Mathematical Modeling

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> *restart*; *with*(*Optimization*);

$$[ImportMPS, Interactive, LPSolve, LSSolve, Maximize, Minimize, NLPSolve, QPSolve] \tag{1}$$

"Theory of games" by B. Kolman
Exercize 6.
a)

$$\begin{array}{cccc} & C & & \\ R & \begin{vmatrix} -3, 4 \\ \mathbf{3}, 5 \end{vmatrix} & & \begin{array}{c} -1 \\ \mathbf{3} \end{array} \\ & \mathbf{3} \;\; 5 & & \end{array}$$

Sadle point a21 = 3 = Value Of the Game

> *with*(*LinearAlgebra*) :

> $A := Matrix([[-3, 4], [3, 5]])$

$$A := \begin{bmatrix} -3 & 4 \\ 3 & 5 \end{bmatrix} \tag{2}$$

> $p := Matrix([0.0, 1.0])$

$$p := \begin{bmatrix} 0. & 1.0 \end{bmatrix} \tag{3}$$

> $q := Vector([1.0, 0.0])$

$$q := \begin{bmatrix} 1.0 \\ 0. \end{bmatrix} \tag{4}$$

> $p := Matrix([0.0, 1.0])$

$$p := \begin{bmatrix} 0. & 1.0 \end{bmatrix} \tag{5}$$

> $q := Vector([1.0, 0.0])$

$$q := \begin{bmatrix} 1.0 \\ 0. \end{bmatrix} \tag{6}$$

> $p \cdot A \cdot q$

$$\begin{bmatrix} 3. \end{bmatrix} \tag{7}$$

Answer:

Value Of the Game = 3

<u>The optimal strategy for the column player is :</u>

$$\boldsymbol{q} := \begin{bmatrix} \mathbf{1.0} \\ \mathbf{0.0} \end{bmatrix}$$

<u>The optimal strategy for the row player is :</u>

$$\boldsymbol{p} := \begin{bmatrix} \mathbf{0.0} & \mathbf{1.0} \end{bmatrix}$$

###

b)

$$\begin{array}{cccc} & & \mathbf{C} & \\ \mathbf{R} & \left| \begin{matrix} -1 & -3 & -2 \\ 3 & \mathbf{-1} & 4 \\ -1 & -2 & 5 \end{matrix} \right| & & \begin{matrix} -1 \\ -1 \\ -1 \end{matrix} \\ & \begin{matrix} 3 & \mathbf{-1} & 5 \end{matrix} & & \end{array}$$

Sadle point a22 = -1 = Value Of the Game

> $A := Matrix([[-1,-3,-2],[3,-1,4],[-1,-2,5]])$

$$A := \begin{bmatrix} -1 & -3 & -2 \\ 3 & -1 & 4 \\ -1 & -2 & 5 \end{bmatrix} \quad \textbf{(8)}$$

> $p := Matrix([0.0, 1.0, 0.0])$

$$p := \begin{bmatrix} 0. & 1.0 & 0. \end{bmatrix} \quad \textbf{(9)}$$

> $q := Vector([0.0, 1.0, 0.0])$

$$q := \begin{bmatrix} 0. \\ 1.0 \\ 0. \end{bmatrix} \quad \textbf{(10)}$$

> $p \cdot A \cdot q$

$$\begin{bmatrix} -1. \end{bmatrix} \quad \textbf{(11)}$$

Answer:

Value Of the Game = -1

<u>The optimal strategy for the column player is :</u>

$$\boldsymbol{q} = \begin{bmatrix} \mathbf{0.0} \\ \mathbf{1.0} \\ \mathbf{0.0} \end{bmatrix}$$

<u>The optimal strategy for the row player is :</u>

$$\boldsymbol{p} = \begin{bmatrix} \mathbf{0.0} & \mathbf{1.0} & \mathbf{0.0} \end{bmatrix}$$

##

c)

		C		
R	|-2 3 **-2** 4|	-2	->	|2 7 2 8|
	|-1 2 **-2** 4|	-2		|3 6 2 8|
	|-2 3 -3 5|	-3		|2 7 1 9|
	|-1 2 -3 1|	-3		|3 6 1 5|
	-1 3 **-2** 5			

Sadle points: a13 = -2 , a23=-2 Value Of the Game -2

> $A := Matrix([[-2, 3, -2, 4], [-1, 2, -2, 4], [-2, 3, -3, 5], [-1, 2, -3, 1]])$

1.

$$A := \begin{bmatrix} -2 & 3 & -2 & 4 \\ -1 & 2 & -2 & 4 \\ -2 & 3 & -3 & 5 \\ -1 & 2 & -3 & 1 \end{bmatrix} \quad (12)$$

> $p1 := Matrix([1.0, 0.0, 0.0, 0.0])$

$$p1 := \begin{bmatrix} 1.0 & 0. & 0. & 0. \end{bmatrix} \quad (13)$$

> $p2 := Matrix([0.0, 1.0, 0.0, 0.0])$

$$p2 := \begin{bmatrix} 0. & 1.0 & 0. & 0. \end{bmatrix} \quad (14)$$

> $q := Vector([0.0, 0.0, 1.0, 0.0])$

$$q := \begin{bmatrix} 0. \\ 0. \\ 1.0 \\ 0. \end{bmatrix} \quad (15)$$

> $p1 \cdot A \cdot q$

$$\begin{bmatrix} -2. \end{bmatrix} \quad (16)$$

> $p2 \cdot A \cdot q$

$$\begin{bmatrix} -2. \end{bmatrix} \quad (17)$$

Answer:

Value Of the Game = -2

<u>The optimal strategy for the column player is :</u>

$$q = \begin{bmatrix} 0.0 \\ 0.0 \\ 1.0 \\ 0.0 \end{bmatrix}$$

<u>The optimal strategy for the row player is :</u>

$$p1 = \begin{bmatrix} 1.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$
$$p2 = \begin{bmatrix} 0.0 & 1.0 & 0.0 & 0.0 \end{bmatrix}$$

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Problem 10.

|4 8|
|6 -2|

Value Of The game = (a11*a22-a12*a21)/(a11+a22-a12-a21) =
(4*(-2) - 6*8) / (4-2-6-8) = -56/ -12 = 14/3 = 4.67
p1 = (a22-a21)/(a11+a22-a12-a21) = (-2-6)/(-12) = 2/3
p2 = (a11-a12)/(a11+a22-a12-a21) = (4-8)/(-12) = 1/3
q1 = (a22-a12)/(a11+a22-a12-a21) = (-2-8)/(-12) = 5/6
q2 = (a11-a21)/(a11+a22-a12-a21) = (4-6)/(-12) = 1/6
Check if Value Of the Game is correct:

```
> A:= Matrix([[4,8],[6,-2]])
```

$$A := \begin{bmatrix} 4 & 8 \\ 6 & -2 \end{bmatrix} \tag{18}$$

> $p := Matrix\left(\left[\frac{2}{3}, \frac{1}{3}\right]\right)$

$$p := \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \end{bmatrix} \tag{19}$$

> $q := Vector\left(\left[\frac{5}{6}, \frac{1}{6}\right]\right)$

$$q := \begin{bmatrix} \frac{5}{6} \\ \frac{1}{6} \end{bmatrix} \tag{20}$$

> $p \cdot A \cdot q$

$$\begin{bmatrix} \frac{14}{3} \end{bmatrix} \tag{21}$$

>

Answer:
Value of the Game = 14/3 = 4.67
Optimal strategy for row player

$$\boldsymbol{p} := \begin{bmatrix} \mathbf{\frac{2}{3}} & \mathbf{\frac{1}{3}} \end{bmatrix}$$

Optimal strategy for column player

$$\boldsymbol{q} := \begin{bmatrix} \mathbf{\frac{5}{6}} \\ \mathbf{\frac{1}{6}} \end{bmatrix}$$

##

Problem 12.

|-2 3| -> |1 6|
|4 5| |7 8|
|5 2| |8 5|

Optimal strategy for column player:
Objective: maximaze : $x1 + x2$
Constrains: 1*x1 + 6 * x2 <=1
7*x1 + 8 * x2 <=1
8*x1 + 5 * x2 <=1
x1>=0,x2>=0

> $objC := x1 + x2$

$$objC := x1 + x2 \quad (22)$$

> $consC := [1 \cdot x1 + 6 \cdot x2 \le 1, 7 \cdot x1 + 8 \cdot x2 \le 1, 8 \cdot x1 + 5 \cdot x2 \le 1, x1 \ge 0, x2 \ge 0]$

$$consC := [x1 + 6\,x2 \le 1, 7\,x1 + 8\,x2 \le 1, 8\,x1 + 5\,x2 \le 1, 0 \le x1, 0 \le x2] \quad (23)$$

> $LPSolve(objC, consC, maximize)$

$$[0.137931034482759, [x1 = 0.103448275862069, x2 = 0.0344827586206896]] \quad (24)$$

> $value\_of\_the\_game := 1/0.13793$

$$value\_of\_the\_game := 7.250054375 \quad (25)$$

original game value = 7.25-3 = 4.25

The optimal strategy for the column player is :
q = [(x1=0.103448275862069) * *value_of_the_game*,
(x2=0.0344827586206896) * *value_of_the_game*]

> $q := Vector([0.103448 * value\_of\_the\_game, 0.034483 * value\_of\_the\_game])$

$$q := \begin{bmatrix} 0.7500036250 \\ 0.2500036250 \end{bmatrix} \quad (26)$$

Optimal strategy for the row player:
objective : minimize y1 + y2 +y3
constarins: $1 \cdot y1 + 7 \cdot y2 + 8 \cdot y3 \ge 1,$
$6 \cdot y1 + 8 \cdot y2 + 5 \cdot y3 \ge 1,$
$y1 \ge 0, y2 \ge 0, y3 \ge 0$

> $objR := y1 + y2 + y3$

$$objR := y1 + y2 + y3 \quad (27)$$

> $consR := [1 \cdot y1 + 7 \cdot y2 + 8 \cdot y3 \ge 1, 6 \cdot y1 + 8 \cdot y2 + 5 \cdot y3 \ge 1, y1 \ge 0, y2 \ge 0, y3 \ge 0]$

$$consR := [1 \le y1 + 7\,y2 + 8\,y3, 1 \le 6\,y1 + 8\,y2 + 5\,y3, 0 \le y1, 0 \le y2, 0 \le y3] \quad (28)$$

> $LPSolve(objR, consR)$

$$[0.137931034482759, [y1 = 0., y2 = 0.103448275862069, y3 = 0.0344827586206897]] \quad (29)$$

> $value\_of\_the\_game := 1/0.13793$

$$value\_of\_the\_game := 7.250054375 \quad (30)$$

So original game value = 7.25-3 = 4.25
The optimal strategy for the row player is :
p = [(y1=0.0) * *value_of_the_game*,
(y2=0.103448) * *value_of_the_game*,
(y3=0.034483) * *value_of_the_game*]

> $p := Matrix([0.0 \cdot value\_of\_the\_game, 0.103448 \cdot value\_of\_the\_game, 0.034483 \cdot value\_of\_the\_game])$

$$p := \begin{bmatrix} 0. & 0.7500036250 & 0.2500036250 \end{bmatrix} \quad (31)$$

Check if correct:

> $A := Matrix([[-2, 3], [4, 5], [5, 2]])$

$$A := \begin{bmatrix} -2 & 3 \\ 4 & 5 \\ 5 & 2 \end{bmatrix} \tag{32}$$

```
> p.A.q
```

$$\begin{bmatrix} 4.25006162521025 \end{bmatrix} \tag{33}$$

Answer:

Original game value = 7.25-3 = 4.25

Optimal strategy for row player

$$\boldsymbol{p} := \begin{bmatrix} \mathbf{0.} & \mathbf{0.75} & \mathbf{0.25} \end{bmatrix}$$

Optimal strategy for column player

$$\boldsymbol{q} := \begin{bmatrix} \mathbf{0.75} \\ \mathbf{0.25} \end{bmatrix}$$

##

Problem 1

. Do exercise 1 and find the optimal mixed strategy as in exercise 10
(first notice that the game is not strictly determined) Each of the two players shows two or tree fingers. If the sum of the fingers shown is even, then R pays C amount equal to the sum of the numbers shown; if the sum is odd, then C pays R an amount equal to the sum of the numners shown.

		C	
		2	3
R	2	|-4	5|
	3	|5	-6 |

Value Of The game = (a11*a22-a12*a21)/(a11+a22-a12-a21) =
((-4)*(-6) - 5*5) / (-4-6-5-5) = -1/ -20 = 0.05
p1 = (a22-a21)/(a11+a22-a12-a21) = (-6-5)/(-20) = 11/20
p2 = (a11-a12)/(a11+a22-a12-a21) = (-4-5)/(-20) = 9/20
q1 = (a22-a12)/(a11+a22-a12-a21) = (-6-5)/(-20) = 11/20
q2 = (a11-a21)/(a11+a22-a12-a21) = (-4-5)/(-20) = 9/20
Check if Value Of the Game is correct:

```
>  A:= Matrix([[-4,5],[5,-6]])
```

$$A := \begin{bmatrix} -4 & 5 \\ 5 & -6 \end{bmatrix} \tag{34}$$

> $p := Matrix\left(\left[\frac{11}{20}, \frac{9}{20}\right]\right)$

$$p := \begin{bmatrix} \frac{11}{20} & \frac{9}{20} \end{bmatrix} \tag{35}$$

> $q := Vector\left(\left[\frac{11}{20}, \frac{9}{20}\right]\right)$

$$q := \begin{bmatrix} \frac{11}{20} \\ \frac{9}{20} \end{bmatrix} \tag{36}$$

> evalf($p \cdot A \cdot q$)

$$\begin{bmatrix} 0.05000000000 \end{bmatrix} \tag{37}$$

Answer:
Value of the Game = 0.05
Optimal strategy for row player

$$p := \begin{bmatrix} \frac{11}{20} & \frac{9}{20} \end{bmatrix}$$

Optimal strategy for column player

$$q := \begin{bmatrix} \frac{11}{20} \\ \frac{9}{20} \end{bmatrix}$$

###

Problem 2.
. Do exercise 2 and find the optimal mixed strategy as in exercise 12 (first notice that the game is not strictly determined)

		C						
		Stone	Scissors	Paper				
	Stone	0	1	-1	->	2	3	1
R	Scissors	-1	0	1		1	2	3
	Paper	1	-1	0		3	1	2

Optimal strategy for column player:
Objective: maximaze : $x1 + x2 + x3$
Constrains: 2*x1 + 3 * x2 + 1 *x3 <=1
1*x1 + 2 * x2 + 3 *x3 <=1
3*x1 + 1 * x2 + 2*x3 <=1
x1>=0, x2>=0, x3>=0

> objC := $x1 + x2 + x3$

$$objC := x1 + x2 + x3 \tag{38}$$

> consC := $[2 \cdot x1 + 3 \cdot x2 + 1 \cdot x3 \le 1, 1 \cdot x1 + 2 \cdot x2 + 3 \cdot x3 \le 1, 3 \cdot x1 + 1 \cdot x2 + 2 \cdot x3 \le 1, x1 \ge 0, x2 \ge 0, x3 \ge 0]$

$$consC := [2\,x1 + 3\,x2 + x3 \le 1, x1 + 2\,x2 + 3\,x3 \le 1, 3\,x1 + x2 + 2\,x3 \le 1, 0 \le x1, 0 \le x2, 0 \le x3] \tag{39}$$

> LPSolve(objC, consC, maximize)

$$[0.500000000000000, [x1 = 0.166666666666667, x2 = 0.166666666666667, x3 = 0.166666666666667]] \tag{40}$$

> value_of_the_game := 1 / 0.5

$$value\_of\_the\_game := 2.000000000 \tag{41}$$

original game value = 2-2 = 0

The optimal strategy for the column player is :
q = [(x1=0.166666666666667) * value_of_the_game,
(x2=0.166666666666667) * value_of_the_game]

> q := Vector([0.166666666666667 * value_of_the_game, 0.166666666666667 * value_of_the_game, 0.166666666666667 · value_of_the_game])

$$q := \begin{bmatrix} 0.3333333334 \\ 0.3333333334 \\ 0.3333333334 \end{bmatrix} \tag{42}$$

Optimal strategy for the row player:
objective : minimize y1 + y2 +y3
constarins: $2 \cdot y1 + 1 \cdot y2 + 3 \cdot y3 \geq 1,$
$3 \cdot y1 + 2 \cdot y2 + 1 \cdot y3 \geq 1,$
$1 \cdot y1 + 3 \cdot y2 + 2 \cdot y3 \geq 1,$
$y1 \geq 0, y2 \geq 0, y3 \geq 0$

> $objR := y1 + y2 + y3$

$$objR := y1 + y2 + y3 \tag{43}$$

> $consR := [2 \cdot y1 + 1 \cdot y2 + 3 \cdot y3 \geq 1, 3 \cdot y1 + 2 \cdot y2 + 1 \cdot y3 \geq 1, 1 \cdot y1 + 3 \cdot y2 + 2 \cdot y3 \geq 1, y1 \geq 0, y2 \geq 0, y3 \geq 0]$

$$consR := [1 \leq 2\,y1 + y2 + 3\,y3, 1 \leq 3\,y1 + 2\,y2 + y3, 1 \leq y1 + 3\,y2 + 2\,y3, 0 \leq y1, 0 \leq y2, 0 \leq y3] \tag{44}$$

> $LPSolve(objR, consR)$

$$[0.500000000000000, [y1 = 0.166666666666667, y2 = 0.166666666666667, y3 = 0.166666666666667]] \tag{45}$$

> $value\_of\_the\_game := 1/0.5$

$$value\_of\_the\_game := 2.000000000 \tag{46}$$

p = [(y1=0.166666666666667) * *value_of_the_game*,
(y2=0.166666666666667) * *value_of_the_game*,
(y3=0.166666666666667) * *value_of_the_game*,

> $p := Matrix([0.166666666666667 \cdot value\_of\_the\_game, 0.166666666666667 \cdot value\_of\_the\_game, 0.166666666666667 \cdot value\_of\_the\_game])$

$$p := \begin{bmatrix} 0.3333333334 & 0.3333333334 & 0.3333333334 \end{bmatrix} \tag{47}$$

>

Check if correct:

> $A := Matrix([[-0, 1, -1], [-1, 0, 1], [1, -1, 0]])$

$$A := \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \tag{48}$$

> `p.A.q`

$$\begin{bmatrix} 0. \end{bmatrix} \tag{49}$$

Answer:
Original game value =2-2 = 0
Optimal strategy for row player

$$p = \begin{bmatrix} 0.33334 & 0.33334 & 0.33334 \end{bmatrix}$$

Optimal strategy for column player

$$q = \begin{bmatrix} 0.33334 \\ 0.33334 \\ 0.33334 \end{bmatrix}$$

##

Problem 6 from Homework

Consider a football game where the row team plays offense and the column team plays defense. The offense team can call two strategies (run or pass), the defense team can call three strategies (run defense, pass defense, or blitz). Below is the payoff matrix (yards gained by the offense team).

Find the optimal mixed strategy of this zero-sum game (as in exercise 12).

		Defense						
		Run D	**Pass D**	**Blitz**				
Offense	**Run**	**-3**	**4**	**3**	**->**	**1**	**8**	**7**
	Pass	**9**	**-3**	**5**		**13**	**1**	**9**

Optimal strategy for column player:

Objective: maximaze : $x1 + x2 + x3$

Constrains: 1 *x1 + 8 * x2 + 7 *x3 <=1

13*x1 + 1 * x2 + 9 *x3 <=1

x1>=0, x2>=0, x3>=0

> $objC := x1 + x2 + x3$

$$objC := x1 + x2 + x3 \quad (50)$$

> $consC := [1 \cdot x1 + 8 \cdot x2 + 7 \cdot x3 \le 1, 13 \cdot x1 + 1 \cdot x2 + 9 \cdot x3 \le 1, x1 \ge 0, x2 \ge 0, x3 \ge 0]$

$$consC := [x1 + 8\,x2 + 7\,x3 \le 1, 13\,x1 + x2 + 9\,x3 \le 1, 0 \le x1, 0 \le x2, 0 \le x3] \quad (51)$$

> $LPSolve(objC, consC, maximize)$

$$[0.184466019417476, [x1 = 0.0679611650485437, x2 = 0.116504854368932, x3 = 1.38777878078145\ 10^{-17}]] \quad (52)$$

> $value\_of\_the\_game := 1 / 0.184466019417476$

$$value\_of\_the\_game := 5.421052632 \quad (53)$$

original game value = 5.421052632-4 = 1.421052632

<u>The optimal strategy for the column player is :</u>

q = [(x1=0.0679611650485437) * *value_of_the_game*,
(x2=0.116504854368932) * *value_of_the_game*,
($x3 = 0$) * *value_of_the_game*

> $q := Vector([0.06796116504854377 * value\_of\_the\_game, 0.116504854368932 * value\_of\_the\_game, 0.0 \cdot value\_of\_the\_game])$

$$q := \begin{bmatrix} 0.3684210527 \\ 0.6315789476 \\ 0. \end{bmatrix} \quad (54)$$

Optimal strategy for the row player:

objective : minimize y1 + y2

constarins: $1 \cdot y1 + 13 \cdot y2 \ge 1,$

$8 \cdot y1 + 1 \cdot y2 \ge 1,$

$7 \cdot y1 + 9 \cdot y2 \ge 1,$

$y1 \ge 0, y2 \ge 0$

> $objR := y1 + y2$

$$objR := y1 + y2 \quad (55)$$

> $consR := [1 \cdot y1 + 13 \cdot y2 \ge 1, 8 \cdot y1 + 1 \cdot y2 \ge 1, 7 \cdot y1 + 9 \cdot y2 \ge 1, y1 \ge 0, y2 \ge 0]$

$$consR := [1 \le y1 + 13\,y2, 1 \le 8\,y1 + y2, 1 \le 7\,y1 + 9\,y2, 0 \le y1, 0 \le y2] \quad (56)$$

> $LPSolve(objR, consR)$

$$[0.184466019417476, [y1 = 0.116504854368932, y2 = 0.0679611650485438]] \tag{57}$$

> $value\_of\_the\_game := 1/0.184466019417476$

$$value\_of\_the\_game := 5.421052632 \tag{58}$$

So original game value = 5.421052632-3 = 1.421052632

The optimal strategy for the row player is :

p = [(y1=0.116504854368932) * *value_of_the_game*,
(y2=0.0679611650485438) * *value_of_the_game*]

> $p := Matrix([0.116504854368932 \cdot value\_of\_the\_game, 0.0679611650485438 \cdot value\_of\_the\_game])$

$$p := \begin{bmatrix} 0.6315789476 & 0.3684210527 \end{bmatrix} \tag{59}$$

```
> A:= Matrix([[-3,4,6],[9,-3,-5]])
```

$$A := \begin{bmatrix} -3 & 4 & 6 \\ 9 & -3 & -5 \end{bmatrix} \tag{60}$$

> $p \cdot A \cdot q$

$$\begin{bmatrix} 1.42105263243158 \end{bmatrix} \tag{61}$$

Answer:

Original game value =5.42-3 = **1.42**

Optimal strategy for row player

$$\boldsymbol{p} = \begin{bmatrix} \mathbf{0.63157} & \mathbf{0.36842} \end{bmatrix}$$

Optimal strategy for column player

$$\boldsymbol{q} = \begin{bmatrix} \mathbf{0.36842} \\ \mathbf{0.63159} \\ \mathbf{0.} \end{bmatrix}$$

>