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Filtered integration rules for finite weighted Hilbert transforms

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ABSTRACT

A product quadrature rule, based on the filtered de la Vallée Poussin polynomial approximation, is proposed for evaluating the finite weighted Hilbert transform in $[-1, 1]$. Convergence results are stated in weighted uniform norm for functions belonging to suitable Besov type subspaces. Several numerical tests are provided, also comparing the rule with other formulas known in literature.

1. Introduction

The numerical computation of the Hilbert transform of a function plays an important role in several fields, since many mathematical models in applied sciences lead to it (see e.g. [1,2] and the references therein). Depending on the specific application, we may consider bounded or unbounded integration domains where the function may have several degrees of smoothness (see e.g. the treatise by F. King in two volumes [2] on many aspects of the Hilbert transform and its possible variants). Here we consider the case of locally continuous functions f on the reference domain $[-1, 1]$ and, for any Jacobi weight

$$u(x) = v^{a,b}(x) := (1-x)^a(1+x)^b, \qquad a, b > -1,$$

we focus on the numerical approximation of the finite weighted Hilbert transform of f (the latter also known as "density function") defined as the following principal value integral

$$\mathcal{H}^u f(t) := \int_{-1}^{1} \frac{f(x)}{x-t} u(x)dx = \lim_{\epsilon\to 0} \int_{|x-t|\geq\epsilon} \frac{f(x)}{x-t} u(x)dx, \qquad -1 < t < 1, \tag{1}$$

being $\mathcal{H} \equiv \mathcal{H}^u$ in the case $u = 1$.

Incidentally, such transform appears in Cauchy singular integral equations, which, in turn, arise in several mathematical models (see for instance [3–10]). Moreover, $\mathcal{H}^u f$ is related to hypersingular operators that can be defined as derivatives of the Hilbert transform and appear in integro-differential equations (see e.g. [11–14]).

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Due to the relevance of the problem, there exists a wide literature on quadrature rules for $\mathcal{H}^u f$. We cite for instance [15–24]. In particular, concerning the unweighted case, i.e. the classical finite Hilbert transform, quadrature rules based on equidistant nodes of $[-1, 1]$ have been recently considered in [25,26].

In this paper, as system of nodes, we shall consider the zeros of Jacobi polynomials $\{p_n(w)\}_n$ associated to suitable Jacobi weights $w = v^{\alpha,\beta}$, say

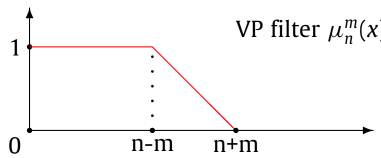
$$X_n(w) := \{x_k := x_{n,k}(w) \ : \ k = 1, \dots, n\}, \qquad n \in \mathbb{N}. \tag{2}$$

Based on Jacobi systems of nodes, several quadrature rules are known in literature (see e.g. [15,16,21,27,28] and the references therein). In particular, for any $n \in \mathbb{N}$, we recall the product rule (L-rule) obtained by replacing f with the Lagrange polynomial $L_n(w, f)$ interpolating f at $X_n(w)$ (see e.g. [15,21,27]).

Here we propose a similar product rule (that we will call VP-rule), but instead of the Lagrange polynomial, we employ the filtered de la Vallée Poussin (VP) polynomial of f [29,30]

$$V_n^m(w, f, x) = \sum_{k=1}^{n} f(x_k)\Phi_{n,k}^m(x), \qquad n > m \in \mathbb{N}, \qquad x \in [-1, 1],$$

based on the same n nodes required for $L_n(w, f)$. Differently from $L_n(w, f)$, the polynomial $V_n^m(w, f)$ does not interpolate f except in some special cases [29,31,32]. Its main characteristic is the dependence on the additional degree-parameter $0 < m < n$ which determines the action ray of the VP filter



$$\text{VP filter } \mu_n^m(x) = \begin{cases} 1 & \text{if } \ 0 \le x \le n-m, \\ \frac{n+m-x}{2m} & \text{if } \ n-m < x < n+m. \end{cases}$$

Such filter function defines the fundamental VP polynomials as follows [33]

$$\Phi_{n,k}^m(x) = \lambda_k \sum_{j=0}^{n+m-1} \mu_n^m(j) p_j(w, x_k) p_j(w, x),$$

being

$$\lambda_k := \lambda_{n,k}(w) = \left(\sum_{j=0}^{n-1} [p_j(w, x_k)]^2 \right)^{-1}, \qquad k = 1, 2, \dots, n, \tag{3}$$

the Christoffel numbers related to the weight w.

Note that in the limiting case $m = 0$ the previous VP polynomial $V_n^m(w, f)$ coincides with the Lagrange polynomial $L_n(w, f)$ and therefore it yields the same quadrature rule.

The aim of the present paper is to show that, by using $V_n^m(w, f)$ with $0 < m < n$, we can take advantage of the additional parameter m which can be suitably modulated in order to improve the quadrature error of the L-rule.

Indeed, it is already known (see e.g. [29,32,34]) that corresponding to suitable choices of $0 < m < n$, the polynomial $V_n^m(w, f)$ provides a pointwise approximation better than the one offered by $L_n(w, f)$, especially in presence of Gibbs phenomena. In fact, if f presents some "pathologies" (peaks, cusps, etc.) localized in isolated points, the Gibbs phenomenon affects $L_n(w, f)$ with overshoots and oscillations that spill over the whole interval. On the contrary such phenomenon appears strongly reduced by using $V_n^m(w, f)$.

The experimental results will show that such an improvement is inherited by VP-rules, that for these kinds of density functions may provide a performance better than the L-rule.

In our experiments we will also compare VP-rules with another class of quadrature formulas based on Jacobi zeros, i.e. the Modified Gaussian rules (shortly denoted by MG rules), proposed in [17] and also studied in [15]. Such formulas provide higher performance than L-rule, but we will show some cases where VP-rules produce an even better quadrature error. Moreover, we remark that differently from both L and VP-rules, MG rules require a variable number of nodes depending on the singular point t and consequently their employment, for instance in numerical methods for singular integral equations, appears more complicated than the previous product rules. Finally, we make also some comparisons with the methods proposed in [20] and [23]. The first is always based on Jacobi zeros, while the second one is an "analytic function based method" employing a Sinc series.

From a theoretical point of view, in this paper we provide new estimates of the VP polynomial approximation in some Besov type spaces characterized by a Dini-type condition. Using such estimates we prove the convergence of the proposed quadrature rule and state several asymptotic bounds for the quadrature error.

The outline of the paper is the following. Section 2 contains preliminary notations and results concerning the approximation spaces. Moreover, in a dedicated subsection, the mapping properties of the Hilbert transform in such spaces are studied. Section 3 is devoted to the VP approximation, while the new quadrature rule is introduced in Section 4, where we state the main results about it. Finally, Section 5 concerns the numerical tests and Section 6 is devoted to the proofs.

2. Notations and preliminary results

Consider a fixed Jacobi weight

$$v(x) = v^{\gamma,\delta}(x) := (1-x)^{\gamma}(1+x)^{\delta}, \qquad x \in [-1,1], \quad \gamma, \delta \geq 0.$$

Throughout the paper the space of all functions f continuous on $(-1,1)$ and satisfying

$$\lim_{x\to+1} f(x)v(x) = 0 \quad \text{if } \gamma > 0, \qquad \text{and} \qquad \lim_{x\to-1} f(x)v(x) = 0 \quad \text{if } \delta > 0, \tag{4}$$

is denoted by C_v^0 and equipped with the norm

$$\|f\|_{C_v^0} := \|fv\|_\infty = \max_{x\in[-1,1]} |f(x)|v(x).$$

It is well-known (see for instance [35]) that the Weierstrass approximation theorem holds in this Banach space and we have

$$f \in C_v^0 \iff \lim_{n\to\infty} E_n(f)_v = 0,$$

where $E_n(f)_v$ denotes the error of best approximation of $f \in C_v^0$ in the space $\mathbb{P}_n$ of all algebraic polynomials of degree at most n, namely

$$E_n(f)_v := \inf_{P\in\mathbb{P}_n} \|(f-P)\|_{C_v^0}.$$

The rate of convergence of such error, as $n \to \infty$, depends on the smoothness of the function f and it is well characterized by the following moduli of smoothness introduced in [36] by Z. Ditzian and V. Totik

$$\Omega_\varphi^k(f,t)_v = \sup_{0<h\leq t} \|v\Delta_{h\varphi}^k f\|_{L^\infty[-1+2h^2k^2,1-2h^2k^2]},$$

$$\omega_\varphi^k(f,t)_v = \Omega_\varphi^k(f,t)_v + \inf_{q\in\mathbb{P}_{k-1}} \|(f-q)v\|_{L^\infty[-1,-1+t^2k^2]} + \inf_{q\in\mathbb{P}_{k-1}} \|(f-q)v\|_{L^\infty[1-t^2k^2,1]},$$

where

$$\Delta_{h\varphi}^k(f,x) = \sum_{i=0}^k (-1)^i \binom{k}{i} f\left(x + \frac{kh}{2}\varphi(x) - ih\varphi(x)\right), \quad k \in \mathbb{N}, \quad \varphi(x) = \sqrt{1-x^2}.$$

In fact, for $n \in \mathbb{N}$ sufficiently large (say $n \geq n_0$) and $t > 0$ sufficiently small (say $t \leq t_0$), the following Jackson and Stechkin type inequalities hold (see [36,37])

$$E_n(f)_v \leq \mathcal{C}\omega_\varphi^k\left(f, \frac{1}{n}\right)_v \leq \mathcal{C}\int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,t)_v}{t}dt, \qquad \mathcal{C} \neq \mathcal{C}(n,f), \tag{5}$$

$$\Omega_\varphi^k(f,t)_v \leq \omega_\varphi^k(f,t)_v \leq \mathcal{C}t^k \sum_{n=0}^{[1/t]} (n+1)^{k-1} E_n(f)_v, \qquad \mathcal{C} \neq \mathcal{C}(n,f,t), \tag{6}$$

where throughout the paper we use $\mathcal{C}$ in order to denote a positive constant, which may have different values at different occurrences, and we write $\mathcal{C} \neq \mathcal{C}(n,f,\ldots)$ to mean that $\mathcal{C} > 0$ is independent of $n, f, \ldots$.

Note that the previous direct and inverse results yield, for instance, the following equivalences

$$E_n(f)_v = \mathcal{O}(n^{-r}) \iff \omega_\varphi^k(f,t)_v = \mathcal{O}(t^r) \iff \Omega_\varphi^k(f,t)_v = \mathcal{O}(t^r)$$

holding for any $0 < r < k \in \mathbb{N}$.

In literature, several approximation spaces have been introduced in order to classify the smoothness of the functions f w.r.t. the decay of $E_n(f)_v$ as $n \to \infty$. In particular, we recall the following ones

$$B_r(v) := \left\{ f \in C_v^0 \ : \sum_{n=1}^\infty (n+1)^{r-1} E_n(f)_v < \infty \right\}, \qquad r > 0,$$

$$\text{equipped with} \quad \|f\|_{B_r(v)} := \|fv\|_\infty + \sum_{n=1}^\infty (n+1)^{r-1} E_n(f)_v.$$

These Banach spaces belong to the class of Besov type space $B^{\infty}_{r,q}(v)$, introduced in [38] and characterized by the following norm

$$\|f\|_{B^{\infty}_{r,q}(v)} := \|fv\|_\infty + \begin{cases} \left(\sum_{n=1}^{\infty}\left[(n+1)^{r-\frac{1}{q}}E_n(f)_v\right]^q\right)^{\frac{1}{q}}, & \text{if} \quad 1 \le q < \infty, \\ \sup_{n>0}(n+1)^r E_n(f)_v, & \text{if} \quad q=\infty. \end{cases}$$

Hence $B_r(v) \equiv B^{\infty}_{r,1}(v)$.

In the sequel, for $A, B > 0$ we will write $A \sim B$ meaning that $\mathcal{C}^{-1}A \le B \le \mathcal{C}A$, with $\mathcal{C} > 0$ independent of the parameters in A, B. By virtue of (5)–(6) the following equivalence between the norms holds true [38, Th. 3.1],

$$\|f\|_{B_r(v)} \sim \|fv\|_\infty + \int_0^1 \frac{\Omega^k_\varphi(f,t)_v}{t^{r+1}}dt, \qquad k > r > 0, \tag{7}$$

being the constant $\mathcal{C}$, involved in such equivalence, only depending on r. Moreover, in (7) the main part modulus Ω^k_φ can be also replaced by the complete modulus ω^k_φ or by the following K-functional

$$K^k_\varphi(f,t^k)_v := \inf_{g^{(k-1)}\in AC_{loc}} \left[\|(f-g)v\|_\infty + t^k\|g^{(k)}\varphi^k v\|_\infty\right]$$

where AC_{loc} denotes the class of locally absolutely continuous functions on $[-1,1]$ (i.e. absolutely continuous on any $[c,d] \subset (-1,1)$). We also recall that [36]

$$K^k_\varphi(f,t^k)_v \sim \omega^k_\varphi(f,t)_v.$$

For any $r > 0$, the previous spaces $B_r(v)$ are compactly embedded in a larger space $B_0(v)$ that is defined [37] as the "limit" case of the Besov spaces $B_r(v)$ as $r \to 0$, namely

$$B_0(v) := \left\{ f \in C^0_v \ : \sum_{n=1}^{\infty} \frac{E_n(f)_v}{n+1} < \infty \right\},$$

and equipped with the norm

$$\|f\|_{B_0(v)} := \|fv\|_\infty + \sum_{n=1}^{\infty}\frac{E_n(f)_v}{n+1}.$$

Also in this case, by virtue of (5)–(6), we have the following equivalences

$$\|f\|_{B_0(v)} \sim \|fv\|_\infty + \int_0^1 \frac{\omega^k_\varphi(f,t)_v}{t}dt \sim \|fv\|_\infty + \int_0^1 \frac{K^k_\varphi(f,t^k)_v}{t}dt. \tag{8}$$

Moreover, the elements $f \in B_0(v)$ can be easily characterized with the help of the classical modulus of continuity of $g = fv$, defined as

$$\omega(g,t) := \sup_{x,y\in[-1,1],\ |x-y|\le t} |g(x)-g(y)|.$$

Indeed a practical criterion to check whether $f \in C^0_v$ belongs to $B_0(v)$ or not is the following [39, Th.2.6]

$$f \in B_0(v) \Longleftrightarrow \int_0^1 \frac{\omega(fv,t)}{t}dt < \infty,$$

and we also have the equivalence between the norms

$$\|f\|_{B_0(v)} \sim \|fv\|_\infty + \int_0^1 \frac{\omega(fv,t)}{t}dt.$$

Moreover, for all $r \ge 0$, if $v = 1$ ($\gamma = \delta = 0$) we use the notation B_r instead of $B_r(v)$, and have the following

Lemma 2.1. *For all $r \ge 0$ and any $v = v^{\gamma,\delta}$ with $\gamma, \delta \ge 0$ we have that $f \in B_r(v)$ iff $(fv) \in B_r$ and (4) holds. Moreover we have*

$$\|f\|_{B_r(v)} \sim \|fv\|_{B_r}, \qquad \forall f \in B_r(v). \tag{9}$$

Finally the following result states some asymptotic bounds for the error of the best polynomial approximation of $f \in B_r(v)$.

Lemma 2.2. *For any n sufficiently large (say $n \geq n_0$) it results*

$$E_n(f)_v \leq \mathcal{C} \begin{cases} \dfrac{\|f\|_{B_0(v)}}{\log n}, & \text{if } f \in B_0(v), \\ \dfrac{\|f\|_{B_r(v)}}{n^r}, & \text{if } f \in B_r(v), \quad r > 0, \end{cases} \qquad \mathcal{C} \neq \mathcal{C}(n, f). \tag{10}$$

2.1. Mapping properties of the Hilbert transform

The approximation space $B_0(v)$ was firstly introduced in [37] in order to find the "correct and minimal" space for studying the boundedness of the Hilbert transform w.r.t. the uniform norm. More precisely, using the standard notation for the constants

$$c_+ := \max\{0, c\}, \qquad c_- := \max\{0, -c\},$$

by [37, Th. 3.1] we have the following

Theorem 2.3. *Let $u = v^{a,b}$, with $a, b > -1$, be the Jacobi weight defining the Hilbert transform (1) and set $u = \frac{u_+}{u_-}$ with $u_+ := v^{a_+,b_+}$ and $u_- := v^{a_-,b_-}$. If we consider the following weight functions*

$$u_1(x) := u_+(x), \qquad u_2(x) := \frac{u_-(x)}{1 + (u_+u_-)(-1)|\log(1+x)| + (u_+u_-)(1)|\log(1-x)|},$$

then for all $t \in (-1, 1)$ and any $f \in B_0(u_1)$, we have

$$|\mathcal{H}^u f(t)| u_2(t) \leq \mathcal{C} \left(|f(t)| u_1(t) + \|f\|_{B_0(u_1)} \right), \qquad \mathcal{C} \neq \mathcal{C}(f, t). \tag{11}$$

From the previous result it immediately follows that $\mathcal{H}^u$ is a bounded map from $B_0(u_1)$ to $C^0_{u_2}$.

3. Filtered VP discrete approximation

This section concerns the main approximation tool we are going to use in getting our quadrature rule for (1).

Let $n \in \mathbb{N}$ and denote by $p_n(w) \in \mathbb{P}_n$ the n-th orthonormal polynomial (with positive leading coefficient) associated to the Jacobi weight $w(x) = v^{\alpha,\beta}(x)$, with $\alpha, \beta > -1$. Based on the set of zeros of $p_n(w)$, i.e. $X_n(w)$ in (2), the filtered VP polynomial of a function f is defined as follows (see [29,40])

$$V_n^m(w, f, x) = \sum_{k=1}^{n} f(x_k) \Phi_{n,k}^m(x), \qquad 0 < m < n, \tag{12}$$

with

$$\Phi_{n,k}^m(x) = \lambda_k \sum_{j=0}^{n+m-1} \mu_{n,j}^m p_j(w, x) p_j(w, x_k), \tag{13}$$

where λ_k are the Christoffel numbers defined in (3) and $\mu_{n,j}^m$ are the following VP filter coefficients

$$\mu_{n,j}^m := \begin{cases} 1, & \text{if } \ j = 0, \dots, n-m, \\ \dfrac{n+m-j}{2m}, & \text{if } \ n-m < j < n+m. \end{cases} \tag{14}$$

We point out that the polynomial in (12) depends on two degree-parameters: n, which determines the number of nodes, and m which determines the VP filter action range. Both the parameters are involved in the following polynomial preserving property [29]

$$V_n^m(w, P) = P, \qquad \forall P \in \mathbb{P}_{n-m}. \tag{15}$$

In what follows we will assume that the degree-parameters m, n are such that

$$c_1 m \leq n \leq c_2\, m, \qquad \text{for some } c_2 \geq c_1 > 1 \text{ independent of } n \text{ and } m,$$

and we will write $m \approx n$ to express this kind of relation.

We firstly recall the following theorem which gives sufficient conditions for the map $V_n^m(w) : f \in C^0_v \to V_n^m(w, f) \in C^0_v$ to be uniformly bounded w.r.t. n [33].

Theorem 3.1. *Let $v = v^{\gamma,\delta}$ with $\gamma, \delta \geq 0$ and let $f \in C_v^0$. Moreover let $n \in \mathbb{N}$ and $m \approx n$ be arbitrarily fixed and consider the filtered VP polynomial $V_n^m(w, f)$ associated to the Jacobi weight $w = v^{\alpha,\beta}$, $\alpha, \beta > -1$. If we have*

$$\left|\gamma - \delta - \frac{\alpha - \beta}{2}\right| < 1, \tag{16}$$

and

$$\begin{cases} \dfrac{\alpha}{2} - \dfrac{1}{4} < \gamma \leq \dfrac{\alpha}{2} + \dfrac{5}{4} \\ \dfrac{\beta}{2} - \dfrac{1}{4} < \delta \leq \dfrac{\beta}{2} + \dfrac{5}{4} \end{cases} \tag{17}$$

then we get

$$\|V_n^m(w, f)v\|_\infty \leq \mathcal{C}\|fv\|_\infty, \qquad \mathcal{C} \neq \mathcal{C}(n, m, f). \tag{18}$$

We also recall that this result has been recently improved in the special case w is one of the four Chebyshev weights, i.e. $|\alpha| = |\beta| = \frac{1}{2}$, and we refer the reader to [31, Th.3.1] where necessary and sufficient conditions on the weights v and w have been stated.

We point out that, due to (15), the uniform boundedness result (18) is equivalent to the following error estimate

$$E_{n+m-1}(f)_v \leq \|[f - V_n^m(w, f)]v\|_\infty \leq \mathcal{C}E_{n-m}(f)_v, \qquad \mathcal{C} \neq \mathcal{C}(n, m, f). \tag{19}$$

Hence, taking into account that $m \approx n$ implies $(n - m) \sim n$, we can say that if (16)–(17) hold then we have

$$\lim_{m\approx n\to\infty} \|[f - V_n^m(w, f)]v\|_\infty = 0, \qquad \forall f \in C_v^0,$$

being the convergence order comparable with that one of the error of best polynomial approximation $E_n(f)_v$.

In the sequel we are going to investigate the behaviour of VP filtered approximation in the case f belongs to the Besov type spaces $B_r(v)$ introduced in the previous section. We underline that recently in [32], some convergence estimates were stated in Zygmund type spaces (i.e. in the Besov type space $B_{r,q}^\infty(v)$, $q = \infty$, $r > 0$).

Obviously, under the assumption of Theorem 3.1, by (19) and Lemma 2.2 we easily get

$$\|[f - V_n^m(w, f)]v\|_\infty \leq \mathcal{C} \begin{cases} \dfrac{\|f\|_{B_0(v)}}{\log n}, & \text{if } f \in B_0(v), \\ \dfrac{\|f\|_{B_r(v)}}{n^r}, & \text{if } f \in B_r(v), \quad r > 0, \end{cases} \tag{20}$$

where $\mathcal{C} \neq \mathcal{C}(n, f)$.

Moreover, we have the following

Theorem 3.2. *Let $v = v^{\gamma,\delta}$ and $w = v^{\alpha,\beta}$ be such that $\gamma, \delta \geq 0$, $\alpha, \beta > -1$ and assume $V_n^m(w) : C_v^0 \to C_v^0$ is a uniformly bounded map w.r.t $n, m \in \mathbb{N}$ with $m \approx n$. For any $r \geq 0$, also the map $V_n^m(w) : B_r(v) \to B_r(v)$ is uniformly bounded w.r.t. n and we have*

$$\lim_{m\approx n\to\infty} \|f - V_n^m(w, f)\|_{B_r(v)} = 0, \qquad \forall f \in B_r(v), \quad \forall r \geq 0. \tag{21}$$

Moreover, for all $f \in B_r(v)$ with $r > 0$, the following error estimates hold

$$\|f - V_n^m(w, f)\|_{B_0(v)} \leq \mathcal{C}\frac{\log n}{n^r}\|f\|_{B_r(v)}, \qquad r > 0, \tag{22}$$

$$\|f - V_n^m(w, f)\|_{B_s(v)} \leq \frac{\mathcal{C}}{n^{r-s}}\|f\|_{B_r(v)}, \qquad r \geq s > 0, \tag{23}$$

where in both cases $\mathcal{C} \neq \mathcal{C}(n, f)$.

3.1. The limit case $m = 0$: the Lagrange interpolation

As said in the Introduction, for all $n \in \mathbb{N}$, when the parameter m defining the VP operator is set equal to zero, we get the Lagrange interpolating operator at the same system of nodes $X_n(w)$, namely

$$L_n(w, f, x) = \sum_{k=1}^{n} f(x_k)l_{n,k}(x), \qquad l_{n,k}(x) = \lambda_k \sum_{j=0}^{n-1} p_j(w, x_k)p_j(w, x),$$

being λ_k given in (3).

While under suitable conditions for the weights w and v the VP operator $V_n^m(w) : C_v^0 \to C_v^0$ is uniformly bounded w.r.t. n (cf. Theorem 3.1), it is well known that the norm of the Lagrange operator $L_n(w) : C_v^0 \to C_v^0$, the so called weighted Lebesgue constant, does not. Nevertheless, necessary and sufficient conditions for optimal Lebesgue constant, i.e. behaving like $\log n$, are known [35,41]. More precisely, we have the following theorem which collects the analogous of previous estimates stated for the VP approximation.

Theorem 3.3. *Let $v = v^{\gamma,\delta}$ and $w = v^{\alpha,\beta}$ be such that $\gamma, \delta \geq 0$, $\alpha, \beta > -1$. For all $f \in C_v^0$, the conditions*

$$\begin{cases} \dfrac{\alpha}{2} + \dfrac{1}{4} \leq \gamma \leq \dfrac{\alpha}{2} + \dfrac{5}{4} \\[2ex] \dfrac{\beta}{2} + \dfrac{1}{4} \leq \delta \leq \dfrac{\beta}{2} + \dfrac{5}{4} \end{cases} \tag{24}$$

are necessary and sufficient for having

$$\|[f - L_n(w, f)]v\|_\infty \leq \mathcal{C} \log n E_n(f)_v, \quad \mathcal{C} \neq \mathcal{C}(n, f). \tag{25}$$

Moreover, if (24) holds then for all $f \in B_r(v)$ with $r > 0$, we get

$$\|f - L_n(w, f)\|_{B_0(v)} \leq \mathcal{C} \frac{\log^2 n}{n^r} \|f\|_{B_r(v)}, \qquad r > 0, \tag{26}$$

$$\|f - L_n(w, f)\|_{B_s(v)} \leq \mathcal{C} \frac{\log n}{n^{r-s}} \|f\|_{B_r(v)}, \qquad r \geq s > 0, \tag{27}$$

where in both cases $\mathcal{C} \neq \mathcal{C}(n, f)$.

Remark 3.4. Comparing the convergence results between VP and Lagrange approximations we remark that for the same class of function (say $f \in B_r(v)$, $r > 0$) the convergence rate is similar, but in the Lagrange estimates an extra factor $\log n$ appears. Moreover conditions (17) are wider than (24).

4. Filtered VP quadrature rules for the Hilbert transform

In this section we are going to introduce a new class of product integration rules for the Hilbert transform in (1) based on filtered VP approximation. More precisely, by replacing f in (1) with the polynomial $V_n^m(w, f)$, we define

$$\mathcal{H}_{n,m}^u(w, f, t) := \int_{-1}^1 \frac{V_n^m(w, f, x)}{x - t} u(x) dx, \qquad -1 < t < 1. \tag{28}$$

More explicitly, by (12)–(13) we get the following quadrature rule (VP rule)

$$\mathcal{H}_{n,m}^u(w, f, t) = \sum_{j=0}^{n+m-1} \rho_{n,j}^m(w, f) Q_j^u(w, t), \qquad -1 < t < 1,$$

where we set

$$Q_j^u(w, t) := \int_{-1}^1 \frac{p_j(w, x)}{x - t} u(x) dx, \tag{29}$$

$$\rho_{n,j}^m(w, f) := \mu_{n,j}^m \sum_{k=1}^n \lambda_k p_j(w, x_k) f(x_k),$$

and $\mu_{n,j}^m$ defined in (14).

4.1. Computational details

First of all we point out that the coefficients $\rho_{n,j}^m(w, f)$ do not depend on t, and therefore are not influenced by the closeness of t to any point $\{x_i\}_{i=1}^n$. This means that numerical cancellation is avoided in computing these quantities.

About the functions $\{Q_j^u(w, t)\}_j$, we remark that only in some particular cases they are explicitly known (see e.g. [4]) while in the general case they can be computed via a recurrence relation (see [2]). The following proposition states such relation, which is deducible from the well-known recurrence relation for the orthonormal sequence $\{p_j(w)\}_j$

$$b_{j+1} p_{j+1}(w, x) = (x - a_j) p_j(w, x) - b_j p_{j-1}(w, x), \qquad j \geq 0, \tag{30}$$

where $p_{-1}(w, x) = 0$ and $p_0(w, x) = \left(\int_{-1}^1 w(x) dx \right)^{-\frac{1}{2}}$ are the starting values and the explicit expression of the coefficients a_j and b_j can be found for instance in [35, pp. 131–133].

Proposition 4.1. *For all Jacobi weights u, w and any $t \in (-1, 1)$, the functions $Q_j(t) := Q_j^u(w, t)$ defined in (29) satisfy the following three-term recurrence relation*

$$Q_{j+1}(t) = (A_j t + B_j)Q_j(t) - C_j Q_{j-1}(t) + D_j, \qquad j \geq 0,$$

where the starting values are given by

$$Q_{-1}(t) = 0, \qquad Q_0(t) = \left(\int_{-1}^{1} \frac{u(x)}{x-t} dt\right)\left(\int_{-1}^{1} w(t)dt\right)^{-\frac{1}{2}},$$

and the coefficients are defined, by means of the coefficients $\{a_j\}_j$ and $\{b_j\}_j$ in (30), as follows

$$A_j = b_{j+1}^{-1},\ B_j = -\frac{a_j}{b_{j+1}},\ C_j = \frac{b_j}{b_{j+1}},\ D_j = \frac{1}{b_{j+1}} \int_{-1}^{1} p_j(w, x)u(x)dx, \quad j = 0, 1, \ldots.$$

4.2. Error estimates

Let us analyse the error function

$$\mathcal{E}_{n,m}^u(w, f, t) := \left|\mathcal{H}^u f(t) - \mathcal{H}_{n,m}^u(w, f, t)\right|, \qquad -1 < t < 1. \tag{31}$$

Of course the behaviour of such error is influenced by the approximation error provided by the VP polynomial that has been employed. Such dependence is specified in the following convergence theorem

Theorem 4.2. *Under the notation of Theorem 2.3, for all $f \in B_0(u_1)$ and any $t \in (-1, 1)$, we have*

$$\mathcal{E}_{n,m}^u(w, f, t)u_2(t) \leq \mathcal{C}\left|f(t) - V_n^m(w, f, t)\right| u_1(t) + \mathcal{C}\|f - V_n^m(w, f)\|_{B_0(u_1)}, \tag{32}$$

where $\mathcal{C} \neq \mathcal{C}(n, f, t)$. Moreover, if $V_n^m(w) : C_{u_1}^0 \to C_{u_1}^0$ is an uniformly bounded map w.r.t. $m \approx n$, then

$$\lim_{n\to\infty} \mathcal{E}_{n,m}^u(w, f, t)u_2(t) = 0, \qquad \forall f \in B_0(u_1), \tag{33}$$

and the convergence holds uniformly in any compact subinterval of $(-1, 1)$.

The previous theorem assures that if the function f is in $B_0(u_1)$ then the proposed quadrature formula converges on conditions that the weight $w = v^{\alpha,\beta}$ is suitably chosen. In particular Theorems 4.2 and 3.1 provide the following criteria for the choice of the weight $w = v^{\alpha,\beta}$

$$a_+ - b_+ - 1 < \frac{\alpha - \beta}{2} < a_+ - b_+ + 1, \qquad \text{and} \qquad \begin{cases} 2a_+ - \frac{5}{2} \leq \alpha < 2a_+ + \frac{1}{2} \\ 2b_+ - \frac{5}{2} \leq \beta < 2b_+ + \frac{1}{2} \end{cases} \tag{34}$$

where $a_+ = \max\{a, 0\}$, $b_+ = \max\{b, 0\}$, being $u = v^{a,b}$ the weight defining the Hilbert transform.

Moreover, recalling Theorem 3.2, under the hypotheses of Theorem 4.2 the map $V_n^m(w) : B_0(u_1) \to B_0(u_1)$ is uniformly bounded w.r.t. $n \sim (n - m)$. Hence by the invariance property (15) and by (32) we get

$$\mathcal{E}_{n,m}^u(w, f, t)u_2(t) \leq \mathcal{C}|f(t) - V_n^m(w, f, t)|u_1(t) + \mathcal{C}E_n(f)_{B_0(u_1)}, \tag{35}$$

where $E_n(f)_{B_0(u_1)} := \inf_{P\in\mathbb{P}_n} \|f - P\|_{B_0(u_1)}$ and $\mathcal{C} \neq \mathcal{C}(n, m, f)$.

We point out that (35) says that for each t the convergence rate of the VP-rule depends on two components: the pointwise approximation that the VP polynomial of f provides at the specific $t \in (-1, 1)$, and the smoothness of f that influences the convergence rate of the error of best polynomial approximation of f in $B_0(u_1)$.

If f is smoother than in Theorem 4.2, from (32) and (22) we immediately deduce the following corollary.

Corollary 4.3. *Under the same assumptions of Theorem 4.2, for all $f \in B_r(u_1)$, $r > 0$, we get*

$$\mathcal{E}_{n,m}^u(w, f, t)u_2(t) \leq \mathcal{C}|f(t) - V_n^m(w, f, t)|u_1(t) + \mathcal{C}\frac{\log n}{n^r}\|f\|_{B_r(u_1)}, \tag{36}$$

where $\mathcal{C} \neq \mathcal{C}(n, f, t)$.

Obviously, concerning the converge rate of the VP-rule, from (36) and (20) we get that

$$\mathcal{E}_{n,m}^u(w, f, t)u_2(t) \leq \mathcal{C}\frac{\log n}{n^r}, \qquad \mathcal{C} \neq \mathcal{C}(n, t), \tag{37}$$

holds true $\forall f \in B_r(u_1)$, with $r > 0$.

We conclude the section with a comparison between the proposed VP-rule and the analogous one using the Lagrange polynomial (L-rule):

$$\mathcal{H}_n^u f(t) := \int_{-1}^{1} \frac{L_n(w, f, x)}{x - t} u(t)\, dx. \tag{38}$$

This kind of formula was considered by several authors in the case $w = u$ (see for instance [15] and the reference therein). Here, set

$$\mathcal{E}_n^u(w, f, t) := \left| \mathcal{H}^u f(t) - \mathcal{H}_n^u(w, f, t) \right|, \qquad -1 < t < 1,$$

as a consequence of Theorems 2.3 and 3.3, we note that if the exponents of $u = v^{a,b}$ and $w = v^{\alpha,\beta}$ are such that

$$\begin{cases} 2a_+ - \dfrac{5}{2} \le \alpha \le 2a_+ - \dfrac{1}{2} \\[2ex] 2b_+ - \dfrac{5}{2} \le \beta \le 2b_+ - \dfrac{1}{2} \end{cases} \tag{39}$$

then, for all $f \in B_r(u_1)$, $r > 0$, we have

$$\mathcal{E}_n^u(w, f, t) u_2(t) \le \mathcal{C} |f(t) - L_n(w, f, t)| u_1(t) + \mathcal{C} \frac{\log^2 n}{n^r} \, \|f\|_{B_r(u_1)}, \tag{40}$$

with $\mathcal{C} \neq \mathcal{C}(n, f, t)$.

5. Numerical experiments

In this section we report some numerical tests in order to evaluate the performance of the proposed VP-rule, also in comparison with other quadrature formulae.

To be more precise, for several choices of $t \in (-1, 1)$, we compare the absolute errors $e_{n,m}^{VP}$ achieved by our VP-rules, with the errors e_n^L obtained by the L-rule in (38) and the errors e_n^{GM} achieved by using the Modified Gaussian rule (MG-rule) proposed in [15,17]. We also consider an example given in [20] by Hasegawa and Torii, who proposed a product rule (HT-rule) whose absolute error we denote by e_n^{HT} (Example 3). Moreover in Example 6 we will test the same integral considered in [23], where the proposed rule is based on a truncated sum of a Sinc function series. The errors by this method will be denoted by e_n^{Sinc}.

Since the exact values are not available in the general case, we have retained "exact" the values achieved by means of higher degree quadrature formulas, varying the choice of the rule among the previous VP, L and MG rules. In each test we will declare the rule we have used to determine the exact value.

From the wide experimentation we carried on, we have selected six examples, varying the regularity of the functions f and the possible choices of the weight functions u and w, which define the transform $\mathcal{H}^u(f)$ and the VP approximants $V_n^m(w, f)$, respectively.

In all the tests n will denote the number of nodes employed in the considered rules, which also coincide with the number of function evaluations. Only the MG-rule makes exception, since it is based on a variable number of nodes that could be n or $n+1$, being this choice contemplated by the method itself, according to the position of t [15].

The numerical outputs of each example are collected in tables having the same numbering as the examples to which they refer. At each row of the tables, besides the number of nodes n, the value of the additional parameter m chosen in the VP-rule is specified and the best quadrature error is evidenced in bold.

Finally, at the end of the section we present two further tests on the VP-rules, where we focus on the behaviour of the absolute errors as n increases, making several choices of $m \approx n$. More precisely, in Test A we take $m = \lfloor \theta n \rfloor$ with fixed $\theta \in (0, 1)$ and we compare the error's behaviours corresponding to different choices of θ. In Test B, in correspondence of any fixed number of nodes we plotted the behaviour of the best and the worst $e_{n,m}^{VP}$ obtained for $0 < m < n$.

In the sequel, details and comments concerning the six examples and the two tests are given. We point out that all the computations have been performed in double-machine precision ($eps_D \approx 2.22044e-16$) by using MATLAB R2021a.

Example 1.

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{|x - 0.5|^{10.01}}{x - t} \sqrt{\frac{1 - x}{1 + x}} dx.$$

Here $u = v^{\frac{1}{2}, -\frac{1}{2}}$, $u_1 = v^{\frac{1}{2}, 0}$, $u_2 = v^{0, \frac{1}{2}}$ and we chose $w = u$. This choice ensures that (34) and (39) are satisfied (with $\alpha = -\beta = \frac{1}{2}$, $a_+ = \frac{1}{2}$ and $b_+ = 0$). We have $f \in B_r(u_1)$, with $r \le 10.01$ and according to (37), the error goes like $\mathcal{O}\left(\frac{\log n}{n^{10.01}}\right)$. Taking into account that the norm $\|f\|_{B_r(u_1)} \ge 2.5 \times 10^6$, the theoretical rate of convergence is confirmed, also for values of t "close" to the point 0.5 where the function f is less regular. Indeed, for $n = 151$ we can expect 14 exact digits.

Table 1
Example 1.

$t=0.499999999$					$t=0.5$				
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}
51	8	8.53e−14	6.47e−13	**2.84e−14**	51	8	9.24e−14	6.61e−13	**2.13e−14**
151	13	**0.00**	1.22e−12	1.42e−14	151	19	**0.00**	1.23e−12	2.12e−14
$t=0.51111$					$t=0.75$				
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}
51	4	6.39e−14	5.12e−13	**2.84e−14**	51	3	**0.00**	4.33e−13	4.62e−14
151	68	**0.00**	1.61e−12	8.53e−14					

Table 2
Example 2.

$t=-0.1$					$t=0.7$				
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}
50	18	2.59e−05	9.57e−05	**6.01e−06**	50	30	**4.11e−07**	2.71e−04	2.20e−05
101	78	4.25e−06	3.78e−05	**1.45e−06**	150	120	**3.98e−11**	6.77e−06	2.12e−06
201	78	**4.81e−08**	4.24e−06	2.92e−07	250	14	7.74e−06	2.65e−05	**6.29e−07**
301	31	**7.91e−08**	2.17e−06	1.08e−07	350	156	4.30e−07	1.62e−05	**2.67e−07**
400	377	**1.61e−08**	3.00e−06	5.18e−08	450	164	**1.24e−07**	2.37e−06	1.38e−07
500	313	**1.53e−08**	5.32e−07	2.83e−08	550	389	**7.91e−08**	5.24e−06	7.98e−08
$t=0.9$					$t=0.9999$				
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}
50	10	**5.74e−05**	4.22e−04	6.60e−05	50	3	**1.78e−02**	1.78e−02	3.40e−02
150	45	**8.24e−07**	1.90e−04	6.45e−06	250	236	4.25e−03	5.08e−03	**1.65e−03**
250	150	4.50e−06	2.66e−05	**1.87e−06**	350	271	**4.96e−05**	6.53e−03	7.51e−04
350	38	**1.66e−07**	2.82e−05	8.08e−07	550	495	**5.40e−05**	2.63e−03	2.28e−04
450	66	**8.47e−08**	2.36e−05	4.16e−07	650	525	**5.50e−06**	3.40e−04	1.37e−04

In order to compute the absolute errors, we have retained exact the values achieved by means of the MG rule of order $n=1200$ (see Table 1). Moreover since in this case $w=u=v^{a,-a}$, with $0<a<1$, for the computation of the coefficient of the product rules we have used the following explicit expression (see [4, p. 310])

$$Q^u_j(u,t)=\int_{-1}^{1}\frac{p_j(u,x)}{x-t}u(x)dx=\pi\cot(\pi a)p_j(u,t)u(t)-\frac{\pi}{\sin(\pi a)}p_j(u^{-1},t),\quad j\geq 0.$$

Example 2.

$$\mathcal{H}^u f(t)=\int_{-1}^{1}\frac{\log(1-x)}{x-t}(1-x)^{0.4}(1+x)^{0.25}dx.$$

Here $u\equiv u_1=v^{0.4,0.25}$, $u_2\equiv 1$ and we chose $w=v^{-\frac{1}{2},-\frac{1}{2}}$, whose exponents satisfy (34) and (39) with $a_+=0.4$ and $b_+=0.25$

In this case $f\in B_r(u)$, $r<0.8$, and therefore, by (37) the error of the VP-rule is bounded by $\mathcal{O}\left(\frac{\log n}{n^r}\right)$. Also in this test, we have retained exact the values achieved by means of the MG rule of order $n=1200$ and the results are shown in Table 2.

For t approaching to 1, we observe a progressive loss of exact digits, coherent with the error estimate, while the situation appears much more better when t lies in the remaining part of the interval. Overall, the VP-rule seems to provide a good performance, on average better than those offered by the other two rules.

Example 3.

$$\mathcal{H}^u f(t)=\int_{-1}^{1}\frac{f(x)}{x-t}dx,$$

$$f(x)=e^{8(x-1)},\ u=v^{0,0}=u_1,\ f\in B_r,\ \forall r>0,\ w=v^{-\frac{1}{2},-\frac{1}{2}}.$$

This example is taken from [20] where the quadrature rule here denoted by HT-rule has been proposed. Hence, in Table 3 we have an additional column reporting the errors e^{HT}_n of such rule as provided in [20], writing n.a. (not available) in the cases they are not furnished by the authors. Moreover for computing the errors of the other formulas we consider as exact the value of the integral obtained by means of the MG rule with $n=900$.

Table 3
Example 3.

$t=0.2$						$t=0.5$					
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	e^{HT}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	e^{HT}_{n}
25	2	1.4e−13	8.0e−14	**1.4e−15**	9.0e−13	25	2	6.5e−13	2.9e−13	**2.5e−15**	3.0e−13
30	2	**1.9e−15**	4.2e−15	2.7e−15	n.a.	51	27	**3.8e−16**	3.6e−14	5.8e−15	n.a.
101	60	**5.6e−17**	8.1e−14	2.9e−15	n.a.						
$t=0.95$						$t=0.999$					
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	e^{HT}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}	e^{MG}_{n}	e^{HT}_{n}
25	2	1.2e−12	3.5e−13	**1.3e−14**	1.0e−12	25	2	8.0e−13	1.9e−13	**1.8e−15**	n.a.
30	2	**3.4e−15**	1.6e−14	1.4e−14	n.a.	30	2	**8.9e−16**	1.8e−15	1.1e−14	n.a.
51	3	**3.1e−15**	2.2e−14	2.5e−14	n.a.						

Table 4
Example 4.

$t=0.2$				$t=0.4$			
n	m	$e^{VP}_{n,m}$	e^{MG}_{n}	n	m	$e^{VP}_{n,m}$	e^{MG}_{n}
81	48	**7.14e−02**	5.94e+00	81	40	**8.69e−02**	2.81e+00
101	90	**1.58e−02**	1.69e+00	101	9	**8.87e−03**	8.63e−01
201	61	7.08e−03	**3.06e−03**	201	160	**5.93e−04**	1.66e−03
301	30	6.69e−05	**5.92e−06**	401	160	3.31e−05	**2.10e−08**
401	30	7.11e−05	**4.08e−07**	501	53	4.06e−07	**1.47e−08**
501	30	**7.06e−08**	3.96e−07	601	38	**7.67e−09**	1.48e−08
601	34	**6.36e−10**	3.96e−07	701	51	**3.57e−10**	1.47e−08
$t=0.6$				$t=0.9$			
n	m	$e^{VP}_{n,m}$	e^{MG}_{n}	n	m	$e^{VP}_{n,m}$	e^{MG}_{n}
81	55	**6.98e−02**	1.88e+00	81	71	**7.20e−01**	1.35e+00
101	9	8.71e−01	**5.40e−01**	101	9	**2.09e−01**	3.85e−01
201	19	**3.78e−04**	1.11e−03	201	28	**6.28e−04**	6.99e−04
301	34	5.36e−05	**2.01e−06**	301	84	1.10e−04	**1.36e−06**
401	200	8.17e−07	**1.21e−08**	401	28	1.30-08	**3.06e−09**
501	200	1.84e−07	**7.98e−09**	501	28	**9.67e−10**	5.84e−09
601	83	**4.91e−10**	7.98e−09	601	70	**1.64e−09**	5.84e−09
701	79	**5.23e−10**	7.98e−09	701	144	**8.73e−12**	5.84e−09

The function f is very smooth and, as we expect, the VP, L and MG rule give a good performance. We underline that speeding up a little bit n the VP-rule catches the machine precision. On the other hand for small values of n, better results are given by the MG rule.

Example 4.

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{1}{x^2+2^{-10}} \frac{\sqrt[3]{1-x^2}}{x-t} dx,$$

Here $u \equiv u_1 = v^{\frac{1}{3},\frac{1}{3}}$, $u_2 \equiv 1$ and we have chosen $w = v^{1,1}$.

As already remarked the sufficient conditions stated for the convergence of the VP-rule are wider than those of the L-rule. In this case $a_+ = b_+ = \frac{1}{3}$ and the choice of w assures the convergence of the VP-rule, but not that of the L-rule. So we will compare the results by VP-rules only with those by MG-rule. In this test, we have retained exact the values achieved by means of the VP-rule of order $n = 1000$, $m = 500$.

In this example $f \in B_r(u)$, $\forall r > 0$, but the presence of the complex conjugate poles $\pm i2^{-5}$ too close to the integration interval $(-1, 1)$ produces slower convergence, as well as it happens in quadrature rules for ordinary integrals (see e.g. [28,42]). The numerical results in Table 4 confirm this trend for the MG-rule that shows a certain saturation, while it seems that the VP-rule converges faster.

Example 5.

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{f(x)}{x-t} \sqrt{1-x^2} dx,$$

$$f(x) = \frac{1}{1+1000(x+0.5)^2} + \frac{1}{\sqrt{1+1000(x-0.5)^2}}.$$

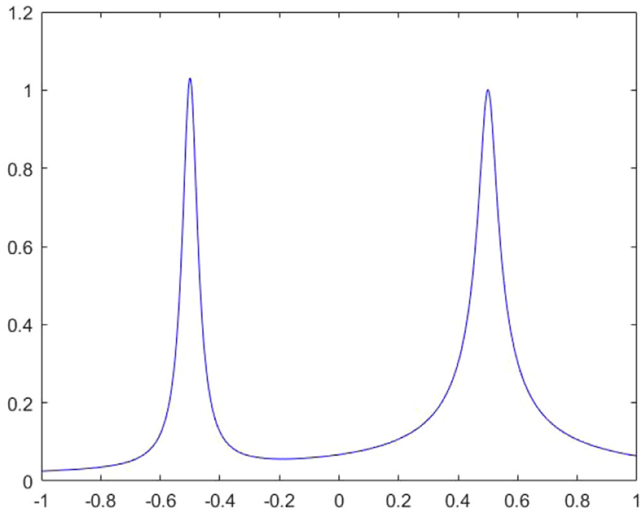


Fig. 1. Example 5: graphic of f.

Table 5
Example 5, $t \in \{0.1,\ 0.2,\ 0.5,\ 0.8\}$.

$t=0.1$				$t=0.2$			
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}
20	12	**1.79e−03**	2.38e−01	20	8	**5.67e−03**	2.18e−01
30	3	**2.82e−03**	3.37e−03	30	9	**2.97e−04**	1.69e−03
40	17	**5.90e−04**	9.98e−02	40	19	**3.90e−03**	1.40e−01
50	16	**5.63e−04**	7.40e−02	50	8	**5.15e−04**	4.84e−02
60	6	**1.73e−05**	1.53e−04	60	24	**7.65e−05**	3.28e−03
70	14	**3.94e−05**	2.62e−02	70	49	**2.96e−04**	4.17e−02
200	45	**1.73e−06**	2.32e−04	200	35	**2.31e−07**	1.32e−04
250	175	**1.24e−09**	4.23e−06	250	24	**7.45e−08**	3.37e−06
300	17	**1.33e−08**	1.36e−06	300	181	**3.74e−07**	9.28e−07
$t=0.5$				$t=0.8$			
n	m	$e^{VP}_{n,m}$	e^{L}_{n}	n	m	$e^{VP}_{n,m}$	e^{L}_{n}
30	5	**3.10e−02**	**3.10e−02**	20	2	**1.11e−02**	1.27e−02
50	3	1.42e−01	**1.25e−01**	30	24	**3.75e−02**	3.86e−02
60	23	**3.92e−03**	**3.92e−03**	50	20	**1.82e−04**	9.12e−03
80	7	4.86e−02	**4.31e−02**	70	13	**5.37e−05**	6.01e−03
90	80	**4.61e−04**	**4.61e−04**	80	8	**5.78e−05**	3.79e−03
150	105	**6.20e−06**	**6.20e−06**	100	60	**1.40e−04**	1.42e−03
200	13	5.06e−04	**4.19e−04**	150	11	**3.41e−06**	3.82e−05
250	16	**5.54e−05**	6.67e−05	250	15	**8.01e−09**	3.11e−07
300	150	**2.11e−10**	**2.11e−10**	300	28	**6.11e−10**	1.94e−07

In this example $u \equiv u_1 = v^{\frac{1}{2},\frac{1}{2}}$, $u_2 \equiv 1$, and we fixed $w = v^{-\frac{1}{2},-\frac{1}{2}}$. The density function f belongs to $B_r(u_1)$, for any $r > 0$. Nevertheless the graphic of f, given in Fig. 1, shows two picks corresponding to $x = \pm 0.5$. In this case the pointwise approximation provided by the VP polynomial is almost everywhere better than the Lagrange approximation and we aim to test if such difference is reflected in the quadrature errors e_n^{VP} and e_n^{L} too. For this reason we exclude MG-rule from this test, limiting the comparison between VP and L rules.

Retaining exact the values achieved by means of the L-rule of order $n = 1200$, the errors e_n^{VP} and e_n^{L} are shown in Table 5 for different choices of $t > 0$. We can observe that for $t = 0.5$ the errors by VP-rule are comparable with those by L-rule, while in the points t far from the pathological points ± 0.5, VP-rule gives better results.

In order to justify this behaviour, we have plotted the error curves $\mathrm{err}_L(\tau) := f(\tau) - L_n(w, f, \tau)$, and $\mathrm{err}_{VP}(\tau) := f(\tau) - V_m^n(w, f, \tau)$ with $\tau \in [-1, 1]$, in the cases $n = 200,\ m = 35$ (Fig. 2, left) and $n = 250,\ m = 16$ (Fig. 3, left).

Note that the selected n, m are those shown in Table 5 for the quadrature errors of the Hilbert transform at $t = 0.2$ and $t = 0.5$ (see the third and second last lines of the corresponding sub-tables, respectively).

On the right of each figure, we have zoomed the same plot given on the left in a range close to $\tau = 0.2$ (Fig. 2, right) and around $\tau = 0.5$ (Fig. 3, right).

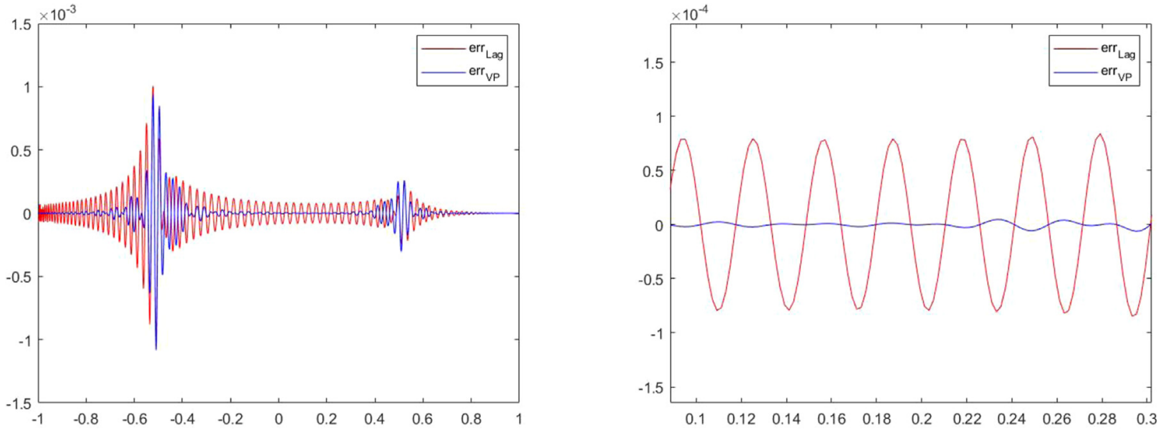


Fig. 2. Example 5: graphics of the errors $\mathrm{err}_L(\tau)$ and $\mathrm{err}_{VP}(\tau)$ for $n = 200,\ m = 35$ (on the left) and zooming of the same plot around $\tau = 0.2$ (on the right).

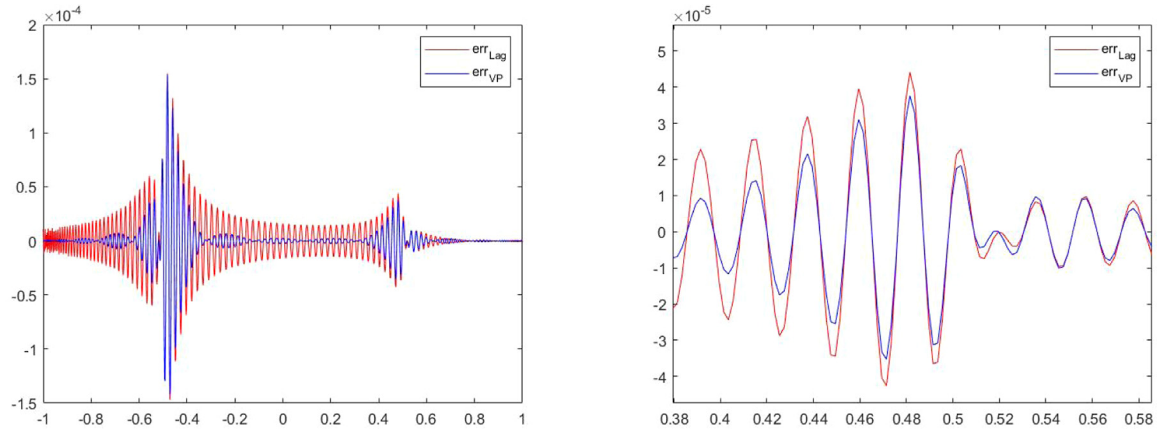


Fig. 3. Example 5: graphics of the errors $f(\tau) - L_n(w, f, \tau)$ and $f(\tau) - V_m^n(w, f, \tau)$ for $n = 250,\ m = 16$ (on the left) and zooming of the same plot around $\tau = 0.5$ (on the right).

As the graphics reveal, the trend of the quadrature errors for $t = 0.2$ and $t = 0.5$ reflect that one of the pointwise Lagrange and de la Vallée Poussin approximation errors at $\tau = 0.2$ and $\tau = 0.5$, respectively, in agreement with the theoretical error estimates (36) and (40).

Example 6. This example can be found in [20] and [23]:

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{1}{x^2+\eta^2}\frac{1}{x-t}dx, \quad \eta = 0.125,\ \eta = 1.$$

Here $u \equiv u_1 = v^{0,0}$, $u_2 = \frac{1}{1+|\log(1+x)|+|\log(1-x)|}$ and we have chosen $w = v^{-\frac{1}{2},-\frac{1}{2}}$. The function $f \in B_r(u)$, $\forall r > 0$, and the only pathology is due to the presence of the complex conjugate poles $\pm i\eta$. Here we compare the results by our procedure with those obtained in [23] and in [20]. Such methods can be applied only in the case $u \equiv 1$ and the first one is especially suggested for analytic functions f. We retained as exact the value of the integral obtained by means of the MG rule with $n = 1024$. The errors e_n^{HT} and e_n^{Sinc} are the ones that are reported in [23, Tables 1–2], where the integrals are evaluated for $t = 0.5$. The results are shown in Tables 6, 7. We observe that both the methods achieve a high precision, even if the Sinc method requires a major number of function evaluations, in favour of the Hasegawa–Torii method, while the results by HT-rule and our rule are comparable among them.

Test A

In this test, we compare the errors obtained by several VP-rules $\{\mathcal{H}_{n,m}^u(w, f, t)\}_n$, that differ for the choice of the additional parameter m. More precisely, we choose $m = \lfloor n\theta \rfloor$ and let θ varying in (0, 1), taking also the limit case $\theta = 0$,

Table 6
Example 6: errors by our method (left) and Sinc method (right), $t = 0.5$.

$\eta = 1$			$\eta = 0.125$			$\eta = 1$		$\eta = 0.125$	
n	m	$e^{VP}_{n,m}$	n	m	$e^{VP}_{n,m}$	n	e^{Sinc}_n	n	e^{Sinc}_n
11	2	3.14e−05	51	5	2.80e−04	317	7.3e−10	1287	4.4e−10
21	4	2.76e−11	151	15	2.18e−07	240	4.4e−12	1895	7.3e−12
30	6	1.31e−14	301	30	3.69e−13	336	2.2e−14	2663	2.1e−14
50	10	eps_D	501	50	3.55e−14	448	9.3e−15	3591	4.2e−13

Table 7
Example 6: errors by HT rule, $t = 0.5$.

$\eta = 1$		$\eta = 0.125$	
n	e^{HT}_n	n	e^{HT}_n
34	7.0e−13	162	2.80e−07
42	2.0e−15	258	3.69e−13
		322	eps_D

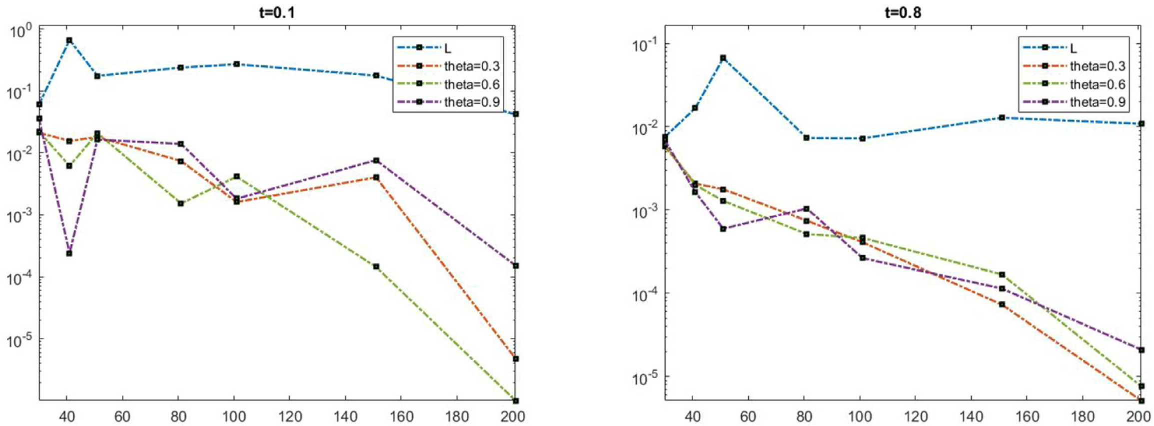


Fig. 4. Absolute errors for Test A, $t = 0.1$ (left), $t = 0.8$ (right).

corresponding to the L-rule. For θ varying in the open interval (0, 1), we have different VP-rules sequences, all of them convergent, but allowing to different errors.

Consider indeed the following example

$$\mathcal{H}^u f(t) = \int_{-1}^{1} \frac{|x|^{\frac{1}{7}}}{x-t} \sqrt[4]{\frac{1-x}{1+x}} dx.$$

Here

$$u = v^{\frac{1}{4},-\frac{1}{4}}, \ w = u, \ u_1 = v^{\frac{1}{4},0}, \ f \in B_r(u_1), \quad r \le \frac{1}{7}.$$

In this case, the function is very smooth, except that in a small interval around 0 where a cusp holds. Here the exact value has been computed by the MG rule with $n = 1200$. In Fig. 4 we show for a fixed t, the graphics of the errors in log-scale, for increasing values of n and for fixed ratios $\theta = \frac{m}{n} \in \{0.3, \ 0.6, 0.9\}$.

As we can see, all the sequences $\{\mathcal{H}^u_{n,m}(w, f, t)\}_n$ converge faster than the L-rule, for any choice of θ.

Test B

As we have remarked in the previous test, the rule $\mathcal{H}^u_{n,m}(w, f, t)$ for n fixed and m varying, induces different errors. In this experiment, for any fixed n, we determine the value of m for which the smallest error is attained in VP-rule (say it the "optimal" m) and for the same n, the value of m for which the largest error is achieved (say it the "worst" m). Referred to the previous Example 5, we produce for increasing values of n (in linear scale) the plot of the absolute errors (in log-scale) of VP-rules (either the best and the worst) and those due to the L-rule and the MG-rule. Here, as well as in Example 5, we have retained exact the values achieved by means of the L-rule of order $n = 1000$ (see Fig. 5).

The errors behaviour confirm that the results by the VP-rule are globally better than those achieved by the L-rule, when the function f presents a quick variation in a localized range. As matter of fact, the parameter m can be used to reduce the quadrature error.

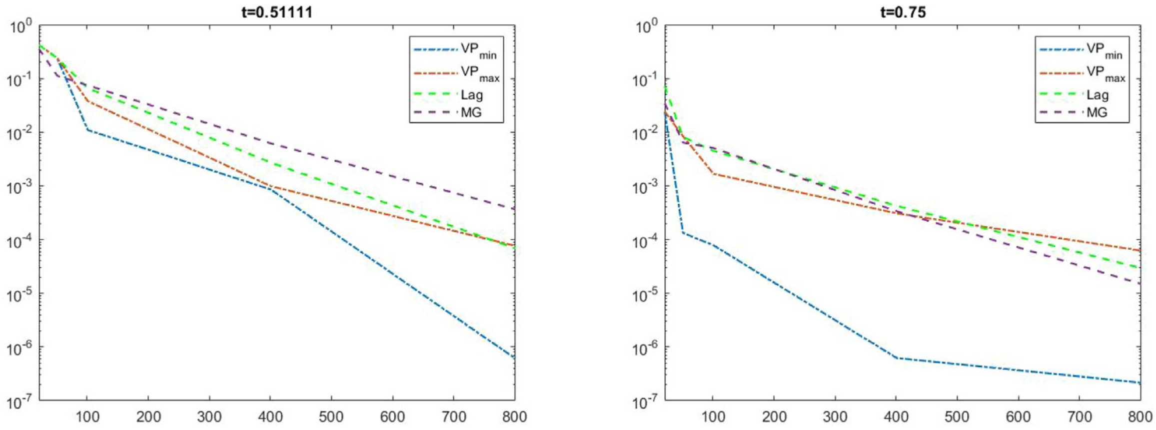


Fig. 5. Absolute errors for Example 5, $t = 0.51111$ (left), $t = 0.75$ (right).

6. Proofs

Proof of Lemma 2.1. In the case $r = 0$, equivalence (9) was proved in [39, Eq.(2.8)]. The proof was mainly based on the following crucial equivalences

$$\sum_{n=1}^{\infty} \frac{E(n)}{n} \sim \sum_{j=0}^{\infty} E(2^j) \sim E(1) + \int_0^1 \frac{E(t^{-\theta})}{t} dt,$$

where $E(x)$ stands for any nonnegative decreasing function defined on $[1, \infty)$ and θ is an arbitrary positive fixed number.

Therefore, in the general case $r > 0$, (9) can be proved with analogous arguments by replacing the previous equivalences with the following ones

$$\sum_{n=1}^{\infty} n^{r-1} E(n) \sim \sum_{j=0}^{\infty} 2^{jr} E(2^j) \sim E(1) + \int_0^1 \frac{E(t^{-\theta})}{t^{r+1}} dt. \quad \square$$

Proof of Lemma 2.2. The statement easily follows from the Jackson type inequalities in (5). Using the first one of such inequalities and recalling that $\omega_\varphi^k(f,t)_v$ is an increasing function of t, for all $f \in B_0(v)$ we get

$$\begin{aligned} E_n(f)_v &\le \mathcal{C}\omega_\varphi^k\left(f, \frac{1}{n}\right)_v = \frac{\mathcal{C}}{\log n}\omega_\varphi^k\left(f, \frac{1}{n}\right)_v \int_{\frac{1}{n}}^1 \frac{dt}{t} \\ &\le \frac{\mathcal{C}}{\log n} \int_{\frac{1}{n}}^1 \frac{\omega_\varphi^k(f,t)_v}{t} dt \le \frac{\mathcal{C}}{\log n} \|f\|_{B_0(v)}, \end{aligned}$$

having used (8) in the last step.

Moreover, starting from the second inequality in (5), for any $f \in B_r(v)$ and $k > r$, we get

$$E_n(f)_v \le \mathcal{C} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,t)_v}{t} dt = \mathcal{C} \int_0^{\frac{1}{n}} \frac{t^r \Omega_\varphi^k(f,t)_v}{t^{r+1}} dt \le \frac{\mathcal{C}}{n^r} \int_0^{\frac{1}{n}} \frac{\Omega_\varphi^k(f,t)_v}{t^{r+1}} dt$$

and the second inequality in (10) follows by (7). $\square$

Proof of Theorem 2.3. We remark that in [37, Th. 3.1] the authors have already proved that

$$|\mathcal{H}^u f(t)| u_2(t) \le \mathcal{C} \|f\|_{B_0(u_1)}, \qquad \mathcal{C} \ne \mathcal{C}(f,t)$$

holds. Here we show how the slightly different estimate in (11) can be easily deduced from the proof given in [37].

Indeed, starting from the following decomposition

$$\begin{aligned} \mathcal{H}^u f(t) &= \int_{-1}^1 \frac{f(x)}{x-t} \frac{u_+(x)}{u_-(x)} dx \\ &= f(t) u_+(t) \int_{-1}^1 \frac{u_-^{-1}(x)}{x-t} dx + \int_{-1}^1 \frac{f(x)u_+(x) - f(t)u_+(t)}{x-t} \frac{dx}{u_-(x)} \\ &=: J_1(t) + J_2(t), \end{aligned}$$

the first term J_1 can be estimated by means of [37, Lemma 4.4], which yields

$$\left|\int_{-1}^{1}\frac{u_-^{-1}(x)}{x-t}\,dx\right|\le \mathcal{C}u_2^{-1}(t),\quad \mathcal{C}\neq\mathcal{C}(f,t),$$

and consequently

$$|J_1(t)|u_2(t)\le \mathcal{C}|f(t)|u_1(t),\qquad \mathcal{C}\neq\mathcal{C}(f,t).$$

The estimate of the other term J_2 can be achieved taking into account that

$$\begin{aligned} J_2(t) &= \left(\int_{-1}^{\frac{t-1}{2}}+\int_{\frac{t+1}{2}}^{1}\right)\frac{f(x)u_+(x)-f(t)u_+(t)}{x-t}u_-^{-1}(x)\,dx\\ &\quad+\int_{\frac{t-1}{2}}^{\frac{t+1}{2}}\frac{f(x)u_+(x)-f(t)u_+(t)}{x-t}u_-^{-1}(x)\,dx\\ &=: I_2+I_3 \end{aligned}$$

and applying the estimates of these two addenda given in the proof of [37, Th. 3.1]. $\square$

Proof of Theorem 3.2. Let us first prove that for all $r\ge 0$, the map $V_n^m(w):B_r(v)\to B_r(v)$ is uniformly bounded w.r.t. n. To this aim we observe that

$$E_k(V_n^m(w,f))_v=0,\qquad \forall k\ge n+m-1,$$

since $V_n^m(w,f)\in\mathbb{P}_{n+m-1}$. Moreover, for $k=0,\dots,n+m-2$, we note that

$$E_k(V_n^m(w,f))_v\le E_k(V_n^m(w,f)-f)_v+E_k(f)_v\le\|[f-V_n^m(w,f)]v\|_\infty+E_k(f)_v.$$

Consequently, for all $r\ge 0$, since we are assuming that $V_n^m(w)$ is uniformly bounded in C_v^0, we get

$$\begin{aligned} &\|V_n^m(w,f)\|_{B_r(v)}=\|V_n^m(w,f)v\|_\infty+\sum_{k=1}^{n+m-2}(k+1)^{r-1}E_k(V_n^m(w,f))_v\\ &\le \mathcal{C}\|fu\|_\infty+\|[f-V_n^m(w,f)]v\|_\infty\sum_{k=1}^{n+m-2}(k+1)^{r-1}+\sum_{k=1}^{n+m-2}(k+1)^{r-1}E_k(f)_v\\ &\le \mathcal{C}\|f\|_{B_r(v)}+\|[f-V_n^m(w,f)]v\|_\infty\sum_{k=1}^{n+m-2}(k+1)^{r-1} \end{aligned}$$

Hence, taking into account that

$$\sum_{k=1}^{n+m-2}(k+1)^{r-1}\le \mathcal{C}\begin{cases}\log n, & \text{if } r=0\\ n^r, & \text{if } r>0\end{cases},\qquad \mathcal{C}\neq\mathcal{C}(n),$$

from (20) we deduce

$$\|V_n^m(w,f)\|_{B_r(v)}\le\mathcal{C}\|f\|_{B_r(v)},\quad \mathcal{C}\neq\mathcal{C}(n,f),\qquad \forall r\ge 0.$$

This equation and (15), for all $P\in\mathbb{P}_{n-m}$, yield

$$\|f-V_n^m(w,f)\|_{B_r(v)}\le\|f-P\|_{B_r(v)}+\|V_n^m(w,f-P)\|_{B_r(v)}\le\mathcal{C}\|f-P\|_{B_r(v)},$$

where $\mathcal{C}\neq\mathcal{C}(n,f,P)$ and (21) follows by taking into account that $(n-m)\sim n$ and that the polynomials are dense in the space $B_r(u)$, $r\ge 0$ (see [39,41]).

Finally, let us prove (22)–(23).

We remark that

$$E_k(f-V_n^m(w,f))_v\left[\begin{array}{ll} =E_k(f)_v, & \text{if } k\ge n+m-1,\\ \\ \le\|[f-V_n^m(w,f)]v\|_\infty, & \text{if } k<n+m-1. \end{array}\right.$$

Consequently, for all $s\ge 0$, we have

$$\begin{aligned} &\|f-V_n^m(w,f)\|_{B_s(v)}=\|[f-V_n^m(w,f)]v\|_\infty+\sum_{k=0}^{\infty}(k+1)^{s-1}E_k(f-V_n^mf)_v\\ &\le\|[f-V_n^m(w,f)]v\|_\infty\left[1+\sum_{k=1}^{n+m-2}(k+1)^{s-1}\right]+\sum_{k=n+m-1}^{\infty}(k+1)^{s-1}E_k(f)_v \end{aligned}$$

Therefore, using (20) and (10), in the case $s = 0$ we get

$$\|f - V_n^m(w,f)\|_{B_0(v)} \le \mathcal{C}\frac{\|f\|_{B_r(v)}}{n^r}\left[1 + \sum_{k=1}^{n+m-2}\frac{1}{k+1}\right] + \mathcal{C}\|f\|_{B_r(v)}\sum_{k=n+m-1}^{\infty}\frac{1}{(k+1)^{r+1}} \le \frac{\|f\|_{B_r(v)}}{n^r}(\mathcal{C} + \mathcal{C}\log n)$$

as well as, for all $s > 0$, we have

$$\|f - V_n^m(w,f)\|_{B_s(v)} \le \mathcal{C}\frac{\|f\|_{B_r(v)}}{n^r}\left[1 + \sum_{k=1}^{n+m-2}(k+1)^{s-1}\right] + \mathcal{C}\|f\|_{B_r(v)}\sum_{k=n+m-1}^{\infty}(k+1)^{s-r-1} \le \frac{\|f\|_{B_r(v)}}{n^r}\left(\mathcal{C} + \mathcal{C}n^s\right) \quad \square$$

Proof of Theorem 3.3. We point out that the equivalence between (24) and (25) can be found in [35, Th. 4.3.1] while (27) has been stated in [41]. Finally, the proof of (26) can be carried out by similar arguments used in the proof of Theorem 3.2 about $V_n^m(w,f)$. Indeed by (25) and (10) we deduce

$$\begin{aligned}\|f - L_n(w,f)\|_{B_0(v)} &\le \mathcal{C}\|[f - L_n(w,f)]v\|_\infty\left[1 + \sum_{k=1}^{n-1}(k+1)^{-1}\right] + \sum_{k=n}^{\infty}\frac{E_k(f)_v}{k+1}\\ &\le \mathcal{C}\|f\|_{B_r(v)}\frac{\log n}{n^r}\left[1 + \sum_{k=1}^{n-1}(k+1)^{-1}\right] + \mathcal{C}\frac{\|f\|_{B_r(v)}}{n^r}\\ &\le \mathcal{C}\frac{\log^2 n}{n^r}\|f\|_{B_r(v)}. \quad \square\end{aligned}$$

Proof of Proposition 4.1. We start recalling that for $u = v^{a,b}$, with $a + b \neq -1$, it results [43, n. 3.228]:

$$\begin{aligned}\mathcal{H}^u\mathbf{1}(t) &= \int_{-1}^{1}\frac{u(x)}{x-t}dx = u(t)\pi\cot\pi(a+1)\\ &- \frac{2^{a+b}\Gamma(a)\Gamma(b+1)}{\Gamma(a+b+1)}\,{}_2F_1\left(-a-b,\,1;\,1-a;\,\frac{1-t}{2}\right),\end{aligned}$$

where ${}_2F_1$ is the generalized hypergeometric function of order $2,\ 1$.

By the recurrence relation (30), it easily follows

$$\begin{aligned}Q_0(t) &= \frac{\mathcal{H}^u\mathbf{1}(t)}{\sqrt{\int_{-1}^{1}w(x)dx}}\\ b_1Q_1(t) &= (t-a_0)Q_0(t) + d_0\\ b_{j+1}Q_{j+1}(t) &= (t-a_j)Q_j(t) - b_jQ_{j-1}(t) + d_j, \quad j \ge 1,\end{aligned}$$

where for $j = 0, 1, \dots,\ d_j = \int_{-1}^{1}p_j(w,x)u(x)dx$. Therefore, setting

$$\begin{aligned}&A_0 = b_1^{-1},\ B_0 = \frac{a_0}{b_1},\\ &A_j = b_{j+1}^{-1},\ B_j = -\frac{a_j}{b_{j+1}},\ C_j = \frac{b_j}{b_{j+1}}\quad D_j = \frac{d_j}{b_{j+1}}, \quad j = 0, 1, \dots\end{aligned}$$

we have

$$\begin{aligned}&Q_0(t) = \frac{\mathcal{H}^u\mathbf{1}(t)}{\sqrt{\int_{-1}^{1}w(x)dx}}, \quad Q_1(t) = (A_0t + B_0)Q_0(t) + D_0\\ &Q_{j+1}(t) = (A_jt + B_j)Q_j(t) - C_jQ_{j-1}(t) + D_j, \quad j \ge 0. \quad \square\end{aligned}$$

Proof of Theorem 4.2. We remark that the definition (31), (28) and (1) yield

$$\mathcal{E}_{n,m}^u(w,f,t) := \left|\mathcal{H}^uf(t) - \mathcal{H}_{n,m}^u(w,f,t)\right| = \left|\mathcal{H}^u[f - V_n^m(w,f)](t)\right|, \quad t \in (-1,1).$$

Therefore, by applying Theorem 2.3, for all $t \in (-1,1)$, we immediately get (32). Consequently (33) follows from (19) and (21). $\square$

7. Conclusions

The paper provides a class of new quadrature formulas for the finite weighted Hilbert transform $\mathcal{H}^u$ defined in (1). Such rules have been obtained by replacing the "density function" f with certain discrete approximation polynomials $V_n^m(w, f)$, based on a de la Vallée Poussin filter and involving the values of f at the Jacobi zeros of order n, associated to a Jacobi weight w.

The main contributions of the paper regard both the theoretical and numerical fields. From the theoretical point of view, new estimates of VP polynomial approximation in Besov type spaces are derived. Moreover sufficient conditions on w and m are stated for the convergence of the quadrature rules when f satisfies a Dini-type condition. In this case pointwise estimates of the error show that the rate of convergence is comparable with that of the weighted best polynomial approximation. From the computational point of view we derived suitable recurrence relations in order to construct the coefficients of the rules. Moreover we compared our rules with others proposed in literature. In particular we have evidenced how pointwise errors are reduced for a.e. smooth functions, that have some peaks or other isolated pathologies.

As future work, we will focus on special cases of finite weighted Hilbert transforms, arising in Cauchy Singular Integral Equations, looking for more compact quadrature formulas and sharper error estimates.

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