Properties of figures

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MPM2DE-B

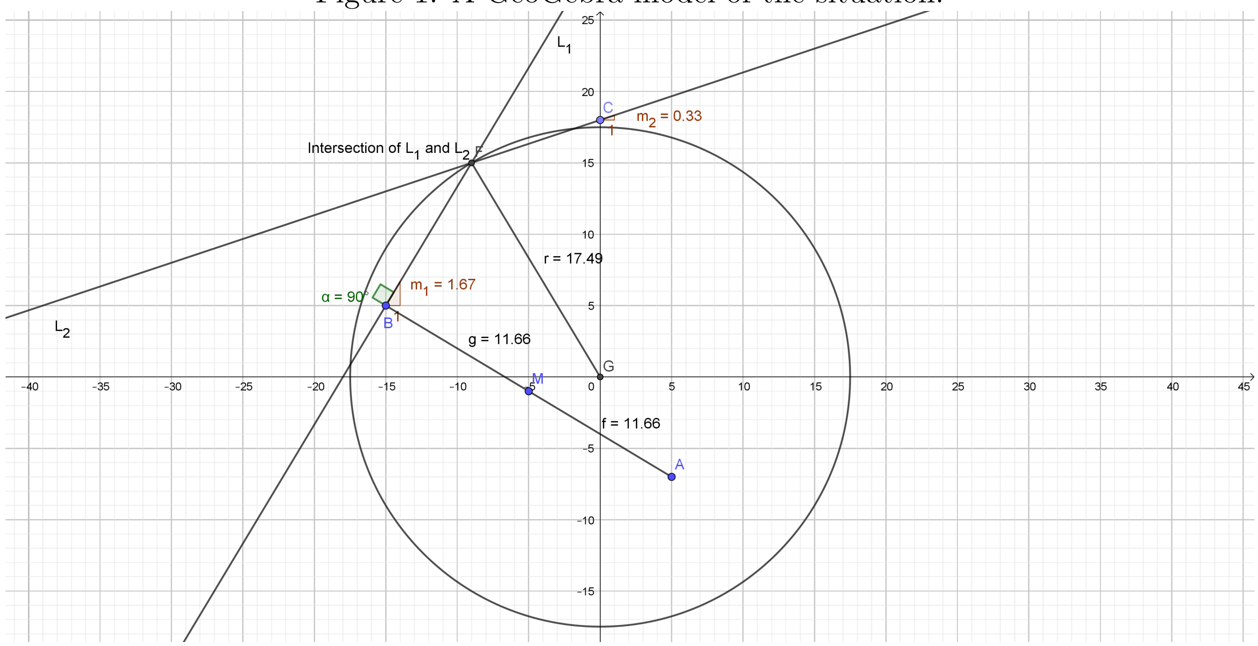
Contents

1 Problem statement

Consider a line segment AB where $A(5, -7)$ and the midpoint of AB is $M(-5, -1)$. If line L_1 passes through B and is perpendicular to AB, and line L_2 is given by $y = \frac{1}{3}x + 18$, determine the equation of the circle centered at the origin that passes through the point of intersection of the lines L_1 and L_2.

2 Graphical solution

Figure 1: A GeoGebra model of the situation.



As can be seen inv Figure 1, the desired circle is centered at the origin and has a radius $r = 17.49$. The equation of the circle is therefore given by:

$$x^2 + y^2 = r^2$$
$$x^2 + y^2 = 305.90 \blacksquare$$

3 Algebraic solution

3.1 Determining B

Let the point B be defined as $B(x_B, y_B)$. M is the midpoint of AB, therefore, by the midpoint formula:

$$\begin{aligned}\left(\frac{x_A + x_B}{2}, \frac{y_A + y_B}{2}\right) &= (x_M, y_M) \\ \left(\frac{5 + x_B}{2}, \frac{-7 + y_B}{2}\right) &= (-5, -1)\end{aligned} \tag{1}$$

Solving for x_B:

$$\begin{aligned}\frac{5 + x_B}{2} &= -5 \\ 5 + x_B &= -10 \\ x_B &= -15\end{aligned} \tag{2}$$

Solving for y_B:

$$\begin{aligned}\frac{-7 + y_B}{2} &= -1 \\ -7 + y_B &= -2 \\ y_B &= 5\end{aligned} \tag{3}$$

From equations (1), (2) and (3), we have determined the point $B(-15, 5)$.

3.2 Determining the slope of L_1

The slope m_{AB} of AB is given by:

$$m_{AB} = \frac{y_B - y_A}{x_B - x_A} \tag{4}$$

L_1 is perpendicular to AB, so its slope m_1 must be the negative reciprocal of m_{AB}.

$$\begin{aligned}m_1 &= -\frac{1}{m_{AB}} \\ m_1 &= \frac{x_B - x_A}{y_B - y_A} \\ m_1 &= -\frac{-15 - 5}{5 - (-7)} \\ m_1 &= -\frac{-20}{12} \\ m_1 &= \frac{5}{3}\end{aligned} \tag{5}$$

3.3 Determining the equation of L_1

We have determined the slope of L_1 and a point that L_1 passes through from (2), (3) and (5), so we can now determine an equation for L_1.

$$\begin{aligned}
y &= mx + b \\
y_B &= m_1 \cdot x_B + b \\
b &= y_B - m_1 \cdot x_B \\
b &= 5 - \frac{5}{3}(-15) \\
b &= 5 + 25 \\
b &= 30
\end{aligned} \tag{6}$$

The equation for L_1 is given by:

$$y = \frac{5}{3}x + 30 \tag{7}$$

3.4 Finding the point of intersection of L_1 and L_2

Now that the equations for both L_1 and L_2 are known, we have a system of linear equations that can be solved to determine the point of intersection $P(x_I, y_I)$ of the two lines.

$$\begin{cases} y_I = \frac{5}{3}x_I + 30 \\ y_I = \frac{1}{3}x_I + 18 \end{cases} \tag{8}$$

This is a very simple system to solve, because both equations have y isolated already. We can therefore just substitute.

$$\begin{aligned}
\frac{5}{3}x_I + 30 &= \frac{1}{3}x_I + 18 \\
\frac{4}{3}x_I &= -12 \\
x_I &= -12 \cdot \left(\frac{3}{4}\right) \\
x_I &= -9
\end{aligned} \tag{9}$$

We can substitute this value of X_I back into one of the equations to determine y_I.

$$y_I = \frac{1}{3} \cdot (-9) + 18 = 15 \tag{10}$$

The point of intersection of the lines L_1 and L_2 is $(x_I, y_I) = (-9, 15)$.

3.5 Determining the equation of the circle

The circle is centered at the origin $O(0,0)$, so the equation of the circle will take the form of $x^2 + y^2 = r^2$, where r is the radius of the circle.
Knowing that the point of intersection of L_1 and L_2, $P(x_I, y_I) = (-9, 15)$, is passed through by the circle, the coordinates $(-9, 15)$ must satisfy the equation of the circle, and we can solve for the unknown value of r_2 in this way.

$$\begin{aligned} x^2 + y^2 &= r^2 \\ x_I^2 + y_I^2 &= r^2 \\ (-9)^2 + 15^2 &= r^2 \\ 81 + 225 &= r^2 \\ r^2 &= 306 \end{aligned} \tag{11}$$

The equation of the circle is therefore:

$$x^2 + y^2 = 306 \ \blacksquare \tag{12}$$

This is corroborated by my graphical solution in GeoGebra from Figure 1: there is an error of only 0.1 in the graphical solution, which is reasonable, due to the accuracy of numbers that GeoGebra displays.