I. In the circulatory system, the red blood cells (RBC) are constantly being destroyed and replaced. Assume that the spleen filters out and destroys a certain fraction f of the cells daily and that the bone marrow produces a number proportional to the number lost on the previous day. If $C(n)$ is the number of RBCs in circulation on day n then the above assumptions can be translated into:

$$C_n = (1-f)\cdot C_{n-1} + a\cdot f\cdot C_{n-2}$$

where $0 \le f \le 1$ and $0 \le a \le 1$.

(i) Comment on the validity of the recursive relation based on biological considerations.

(ii) Estimate the number of cells C_n at time n if $f = 2/5$, $a = 7/40$, and $C_0 = 30$, $C_1 = 25$ (in trillions). Describe the long term behavior as $n \to \infty$.

(iii) The definition of *homeostasis* is that C_n converges to a nonzero constant, as $n \to \infty$. How may the parameters f, a be chosen to achieve this?

(iv) Would this be a good model for homeostasis based on your findings in part (iii)?

(i) The model would not be valid to model RBC. Because RBC must be some const number and not grow to infinity or die to zero. The only some validity can be alived if $a=1$ and f close to zero.

(ii) $C_n=(1-f)\cdot C_{n-1}+af\cdot C_{n-2}$, $f=\frac{2}{5}$, $a=\frac{7}{40}$, $C_0=30$, $C_1=25$, C_n-?

$C_n = C\cdot\lambda^n$

$c\cdot\lambda^n=(1-f)\cdot c\cdot\lambda^{n-1}+af\cdot c\cdot\lambda^{n-2}$

$c\lambda^n\left(1-\frac{1-f}{\lambda}-\frac{af}{\lambda^2}\right)=0 \Rightarrow \lambda^2-\lambda(1-f)-af=0,\ \lambda=\frac{1-f\pm\sqrt{f^2+2af(2a-1)+4af}}{2}$

$$\lambda=\frac{1-\frac{2}{5}\pm\sqrt{\frac{4}{25}+\frac{2\cdot2}{5}\left(2\cdot\frac{7}{40}-1\right)+1}}{2}=\frac{\frac{3}{5}\pm\sqrt{\frac{4}{25}+\frac{4}{5}\left(\frac{7}{20}-\frac{20}{20}\right)+1}}{2}$$

$$\lambda=\frac{\frac{3}{5}\pm\sqrt{\frac{16}{25}}}{2} \Rightarrow \boxed{\lambda_1=0.7}\quad\boxed{\lambda_2=-0.1}$$

Solution will be of the form $C_1\lambda_1^n+C_2\lambda_2^n=C_n$, $C_0=30$, $C_1=25$

$$\begin{cases}C_0=30=C_1\cdot0.7^0+C_2(-0.1)^0\\ C_1=25=C_1\cdot0.7^1+C_2(-0.1)^1\end{cases} \Rightarrow \begin{cases}C_1+C_2=30\\ C_1\cdot0.7+C_2\cdot(-0.1)=25\end{cases} \Rightarrow \left|\begin{array}{cc|c}1&1&30\\0.7&-0.1&25\end{array}\right| \Rightarrow$$

$$\Rightarrow\left|\begin{array}{cc|c}1&1&30\\0.7&-0.1&25\end{array}\right|\Rightarrow\left|\begin{array}{cc|c}0.1&0.1&3\\0.7&-0.1&25\end{array}\right|\Rightarrow\left|\begin{array}{cc|c}0.8&0&28\\0.7&-0.1&25\end{array}\right|\Rightarrow\left|\begin{array}{cc|c}0.2&0&7\\0.7&-0.1&25\end{array}\right|=\left|\begin{array}{cc|c}0.14&0&49\\-0.14&0.2&50\end{array}\right|\Rightarrow$$

$$\Rightarrow\left|\begin{array}{cc|c}0.14&0&49\\0&0.2&-1\end{array}\right|\Rightarrow\left|\begin{array}{cc|c}0.2&0&7\\0&0.2&-1\end{array}\right|\Rightarrow\begin{cases}C_1\cdot0.2=7\\ C_2\cdot0.2=-1\end{cases}\Rightarrow\boxed{C_1=35}\quad\boxed{C_2=-5}$$

Answer:

(ii) continue:

$\boxed{C_n = 35\cdot(0.7)^n - 5(-0.1)^n}$ → solution

$\lim_{n\to\infty} C_n = 0$. Long term behavior of $C_n \to 0$ if $n \to \infty$

(iii) if $|\lambda_1| > 1$ or $|\lambda_2| > 1$ then $C_n \to \infty$ | if both $|\lambda_1| < 1$ and $|\lambda_2| < 1$ then $C_n \to 0$

The only way C_n can be constant if ~~largest~~ $\lambda = 1$ and ~~characteristic~~ λ can be $= 1$ only if $a = 1$ and characteristic equation will be the following: $\lambda - \lambda(1-f) - f = 0 \Rightarrow$

$$\Rightarrow \lambda = \frac{1-f \pm \sqrt{1-2f+f^2+4f}}{2} = \frac{1-f \pm (1+f)}{2} \Rightarrow \boxed{\lambda_1 = 1} \quad \boxed{\lambda_2 = -f}$$

$$\Rightarrow \begin{array}{l} c_1 + c_2 = 30 \\ c_1 - f c_2 = 25 \end{array} \left|\begin{array}{cc|c} 1 & 1 & 30 \\ 1 & -f & 25 \end{array}\right| = \left|\begin{array}{cc|c} 0 & 1+f & 5 \\ 1 & -f & 25 \end{array}\right| = \left|\begin{array}{cc|c} 0 & 1+f & 5 \\ 1 & 1 & 30 \end{array}\right\} \Rightarrow$$

$$c_2(1+f) = 5 \Rightarrow \boxed{c_2 = \frac{5}{1+f}} \quad c_1 + c_2 = 30 \Rightarrow \boxed{c_1 = \frac{25+30f}{1+f}}$$

Answer: $a = 1$, ~~$f \neq 0$~~ $0 < f \le 1$, $\lambda_1 = 1$, $\lambda_2 = -f$

$$\boxed{C_n = \frac{25+30f}{1+f}\cdot(1)^n + \frac{5}{1+f}\cdot(-f)^n}$$

$$\lim_{f\to 1}\left(\frac{5}{1+f}\right)(-f)^n = \pm\frac{5}{1+f} \Rightarrow \lim_{\substack{n\to\infty \\ f\to 1}} C_n \to \frac{25+30f}{1+f} \pm \frac{5}{1+f}$$

$$\lim_{f\to 0}\left(\frac{5}{1+f}\right)(-f)^n = 0 \Rightarrow \lim_{\substack{n\to\infty \\ f\to 0}} C_n = \frac{25+30f}{1+f} = \text{equilibrium}$$

So if we chose f close to 0 the we can get C_n converge to nonzero constant $= \frac{25+30f}{1+f}$

if we chose f close to 1 the C_n will ossilate around $\frac{25+30f}{1+f}$ and will be $\frac{25+30f}{1+f} + \frac{5}{1+f}$ or $\frac{25+30f}{1+f} - \frac{5}{1+f}$

(iv) Unless $a = 1$, the model can not be homeostasis. Red blood cells will be or die when $C_n \to 0$ or grow to very large number when $C_n \to \infty$. It is not good model to model RBC that has to be some constant value for body to be alive

II. An Usher matrix model is a slight variation on a Leslie model, in which there may be nonzero entries on the diagonal. This means that while some individuals in a class will move up to the next class after a time step, many will stay where they are. For example a 3-class structured population can have a matrix

$$\begin{pmatrix} f_1 + p_{1,1} & f_2 & f_3 \\ p_{1,2} & p_{2,2} & 0 \\ 0 & p_{2,3} & p_{3,3} \end{pmatrix}$$

with the parameters $p_{i,j}$ denoting the fraction of the ith age class that moves to the jth age class, and f_i denotes the per-capita fertility rate of age-class i (some values can be zero). Consider now a model given by (Cullen, 1985) that describes a certain coyote population. Three age classes—pup, yearling, and adult—are used and the Usher matrix

$$\begin{pmatrix} .11 & .15 & .15 \\ .3 & 0 & 0 \\ 0 & .6 & .6 \end{pmatrix}$$

describes changes over a time step of 1 year.

(i) Explain what each matrix entry is saying about the population. (Be careful when trying to explain the .11 entry in the upper left corner).

(ii) Find the growth rate and stable stage distribution of the model.

(iii) Will the population grow or decline? Quickly or slowly?

(i)

pup	yearling	adult
0.11	0.15	0.15
0.3	0	0
0	0.6	0.6

0.15 → adult reproduce with rate = 0.15

0.15 → yearling reproduce with rate = 0.15

0.11 → 0.11 pups will stay pups

0 → zero move from adult to yearling

0.6 → 0.6 stay adults

0 → zero stay in yearlings

0.6 → 0.6 moves from yearlings to adults

0 → zero moves from pup to adults

0.3 → 0.3 moves from pup to yearling

(II) Continue:

(ii) Using Maple Eigenvalues = $\begin{bmatrix} 0.000 \\ 0.031 \\ 0.679 \end{bmatrix}$

The largest eigenvalue = $\boxed{0.679}$

domVector = $\begin{bmatrix} 0.284 \\ 0.125 \\ 0.951 \end{bmatrix}$ → corresponding eigenvalue = 0.679

steady probability vector = $\dfrac{\text{domVector}}{\text{Norm(domVector, 1)}}$ =

= $\begin{bmatrix} 0.209 \\ 0.092 \\ 0.699 \end{bmatrix}$

Answer: Growth Rate = eigenvalue = 0.679.

Stable stage distribution =

$\begin{bmatrix} 0.209 \\ 0.092 \\ 0.699 \end{bmatrix}$
→ probability of dist. for pup (20.9%)
→ probability of dist for yearly (9.2%)
→ probability of dist for adults (69.9%)

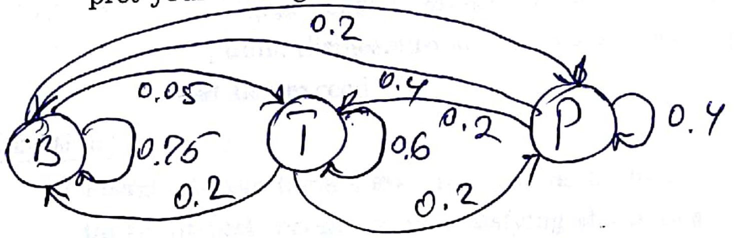
(iii) The population will decline.
The population will decline with the rate $\boxed{0.679}$ on every step.

III. A student center cafeteria has three fast-food centers — serving burgers, tacos, and pizza. A survey of students found the following information concerning lunch: 75% who ate burgers will eat burgers again at the next lunch, 5% will eat tacos next, and 20% will eat pizza next. Of those who ate tacos last, 20% will eat burgers next, 60% will stay with tacos, and 20% will eat pizza next. Of those who ate pizza, 40% will eat burgers next, 20% tacos, and 40% pizza again.

Assume initially that one-third of students ate at each of the burger, taco, and pizza stations.

(i) What percentage of students will be eating burgers after 2 days? After 5 days?

(ii) Find the long-term behavior of the students regarding fast food. Explain and interpret your findings.



	B	T	P
B	0.75	0.20	0.40
T	0.05	0.60	0.20
P	0.20	0.20	0.40

$$T = \begin{vmatrix} 0.75 & 0.20 & 0.40 \\ 0.05 & 0.60 & 0.20 \\ 0.20 & 0.20 & 0.40 \end{vmatrix}$$

$$X_0 = [0.33, 0.33, 0.33]$$

(i) $X_2 = T^2 \times X_0 = \begin{vmatrix} 0.501 \\ 0.246 \\ 0.250 \end{vmatrix}$ → after 2 days $\boxed{50.1\%}$ of students will be eating burgers

$X_5 = T^5 \cdot X_0 = \begin{vmatrix} 0.547 \\ 0.203 \\ 0.250 \end{vmatrix}$ → after 5 days $\boxed{54.7\%}$ of students will be eating burgers

(ii) Largest eigenvalue = 1

dominent eigen vector corresponding to eigenvalue = 1 $= \begin{vmatrix} 0.869 \\ 0.304 \\ 0.301 \end{vmatrix}$

steady vector $= \dfrac{\text{domvector}}{\text{Norm (domvector, 1)}} = \begin{vmatrix} 0.556 \\ 0.194 \\ 0.250 \end{vmatrix}$

(ii) Continue:

Answer:

Long team behavior described by probability distribution steady vector $= \begin{vmatrix} 0.556 \\ 0.194 \\ 0.250 \end{vmatrix}$

in a long run,

55.6% of students will eat Burgers

19.4% of Students will eat tacos

25.0% of Student will eat pizzas

IV. A bank makes four kinds of loans to its personal customers, and these loans yield the following annual interest rates to the bank:

- First mortgage: 5%
- Second mortgage: 8%
- Home improvement: 10%
- Personal overdraft: 5%.

The bank has a maximum foreseeable lending capability of $250 million and is further constrained by the following policies:

(1) First mortgages must be at least 55 percent of all mortgages issued and at least 25 percent of all loans issued (in $ terms).

(2) Second mortgages cannot exceed 25 percent of all loans issues (in $ terms).

(3) To avoid public displeasure and a new windfall tax, the average interest rate on all loans must not exceed 7%.

<u>Questions:</u>

(i) Formulate the bank's loan problem as a linear programming problem so as to maximize interest income while satisfying the policy limitations. (Identify the variables, objective function, and constraints.)

(ii) Use Maple (LPSolve command) to solve the linear programming problem.

(i) Objective:

Maximeze Interest in Come =

$$\Rightarrow \boxed{X_1 \cdot 0.05 + X_2 \cdot 0.03 + X_3 \cdot 0.10 + X_4 \cdot 0.05}$$

First Mortgage Amoun $= X_1$
Second Mortgage Amoun $= X_2$
Home Improvement Amoun $= X_3$
Personel Overdraft Amount $= X_4$

Variables, X_1, X_2, X_3, X_4
Interest Rate

Constraints:

$$\begin{cases} X_1 + X_2 + X_3 + X_4 \geq 250 \\ X_1 \geq 0.55(X_1 + X_2) \\ X_1 \geq 0.25(X_1 + X_2 + X_3 + X_4) \\ X_2 \leq 0.25(X_1 + X_2 + X_3 + X_4) \\ \text{Interest\_Rate} = \dfrac{X_1 \cdot 0.05 + X_2 \cdot 0.08 + X_3 \cdot 0.10 + X_4 \cdot 0.05}{X_1 + X_2 + X_3 + X_4} \leq 0.07 \\ X_1 \geq 0, X_2 \geq 0, X_3 \geq 0, X_4 \geq 0 \end{cases}$$

IV Continue:

LP Solve Results:

Objective $= 17.5$

$x_1 = 62.500$

$x_2 = 51.136$

$x_3 = 69.319$

$x_4 = 67.045$

V. Consider two players (called Odd and Even) playing the Odd-or-Even game: the players simultaneously call out one of the numbers one or two. Player I wins if the sum of the numbers is odd. Player II wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars.

(i) Construct the 2x2 payoff matrix, analyze whether the game is strictly determined and if not, find an optimal mixed strategy for each player. What is the value of the game?

(ii) Assume now that each player calls out one of the numbers 0,1,2. The payoffs are calculated by the same rule. Construct the 3x3 payoff matrix, analyze whether the game is strictly determined and if not, find an optimal mixed strategy for each player. What is the value of the game in this case?

(i)

	1	2	
1	-2	+3	-2
2	+3	-4	-4
	3	3	

$\Rightarrow A = \begin{vmatrix} -2 & 3 \\ 3 & -4 \end{vmatrix}$ $\quad a_{11} = -2 \quad a_{12} = 3 \quad a_{21} = 3 \quad a_{22} = -4$

Game is not strictly determined.

$$\text{Value of the Game} = \frac{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{(-2)(-4) - 3 \cdot 3}{-2-4-3-3} = \frac{8-9}{-12} = \boxed{\frac{1}{12}}$$

Optimal Column strategy

$$q_1 = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{-4-3}{-12} = \boxed{\frac{7}{12}} \qquad q_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{-2-3}{-12} = \boxed{\frac{5}{12}}$$

Optimal Row strategy

$$p_1 = \frac{a_{22} - a_{21}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{-4-3}{-12} = \boxed{\frac{7}{12}} \qquad p_2 = \frac{a_{11} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}} = \frac{-2-3}{-12} = \boxed{\frac{5}{12}}$$

$$p \times A \times q = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix} \begin{vmatrix} \frac{7}{12} \\ \frac{5}{12} \end{vmatrix} = \begin{bmatrix} \frac{1}{12} & \frac{1}{12} \end{bmatrix} \begin{bmatrix} \frac{7}{12} \\ \frac{5}{12} \end{bmatrix} = \frac{1}{12}\left[\frac{7}{12} + \frac{5}{12}\right] = \boxed{\frac{1}{12}}$$

Answer: Game is not strictly determined.

value of The Game $= \boxed{\frac{1}{12}}$

Optimal column strategy $q = \begin{bmatrix} \frac{7}{12} \\ \frac{5}{12} \end{bmatrix}$

Optimal row strategy $p = \begin{bmatrix} \frac{7}{12} & \frac{5}{12} \end{bmatrix}$

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(ii) (V)

	0	1	2	
0	0	1	-2	-2
1	1	-2	3	-2
2	-2	3	-4	-4
	1	3	3	

Game is not strictly determined because saddle point can not be determined

$$A = \begin{vmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{vmatrix} \xrightarrow{+5} \begin{vmatrix} 5 & 6 & 3 \\ 6 & 3 & 8 \\ 3 & 8 & 1 \end{vmatrix}$$

~~Optimal strategy for Column player:~~

Objective: $x_1 + x_2 + x_3 \rightarrow$ maximaze

+5

	0	1	2
0	5	6	3
1	6	3	8
2	3	8	1

Constrains:

$$\begin{cases} 5x_1 + 6x_2 + 3x_3 \le 1 \\ 6x_1 + 3x_2 + 8x_3 \le 1 \\ 3x_1 + 8x_2 + 1 \cdot x_3 \le 1 \\ x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0 \end{cases}$$

LP Solve Results: $[0.2, [x_1 = 0.05,\ x_2 = 0.10,\ x_3 = 0.05]$

value of the Game $= \frac{1}{0.2} = 5$, $x_1 = 0.05$, $x_2 = 0.10$, $x_3 = 0.05$

Optimal strategy for column player:

$$q = \begin{vmatrix} x_1 \cdot \text{value of the Game} \\ x_2 \cdot \text{value of the Game} \\ x_3 \cdot \text{value of the Game} \end{vmatrix} = \begin{bmatrix} 0.05 \cdot 5 \\ 0.10 \cdot 5 \\ 0.05 \cdot 5 \end{bmatrix} = \boxed{\begin{vmatrix} 0.25 \\ 0.50 \\ 0.25 \end{vmatrix}}$$

Optimal strategy for row player:

Objective $y_1 + y_2 + y_3 \rightarrow$ minimize

constrains:

$$\begin{cases} 5y_1 + 6y_2 + 3y_3 \ge 1 \\ 6y_1 + 3y_2 + 8y_3 \ge 1 \\ 3y_1 + 8y_2 + 1 \cdot y_3 \ge 1 \\ y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ge 0 \end{cases}$$

LPSolve Results: $[0.2, [y_1 = 0.05,\ y_2 = 0.10,\ y_3 = 0.05]$

value of the Game $= 1/0.2 = 5$, $y_1 = 0.05$, y_2 0.10, $y_3 = 0.05$

Optimal strategy for raw player:

$p = [y_1 \cdot \text{value of the Gem},\ y_2 \cdot \text{value of the Game},\ y_3 \cdot \text{value of the Game}]$

$p = [0.25\ \ 0.5\ \ 0.25]$

$$pA \cdot q = [0.25\ \ 0.5\ \ 0.25] \begin{vmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{vmatrix} \begin{vmatrix} 0.25 \\ 0.50 \\ 0.25 \end{vmatrix} = 0 \Rightarrow$$ continue on last page

(V) Continue (ii)

Answer: Game is not strictly determined

Original value of the Game = 5-5 = $\boxed{0}$

The Game is fair.

Optimal strategy for column player:

$$q = \begin{bmatrix} 0.25 \\ 0.50 \\ 0.25 \end{bmatrix}$$

Optimal strategy for row player

$$p = [0.25 \quad 050 \quad 0.25]$$