859792, 27.05198146, 27.06361298, 27.05547091, 27.06117036, 27.05718075, 27.05997348, 27.05801857, 27.05938700, 27.05842910, 27.05909963, 27.05863026, 27.05895882, 27.05872883, 27.05888982, 27.05877712, 27.05885601, 27.05880079, 27.05883945, 27.05881239, 27.05883133, 27.05881807, 27.05882735, 27.05882085, 27.05882540, 27.05882222, 27.05882445, 27.05882289, 27.05882398, 27.05882321, 27.05882375, 27.05882338, 27.05882364, 27.05882345, 27.05882358 **(17)**

> $f := 0.99$

$$f := 0.99 \quad \mathbf{(18)}$$

> $rbc := rsolve(\{c(n) = (1 - f) \cdot c(n-1) + a \cdot f \cdot c(n-2), c(0) = 30, c(1) = 25, \}, c(n))$

$$rbc := \frac{500\left(-\frac{99}{100}\right)^n}{199} + \frac{5470}{199} \quad \mathbf{(19)}$$

> $seq(evalf(rbc), n = 0..50)$

30., 25., 29.95000000, 25.04950000, 29.90099500, 25.09801495, 29.85296520, 25.14556445, 29.80589119, 25.19216772, 29.75975396, 25.23784358, 29.71453485, 25.28261049, 29.67021561, 25.32648655, 29.62677832, 25.36948946, 29.58420543, 25.41163662, 29.54247974, 25.45294505, 29.50158440, 25.49343145, 29.46150287, 25.53311216, 29.42221896, 25.57200323, 29.38371680, 25.61012037, 29.34598084, 25.64747897, 29.30899582, 25.68409414, 29.27274680, 25.71998067, 29.23721914, 25.75515305, 29.20239848, 25.78962550, 29.16827075, 25.82341196, 29.13482216, 25.85652606, 29.10203920, 25.88898119, 29.06990862, 25.92079047, 29.03841744, 25.95196673, 29.00755293 **(20)**

##

Problem 2

Usher Matrix

>

> $L := Matrix([[0.11, 0.15, 0.15], [0.3, 0.0, 0.0], [0.0, 0.6, 0.6]])$

$$L := \begin{bmatrix} 0.11 & 0.15 & 0.15 \\ 0.3 & 0. & 0. \\ 0. & 0.6 & 0.6 \end{bmatrix} \tag{21}$$

> $X0 := \langle 1, 1, 1 \rangle$

$$X0 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{22}$$

> $X10 := MatrixPower(L, 10)\,X0$

$$X10 := \begin{bmatrix} 0.0121963762712486 \\ 0.00538807878796246 \\ 0.0408829895293162 \end{bmatrix} \tag{23}$$

> $\dfrac{X10}{Norm(X10, 1)}$

$$\begin{bmatrix} 0.208601151599528 \\ 0.0921551955335739 \\ 0.699243652948870 \end{bmatrix} \tag{24}$$

> $(evalues, evectors) := Eigenvectors(L)$

$$evalues, evectors := \begin{bmatrix} -3.05311331771918\ 10^{-16} + 0.\,I \\ 0.0309243915380240 + 0.\,I \\ 0.679075608461976 + 0.\,I \end{bmatrix}, \tag{25}$$

$[[4.44089209850062\ 10^{-16} + 0.\,I, 0.0707587979320973 + 0.\,I,$
$0.283621179138579 + 0.\,I],$
$[-0.707106781186547 + 0.\,I, 0.686436767996811 + 0.\,I, 0.125297319887963 + 0.\,I],$
$[0.707106781186547 + 0.\,I, -0.723738734666935 + 0.\,I, 0.950715314051970 + 0.\,I]]$

> $domvector := Column(evectors, 3)$

$$domvector := \begin{bmatrix} 0.283621179138579 + 0.\,I \\ 0.125297319887963 + 0.\,I \\ 0.950715314051970 + 0.\,I \end{bmatrix} \tag{26}$$

> $steady := \dfrac{domvector}{Norm(domvector, 1)}$

$$steady := \begin{bmatrix} 0.208601151595946 + 0.\,I \\ 0.0921551955319388 + 0.\,I \\ 0.699243652936940 + 0.\,I \end{bmatrix} \tag{27}$$

##

Problem 3

Marcov Chains

> $T := Matrix([[0.75, 0.2, 0.4], [0.05, 0.6, 0.2], [0.2, 0.2, 0.4]])$

$$T := \begin{bmatrix} 0.75 & 0.2 & 0.4 \\ 0.05 & 0.6 & 0.2 \\ 0.2 & 0.2 & 0.4 \end{bmatrix} \quad \textbf{(28)}$$

> $X0 := \left\langle \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right\rangle$

$$X0 := \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix} \quad \textbf{(29)}$$

> $X2 := MatrixPower(T, 2).X0$

$$X2 := \begin{bmatrix} 0.500833333333333 \\ 0.245833333333333 \\ 0.253333333333333 \end{bmatrix} \quad \textbf{(30)}$$

> $X5 := MatrixPower(T, 5).X0$

$$X5 := \begin{bmatrix} 0.546752812500000 \\ 0.203220520833333 \\ 0.250026666666667 \end{bmatrix} \quad \textbf{(31)}$$

> $(evalues, evectors) := Eigenvectors(T)$

$$evalues, evectors := \begin{bmatrix} 1.00000000000000 + 0.\ \mathrm{I} \\ 0.550000000000001 + 0.\ \mathrm{I} \\ 0.200000000000000 + 0.\ \mathrm{I} \end{bmatrix}, \quad \textbf{(32)}$$

$[[-0.868744485526139 + 0.\ \mathrm{I}, -0.707106781186548 + 0.\ \mathrm{I},$
$-0.464990554975277 + 0.\ \mathrm{I}],$
$[-0.304060569934149 + 0.\ \mathrm{I}, 0.707106781186547 + 0.\ \mathrm{I}, -0.348742916231458 + 0.\ \mathrm{I}],$
$[-0.390935018486762 + 0.\ \mathrm{I}, 3.05363696291945\ 10^{-16} + 0.\ \mathrm{I}, 0.813733471206735$
$+ 0.\ \mathrm{I}]]$

>

> $domvector := Column(evectors, 1)$

$$domvector := \begin{bmatrix} -0.868744485526139 + 0.\ \mathrm{I} \\ -0.304060569934149 + 0.\ \mathrm{I} \\ -0.390935018486762 + 0.\ \mathrm{I} \end{bmatrix} \quad \textbf{(33)}$$

> $steady := \dfrac{domvector}{Norm(domvector, 1)}$

$$steady := \begin{bmatrix} -0.555555555544873 + 0.\ I \\ -0.194444444440706 + 0.\ I \\ -0.249999999995193 + 0.\ I \end{bmatrix} \quad \textbf{(34)}$$

###

Problem 4
Linear Programming

First Mortgage amount = x1
Second Mortgage amount = x2
Home Improvement amount = x3
Personal Overdraft amount =x4

Interest Rates	x1	x2	x3	x4
	0.05	0.08	0.1	0.05

Average Interest Rate : R = (x1* 0.05 + x2 * 0.08 +x3 * 0.1 + x4 * 0.05)/(x1+x2+x3+x4)
Objective interst income = x1* 0.05 + x2 * 0.08 +x3 * 0.1 + x4 * 0.05
Constrains

x1 + x2 + x3 + x4 <=250
x1 >= 0.55 *(x1+x2) or $0.55\,x1 + 0.55\,x2 - x1 \le 0 \Rightarrow 0.55\,x2 - 0.45 \cdot x1 \le 0$
x1 >= 0.25 *(x1 + x2 + x3 + x4) => $0.25\,x2 + 0.25\,x3 + 0.25\,x4 - 0.75 \cdot x1 \le 0$
x2 <= 0.25 *(x1 + x2 + x3 + x4) => $0.25\,x1 - 0.75\,x2 + 0.25\,x3 + 0.25\,x4 - x2 \ge 0$
$(x1 \cdot 0.05 + x2 \cdot 0.08 + x3 \cdot 0.1 + x4 \cdot 0.05) / (x1 + x2 + x3 + x4) \le 0.07$
$x1 \ge 0, x2 \ge 0, x3 \ge 0, x4 \ge 0$

$$-0.45\,x1 + 0.55\,x2 \le 0 \Rightarrow -0.45\,x1 + 0.55\,x2 \le 0$$
$$0.25\,x2 + 0.25\,x3 + 0.25\,x4 - 0.75\,x1 \le 0$$
$$0 \le 0.25\,x1 - 1.75\,x2 + 0.25\,x3 + 0.25\,x4$$
$$\frac{0.05\,x1 + 0.08\,x2 + 0.1\,x3 + 0.05\,x4}{x1 + x2 + x3 + x4} \le 0.07$$
$$0 \le x1, 0 \le x2, 0 \le x3, 0 \le x4 \quad \textbf{(35)}$$

> restart; with(Optimization);

$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve]$ **(36)**

> $obj := x1 \cdot 0.05 + x2 \cdot 0.08 + x3 \cdot 0.10 + x4 \cdot 0.05$

$$obj := 0.05\,x1 + 0.08\,x2 + 0.10\,x3 + 0.05\,x4 \quad \textbf{(37)}$$

> $constrains := [x1 + x2 + x3 + x4 \le 250, \ x1 \ge 0.55 \cdot (x1 + x2), x1 \ge 0.25 \cdot (x1 + x2 + x3 + x4), \ x2 \le 0.25 * (x1 + x2 + x3 + x4), \ (x1 \cdot 0.05 + x2 \cdot 0.08 + x3 \cdot 0.1 + x4 \cdot 0.05) \le 0.07 \cdot (x1 + x2 + x3 + x4), x1 \ge 0, x2 \ge 0, x3 \ge 0, x4 \ge 0]$

$constrains := [x1 + x2 + x3 + x4 \le 250, 0.55\,x1 + 0.55\,x2 \le x1, 0.25\,x1 + 0.25\,x2 + 0.25\,x3 + 0.25\,x4 \le x1, x2 \le 0.25\,x1 + 0.25\,x2 + 0.25\,x3 + 0.25\,x4, 0.05\,x1 + 0.08\,x2 + 0.1\,x3 + 0.05\,x4 \le 0.07\,x1 + 0.07\,x2 + 0.07\,x3 + 0.07\,x4, 0 \le x1, 0 \le x2, 0 \le x3, 0 \le x4]$ **(38)**

> LPSolve(obj, constrains, maximize)

$[17.5000000000000, [x1 = 62.5000000000000, x2 = 51.1363636363636, x3 = 69.3181818181818, x4 = 67.0454545454545]]$ **(39)**

###

Problem 5
Game Theory

zero-sum games

ii)

| 0   1 -2| -> |5 6 3|
| 1 -2 3| |6 3 8|
|-2   3 -4| |3 8 1|

Optimal strategy for column player:

Objective: maximaze : $x1 + x2 + x3$

$$x1 + x2 + x3 \quad \textbf{(40)}$$

> $objC := x1 + x2 + x3$

$$objC := x1 + x2 + x3 \quad \textbf{(41)}$$

> $consC := [5 \cdot x1 + 6 \cdot x2 + 3 \cdot x3 \leq 1, 6 \cdot x1 + 3 \cdot x2 + 8 \cdot x3 \leq 1, 3 \cdot x1 + 8 \cdot x2 + 1 \cdot x3 \leq 1, x1 \geq 0, x2 \geq 0, x3 \geq 0]$

$$consC := [5\, x1 + 6\, x2 + 3\, x3 \leq 1, 6\, x1 + 3\, x2 + 8\, x3 \leq 1, 3\, x1 + 8\, x2 + x3 \leq 1, 0 \leq x1, 0 \leq x2, 0 \leq x3] \quad \textbf{(42)}$$

> $LPSolve(objC, consC, maximize)$

$$[0.200000000000000, [x1 = 0.0500000000000000, x2 = 0.100000000000000, x3 = 0.0500000000000000]] \quad \textbf{(43)}$$

> $value\_of\_the\_game := 1/0.2$

$$value\_of\_the\_game := 5.000000000 \quad \textbf{(44)}$$

original game value = 5-5 = 0

The optimal strategy for the column player is :

> $q := Vector([0.05 * value\_of\_the\_game, 0.10 * value\_of\_the\_game, 0.05 \cdot value\_of\_the\_game])$

$$q := \begin{bmatrix} 0.2500000000 \\ 0.5000000000 \\ 0.2500000000 \end{bmatrix} \quad \textbf{(45)}$$

Optimal strategy for the row player:

objective : minimize y1 + y2 +y3

> $objR := y1 + y2 + y3$

$$objR := y1 + y2 + y3 \quad \textbf{(46)}$$

> $consR := [5 \cdot y1 + 6 \cdot y2 + 3 \cdot y3 \geq 1, 6 \cdot y1 + 3 \cdot y2 + 8 \cdot y3 \geq 1, 3 \cdot y1 + 8 \cdot y2 + 1 \cdot y3 \geq 1, y1 \geq 0, y2 \geq 0, y3 \geq 0]$

$$consR := [1 \leq 5\, y1 + 6\, y2 + 3\, y3, 1 \leq 6\, y1 + 3\, y2 + 8\, y3, 1 \leq 3\, y1 + 8\, y2 + y3, 0 \leq y1, 0 \leq y2, 0 \leq y3] \quad \textbf{(47)}$$

> $LPSolve(objR, consR)$

$$[0.200000000000000, [y1 = 0.0500000000000001, y2 = 0.100000000000000, y3 = 0.0500000000000000]] \quad \textbf{(48)}$$

> $value\_of\_the\_game := 1/0.2$

$$value\_of\_the\_game := 5.000000000 \quad \textbf{(49)}$$

> $p := Matrix([0.05 \cdot value\_of\_the\_game, 0.10 \cdot value\_of\_the\_game, 0.05 \cdot value\_of\_the\_game])$

$$p := \begin{bmatrix} 0.2500000000 & 0.5000000000 & 0.2500000000 \end{bmatrix} \quad \textbf{(50)}$$

> $A := Matrix([[0, 1, -2], [1, -2, 3], [-2, 3, -4]])$

(51)

$$A := \begin{bmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{bmatrix} \tag{51}$$

> $p \cdot A \cdot q$

$$\begin{bmatrix} 0. \end{bmatrix} \tag{52}$$

Answer:
Game is not strictly determined
Value Of The Game = 0
Optimal strategy for column player:

$$q := \begin{bmatrix} 0.2500000000 \\ 0.5000000000 \\ 0.2500000000 \end{bmatrix}$$

Optimal strategy for the row player:

$$\begin{bmatrix} 0.2500000000 & 0.5000000000 & 0.2500000000 \end{bmatrix}$$