A brief history of discounting

1. Historical platforms for rates and discounting
2. Current platforms for rates and discounting
    1. Collateralized transactions
    2. Uncollateralized transactions

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1 History

Classical discounting framework up to mid 1990s

- Yield curve: for a given currency, the term structure of interest rates for all maturities
  - Discount Factor: PV of 1 currency unit delivered at time T

continuous rate

$$DF(T) = e^{-R(T)T} \qquad R(T) = -\frac{\log DF(T)}{T}$$

  - The curve R(T) gives all relevant rate information

yield curve

$$T \longrightarrow R(T)$$

Feed models
rates for forwards/futures
discount factors for option payoffs
short forward rates for rate exotics

$$f(T) = -\frac{\partial \log DF(T)}{\partial T}$$

Value rate transactions
“risk free” loans
swaps, spot or forward starting

Tn
receive redemption + interest

$-N$

$$N\left[1+F(T_0,T_n)(T_n-T_0)\right]$$

rate

“coverage”

T0
lend notional

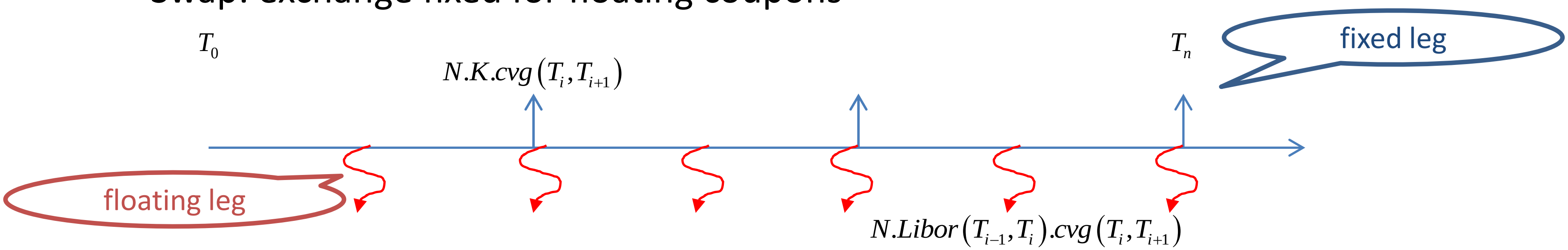
For the transaction to be fair

$$NDF(T_0) = N\left[1+F(T_0,T_n)(T_n-T_0)\right]DF(T_n)$$

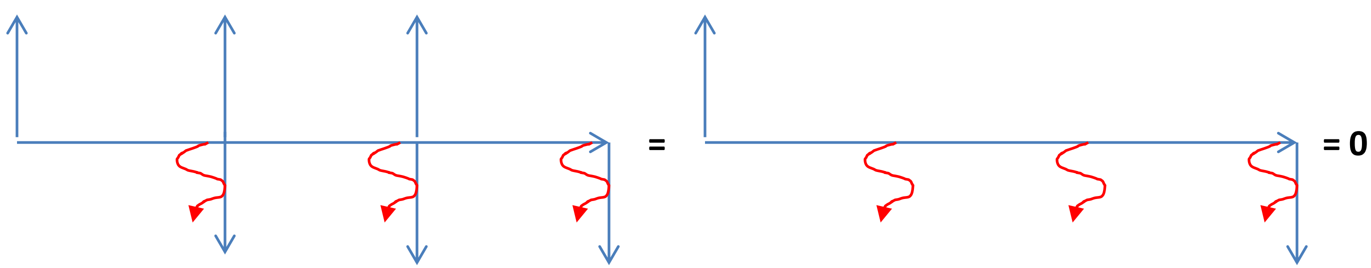
$$F(T_0,T_n) = \frac{DF(T_0)-DF(T_n)}{(T_n-T_0)DF(T_n)}$$

Interest rate swaps

- Swap: exchange fixed for floating coupons



- Floating leg, ***assuming Libor is the interest rate***, is same as rolling deposits



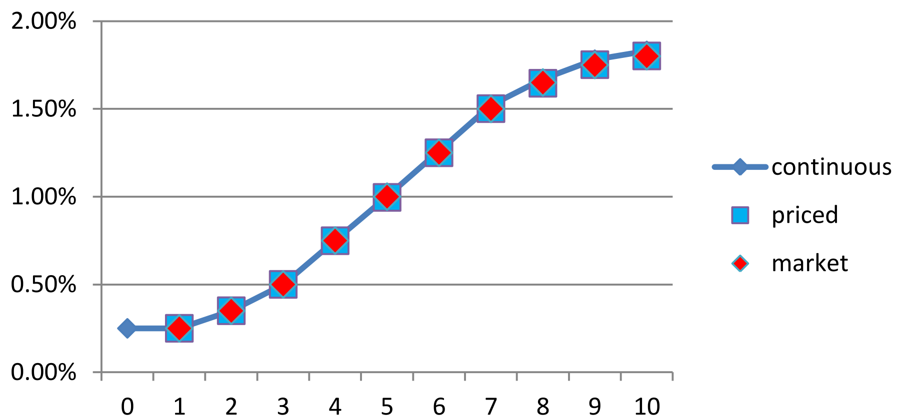
- Hence $float = N\left(DF\left(T_0\right) - DF\left(T_n\right)\right)$ and for the fixed leg $fix = NK\sum_{i=1}^{n} DF\left(T_i\right)cvg\left(T_{i-1},T_i\right)$

annuity A
cvg = day count

- Hence for a par swap $K = \dfrac{DF\left(T_0\right) - DF\left(T_n\right)}{A\left(T_0,T_n\right)}$

Construction of the curve

- For a given set of n liquid standard swaps, assume some interpolation scheme on R
  - Set R1 to match the market value of swap 1
  - Then set R2 to match the market value of swap 2 with R1 constant
  - Etc.



- Then use the curve to value a portfolio of custom swaps and other products
- And compute rate risk by finite differences
  - Bump the value of the first swap
  - Recalibrate the curve
  - Reprice the book
  - Loop over swaps used for calibration

Instruments for curve construction

- Short term: 1d to 3m-6m
  - Deposits
  - Work like single period swaps
  - Valued with swap formula with one period $K = \dfrac{DF(T_1) - DF(T_2)}{cvg \cdot DF(T_2)}$

- Long term: 2y to up to 50y
  - Swaps

- Medium term: 3m-6m to 2y
  - Libor futures
  - If futures same as Libor Forwards or FRA (Forward Rate Agreements)
    - Work like single period forward starting swaps
    - Valued with the same formula
  - Problem: futures not the same as FRAs
  - We need to “turn” futures into FRAs first

Future and Forward rates

- The only exciting aspect for quants in the world of yiled curves in those days
  - “Convexity” bias links higher future rates to covariance between forward and discount rates
  - Active field of research in those days with may academic and professional publications

- Intuition for the convexity bias
  - I buy the future rate (= sell the future in market language)
  - Increase in rates gives positive margin flow **that I invest at a higher rate**
  - Decrease in rates gives negative margin **that I borrow at a lower rate**
  - So buying future rates is better than buying forward rates: no margin flows in that case
  - Hence, in rate terms, Fut > FRA

- Impact on the construction of curves
  - Market quotes Futures, curve construction takes FRAs
  - Need to “remove” convexity bias before construction

Future/Forward convexity

- FRA
  - Single cash flow on T_{pay} : $\left[L_{T_{fix}}(T_{start}, T_{end}) - FRA\right] \cdot cvg$
  - On par when $E^{RN}\left[\exp\left(-\int_0^{T_{pay}} r_s\, ds\right) FRA\right] = E^{RN}\left[\exp\left(-\int_0^{T_{pay}} r_s\, ds\right) L_{T_{fix}}\right]$
  - That is $E^{RN}\left[\exp\left(-\int_0^{T_{pay}} r_s\, ds\right)\right] FRA = E^{RN}\left[\exp\left(-\int_0^{T_{pay}} r_s\, ds\right) L_{T_{fix}}\right] E^{T_{pay}}\left[L_{T_{fix}}\right], FRA = E^{T_{pay}}\left[L_{T_{fix}}\right]$
  - Where $Q^{T_{pay}}$ is the martingale measure associated with the numeraire $DF(\square, T_{pay}) = E^{RN}\left[\exp\left(-\int_{\bullet}^{T_{pay}} r_s\, ds\right)\right]$
  - Also called Tpay-forward measure

- Future
  - Continuous margin flows $dFut_t$ at time *t*
  - We can buy/sell the future on par at any time *t*, so expected discounted future flows must be 0
  - $E_t^{RN}\left[\int_t^{t+\Delta t} \exp\left(-\int_t^u r_v^v\, dv\right) dFut_u\right] = 0$ so $E_t^{RN}\left[dFut_t\right] = 0$ and since $Fut_{T_{fix}} = L_{T_{fix}}$ finally $Fut = E^{RN}\left[L_{T_{fix}}\right]$

Convexity: change of numeraire formulas

- Radon-Nykodym derivative from forward-neutral to risk-neutral: $\dfrac{dQ^{RN}}{dQ^{T_{pay}}} = \dfrac{DF\left(0,T_{pay}\right)}{\exp\left(-\int_0^{T_{pay}} r_s^X\,ds\right)}$
- Future: $Fut = E^{RN}\left[L_{T_{fix}}\right] = E^{T_{pay}}\left[\dfrac{dQ^{RN}}{dQ^{T_{pay}}} L_{T_{fix}}\right] = E^{T_{pay}}\left[\dfrac{DF\left(0,T_{pay}\right)}{\exp\left(-\int_0^{T_{pay}} r_s^X\,ds\right)} L_{T_{fix}}\right]$
- We assume a normal diffusion on the *FRA*
- Then by Girsanov's theorem: $E^{RN}\left[L_{T_{fix}}\right] = E^{T_{pay}}\left[L_{T_{fix}}\right] - \int_0^{T_{fix}} \left\langle d\log DF\left(s,T_{pay}\right), dFRA_s\right\rangle$
- If we assume constant volatility for the FRA
  And constant volatility for all discount rates
  And constant correlation
  We finally get:

$$Fut = FRA + \rho\left(FRA, r\right)\cdot\sigma_N^F\cdot\sigma_N^r\cdot T_{fix}\cdot\left(T_{pay} - \frac{T_{fix}}{2}\right)$$

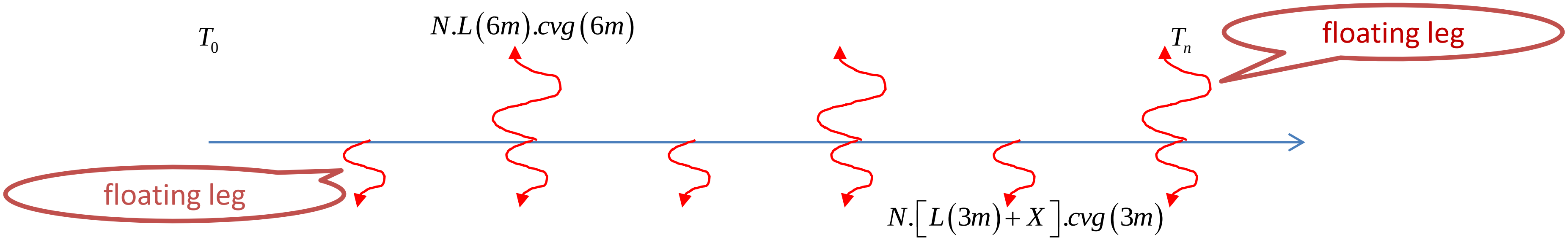
- Or lift simplifying assumptions and use of a multi-factor interest rate model

That’s how we did it in 1995

- A simple methodology
  - Lightweight: doable on Excel without code
  - Simple maths
  - Simple algorithmics: one-dimensional bootstrap (assuming “local” interpolation like linear)
  - Little quant involvement
- Based on one assumption: Libor is the interest rate
  - We can invest and borrow risk free at Libor
  - Libor represents interest rates and interest rates only
- That turns out to be wrong
  - Libor is an index of the Interbank Offered rate
  - Incorporates information other than interest rates
  - After crisis, Libor disconnected from interest rates, which are frozen near zero

Basis swaps

- Swap floating for floating on different frequencies, say 3m for 6m



  - Both legs are supposed to have same value : floating leg PV does not depend on frequency

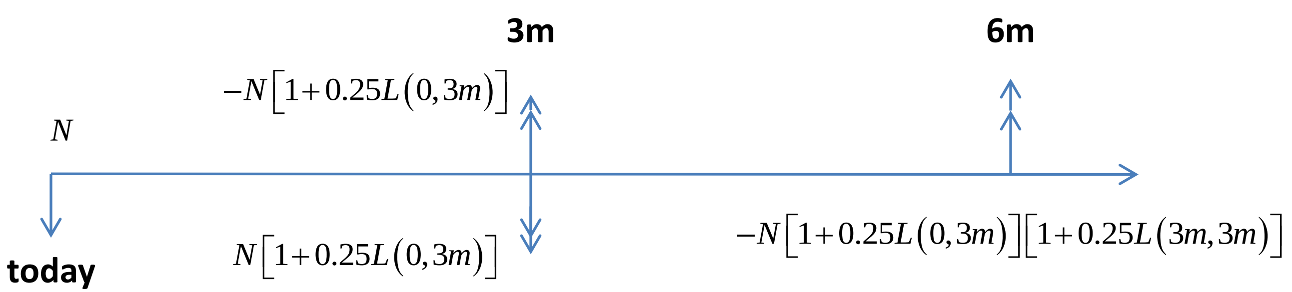
$$float = N\left(DF\left(T_0\right) - DF\left(T_n\right)\right)$$

  - Basis swap theoretically on par
- Yet, 6m systematically trades at significant premium: up to several tens of bp!
- Explanation: credit and liquidity for underlying Libor transactions
  - Transactions underlying to Libor are “real” transactions with exchange of capital
  - They are unsecured (although the basis swap is typically collateralized)
  - Hence they carry credit risk on capital and interest
    (swaps have no risk on capital and generally none on interest either)
  - Credit and liquidity premia on underlying Libor transactions explain the basis swap
- Reflect fact that Libor is **not** the risk-free rate

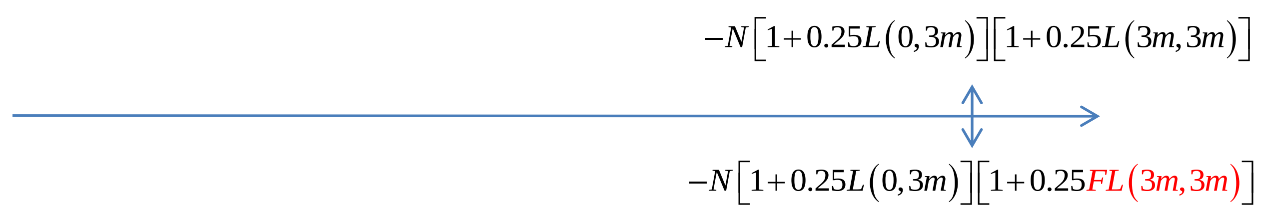
Intuition behind basis swaps

- Example: bank B borrowing at Libor, funding for 6m – 2 possibilities
  1. Borrow for 6m at Libor 6m

  2. Or borrow for 3m at Libor 3m, then borrow capital and interests for another 3m

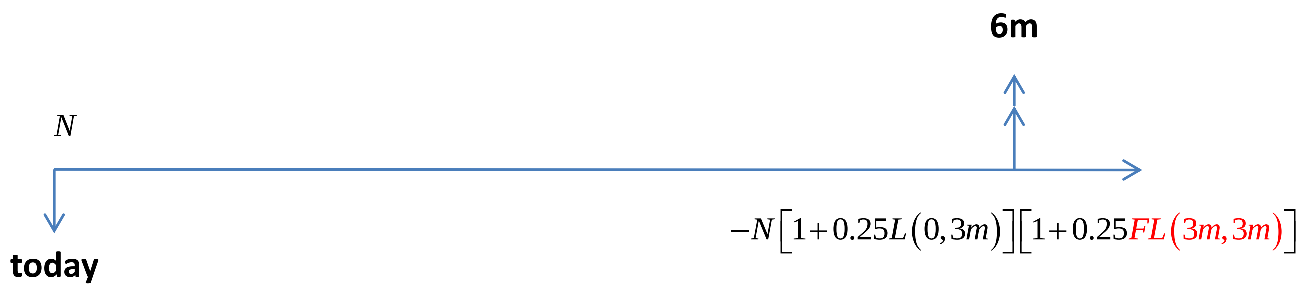


  – And lock the rate of the rollover loan with a 3m Libor 3m forward



Intuition behind basis swaps (2)

- Finally the cash flows in the "3m+3m3m" case resolve to



- Its effective 6m rate is $2\left\{\left[1+0.25L(0,3m)\right]\left[1+0.25FL(3m,3m)\right]-1\right\}$
  - According to school books, it must be same as straight 6m rate, therefore
    $$\left[1+0.25L(0,3m)\right]\left[1+0.25FL(3m,3m)\right]=\left[1+0.5L(0,6m)\right]$$
  - Or $FL(3m,3m)=4\left(\dfrac{1+0.5L(0,6m)}{1+0.25L(0,3m)}-1\right)$
  - Which is same as basis swaps being on par

Intuition behind basis swaps (3)

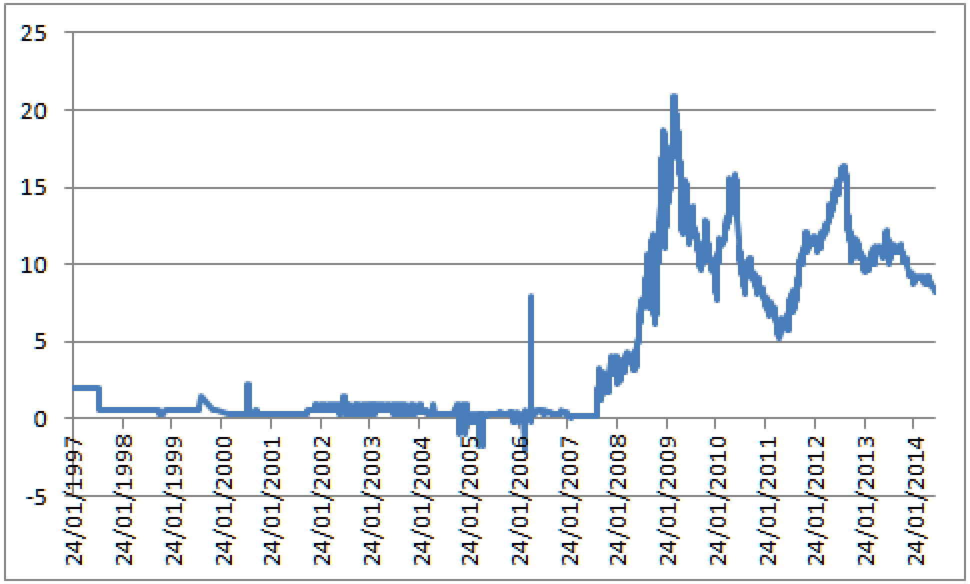
- Now suppose B is in financial difficulty in 3m
  1. In case B borrowed for 6m, it keeps funding for another 3m,
     lender can only hope that B survives another 3m to repay capital and interests
  2. But in case it borrowed for 3m with a forward 3m3m:
     - It may find it difficult to roll the loan for another 3m → liquidity risk
     - And/or pay large premium, say Libor3m+200, not Libor → credit risk
     - Forward Libor only protects against Libor fluctuations, not against own borrowing capacity or spread
  - In this case, B is in a better situation in case 1
  - Whereas the lender is very clearly in a better situation in case 2

- Hence 6m rate must be higher than "3m+3m3m"

$$L(0,6m) > 2\left\{\left[1+0.25L(0,3m)\right]\left[1+0.25FL(3m,3m)\right]-1\right\}$$

  - It is the contrary that would offer arbitrage opportunities
  - And this explains the basis swap as a credit/liquidity premium on underlying Libor transactions

Basis swaps

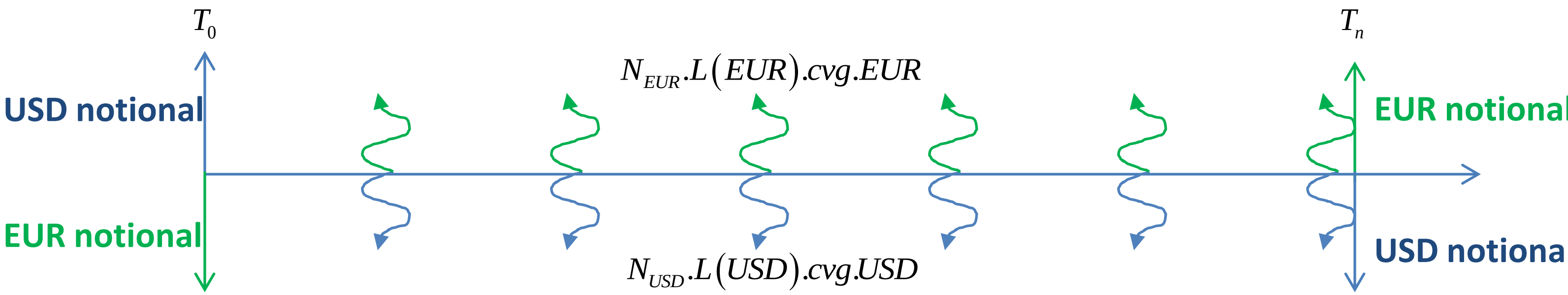


USD Libor 6m vs 3m 5y BS

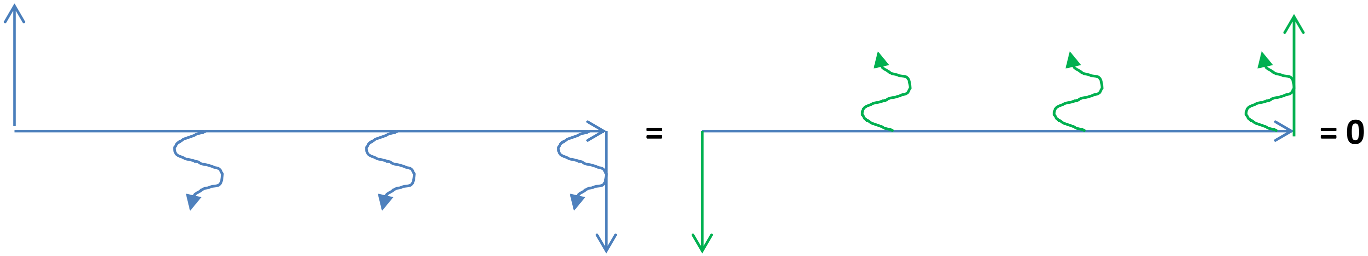
USD Libor 3m vs Fed Funds 5y BS

Currency basis swaps

- Swap floating for floating on different currencies, like USD vs EUR



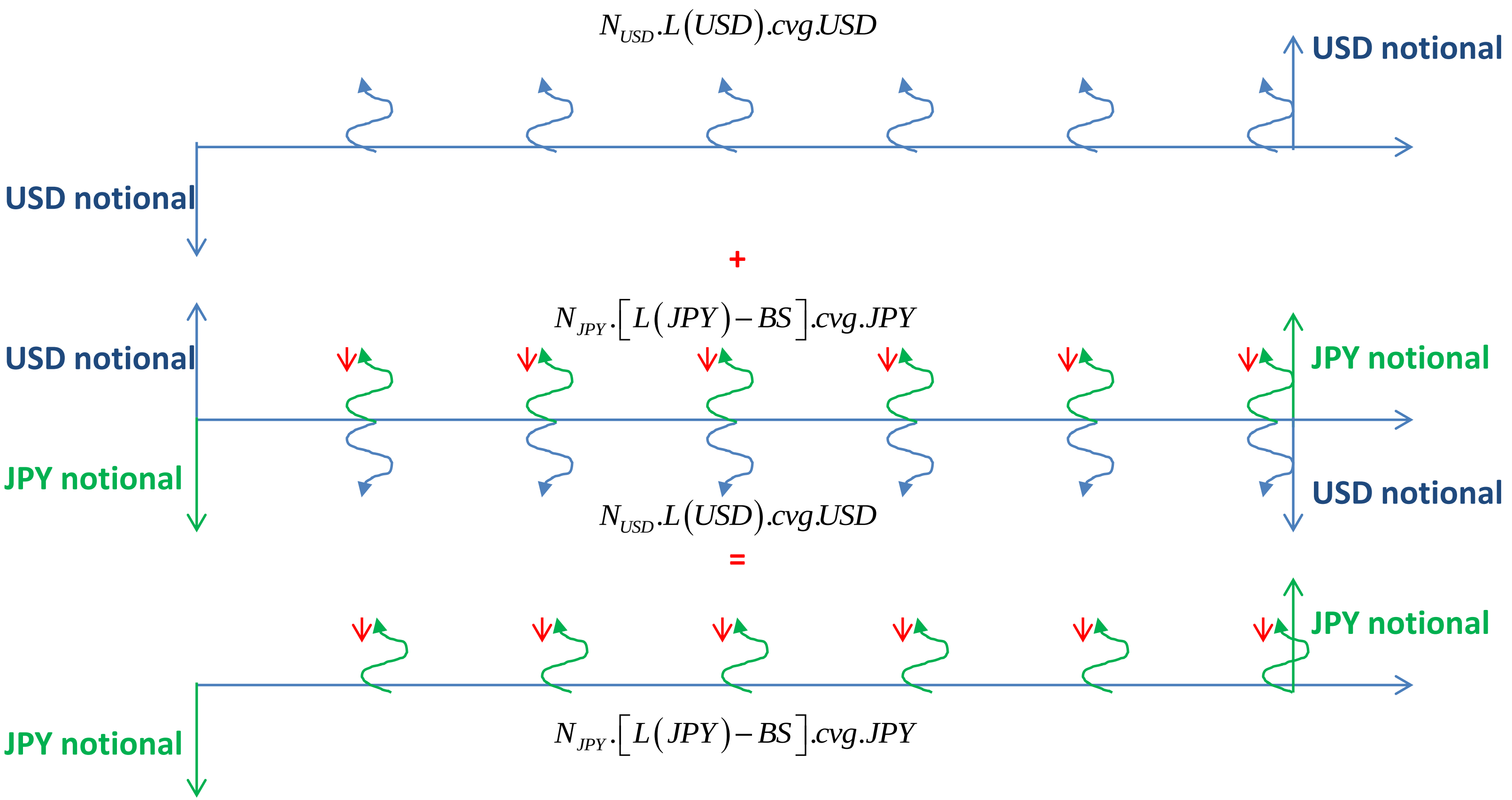
- In theory the swap is on par



- In practice EUR libor trades at significant premium over USD libor
  - European banks are seen as less creditworthy and pay credit premium over US banks
  - European banks also pay a premium to access USD liquidity

Synthetic cross currency transactions

- If a US bank borrows USD at USD Libor…
- …and adds-on a USDJPY basis swap…
- …then it ends up borrowing JPY at JPY Libor – BS

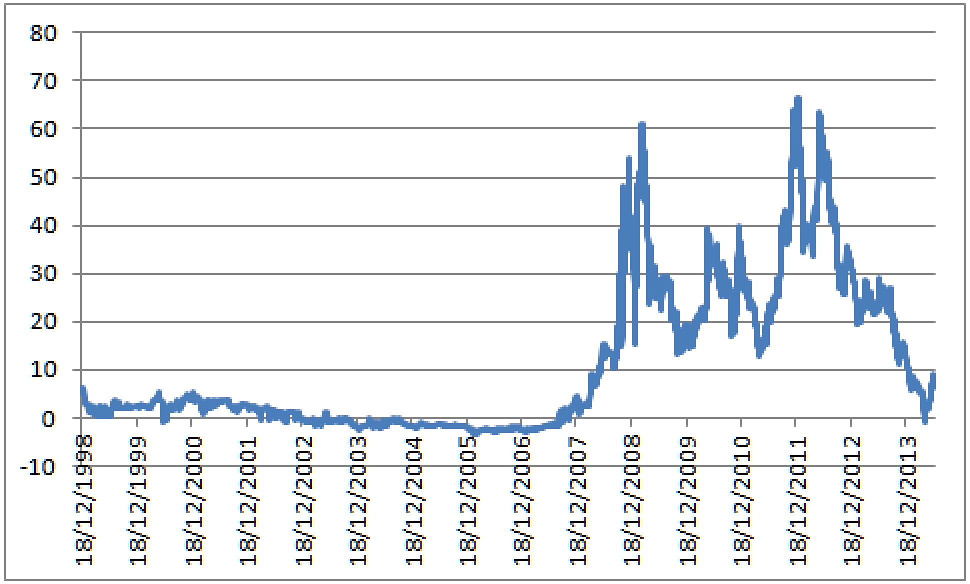


Intuition behind currency basis swaps

- European bank B with EUR Libor funding, needs USD funding – 2 possibilities
  1. Raise USD on markets
     - B has lower credit than US banks that borrow at USD Libor
     - B pays a premium over USD Libor
  2. Borrow EUR at EUR Libor and basis swap into USD
     - Must end up paying same rate or there is an arbitrage
     - Hence the xCCY basis swap

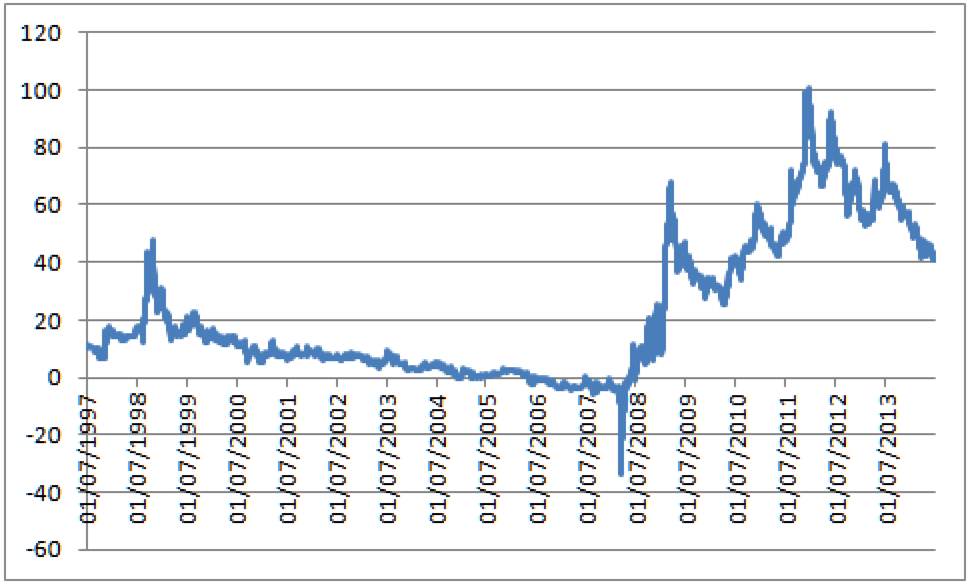
- Similarly US bank C with USD Libor funding , needs EUR funding, can
  1. Either raise EUR, paying less than EUR Libor for better credit
  2. Or borrow USD at USD Libor and basis swap into EUR
  – Must end up paying same rate, hence basis spread necessary to account for difference in credit
  – Note basis spread always reflects a relative difference between different pools of banks
     - If some US bank funds at Libor + 100 in USD
     - Then it funds at Libor + 100 – BS in EUR
     - I borrow in the rate of the currency, but on the basis of my credit

Currency basis swaps



EURUSD 5y xCCY BS

USDJPY 5y xCCY BS



The 1997 Asian crisis

- 1997 collapse of Thailand rose a credit crisis that propagated in Asia 1997-1999

- Consequently Asian banks’ credit deteriorated while liquidity premia raised
- Asian basis swaps raised from negligible to ~50bp

- We can no longer discount USD flows at USD Libor and JPY flows at USD JPY
  - This is inconsistent and actually arbitrageable
  - And note that with low JPY rates and high USDJPY basis US banks fund in JPY at JPY Libor – BS < 0 !!

- The banks who first understood this arbed those slower to react

- Eventually new standards emerged for rate transactions

The post 1997 rate machinery

- The industry agreed to use USD Libor 3m as reference for discounting
  - Dominated by US and European banks with EURUSD BS insignificant at the time
  - USD Libor seen as risk-free reference because “US banks can’t fail”
  - Economically, Japanese houses should really discount at JPY Libor but we cut some corners to avoid operational nightmare of counterparties to a transaction having different valuations
  - A somewhat fishy machinery, did the job until the global financial crisis

- Means we do “as if” everybody’s borrows and invests at Libor 3m in USD

- And with xCCY BS add-on we all fund in other ccies at “CCY Libor – xUSD BS”

- Refers to a theoretical discount curve for the currency called “cash curve”
  = CCY Rate curve in the USD Libor 3m basis

The post 1997 rate machinery (2)

- Libor swaps no longer reference for discount (except vs 3m in the US), instead we used a "cash curve" per currency = "Libor – BS"
- Floating leg pv no longer $N\left(DF(T_0)-DF(T_n)\right)$ but $N\sum_{i=1}^{n} DF(T_i)FL(T_{i-1},T_i)cvg(T_{i-1},T_i)$
  - Where the forward libors FL are read on a "rate surface" by maturity and Libor frequency
  - And DFs are computed on the cash curve
- Construction (and risk management) of curves (now surfaces) much more involved
  - Manipulates large amounts of market data
  - Construct curves by fitting swaps, BS and xCCY BS together by setting discounts and forward Libors
  - Risk less evident to compute and comprehend: what do we freeze when we bump swap rates?
    - Discount basis?
    - Or final discount rates?
  - No longer takes 30min on a spreadsheet, heavy machinery in C++ with thousands of lines of code
- Not really complicated but very heavy
  - Had to be centralized and dedicated quant teams were put in place
  - An so we invented the "linear quant"

And then we had the global crisis

- And it all went bananas
  - Rates stick to 0 while credit and liquidity premia went nuts
  - Libor (even USD) absolutely disconnected from anything to do with interest rates

- So we needed to rethink the whole discounting machinery from the ground
  - Acknowledge disconnection between Libor and Rate
  - Methodology should work under assumption that Libor is a random number
    - Although a published one, with swaps trading on it

- Some IBs were quicker to react and arbed the street for billions

- Subsequently, IBs moved quants from shrinking exotic businesses into linear rates
- And the quant community came out with new standards for discounting

- Best practice today is theoretically clean and incorporates lessons from the crisis
- It is also somehow complex and keeps thousands of quants busy on a daily basis

2 Current practice

Current practice for discounting

- 2 types of transactions: collateralized and uncollateralized

- Collateralized transactions
  - Discount flows at **collateral rates**
  - For floating legs, read forward rates on rate surfaces
  - For cash-flows in currencies other than collateral's (domestic) currency
    "Convert" flows in domestic currency using a forward Fx curve built from Fx and xCCY swaps
  - Advanced:
    - **Collateral convexity adjustment** when a transaction has different collateral to collateral for curve construction
    - **Collateral option** and "cheapest to deliver" when CSA allows to change collateral

- Uncollateralized transactions
  - Makes no sense to discount flows at any kind of "risk free" rate since we know counterparties fail
  - Account for potential loss subsequent to counterparty default ➔ CVA
  - Possible even account for own credit and funding ➔ DVA, FVA, … ➔ xVA

2.1 Collateralized transactions

Collateral discounting: example 1

- Transaction = purchase option
- Without collateral
  - Must pay premium
  - Cost of funding premium: some funding rate *r*
  - Discount payoff at rate *r*
- With collateral
  - Premium reverts immediately as collateral
  - No need to fund the premium
  - But must compensate collateral at *collateral rate c*
  - Hence, discount payoff at rate ***c***

$$rV = \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + V_t$$

net funding cost

$$cV = \mu S V_S + \frac{1}{2}\sigma^2 S^2 V_{SS} + V_t$$

Collateral discounting: example 2

- Transaction = swap with negative PV
- Without collateral
  - Negative PV = borrowing money
  - I expect to pay negative coupons in the future
  - In the meantime, I invest the money at some funding rate *r*
  - Discount expected cash flows at rate *r*
- With collateral
  - Pay the negative PV straight away as collateral
  - My collateral is compensated at *collateral rate c*
  - Hence, discount expected cash flows at rate ***c***

Collateral discounting: maths

- Cash flow CF delivered at future time *T*, discounted at some funding rate *r*

$$V_T = CF, V_t = E_t\left[\exp\left(-\int_t^T r_s ds\right)V_T\right]\ldots$$

- Add flows linked to collateral
  - Collateral posting of C=-V is not a cash flow
  - The cash-flow is the cost *C.r.dt* of funding collateral net of its compensation *C.c.dt*

$$V_t = E_t\left\{\exp\left(-\int_t^T r_s ds\right)V_T - \int_t^T \exp\left(-\int_t^s r_u du\right)V_s\left(c_s - r_s\right)ds\right\}, t_0 \le t, E_{t_0}\left[V_t\right] = E_{t_0}\left\{\exp\left(-\int_t^T r_s ds\right)V_T - \int_t^T \exp\left(-\int_t^s r_u du\right)V_s\left(c_s - r_s\right)ds\right\}$$

  - Differentiate $E_{t_0}\left[dV_t\right] = E_{t_0}\left\{r_t \exp\left(-\int_t^T r_s ds\right)V_T - \int_t^T r_t \exp\left(-\int_t^s r_u du\right)V_s\left(c_s - r_s\right)ds + V_t\left(c_t - r_t\right)ds\right\} = E_{t_0}\left[c_t V_t\right]$
  - So $E_{t_0}\left[V_t\right] = E_{t_0}\left[\exp\left(-\int_t^T c_s ds\right)V_T\right], t_0 \to t, V_t = E_t\left[\exp\left(-\int_t^T c_s ds\right)V_T\right] = \left[\exp\left(-\int_t^T c_s ds\right)CF\right]$

- The funding rate r disappears from the equation, discount at collateral rate c
- Note that collateral changes discount rate but not probability measure

Valuation of collateralized trades: IRS

- OIS
  - OIS = Overnight Indexed Swap
  - Trades compounded 1d rate Vs Libor
  - EONIA in EUR, Fed Funds in the US, …
  - Close to a risk free rate due to very short maturity
- Example / norm for IB swaps: cash collateral in the currency of the swap, earns OIS
  - Project floating cash flows from the forward curve surface
  - Discount cash flows on the OIS curve
  - Discount curve: “Disc = OIS = Libor – LIBOR/OIS BS”
  - Build discount and forward curves together to fit market quotes for Libor swaps and OIS basis swaps

Valuation of collateralized trades: IRS (2)

- Example 2: cash collateral (earning OIS) in a different currency
  - Say: BNP and SocGen trade a USD swap with EUR cash collateral
  - Project floating cash flows from the USD forward curve surface
  - Discount cash-flows with USD rates but on the EUR OIS basis
  - "Disc = USD Libor + EURUSD Libor BS - EUR Libor/OIS BS"
  - Intuition:
    - EUR collateral creates a context for the transaction where EUR funding = EUR OIS
    - So USD funding = EUROIS basis swapped into USD
    - Hence, USD CF are discounted on the EUR OIS curve basis swapped into USD
    - No direct BS from EUROIS into USD ➔ we combine EUR Libor/OIS and EURUSD Libor BS

Valuation of collateralized trades equity and forex forwards

- European equity option, OTC, fully collateralized
  - Depends on rates through discounting and forward
  - Textbook forward (without dividends): $F(t,T) = \frac{S_t}{DF(t,T)}$
  - Depends on discount factor ➔ forward changes with collateral?
  - No: equity is compounded at **repo** rate
    Repo = funding rate when posting equity as collateral
    Therefore depends on the equity but not on derivative transaction
- What about Fx?
  - Fx forward: $F(t,T) = S_t \frac{DF_{for}(t,T)}{DF_{dom}(t,T)} = S_t \exp\left\{\left[R_d(t,T) - R_f(t,T)\right](T-t)\right\}$
  - Changes when collateral switched from domestic cash to foreign cash?
  - No: change of basis shifts both rates by the same amount, hence rate difference remains constant
    In other terms, each DF changes but not the ratio

Valuation of collateralized trades equity and forex forwards

- FRA and Libor futures

  - With discount rate *r*, FRA solves in *X*: $E^{RN}\left\{\exp\left[-\int_0^{T_2} r_s ds\right]\left[Lib(T_1,T_2)-X\right]\right\}=0$
  - In the context where the curve was built, *r* is OIS of the currency of the Libor
  - If we change collateral to *r2=r+b*, then FRA now solves

    $$E\left\{\exp\left[-\int_0^{T_2}(r_s+b_s)ds\right]\left[Lib(T_1,T_2)-X_b\right]\right\}=0$$

  - So if the basis *b* between *r* and *r2* is independent from the FRA (as a particular case, deterministic), the FRA remains unchanged when collateral changes,
    otherwise it is subject to a convexity bias

- Swaps and forward swaps

  - Forward swap = average of FRAs weighted by DFs: $F=\frac{\sum_{i=1}^{n} DF(T_i)FL(T_{i-1},T_i)cvg(T_{i-1},T_i)}{Annuity}$
  - Depends on discount factors, hence changes with collateral
  - Lower collateral rate pushes weights to the back of the curve, generally increase par swap rate
  - USD swaps are lower with EUR collateral

Multi-Curve Linear markets: domestic case

- Multi-Curve Linear market = collection of curves of forward rates, one for each floating rate reference: OIS, Libor 3m, 6m, 12m, etc.

- Each curve implements some interpolation scheme provides forward rates for all maturities
  - Forward OIS rates can be read on the OIS curve for all future dates
  - Forward Libor 3m rates can be read on the Libor 3m curve
  - In case something like the 4m Libor is needed, interpolated from the Libor 3m and Libor 6m curves

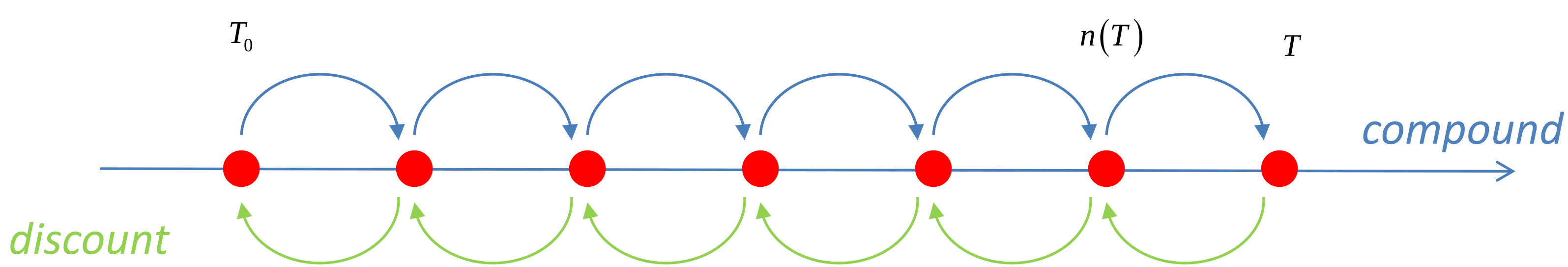
- Forward rates do not depend on collateral **if we ignore convexity adjustments**

Multi-Curve Linear Markets (2)

- For a transaction with cash collateral in the domestic currency, the curve that corresponds to collateral rate ➔ in general OIS is used for discounting cash flows

$$DF_X(T) = \frac{1}{\prod_{i=0}^{n(T)} \left[1 + F_X(T_i, T_{i+1}) cvg(T_i, T_{i+1})\right]} \frac{1}{\left[1 + F_X\left(T_{n(T)}, T\right) cvg\left(T_{n(T)}, T\right)\right]}$$

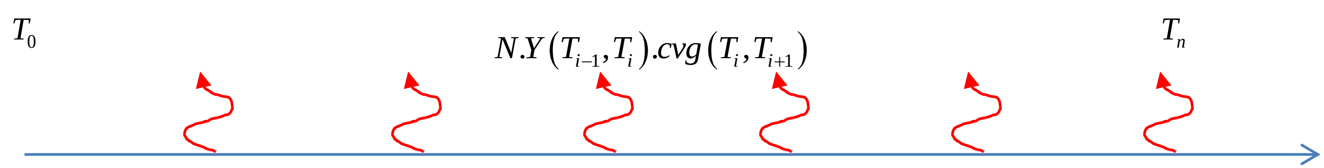
- Where
  - X is the reference curve: OIS, Libor 3m, 6m, 12m, …
  - (T_i) is the timeline for the floating frequency of X
    - For OIS, every business day from today
    - For Libor 3m, every 3m
    - Etc
  - $n(T) = \max\{i, T_i < T\}$
  - *F* is the forward rate X picked on the curve by maturity
  - *cvg* is the conventional day count for the floating rate X



Valuation with a MCL market

- Discounting with cash collateral with collateral rate reference *X*
- Fixed cash flow *CF* paid at date *T* $PV = DF_X(T)CF$
- Floating cash flow with floating rate *Y* for period (T_1, T_2) $PV = DF_X(T_2)F_Y(T_1, T_2)cvg(T_1, T_2)$
  we pick the forward rate on the *Y* curve (e.g. Libor3m) and discount it on the *X* curve (in general, OIS)
- Fixed swap leg

$$PV = NK\sum_{i=1}^{n} DF_X(T_i)cvg(T_{i-1}, T_i) = NK \cdot annuity_X$$



- Floating swap leg
  Ref Y (e.g. Libor 3m)

$$PV = N\sum_{i=1}^{n} DF_X(T_i)F_Y(T_{i-1}, T_i)cvg(T_{i-1}, T_i)$$

Building a MCL market

- Build MCL market = set forward rates for all references
  so as to zero the PV of selected par swaps and basis swaps

- Swaps are cash collateralized with collateral rate = OIS in the currency

- First build OIS and main Libor curve (3m for USD, 6m for EUR, …)
  so as to fit Libor swaps and OIS/Libor basis swaps together

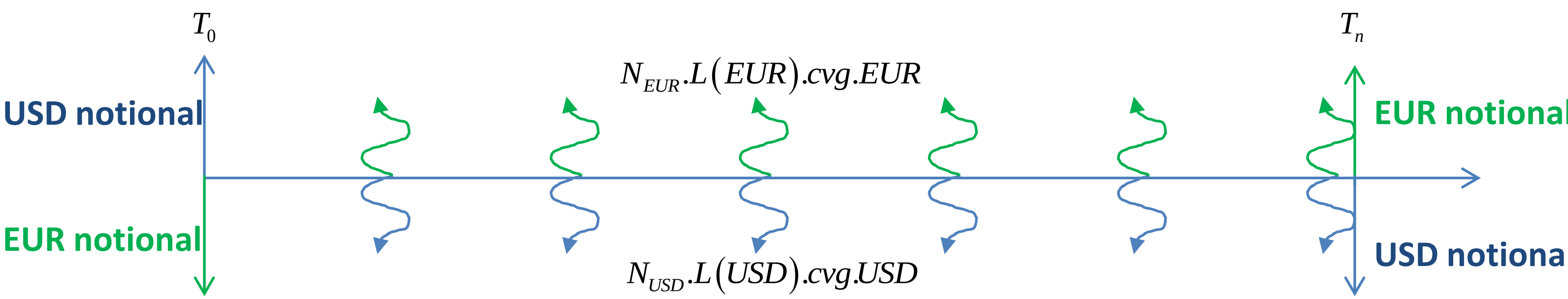
  - Bi-dimensional bootstrap
  - In case of a smooth interpolation, bootstrap cannot be used and global fit is necessary

  - Build short term out of spot Libor and OIS, OIS depos and OIS swaps, and Libor futures
  - Incorporate Central Bank meeting dates and end of year effects in interpolation

- Other curves *Y* are fitted to Libor/*Y* basis swaps
  by setting forward *Y*s
  with DFs and forward Libors known at this stage

Cross Currency MCL markets

- MCL markets cannot value cash flows in another currency
- xCCY BS CCY1 vs CCY2 (assuming both MCL have been constructed)



- Assume USD Fed Funds collateral
- We know how to PV cash-flows on the USD leg
- But for the EUR leg, we can PV only with EUR collateral
  For PV under USD collateral, we need another curve
  ➔ Discount curve for EUR cash flows under USD collateral DF_{USD}^{EUR}

Forward Fx Curves

- Forward Fx (under collateral *X*): $F(T) = S\,\dfrac{DF_X^{for}(T)}{DF_X^{dom}(T)}$

- We saw that *F* is independent of *X*
  since *X* affects identically the numerator and denominator

- So $F(T) = S\,\dfrac{DF_{dom}^{for}(T)}{DF_{dom}^{dom}(T)}$

- Since *S* and $DF_{dom}^{dom}(T)$ are known from domestic markets,
  The curve of forward Fx and the curve of DF_{dom}^{for} carry the same information

- It is standard practice to store forward Fx curves in xCCY MCL markets
  More precisely we usually store forward Fx *factors* = forward Fx / spot Fx

Valuation with a xCCY market

- Discount cash flows in the currency of the collateral same as domestic case

- Discount cash flows in a foreign currency:
  - "Convert" in the domestic currency with the forward fx for payment date
  - Discount as a domestic cash flow on the collateral rate curve

- Fixed cash flow paid at date T in foreign currency $PV_{dom} = DF_{dom}(T) FFX(T) CF_{for}$

- Floating foreign cash flow with floating rate *Y* for period (T_1, T_2)

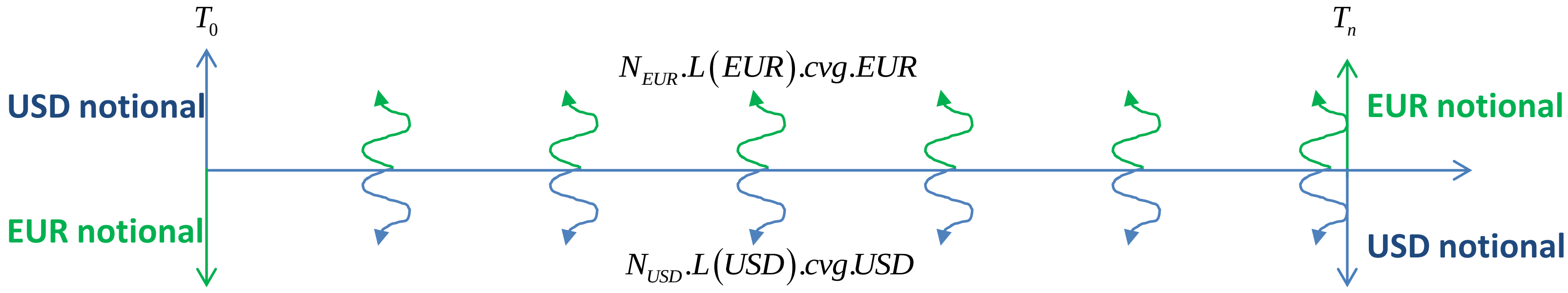
$$PV_{dom} = DF_{dom}(T_2) FFX(T_2) F_Y(T_1, T_2) cvg(T_1, T_2)$$

we pick the forward rate on the foreign *Y* curve (e.g. some Libor)
we convert in domestic currency by applying the forward fx read on the xCCY curve for the payment date
we discount on the domestic *X* curve (in general, OIS)

Building a xCCY market

- Build xCCY market = set forward Fx curve
- As usual, apply interpolation scheme to provide fFx factors of all maturities

- Set the fFx curve so as to zero the PV of market traded par xCCY BS



- After domestic MCL have been constructed, USD leg has fixed value
- Apply $PV_{dom} = DF_{dom}(T_2)FFX(T_2)F_Y(T_1,T_2)cvg(T_1,T_2)$ to value EUR leg
- Set $FFX(.)$ so that the EUR leg matches the USD leg
- Repeat for all quoted maturities in xCCY swaps

Comparing 1997 and current setups

- 1997 setup = particular case of current
  - With implicit assumption = all transactions have USD collateral with collateral rate = USD Libor 3m
  - Which is never the case

- But prior to the crisis, USD OIS/Libor and EURUSD xCCY BS too low to really matter
  - So the assumption was “almost” right
  - In any case, spreads were too low to provide arbitrage opportunities
  - And banks were reluctant to undertake the massive developments required for the current setup

Collateral convexity: intuition

- I buy a forward F under some CSA X
  - Full collateral X
  - Collateral compensated at some rate x
- Then we change to collateral Y
  - Compensated at rate $y=x+b$ where b is the basis spread between X and Y
  - Suppose b is positively correlated to F
  - When F raises I receive collateral that I must compensate at a higher than expected rate
  - When F drops I post collateral that earns a lower than expected rate
  - So I am worst off with collateral Y
  - Hence, under the new CSA, my forward is lower

Collateral convexity: maths

- Consider a floating cash-flow referenced by some index *L* under collateral *X*
  - We have $PV = E^{RN}\left[\exp\left(-\int_0^{T_{pay}} r_s^X\, ds\right) L_{T_{fix}}\right] = DF^X\left(0,T_{pay}\right) E^{DF^{X(.,T_{pay})}}\left[L_{T_{fix}}\right]$
  - Where $DF^X\left(t,T\right) = E_t^{RN}\left[\exp\left(-\int_t^T r_s^X\, ds\right)\right] = \exp\left(-\int_t^T f^X\left(t,u\right) du\right)$
- When we construct a market under collateral *X*
  - We extract forwards $F_L^X\left(0,T_{fix},T_{pay}\right) = E^{DF^{X(.,T_{pay})}}\left[L_{T_{fix}}\right]$
  - They are not risk-neutral expectations but “forward-*X*” neutral expectations
- But when we value this cash-flow under collateral *Y*:

$$PV = E^{RN}\left[\exp\left(-\int_0^{T_{pay}} r_s^Y\, ds\right) L_{T_{fix}}\right] = DF^Y\left(0,T_{pay}\right) E^{DF^{Y(.,T_{pay})}}\left[L_{T_{fix}}\right]$$

still the
risk-neutral measure

different discount

different forward

Collateral convexity: change of numeraire

- Change of collateral from *X* to *Y*: forward changes from $E^{DF^{X(.,T_{pay})}}\left[L_{T_{fix}}\right]$ to $E^{DF^{Y(.,T_{pay})}}\left[L_{T_{fix}}\right]$
  - Radon-Nykodym $\frac{dQ^{DF^{Y(.,T_{pay})}}}{dQ^{DF^{X(.,T_{pay})}}} = \frac{\exp\left[-\int_0^{T_{pay}}\left(r_s^Y - r_s^X\right)ds\right]}{DF^Y\left(0,T_{pay}\right)/DF^X\left(0,T_{pay}\right)} = \frac{\exp\left[-\int_0^{T_{pay}} b_s^{Y-X}\,ds\right]}{E_t^{DF^{X(.,T_{pay})}}\left\{\exp\left[-\int_0^{T_{pay}} b_s^{Y-X}\,ds\right]\right\}}$
  - *b=ry-rx*: basis spread

- We assume a normal diffusion for *F, t*hen by Girsanov's theorem:

$$E^{DF^{Y(.,T_{pay})}}\left[L_{T_{fix}}\right] = E^{DF^{X(.,T_{pay})}}\left[L_{T_{fix}}\right] + \int_0^{T_{fix}}\left\langle d\log\frac{DF^Y\left(s,T_{pay}\right)}{DF^X\left(s,T_{pay}\right)}, dF^X{}_s\right\rangle = F^X - \int_0^{T_{fix}}\left\langle dB^{Y-X}\left(s,T_{pay}\right)\left(T_{pay}-s\right), dF^X{}_s\right\rangle, B^{Y-X}(t,T) \equiv R^Y(t,T) - R^X(t,T)$$

- If we also assume constant (normal) volatility for *F* and all bases, and correlation:

$$F^Y = F^X - \rho(F,B)\cdot\sigma_N^F\cdot\sigma_N^B\cdot T_{fix}\left(T_{pay} - \frac{T_{fix}}{2}\right)$$

- For 10Y3m Libor with nvol = 0.40%, basis nvol = 0.20% and correl = 25% we get ~1bp

- The formula is in T^2 so for a 20Y we get ~4bp

Collateral option

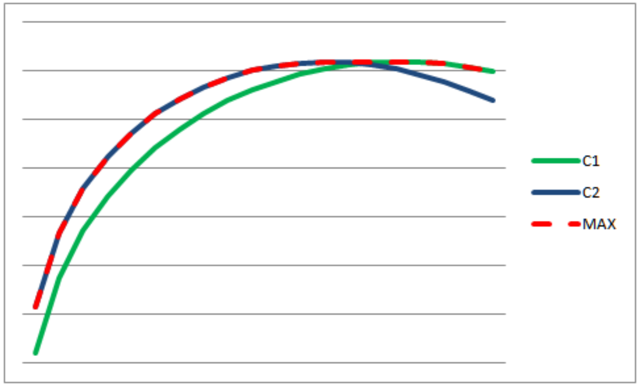
- Some CSAs allow to post collateral in a choice of currencies and/or assets
  - Example: USD swap with choice of USD or EUR collateral

  - Giving an option that belongs to the party posting collateral
  - That party has negative PV
  - It will want to apply maximum discount
  - By posting higher yielding (= cheapest to deliver) collateral

  - So it is always the highest rate **= highest basis** that prevails
  - Right now it is best posting EURs than USDs
  - But according to forward currency basis curves, will reverse in the future

Collateral option (2)

- Value of the collateral option: general case
  - $PV = E^{RN}\left\{\exp\left[-\int_0^T \left(c_s^{\,1} \vee c_s^{\,2}\right) ds\right] CF\right\}$
  - Note $PV = E^{RN}\left\{\exp\left[-\int_0^T c_s^{\,1} ds\right] CF\right\} - E^{RN}\left\{\int_0^T \exp\left[-\int_0^s c_u^{\,1} du\right] PV_s \left[\left(c_s^{\,1} \vee c_s^{\,2}\right) - c_s^{\,1}\right] ds\right\}$
  - Where LHS is the base PV discounted with collateral 1 and RHS is the adjustment for collateral option
  - Note $\left(c_s^{\,1} \wedge c_s^{\,2}\right) - c_s^{\,1} = \left(c_s^{\,2} - c_s^{\,1}\right)^+ = \left[b_s^{\,12}\right]^+$ is the basis from collateral 1 to collateral 2
  - And finally the collateral option is worth $E^{RN}\left\{\int_0^T \exp\left[-\int_0^s c_u^{\,1} du\right] PV_s \left[b_s^{\,12}\right]^+ ds\right\}$
- Value of the collateral option: deterministic case
  - $PV = DF(T)\,E[CF]$
  - With $DF(T) = \exp\left\{-\int_0^T \left[f^1(0,u) \vee f^2(0,u)\right] du\right\}$

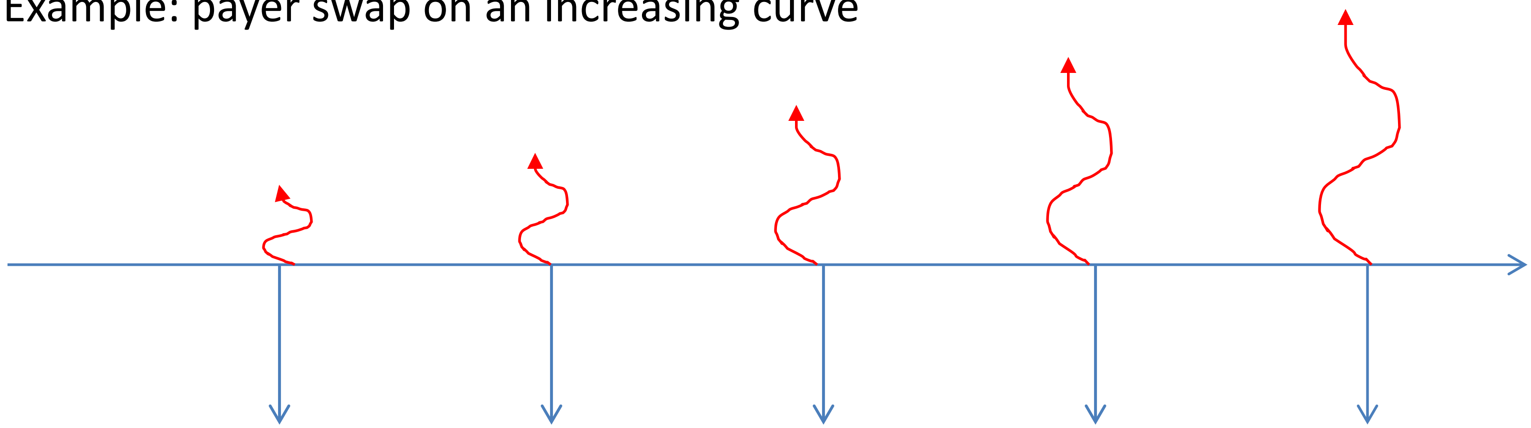


2.2 Uncollateralized transactions

Counterparty Value Adjustments

- Uncollateralized transaction
  - Can produce loss if counterparty defaults while PV is positive
  - But does not produce a gain if default of negative PV

- Example: payer swap on an increasing curve



  - Negative cash-flows followed by positive expected (actually forward) cash flows
  - Synthetic loan to counterparty
  - With loss arising from default

CVA

- On default, PV is netted across transactions with a counterparty or "netting set"
- When we trade without collateral we give away
  a put on the netting set contingent to default
- This is a "real option" with value

$$CVA = E^{RN}\left\{\sum_{i=1}^{n} DF(T_i)(1-RR)\left[S(T_{i-1})-S(T_i)\right]\max\left(0, PV(T_i)\right)\right\}$$

CDS implied probability to default between Ti-1 and Ti

  - DF: here means $DF(T) \equiv \exp\left[-\int_0^T r_s ds\right]$
  - RR: recovery rate
  - S: survival probability
  - E: risk-neutral expectation
- Option pricing, not estimation
- Model parameters must be calibrated
  - For instance, *S* must be implied from CDS whenever possible
  - Because CVA is an option that is meant to be hedged
  - For instance, the loss from the payer swap can be avoided by buying CDS on the counterparty
  - CVA = cost of the dynamic hedge, computed as a RN expectation as per Feynman-Kac theorem

Uncollateralized transactions

- Following Lehman, we can no longer discount cash flows at some “risk free” rate
  - Must incorporate CVA
  - CVA is computed per netting set, not per transaction
- Current practice:
  - Value each transaction as risk-free: base value
  - Compute CVA on the netting set
  - Allocate CVA charge across transactions: incremental or marginal method

CVA allocation: incremental method

- Idea: allocate to each new trade the additional CVA due to trade compared to former CVA without the trade
- Denote CVA_n the CVA for some netting set with the first *n* trades in the set (in chronological order)
- Incremental CVA for trade *n*: $CVA_n - CVA_{n-1}$ may be positive or negative
- Benefit: the contribution for each trade is charged in accordance to change in CVA and then remains constant
- Drawback: allocation depends on the order of the trades
  The same transaction is charged very differently depending on booking time

CVA allocation: marginal method

- Marginal CVA for some trade =
  notional times change in CVA for a marginal increase in notional

- We remind that $CVA = E^{RN}\left\{\sum_{i=1}^{n} DF(T_i)(1-RR)\left[S(T_{i-1})-S(T_i)\right]\left[PV(T_i)\right]^{+}\right\}$

- And $PV = \sum_{j=1}^{m} N_j V_j$ with *m* trades in the NS, with notionals *N* and unit values *V*

- So the marginal CVA for trade *j* is $N_j \dfrac{\partial CVA}{\partial N_j} = E^{RN}\left\{\sum_{i=1}^{n} DF(T_i)(1-RR)\left[S(T_{i-1})-S(T_i)\right] 1_{\{PV(T_i)>0\}} N_j V_j(T_i)\right\}$

- And immediately (also as a consequence of Euler's theorem)
  we see that the CVA is the sum of marginal CVAs $CVA = \sum_{j=1}^{m} N_j \dfrac{\partial CVA}{\partial N_j}$

- Benefit: intellectually sound, easy and fast, charge does not depend on order

- Drawback: all charges change retroactively when new trades are introduced!

Calculation of CVA

- Challenges…
  - The most exotic option ever: credit contingent put on a whole netting set
  - Netting sets may have options an even exotics
  - Model dimensionality
  - Calculation time
  - Risk sensitivities

- …such that many IBs need to recourse to crude approximations…

  Like represent a exotics as a vanillas
  for instance Bermuda as simple swap of same expected duration
  or analytically valued European swaption

- …and/or compute on large racks of servers

Efficient calculation of CVA

- We have been working on efficient implementation for CVA for the last couple of years

- And published results in our series of talks "CVA on iPad mini"
  - Used and adapted our experience with exotics and hybrids
  - Used Longstaff-Schwartz regressions in a particular way
  - Incorporated new algorithms such as parallel computing, AAD and branching

- Short version Global Derivatives 2014, Amsterdam

- Long version Aarhus 2014

- Email cvaCentral@aSavine.com for slides

What model for CVA?

- Challenge: model all variables impacting the whole netting set
  - Rate curves in all currencies in the NS
  - Fx for all currencies involved
  - All equity, credit, inflation, etc. indices referred to in the transactions
  - All at once, consistent and calibrated
- Rate models must be multi-factor
  - Otherwise all rates within currency are perfectly correlated
  - And netting effect between reverse transaction with non-matching maturities is overestimated
  - Choice between Markov (muti-factor Cheyette) and non-Markov (BGM) models
  - Models calibrated to todays prices, curves and volatility surfaces
  - Best practice requires stochastic volatility, especially for NS with exotics
- Quants reuse models developed for exotics…
  - CMS spread steepeners: MF rate models with SV
  - Callable Power Duals / chooser TARNs: Fx with LV/SV coupled with rate models
- …But ultra high dimensionality is something new
  - Never in exotics did we need to deal with such a number of variables
  - Even chooser TARNs (typically up to 5 currencies) implemented single factor IR on each ccy

CVA as stochastic discounting

- CVA formula under numeraire N: $CVA = N_0 E^N \left\{ \sum_{i=1}^{n} \frac{1}{N_{T_i}} (1-RR) \left[S(T_{i-1}) - S(T_i) \right] \max\left(0, PV(T_i)\right) \right\}$
- Knowing that: $PV(T_i) = N_{T_i} E_{T_i}{}^N \left[\sum_{j=i}^{n} \frac{CF(T_j)}{N_{T_j}} \right]$
- We rewrite as: $CVA = N_0 E_0{}^N \left\{ \sum_{i=1}^{n} (1-RR) \left[S(T_{i-1}) - S(T_i) \right] 1_{\{PV(T_i)>0\}} \sum_{j=i}^{n} \frac{CF(T_j)}{N_{T_j}} \right\}$
- At this point it is clear that cash-flow evaluation is quadratic
- But if we use Longstaff-Schwartz proxies for the indicators:

$$CVA = N_0 E_0{}^N \left\{ \sum_{i=1}^{n} (1-RR) \left[S(T_{i-1}) - S(T_i) \right] 1_{\{\tilde{V}_i>0\}} \sum_{j=i}^{n} \frac{CF(T_j)}{N_{T_j}} \right\}$$

- And reverse the order of the sums:

$$CVA = N_0 E_0{}^N \left\{ \sum_{j=1}^{n} \left[\sum_{i=1}^{j} (1-RR) \left[S(T_{i-1}) - S(T_i) \right] 1_{\{\tilde{V}_i>0\}} \right] \frac{CF(T_j)}{N_{T_j}} \right\}$$

CVA reformulation (2)

- Then we can write: $$CVA = E_0^{\,N}\left\{\sum_{j=1}^{n}\frac{\eta_{T_j}}{N_{T_j}}CF\left(T_j\right)\right\}$$

- Where: $$\eta_{T_j} = \sum_{i=1}^{j}(1-RR)\left[S\left(T_{i-1}\right)-S\left(T_i\right)\right]1_{\left\{\tilde{V}_i>0\right\}}$$

- And so: $$\eta_{T_j} = \eta_{T_{j-1}} + (1-RR)\left[S\left(T_{j-1}\right)-S\left(T_j\right)\right]1_{\left\{\tilde{V}_j>0\right\}}$$

- Computing a CVA means discounting cash-flows with an exotic stochastic notional

- The cash-flows are evaluated once, the exotic discount is calculated along the path

- We use proxies for the indicators,
  the cash flows themselves are calculated without approximation

Efficient calculations of CVA (2)

- Recipe = “iPad mini” summary

- Prepare the infrastructure
  - Use scripting for representation of cash-flows
  - Compress trades in a netting set into a “super swap”
  - Use LSM pre-simulations to turn early exercises into path-dependency

- Reformulate CVA problem with LSM proxies and “CVA-discounting”

- Turbo-charge
  - Multi-thread the simulations and LSM
  - Use AAD for risk

xVA

- Common practice: compute, in addition to and in the same way as CVA
  - The quite controversial DVA = "own CVA" or benefit from own default

$$DVA = -E^{RN}\left\{\sum_{i=1}^{n} DF(T_i)(1-myRR)\left[myS(T_{i-1})-myS(T_i)\right]\min\left(0,PV(T_i)\right)\right\}$$

  - Note DVA = -CVA for counterparty

- And/or funding adjustment
  - FBA = benefit from investing excess cash at a my funding rate (buy back my bonds) > risk free rate …
  - …when PV is negative, remember negative PV = synthetically borrow excess cash through transaction

$$FBA = -E^{RN}\left\{\sum_{i=1}^{n} DF(T_i)\, myS(T_{i-1})\left[myR_F(T_{i-1},T_i)-R_{RiskFree}(T_{i-1},T_i)\right](T_{i-1}-T_i)\min\left(0,PV(T_i)\right)\right\}$$

  - Note funding spread reflects CDS, hence FBA redundant with DVA (and meant to replace it)
  - We also have FCA, cost from borrowing at higher rate when PV is positive

$$FCA = E^{RN}\left\{\sum_{i=1}^{n} DF(T_i)\, myS(T_{i-1})\left[myR_F(T_{i-1},T_i)-R_{RiskFree}(T_{i-1},T_i)\right](T_{i-1}-T_i)\max\left(0,PV(T_i)\right)\right\}$$

  - And the total Funding Valuation Adjustment FVA = FBA - FCA

xVA (2)

- Finally we get the adjusted value for uncollateralized transactions:
  - PV* = risk free PV + FVA – CVA
  - Note that FVA is linear and easy to compute from transaction PVs,
    while the split into FBA and FCA is non-linear, heavy to compute and model dependent
  - FBA and FCA usually computed together with CVA (same methodology) as FVA split is useful info
  - Note applying FVA is same as changing discount basis from risk free (OIS) to my funding
  - And then we still need to apply CVA

- For a clear and transparent presentation on FVA, FBA, FCA see Burgard (2014)

- Finally banks are recently most interested in KVA: Capital Valuation Adjustment
  - Credit capital charge is sort of like CVA, but more complicated
  - Can be calculated with similar methodologies
  - KVA = capital cost of making a transaction
  - Out of scope in this presentation
  - For a clear and complete presentation, see Flyger (2014)

Thank you

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