Numerical Analysis

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Section 1.1

Intermediate value Theorem

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$$3x^3 + x^2 = x + 5$$
$$3x^3 + x^2 - x - 5 = 0$$

$f(0) = -5 < 0$

$f(1) = -2 < 0$ root between $1 \div 2$

$f(2) = 33 > 0$

$\Rightarrow$ $\boxed{\text{interval} = [1, 2]}$

3b $\quad \frac{b-a}{2^{n+1}} \le \frac{1}{8}$ In our case $n=2$

$\Downarrow$ $b - a \le \frac{2^{2+1}}{8} \Rightarrow b - a \le 1 \Rightarrow$

interval $[1,2] = b - a = 1$ within $\frac{1}{8}$

step $\boxed{0}$: $\frac{1+2}{2}$ = midpoint $= \frac{3}{2}$ $f(\frac{3}{2}) = 5.875 > 0$

$f(1) < 0$ $\quad f(1.5) > 0$ $\quad f(2) > 0$ $\Rightarrow$ next interval $[1, 1.5]$

Step $\boxed{1}$: $\frac{1+1.5}{2}$ = midpoint $= 1.25$ $\quad f(1.25) = 1.172 > 0$

$f(1) < 0$ $\quad f(1.25) > 0$ $\quad f(1.5) > 0$ next interval $[1.0, 1.25]$

Step $\boxed{2}$: $\frac{1+1.25}{2} = 1.125 > 0$ $\quad f(1.125) = -0.59 < 0$

$\Rightarrow f(1) < 0$ $\quad \boxed{f(1.125) < 0}$, $\quad f(1.25) > 0 \Rightarrow$

$\boxed{\text{Answer:}}$ After step 2 the solution will be $= \boxed{1.125}$ and it will be within $\frac{1}{8}$ of the true root

$\boxed{5}$ (a) $x^4 = x^3 + 10$ find $[a\ b]$ of lenght $= 1$ eq. has a solution?

$x^4 - x^3 - 10 = 0$

$\left.\begin{array}{l} f(0) = -10 \\ f(1) = -10 \\ f(2) = -2 \\ f(3) = 44 \end{array}\right\}$ $\Rightarrow$ There's a solution betwen 2 and 3 $\Rightarrow [a\ b] = [2;3]$ and $[a\ b]$ = is a lenght $3 - 2 = 1$

(b) $\frac{b-a}{2^{n+1}} < 10^{-10} \Rightarrow \frac{3-2}{2^{n+1}} < 10^{-10} \Rightarrow$

$\Rightarrow 10^{+10} < 2^{n+1} \Rightarrow n + 1 > 10 \log_2 10 \Rightarrow$

Number of steps $n > 10 \log_2 10 - 1 = 32.22$

$\boxed{\text{Answer: Number of steps} = 33}$

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$\boxed{\text{Section 0.5}}$

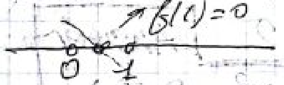
$\boxed{1a}$ use the Intermediate Value Theorem to prove that $f(c) = 0$ for some $0 < c < 1$

$f(x) = x^3 - 4x + 1$ $c \in (0, 1)$ prove $f(c) = 0$

$f(0) = 0 - 0 + 1 = 1 > 0$, $f(1) = 1 - 4 + 1 = -4 < 0$

$f(0) \cdot f(1) < 0 \Rightarrow$

$\Rightarrow$ All values between -4 and 1 including zero must be taken on by $f(x)$ by Intermediate value Theorem $[a\ b] = [0, 1]$



function changing value from (+) to (−) and therefore interval $[0, 1]$ contains the root.

$\boxed{4a}$ Find the Taylor polynomial of degree 2 about the point $x_0 = 0$

ⓐ $f(x) = e^{x^2}$

$f'(x) = e^{x^2} \cdot 2x = 2(e^{x^2} \cdot x)$

$f''(x) = 2(e^{x^2} \cdot 2x \cdot x + e^{x^2}) = 2 \cdot e^{x^2}(2x^2+1)$

$P_2(x) = f(x_0) + f'(x_0) \cdot (x-x_0) + f'' \frac{(x-x_0)^2}{2!}$

$f(x_0) = e^0 = 1$

$f'(x_0) = 2 \cdot e^0 \cdot 0 = 0$

$f''(x_0) = 2 \cdot e^0(2 \cdot 0 + 1) = 2$

$P_2(x) = 1 + \frac{2 \cdot x^2}{2}$

Answer

$\boxed{1 + x^2 = P_2(x)}$

$\boxed{7}$ $f(x) = \ln x$ $\quad x_0 = 1$

ⓐ find Taylor polynomial degree 4

$f'(x) = \frac{1}{x}$ $\quad f''(x) = (-1) \cdot x^{-2}$ $\quad f'''(x) = 2 \cdot x^{-3}$

$f^{IV}(x) = -6 \cdot x^{-4}$

$f(1) = 0$ $\quad f'(1) = 1$ $\quad f''(1) = -1$ $\quad f'''(1) = 2$ $\quad f^{IV}(x) = -6$

$P_4(x) = (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!} =$

$= (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$

7 a continue | a Answer

$$\boxed{P_4(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4}$$

b $f(0.9) - ?$ $f(1.1) - ?$

$f(0.9) = -0.1 - \frac{1}{2}(-0.1)^2 + \frac{1}{3}(-0.1)^3 - \frac{1}{4}(-0.1)^4 =$

$= -0.1 - 0.005000 - 0.000333 - 0.000025 = \boxed{-0.105358}$

$f(1.1) = 0.1 - \frac{1}{2}(0.1)^2 + \frac{1}{3}(0.1)^3 - \frac{1}{4}(0.1)^4 =$

$= 0.100000 - 0.005000 + 0.000333 - 0.000025 = \boxed{0.095308}$

c $R_K(x) = \frac{f^{(K+1)}(c)}{(K+1)}(x-x_0)^{K+1}$; $f^{V}(x) = 24x^{-5}$

$R_4(x) = \frac{24 \cdot c^{-5}}{5!}(x-1)^5 = \frac{1}{5c^5}(x-1)^5$

$\underline{x = 1.1}$ $x > 1 \Rightarrow c \in (1, x)$ $1 < c < x \Rightarrow$

$\Rightarrow x^{-5} < c^{-5} < 1^{-5} \Rightarrow \frac{1}{c^5} < 1 \Rightarrow$

$|R_4(x)| \leq \frac{1}{5}(x-1)^5$

$\underline{x = 0.9}$ $x < 1 \Rightarrow x < c < 1 \Rightarrow 1^{-5} < c^{-5} < x^{-5}$

$\Rightarrow \frac{1}{c^5} < \frac{1}{x^5}$

$|R_4(x)| \leq \frac{1}{5x^5}(x-1)^5$

$$\underbrace{\frac{|\ln x - P_4(x)|}{|\ln x|}}_{\text{relative Error}} \leq \begin{cases} \frac{1}{5}(x-1)^5 / |\ln x|, & x > 1 \\ \frac{1}{5x^5}(x-1)^5 / |\ln x|, & 0 < x < 1 \end{cases}$$

for $x=0.9$

$|\text{relative Err}(0.9)| = \frac{1}{5 \cdot 0.9^5}(0.9-1)^5 / |\ln 0.9|$

$= 0.000032147 = \boxed{3.2 \cdot 10^{-5}} \to$ upper bound

for $x=1.1$

$|\text{Relative Err}(1.1)| = \frac{1}{5}(1.1-1)^5 / \ln 1.1 =$

$= 0.000020984 \approx \boxed{2.1 \cdot 10^{-5}} \to$ upper bound

By using the above result we can see that relative error for 1.1 less then for 0.9 and we whould expect the $P_4(1.1)$ closer to $\ln(1.1)$ then $P_4(0.9)$ to $\ln(0.9)$

(d) $\text{Rel Err actual}_{(1.1)} = \frac{|\ln(1.1) - P_4(1.1)|}{|\ln(1.1)|}$

$= \frac{|0.09531 - 0.095308|}{|0.095308|} = \boxed{2.1 \cdot 10^{-5}}$

$\text{Rel Err actual}(0.9) = \frac{|\ln(0.9) - P_4(0.9)|}{|\ln(0.9)|}$

$= \frac{|-0.105360 + 0.105358|}{|-0.105360|} = \boxed{1.9 \cdot 10^{-5}}$

we can see that our expected error wrong

$RE(1.1) = 2.1 \cdot 10^{-5} < \text{Upper bound} = 3.2 \cdot 10^{-5}$

$RE(0.9) = 1.9 \cdot 10^{-5} < \text{upper bound} = 2.1 \cdot 10^{-5}$