**Stat 648: Assignment 3 Solutions** (85 points)

**(HTF 6.2)** (14 pts.) For local linear regression:

$$\begin{aligned}\sum_{i=1}^{N}(x_i-x_0)l_i(x_0) &= b(x_o)^T(\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}(x_0)\begin{bmatrix} x_1-x_0 \\ x_2-x_0 \\ \vdots \\ x_N-x_0 \end{bmatrix} \\ &= b(x_o)^T(\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B}\begin{bmatrix} -x_0 \\ 1 \end{bmatrix} \\ &= b(x_o)^T\begin{bmatrix} -x_0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & x_0 \end{bmatrix}\begin{bmatrix} -x_0 \\ 1 \end{bmatrix} = 0\end{aligned}$$

For local polynomial regression of any degree:
First, $b(x_0)^T = [1\ x_0\ x_0^2\ \cdots\ x_0^k]$ and $\mathbf{B} = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^k \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^k \end{bmatrix}$. So,

$$\begin{aligned}b_0(x_0) = \sum_{i=1}^{N}(x_i-x_0)^0 l_i(x_0) &= \sum_{i=1}^{N} l_i(x_0) \\ &= b(x_o)^T(\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}(x_0)\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ &= b(x_o)^T(\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B}\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ &= b(x_o)^T\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 1\end{aligned}$$

For local polynomial regression of degree $k$:
Suppose $j \in \{1, 2, \ldots, k\}$. By the Binomial Theorem,

$$(x_i-x_0)^j = \sum_{m=0}^{j}\binom{j}{m}(-x_0)^{j-m}x_i^m = [1\ x_i\ x_i^2\ \cdots\ x_i^k]\begin{bmatrix} \binom{j}{0}(-x_0)^j \\ \binom{j}{1}(-x_0)^{j-1} \\ \vdots \\ \binom{j}{j}(-x_0)^0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

$$\text{So, } \begin{bmatrix} (x_1-x_0)^j \\ (x_2-x_0)^j \\ \vdots \\ (x_N-x_0)^j \end{bmatrix} = \mathbf{B} \begin{bmatrix} \binom{j}{0}(-x_0)^j \\ \binom{j}{1}(-x_0)^{j-1} \\ \vdots \\ \binom{j}{j}(-x_0)^0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \text{ and}$$

$$b_0(x_0) = \sum_{i=1}^{N}(x_i-x_0)^j l_i(x_0) = b(x_o)^T(\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B})^{-1}\mathbf{B}^T\mathbf{W}(x_0)\mathbf{B} \begin{bmatrix} \binom{j}{0}(-x_0)^j \\ \binom{j}{1}(-x_0)^{j-1} \\ \vdots \\ \binom{j}{j}(-x_0)^0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= \sum_{m=0}^{j} \binom{j}{m}(-x_0)^{j-m}x_0^m = (x_0-x_0)^j = 0$$

For a local polynomial regression of degree $k$ the bias depends only on terms that are of degree $k+1$ or higher.

**(HTF 7.3)** (12 pts.) (a) Let $\hat{f}^{-i}$ be the mimimizer of

$$\sum_{i\le j\le N, j\neq i}(y_j - f(x_j))^2 + \text{penalty}(f)$$

where the penalty is 0 for least squares and $\lambda\int\{f''(t)\}^2dt$ for cubic smoothing splines, i.e. $\hat{f}^{-i}$ is the minimizer for the fit of all but the $i^{th}$ observation. Now consider the case where all $x$'s are used and $y_i$ is replaced with $\hat{f}^{-i}(x_i)$. So $\mathbf{X}$ remains the same and $\mathbf{y}$ is replaced with $\mathbf{y}^{-i} = [y_1,\ldots,y_{i-1},\hat{f}^{-i}(x_i),y_{i+1},\ldots,y_N]^T$. Then $\hat{f}^{-i}$ will also be the minimizer of

$$\sum_{i\le j\le N}(y_j - f(x_j))^2 + \text{penalty}(f)$$

since $y_i^{-i} - \hat{f}^{-i}(x_i) = 0$ adds nothing to the sum of squares. Thus, $\hat{f}^{-i} = \mathbf{S}\mathbf{y}^{-i}$ and

$$\begin{aligned} \hat{f}^{-i}(x_i) &= \sum_{j\neq i} S_{ij}y_j + S_{ii}\hat{f}^{-i}(x_i) \\ &= \sum_{j=1}^{N} S_{ij}y_j - S_{ii}y_i + S_{ii}\hat{f}^{-i}(x_i) \\ &= \hat{f}(x_i) - S_{ii}y_i + S_{ii}\hat{f}^{-i}(x_i) \end{aligned}$$

So,

$$-\hat{f}(x_i) = -\hat{f}^{-i}(x_i) - S_{ii}y_i + S_{ii}\hat{f}^{-i}(x_i)$$

and

$$\begin{aligned} y_i - \hat{f}(x_i) &= y_i - \hat{f}^{-i}(x_i) - S_{ii}y_i + S_{ii}\hat{f}^{-i}(x_i) \\ &= (1 - S_{ii})(y_i - \hat{f}^{-i}(x_i)) \end{aligned}$$

Therefore, the result holds.

(b) For the least squares case $\mathbf{S}$ is idempotent and for the cubic smoothing spline case $\mathbf{S} \geq \mathbf{S}^2$. Also, $\mathbf{S}$ is symmetric for both cases. Since $\mathbf{S} \geq \mathbf{S}^2$,

$$S_{ii} \geq \sum_{j=1}^{N} S_{ij}S_{ji} = \sum_{j=1}^{N} S_{ij}^2 = S_{ii}^2 + \sum_{j \neq i} S_{ij}^2.$$

Thus $S_{ii} \geq 0$ and $S_{ii} \geq S_{ii}^2$, i.e. $S_{ii} \leq 1$. Therefore, by part (a) $|y_i - \hat{f}^{-i}(x_i)| \geq |y_i - \hat{f}(x_i)|$.

(c) As long as $\mathbf{S}$ only depends on $\mathbf{X}$ and not $\mathbf{y}$ the result (7.64) will hold.

**(HTF 7.6)** (5 pts.) For $k$-nearest neighbors

$$\hat{\mathbf{y}} = \sum_{x_i \in N_k(x)} \frac{y_i}{k}$$

where $N_k(x)$ is the neighborhood containing the $k$ closest points to $x$. A point is always its own nearest neighbor, so in matrix form we have $\hat{\mathbf{y}} = \mathbf{S}\mathbf{y}$ where every diagonal entry of $\mathbf{S}$ is $1/k$. Thus,

$$\text{df} = tr(\mathbf{S}) = \sum_{i=1}^{N} \frac{1}{k} = \frac{N}{k}.$$

**(8)** i) (7 pts.) We know that

$$\begin{aligned} \gamma_i\phi_i(x) &= \int_0^1 \phi_i(z)K(z,x)dz \\ &= \int_0^1 \phi_i(z)(1 + 2zx + z^2x^2)dz \\ &= \int_0^1 \phi_i(z)dz + 2x\int_0^1 x\phi_i(z)dz + x^2\int_0^1 z^2\phi_i(z)dz \\ &= a_i + b_ix + c_ix^2 \quad \text{for constants } a_i, b_i, c_i. \end{aligned}$$

Thus, $\phi_i(x)$ is quadratic for all $i$.

Since $\phi_i(x) = a_i + b_ix + c_ix^2$, we have

$$\gamma_i(a_i + b_ix + c_ix^2) = \int_0^1 (a_i + b_iz + c_iz^2)(1 + 2zx + z^2x^2)dz.$$

By doing the integration we find that

$$\gamma_i(a_i + b_ix + c_ix^2) = (a_i + 1/2b_i + 1/3c_i) + (a_i + 2/3b_i + 1/2c_i)x + (1/3a_i + 1/4b_i + 1/5c_i)x^2.$$

Thus

$$\begin{aligned} 0 &= (1/5-\gamma_i)c_i + 1/4 b_i + 1/3 a_i \\ 0 &= 1/2 c_i + (2/3-\gamma_i) b_i + a_i \\ 0 &= 1/3 c_i + 1/2 b_i + (1-\gamma_i) a_i \end{aligned}$$

or

$$\begin{bmatrix} (1-\gamma_i) & \frac{1}{2} & \frac{1}{3} \\ 1 & (\frac{2}{3}-\gamma_i) & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & (\frac{1}{5}-\gamma_i) \end{bmatrix} \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

In order for this system of equations to have a nonzero solution, the determinant of

$$A \equiv \begin{bmatrix} (1-\gamma_i) & \frac{1}{2} & \frac{1}{3} \\ 1 & (\frac{2}{3}-\gamma_i) & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{4} & (\frac{1}{5}-\gamma_i) \end{bmatrix}$$

must be 0. Thus, the $\gamma_i$'s are the eigenvalues of A, i.e.

$$\begin{aligned} \gamma_1 &= 1.71292 \\ \gamma_2 &= 0.15014 \\ \gamma_3 &= 0.00360 \end{aligned}$$

Now the eigenvectors of the matrix provide scaled coefficients for the quadratic eigenfunctions. We also know that $\int_0^1 \phi_i(x)^2 dx = 1$ for $\phi_i(x) = a_i + b_i x + c_i x^2$. Therefore,

$$a_i^2 + a_i b_i + 2/3 a_i c_i + 1/3 b_i^2 + 1/2 b_i c_i + 1/5 c_i^2 = 1.$$

Using R, the eigenfunctions can be found to be

$$\begin{aligned} \phi_1(x) &= -0.5644618 - 0.6502907x - 0.2318203x^2 \\ \phi_2(x) &= -1.698942 + 1.999564x + 1.332234x^2 \\ \phi_3(x) &= 2.407276 - 13.695943x + 13.348086x^2 \end{aligned}$$

To verify that

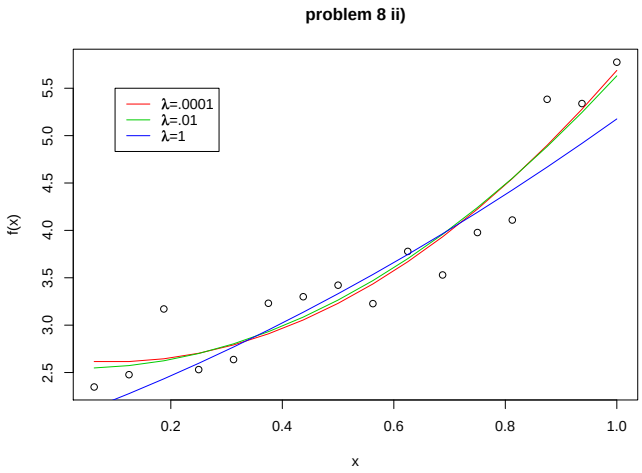
$$K(z,x) = \sum_{i=1}^{3} \gamma_i \phi_i(z)\phi_i(x) = \sum_{i=1}^{3} \gamma_i (a_i + b_i z + c_i z^2)(a_i + b_i x + c_i x^2)$$

we need to show that $\sum_{i=1}^3 \gamma_i c_i^2 = 1$, $\sum_{i=1}^3 \gamma_i b_i^2 = 2$, $\sum_{i=1}^3 \gamma_i a_i^2 = 1$, $\sum_{i=1}^3 \gamma_i a_i b_i = 0$, $\sum_{i=1}^3 \gamma_i a_i c_i = 0$, and $\sum_{i=1}^3 \gamma_i b_i c_i = 0$. These were all verified in R.

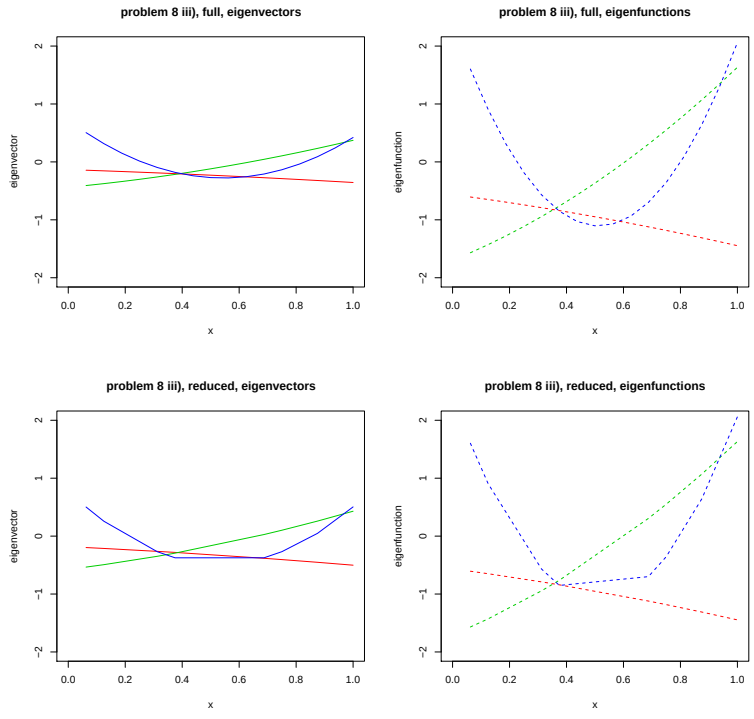
ii) (6 pts.) Plots of the fits are given below.

To find the degrees of freedom: We have $f_\lambda(x_j) = \sum_{i=1}^N b_{\lambda i} K(x_j, x_i) = (\mathbf{K}\mathbf{b}_\lambda)_j$ (i.e. the $j^{th}$ element of the column $\mathbf{K}\mathbf{b}_\lambda$). Then, $\hat{\mathbf{Y}} = \mathbf{K}\mathbf{b}_\lambda = \mathbf{K}(\mathbf{K}+\lambda\mathbf{I})^{-1}\mathbf{Y}$ and the df's can be computed as the trace of $\mathbf{K}(\mathbf{K}+\lambda\mathbf{I})^{-1}$.

The degrees of freedom are found to be 2.998098, 2.838628, and 1.721616.




iii) (4 pts.) The (nonzero) eigenvalues for the full matrix $\mathbf{K}$ are 28.976667, 2.377899, and 0.053760. For the reduced $\mathbf{K}$ that uses only observations $i = 1,2,5,6,11,12,14,16$ the eigenvalues are 14.756499, 1.485195, and 0.031057. Plots of the eigenvectors and eigenfunctions against the $x$'s are given below. The plots are similar in shape but different in scale, suggesting that the eigenvectors are approximately proportional to the eigenfunctions. The eigenfunctions of $K$ are larger than the eigenvectors of $\mathbf{K}$, while the eigenvalues for $K$ are smaller than the eigenvalues for $\mathbf{K}$ (but still approximately proportional to one another).



iv) (7 pts.)

$$\begin{aligned}\int_0^1 g(x)K(z,x)dx &= \int_0^1 g(x)\sum_{i=1}^{3}\gamma_i\phi_i(x)\phi_i(z)dx \\ &= \sum_{i=1}^{3}\gamma_i\phi_i(z)\int_0^1 g(x)\phi_i(x)dx = 0.\end{aligned}$$

So, $g$ is in the null space of the kernel $K$.

Next, we know that

$$\int_0^1 g(x)K(x,z)dx = \int_0^1 (a+bx+cx^2+dx^3)(1+2zx+z^2x^2)dx = 0.$$

By setting $a=1$, we find that $b=-12$, $c=30$, and $d=-20$. So $g(x) = 1-12x+30x^2-20x^3$.

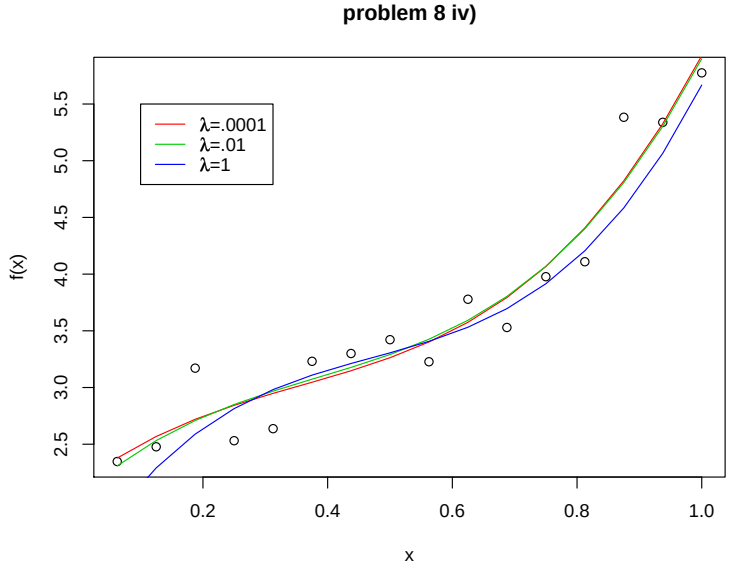
Now, for $f \in \mathcal{H}_K$ we can write $f(x) = c_1\phi_1(x)+c_2\phi_2(x)+c_3\phi_3(x)$, and we can minimize $\sum_{i=1}^{16}(y_i-(\alpha g(x_i)+f(x_i)))^2+\lambda\|f\|_{\mathcal{H}_K}$ by finding

$$\min_{\alpha,c_1,c_2,c_3}\left[\sum_{i=1}^{16}(y_i-\alpha g(x_i)-c_1\phi_1(x_i)-c_2\phi_2(x_i)-c_3\phi_3(x_i))^2+\lambda\sum_{j=1}^{3}c_i^2/\gamma_i\right]$$

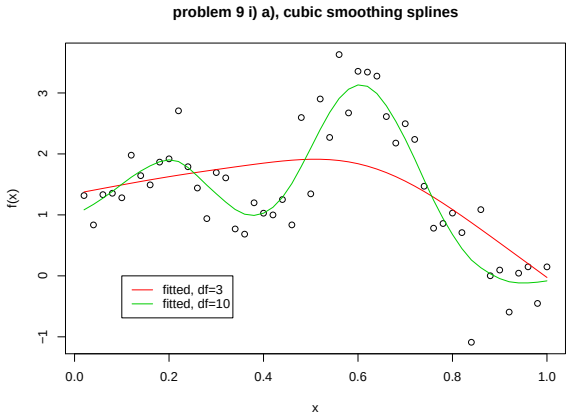
By using R, we arrive at the following:

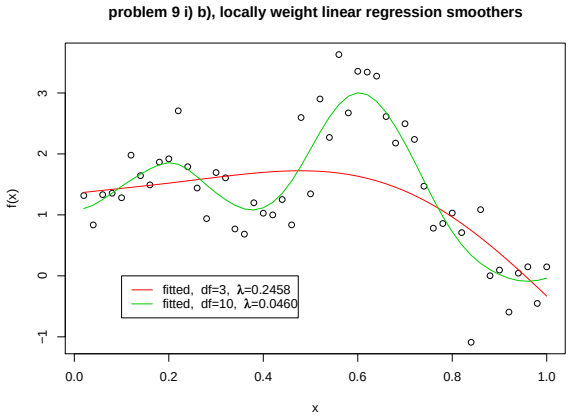
| $\lambda$ | $\alpha$ | $\alpha g(x)$ | $f(x)$ |
|---|---|---|---|
| 0.0001 | -0.35707 | $-0.35708+4.28490x-10.71226x^2+7.14151x^3$ | $2.49579-0.00622x+3.07956x^2$ |
| 0.01 | -0.38249 | $-0.38249+4.58993x-11.47484x^2+7.64989x^3$ | $2.41171+0.41200x+2.69302x^2$ |
| 1 | -0.54731 | $-0.54731+6.56776x-16.41940x^2+10.94627x^3$ | $1.96000+2.22356x+0.93597x^2$ |

The fitted curves are given in the figure below.

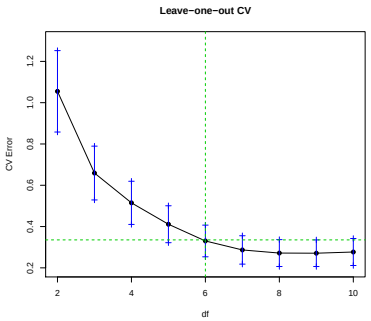


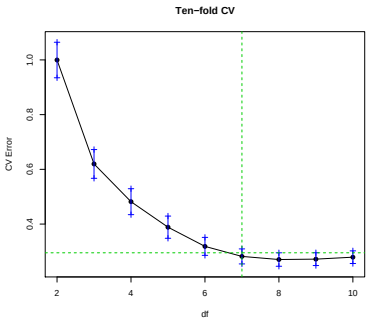
**(9)** i) (6 pts.) Plots are given below for both a) and b). Note that for part b), the degrees of freedom 3 and 10 correspond to $\lambda$ values of 0.2458 and 0.0460, respectively.

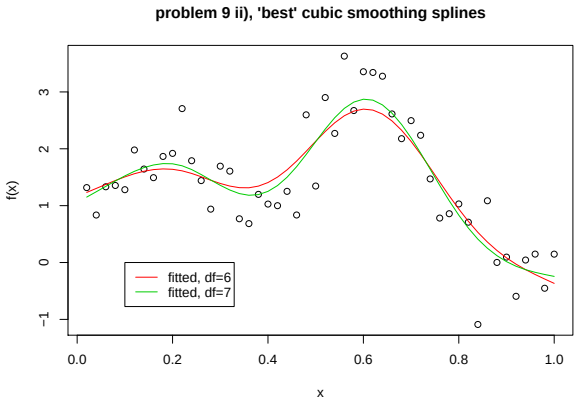



ii) (6 pts.) Plots of the CV error are given below. For leave-one-out CV we choose df=6. For ten-fold CV we choose df=7. The fitted curves for these df's are also provided below.



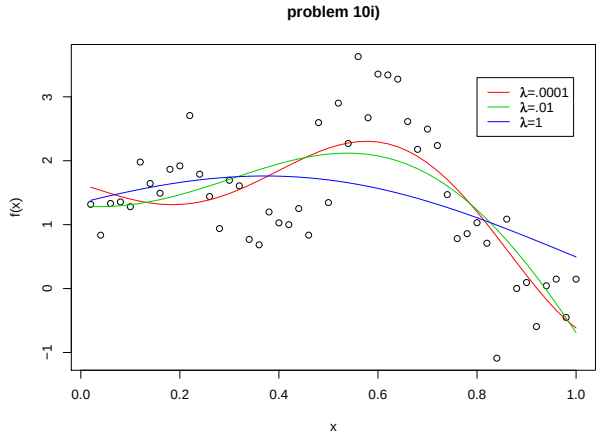



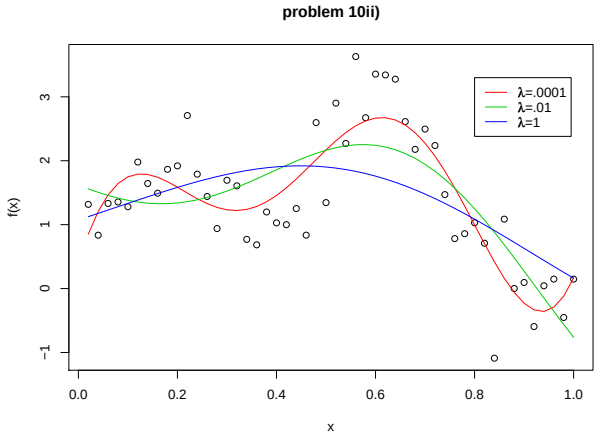
**(10)** Plots for i)-iii) are given below.

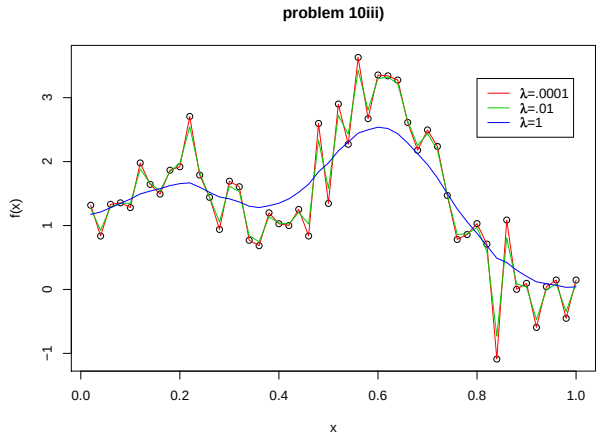
i) (6 pts.) degrees of freedom: 4.977618, 3.679214, and 2.158616

ii) (6 pts.) degrees of freedom: 5.906136, 4.367702, and 2.564012

iii) (6 pts.) degrees of freedom: 49.877790, 40.978458, and 7.293336



### Code for computing problems

```
######problem 8######
####i)
J=matrix(c(1,1,1/3,1/2,2/3,1/4,1/3,1/2,1/5),nrow=3)
#eigenvectors & eigenvalues
e=eigen(J)$values
E=eigen(J)$vectors

#finds the eigenfunctions
E2=E
for(i in 1:3){
con=1/sqrt(E[1,i]^2+E[1,i]*E[2,i]+2/3*E[1,i]*E[3,i]+1/3*E[2,i]^2+1/2*E[2,i]*E[3,i]+1/5*E[3,i]^2)
E2[,i]=E[,i]*con}

#verify the claim
sum(e*E2[3,]^2)
sum(e*E2[2,]^2)
sum(e*E2[1,]^2)
sum(e*E2[1,]*E2[3,])
sum(e*E2[1,]*E2[2,])
sum(e*E2[2,]*E2[3,])

####ii)
K=function(x,z){(1+z*x)^2}
Km=matrix(nrow=16,ncol=16)
for(i in 1:16){
for(j in 1:16){
Km[i,j]=K(p8[i,1],p8[j,1])}}

I=diag(16)
lambda=c(.0001,.01,1)
df=rep(0,3)

plot(p8[,1],p8[,2],main="problem 8 ii)",xlab="x",ylab="f(x)")
for(j in 1:3){
lam=lambda[j]
M=solve(Km+lam*I)
b=M%*%p8[,2]
lines(p8[,1],Km%*%b,col=(j+1))
df[j]=sum(diag(Km%*%M))}
legend(.1,5.5,legend=c(expression(paste(lambda,"=.0001")),expression(paste(lambda,"=.01")),
expression(paste(lambda,"=1"))),lty=rep(1,3),col=c(2,3,4))

####iii)
phi1=function(x){E2[1,1]+E2[2,1]*x+E2[3,1]*x^2}
phi2=function(x){E2[1,2]+E2[2,2]*x+E2[3,2]*x^2}
phi3=function(x){E2[1,3]+E2[2,3]*x+E2[3,3]*x^2}
eigen(Km)$values
V=eigen(Km)$vectors
X=p8[,1]
par(mfrow=c(1,2))
plot(seq(0,1,,100),seq(-2,2,,100),type="n",xlab="x",ylab="eigenvector",
main="problem 8 iii), full, eigenvectors")
```

```
lines(X,V[,1],col=2)
lines(X,V[,2],col=3)
lines(X,V[,3],col=4)
plot(seq(0,1,,100),seq(-2,2,,100),type="n",xlab="x",ylab="eigenfunction",
main="problem 8 iii), full, eigenfunctions")
lines(X,phi1(X),lty=2,col=2)
lines(X,phi2(X),lty=2,col=3)
lines(X,phi3(X),lty=2,col=4)

#with only data for i=1,2,5,6,11,12,14,16
index=c(1,2,5,6,11,12,14,16)
Kmred=Km[index,index]
eigen(Kmred)$values
V=eigen(Kmred)$vectors
X=p8[index,1]
par(mfrow=c(1,2))
plot(seq(0,1,,100),seq(-2,2,,100),type="n",xlab="x",ylab="eigenvector",
main="problem 8 iii), reduced, eigenvectors")
lines(X,V[,1],col=2)
lines(X,V[,2],col=3)
lines(X,V[,3],col=4)
plot(seq(0,1,,100),seq(-2,2,,100),type="n",xlab="x",ylab="eigenfunction",
main="problem 8 iii), reduced, eigenfunctions")
lines(X,phi1(X),lty=2,col=2)
lines(X,phi2(X),lty=2,col=3)
lines(X,phi3(X),lty=2,col=4)

####iv)
#finding b,c,d for a=1
H=matrix(c(1/2,2/3,1/4,1/3,1/2,1/5,1/4,2/5,1/6),nrow=3)
coef=solve(H,b=c(-1,-1,-1/3))
g=function(x){1+coef[1]*x+coef[2]*x^2+coef[3]*x^3}

#function for finding alpha and f
F=function(t){
a=t[1]
c1=t[2]
c2=t[3]
c3=t[4]
sum((p8[,2]-a*g(X)-c1*phi1(X)-c2*phi2(X)-c3*phi3(X))^2)+lam*sum(t[2:4]^2/e)}
X=p8[,1]

plot(p8[,1],p8[,2],main="problem 8 iv)",xlab="x",ylab="f(x)")

lam=.0001
A.0001=optim(rep(1,4),F)$par
A.0001[1]
#coefficients for f
A.0001[2]*E2[,1]+A.0001[3]*E2[,2]+A.0001[4]*E2[,3]
#coefficients for alpha*g
A.0001[1]*c(1,-12,30,-20)
#coefficients for alpha*g+f
est=c(A.0001[2]*E2[,1]+A.0001[3]*E2[,2]+A.0001[4]*E2[,3],0)+A.0001[1]*c(1,-12,30,-20)
```

```
agf=function(x){est[1]+est[2]*x+est[3]*x^2+est[4]*x^3}
lines(X,agf(X),col=2)

lam=.01
A.01=optim(rep(1,4),F)$par
A.01[1]
#coefficients for f
A.01[2]*E2[,1]+A.01[3]*E2[,2]+A.01[4]*E2[,3]
#coefficients for alpha*g
A.01[1]*c(1,-12,30,-20)
#coefficients for alpha*g+f
est=c(A.01[2]*E2[,1]+A.01[3]*E2[,2]+A.01[4]*E2[,3],0)+A.01[1]*c(1,-12,30,-20)
agf=function(x){est[1]+est[2]*x+est[3]*x^2+est[4]*x^3}
lines(X,agf(X),col=3)

lam=1
A1=optim(rep(1,4),F)$par
A1[1]
#coefficients for f
A1[2]*E2[,1]+A1[3]*E2[,2]+A1[4]*E2[,3]
#coefficients for alpha*g
A1[1]*c(1,-12,30,-20)
#coefficients for alpha*g+f
est=c(A1[2]*E2[,1]+A1[3]*E2[,2]+A1[4]*E2[,3],0)+A1[1]*c(1,-12,30,-20)
agf=function(x){est[1]+est[2]*x+est[3]*x^2+est[4]*x^3}
lines(X,agf(X),col=4)
legend(.1,5.5,legend=c(expression(paste(lambda,"=.0001")),expression(paste(lambda,"=.01")),
expression(paste(lambda,"=1"))),lty=rep(1,3),col=c(2,3,4))




######problem 9######
####i)a)
cubic3=smooth.spline(p9[,1],p9[,2],df=3)
cubic10=smooth.spline(p9[,1],p9[,2],df=10)
plot(p9[,1],p9[,2],main="problem 9 i) a), cubic smoothing splines",xlab="x",ylab="f(x)")
lines(predict(cubic3)$x,predict(cubic3)$y,col=2)
lines(predict(cubic10)$x,predict(cubic10)$y,col=3)
legend(.1,0,legend=c("fitted, df=3","fitted, df=10"),lty=rep(1,2),col=c(2,3))

####i)b)
B=matrix(c(rep(1,50),p9[,1]),ncol=2)
yhat=rep(0,50)
plot(p9[,1],p9[,2],main="problem 9 i) b), locally weight
linear regression smoothers",xlab="x",ylab="f(x)")

t=c(3,10)
for(m in 1:2){
Lfind=function(d){
Kn=function(x,z){dnorm((x-z)/d)}
for(i in 1:50){
for(j in 1:50){
```

```
Knm[i,j]=Kn(p9[i,1],p9[j,1])}}
tr=0
for(i in 1:50){
W=diag(Knm[i,])
S=matrix(c(1,i/50),nrow=1)%*%solve(t(B)%*%W%*%B)%*%t(B)%*%W
tr=tr+S[,i]}
return(tr-t[m])}
lm=uniroot(Lfind,c(.01,3))$root
tr=0
Kn=function(x,z){dnorm((x-z)/lm)}
for(i in 1:50){
for(j in 1:50){
Knm[i,j]=Kn(p9[i,1],p9[j,1])}}
for(i in 1:50){
W=diag(Knm[i,])
S=matrix(c(1,i/50),nrow=1)%*%solve(t(B)%*%W%*%B)%*%t(B)%*%W
yhat[i]=S%*%p9[,2]
tr=tr+S[,i]}
cat(tr,lm,"\n")
lines(p9[,1],yhat,col=m+1)}
legend(.1,0,legend=c(expression(paste("fitted,  df=3,  ",lambda,"=0.2458")),
expression(paste("fitted,  df=10,  ",lambda,"=0.0460"))),lty=rep(1,2),col=c(2,3))


####ii)
##Leave-one-out CV
n=10
perr=matrix(nrow=n-1,ncol=50)
for(i in 2:n){
for(j in 1:50){
dat=p9[-j,]
cub=smooth.spline(dat[,1],dat[,2],df=i)
perr[i-1,j]=(predict(cub,x=j/50)$y-p9[j,2])^2}}
se=apply(perr,1,sd)/sqrt(50)
err=apply(perr,1,mean)
plot(seq(2,n,,100),seq(0.2,1.3,,100),xlab="df",ylab="CV Error",type="n",main="Leave-one-out CV")
points(2:n,err,pch=19)
lines(2:n,err)
points(2:n,err+se,pch=3,col=4)
points(2:n,err-se,pch=3,col=4)
for(i in 1:10){
lines(rep(i+1,2),c((err+se)[i],(err-se)[i]),col=4)}
abline(h=min(err+se),col=3,lty=2)
abline(v=6,col=3,lty=2)

##Ten-fold CV
n=10
perr=matrix(nrow=n-1,ncol=10)
for(i in 2:n){
for(j in 1:10){
dat=p9[-c(j,j+10,j+20,j+30,j+40),]
datl=p9[c(j,j+10,j+20,j+30,j+40),]
cub=smooth.spline(dat[,1],dat[,2],df=i)
```

```
perr[i-1,j]=mean((predict(cub,x=datl[,1])$y-datl[,2])^2)}}
se=apply(perr,1,sd)/sqrt(50)
err=apply(perr,1,mean)
plot(seq(2,n,,100),seq(0.24,1.07,,100),xlab="df",ylab="CV Error",type="n",main="Ten-fold CV")
points(2:n,err,pch=19)
lines(2:n,err)
points(2:n,err+se,pch=3,col=4)
points(2:n,err-se,pch=3,col=4)
for(i in 1:10){
lines(rep(i+1,2),c((err+se)[i],(err-se)[i]),col=4)}
abline(h=min(err+se),col=3,lty=2)
abline(v=7,col=3,lty=2)

##plots of selected fits
cubic6=smooth.spline(p9[,1],p9[,2],df=6)
cubic7=smooth.spline(p9[,1],p9[,2],df=7)
plot(p9[,1],p9[,2],main="problem 9 ii), 'best' cubic smoothing splines",xlab="x",ylab="f(x)")
lines(predict(cubic6)$x,predict(cubic6)$y,col=2)
lines(predict(cubic7)$x,predict(cubic7)$y,col=3)
legend(.1,0,legend=c("fitted, df=6","fitted, df=7"),lty=rep(1,2),col=c(2,3))


######problem 10####
Ka=function(x,z){exp(-(x-z)^2)}
Kb=function(x,z){exp(-2*(x-z)^2)}
Kc=function(x,z){exp(-2*abs(x-z))}

Kam=Kbm=Kcm=matrix(nrow=50,ncol=50)
for(i in 1:50){
for(j in 1:50){
Kam[i,j]=Ka(p9[i,1],p9[j,1])
Kbm[i,j]=Kb(p9[i,1],p9[j,1])
Kcm[i,j]=Kc(p9[i,1],p9[j,1])}}

I=diag(50)
lambda=c(.0001,.01,1)

dfa=dfb=dfc=rep(0,3)
plot(p9[,1],p9[,2],main="problem 10i)",xlab="x",ylab="f(x)")
for(j in 1:3){
lam=lambda[j]
M=solve(Kam+lam*I)
b=M%*%p9[,2]
lines(p9[,1],Kam%*%b,col=(j+1))
TT=Kam%*%M
dfa[j]=sum(diag(Kam%*%M))}
legend(.8,3.3,legend=c(expression(paste(lambda,"=.0001")),expression(paste(lambda,"=.01")),
expression(paste(lambda,"=1"))),lty=rep(1,3),col=c(2,3,4))

plot(p9[,1],p9[,2],main="problem 10ii)",xlab="x",ylab="f(x)")
for(j in 1:3){
lam=lambda[j]
M=solve(Kbm+lam*I)
```

```
b=M%*%p9[,2]
lines(p9[,1],Kbm%*%b,col=(j+1))
dfb[j]=sum(diag(Kbm%*%M))}
TT2=Kbm%*%M
legend(.8,3.3,legend=c(expression(paste(lambda,"=.0001")),expression(paste(lambda,"=.01")),
expression(paste(lambda,"=1"))),lty=rep(1,3),col=c(2,3,4))

plot(p9[,1],p9[,2],main="problem 10iii)",xlab="x",ylab="f(x)")
for(j in 1:3){
lam=lambda[j]
M=solve(Kcm+lam*I)
b=M%*%p9[,2]
lines(p9[,1],Kcm%*%b,col=(j+1))
dfc[j]=sum(diag(Kcm%*%M))}
TT3=Kcm%*%M
legend(.8,3.3,legend=c(expression(paste(lambda,"=.0001")),expression(paste(lambda,"=.01")),
expression(paste(lambda,"=1"))),lty=rep(1,3),col=c(2,3,4))
```