Bruno Dupire – Shaping derivatives markets for 30 years

Antoine Savine

Introduction: Pillars of Modern Finance

From Black and Scholes (1973) to Dupire (1992-1996)

Defining works of modern finance

- Black and Scholes (1973)

  Condition for absence of arbitrage

$$\frac{\partial C}{\partial t} = -\frac{\partial^2 C}{2\partial S^2}\sigma^2 S_t^{\,2} \text{ or } \vartheta_t = -\frac{\Gamma_t}{2}\sigma^2 S_t^{\,2}$$

- Heath-Jarrow-Morton (1992)

  Condition for respect of initial yield curve

$$df\left(t,T\right) = \sigma\left(t,T\right)\left(\int_t^T \sigma\left(t,u\right)du - dW^Q\right)$$

- Dupire (1992-1996)

  Condition for respect of initial call prices

$$E\left[S_T^{\,2}\sigma_T^{\,2} \middle| S_T = K\right] = \frac{2\partial C}{\partial T}\bigg/\frac{\partial^2 C}{\partial K^2} = \frac{2C_T}{C_{KK}}$$

= "2 calendar spreads over 1 butterfly"
= forward variance (in normal terms)
= conditional variance tradable through European options

Black and Scholes (1973)

- Ground breaking paradigm: **replication:** options are hedged with trading strategies

  Using "Greek" notations $\Delta \equiv \partial C/\partial S, \Gamma \equiv \partial^2 C/\partial S^2, \vartheta \equiv \partial C/\partial t$ in a simplified world where rates = dividends = 0

  From Ito's lemma: $dC(S_t,t) = \Delta dS_t + \left(\vartheta - \frac{\Gamma}{2}\frac{(dS_t)^2}{dt}\right)dt$ after delta-hedge: $dC(S_t,t) - \Delta dS_t = \left(\vartheta - \frac{\Gamma}{2}\underbrace{\frac{(dS_t)^2}{dt}}_{\equiv\sigma^2 S_t^2}\right)dt \underset{AF}{=} 0$

- Black & Scholes arbitrage-free PDE: $\vartheta_t = -\frac{\Gamma_t}{2}\sigma_t^2 S_t^2$ and by Feynman-Kac: $C_t = E^{\frac{dS_t}{S_t}=0dt+\sigma dW_t}\left[C_T | S_t\right]$

- (Rather simplistic) model: volatility is known

  The solution is analytic: $C_t = E\left[(S_T - K)^+ | S_t\right] = S_t N(d_1) - KN(d_2)$

  Replication startegy: hold $N(d_1)$ stocks, borrow $KN(d_2)$ dollars

  Hence value: $S_t N(d_1) - KN(d_2)$

Dupire (1992-1996)

- Ground breaking paradigm: **calibration:** models must respect market prices of calls
  - Necessary and sufficient condition (Dupire, Unified Theory of Volatility, 1996): $E\left[S_T{}^2\sigma_T{}^2 \middle| S_T = K\right] = \dfrac{2C_T}{C_{KK}}$
  - Applies to a wide a class of diffusion models - demonstration:

  By Tanaka's formula: $d\left(S_T - K\right)^+ = 1_{\{S_T > K\}} dS_T + \dfrac{1}{2}\delta_{S_T - K} S_T{}^2 \sigma_T{}^2 dT$

  Taking (RN) expectations on both sides: $dE^Q\left[\left(S_T - K\right)^+\right] = 0 + \dfrac{1}{2}E^Q\left[\delta_{S_T-K} S_T{}^2\sigma_T{}^2\right] dT$

  So: $\underbrace{2\dfrac{dE^Q\left[\left(S_T-K\right)^+\right]}{dT}}_{C_T} = E^Q\left[\delta_{S_T-K}S_T{}^2\sigma_T{}^2\right] = E^Q\left[\delta_{S_T-K}E^Q\left[S_T{}^2\sigma_T{}^2 \middle| S_T\right]\right] = \underbrace{q\left(K,T\right)}_{=C_{KK}} E^Q\left[S_T{}^2\sigma_T{}^2 \middle| S_T = K\right] \Leftrightarrow E^Q\left[S_T{}^2\sigma_T{}^2 \middle| S_T = K\right] = \dfrac{2C_T}{C_{KK}}$

  - Define forward variance: $\sigma_f{}^2\left(K,T\right) \equiv \dfrac{2C_T}{K^2 C_{KK}}$ then the condition for calibration is written: $E\left[\sigma_T{}^2 \middle| S_T = K\right] = \sigma_f{}^2\left(K,T\right)$

- Local volatility model: simple case where volatility is a known function of S: $\sigma_t \equiv \sigma\left(S_t, t\right)$

  Then the condition becomes (Dupire, Pricing with a Smile, 1992): $\sigma^2\left(K,T\right) = \sigma_f{}^2\left(K,T\right) \equiv \dfrac{2C_T}{K^2 C_{KK}}$

Calibration or estimation?

- Like Black-Scholes, Dupire’s work is about the approach, not the model or the formula
- Black and Scholes established the key concept of replication
  But left open the question of the parameters
- Practical application of a derivatives model
  (with modern lingo borrowed from Machine Learning):
  - First, learn model parameters from market data
  - Then, apply the model to a more complicated risk management problem: OTM option, exotic option, CVA, ...
- Calibration approach: imply parameters from market prices of Europeans
  - Natural approach for every derivatives professional
- Estimation aproach: statistically estimate parameters from historical data
  - Natural approach for everyone else
- Why is calibration the natural choice on capital markets?

Calibration and Risk Management

From the Fundamental Theorem of Derivatives Trading to Dupire's Sigma-Zero formula

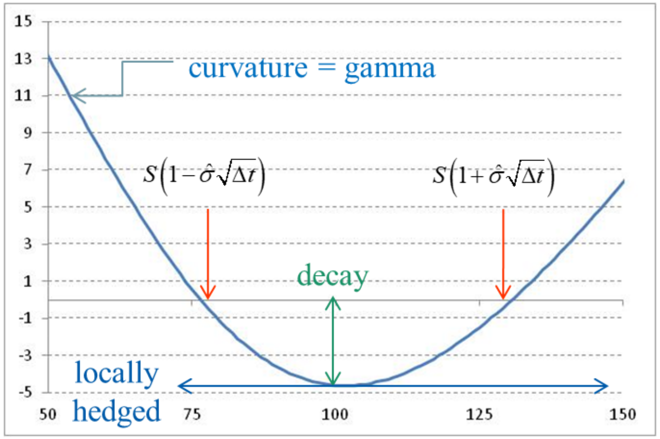
The Fundamental Theorem of Derivatives Trading

- Address pricing problem from hedging and risk management perspective
- Assuming diffusion price process (ignoring jumps)
- Buy C(K,T) and risk manage with Black and Scholes and some implied volatility σ–hat
- What are the residual risks?

- Apply Ito to Black and Scholes's valuation formula: $d\hat{C}(S_t,t)=\hat{\Delta}_t dS_t+\hat{\vartheta}_t dt+\frac{\hat{\Gamma}_t}{2}\underbrace{(dS_t)^2}_{\sigma_t^2 S_t^2 dt}$, σ_t : realized volatility
- After delta-hedge: $d\hat{C}(S_t,t)-\hat{\Delta}_t dS_t=\left(\hat{\vartheta}_t+\frac{\hat{\Gamma}_t}{2}\sigma_t^2 S_t^2\right)dt$
- Applying Black and Scholes PDE:

$$\hat{\vartheta}_t=-\frac{\hat{\Gamma}_t}{2}\hat{\sigma}^2 S_t^2 \text{ so } d\hat{C}(S_t,t)-\hat{\Delta}_t dS_t=\frac{\hat{\Gamma}_t}{2}\left(\sigma_t^2-\hat{\sigma}^2\right)S_t^2 dt$$

- In English:
  mis-replication ~ realised – implied variance



The Fundamental Theorem of Derivatives Trading

- Integrating over hedge period [0,T]: $\underbrace{\underbrace{\hat{C}(S_T,T)}_{payoff} = \underbrace{\hat{C}(S_0,0)}_{premium} + \underbrace{\int_0^T \hat{\Delta}_t dS_t}_{delta-hedge}}_{textbook-replication} + \underbrace{\frac{1}{2}\int_0^T \hat{\Gamma}_t \left(\sigma_t^2 - \hat{\sigma}^2\right) S_t^2 dt}_{mis-replication}$

- In English:
  - Delta-hedging is only risk-free when model correctly predicts realized volatility
  - PnL has an additional term: weighted sum (weighted by gamma) of realized – predicted (normal) variance
  - Replication error = finite variation process => slowly ”bleeds” but does not ”explode”

- This was called ”Robustness of Black-Scholes” (El-Karoui and al, 1998) and ”Fundamental Theorem of Derivatives Trading” (Poulsen and al, 2015)
- This is a central formula for understanding and managing derivatives risk
- Bruno Dupire heavily relied on this approach since the early 1990s

The Sigma-Zero formula (Dupire, 1996)

- Consider the FTDT: $\hat{C}(S_T,T)=\hat{C}(S_0,0)+\int_0^T \hat{\Delta}_t dS_t+\frac{1}{2}\int_0^T \hat{\Gamma}_t\left(\sigma_t^2-\hat{\sigma}^2\right)S_t^2 dt$

- Applying expectations (under Q = risk neutral real world probability):

$$E^Q\left[C(S_T,T)\right]=C(S_0,0)+0+E^Q\left[\int_0^T \frac{\Gamma_t}{2}\left(\sigma_t^2-\hat{\sigma}^2\right)S_t^2 dt\right] \Leftrightarrow \underbrace{E^Q\left[\hat{C}(S_T,T)\right]}_{\text{correct price under Q}}-\underbrace{\hat{C}(S_0,0)}_{\text{price under } BS(\hat{\sigma})}=E^Q\left[\underbrace{\int_0^T \frac{\hat{\Gamma}_t}{2}\left(\sigma_t^2-\hat{\sigma}^2\right)S_t^2 dt}_{mis-replication}\right]$$

- In English: mis-pricing is the expected mis-replication

- More generally:
  - If M1 (Q1) and M0 (Q0) are two arbitrage-free diffusion models (where M1 is a general diffusion and M0 is a non-stochastic volatility model)
  - Then the difference of price between the two models is the expected mis-hedge from delta-heding with M0 in a world driven by M1

$$\begin{aligned}
&E^{Q_1}\left[C_T\right]-E^{Q_0}\left[C_T\right]\\
&=E^{Q_1}\left[\int_0^T \frac{\Gamma^0{}_t}{2}\left(\sigma_t^2-\sigma^0\left(S_t,t\right)^2\right)S_t^2 dt\right]\\
&=E^{Q_1}\left[\int_0^T \frac{\Gamma^0{}_t}{2}\left(E^{Q_1}\left[\sigma_t^2 \middle| S_t\right]-\sigma^0\left(S_t,t\right)^2\right)S_t^2 dt\right]
\end{aligned}$$

The implied volatility formula (Dupire, 1996)

- Consider Sigma-Zero with M0 = Black-Scholes(sigma-hat) and M1(Q) general:

$$E^Q\left[C_T\right]-E^{BS(\hat{\sigma})}\left[C_T\right]=E^Q\left[\int_0^T \frac{\hat{\Gamma}_t}{2}\left(E^Q\left[\sigma_t^2 \mid S_t\right]-\hat{\sigma}^2\right)S_t^2 dt\right]$$

- The model and Black-Scholes agree if and only if:

$$E^Q\left[C_T\right]-E^{BS(\hat{\sigma})}\left[C_T\right]=0$$

$$<=> E^Q\left[\int_0^T \hat{\Gamma}_t \hat{\sigma}^2 S_t^2 dt\right]=E^Q\left[\int_0^T \hat{\Gamma}_t E^Q\left[\sigma_t^2 \mid S_t\right] S_t^2 dt\right]$$

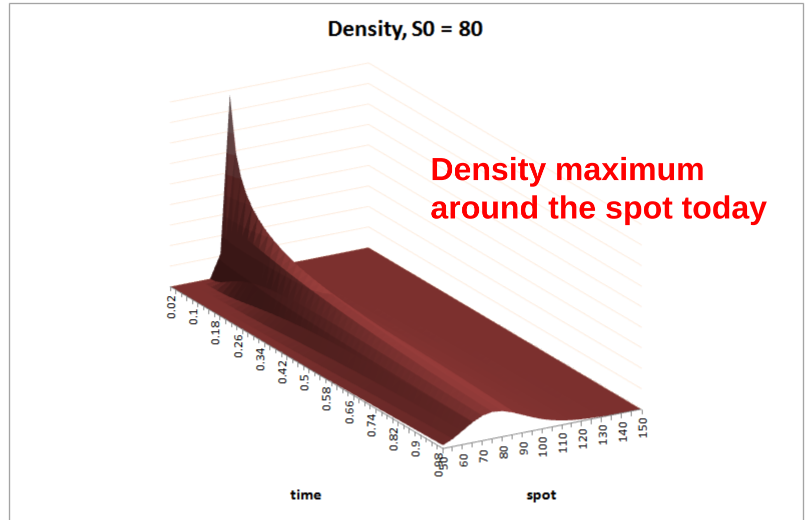
$$<=> \hat{\sigma}^2=\frac{\int_0^T \int E^Q\left[\sigma_t^2 \mid S_t\right] q\left(S_t,t\right)\hat{\Gamma}\left(S_t,t\right)S_t^2 dS_t dt}{\int_0^T \int q\left(S_t,t\right)\hat{\Gamma}\left(S_t,t\right)S_t^2 dS_t dt}$$

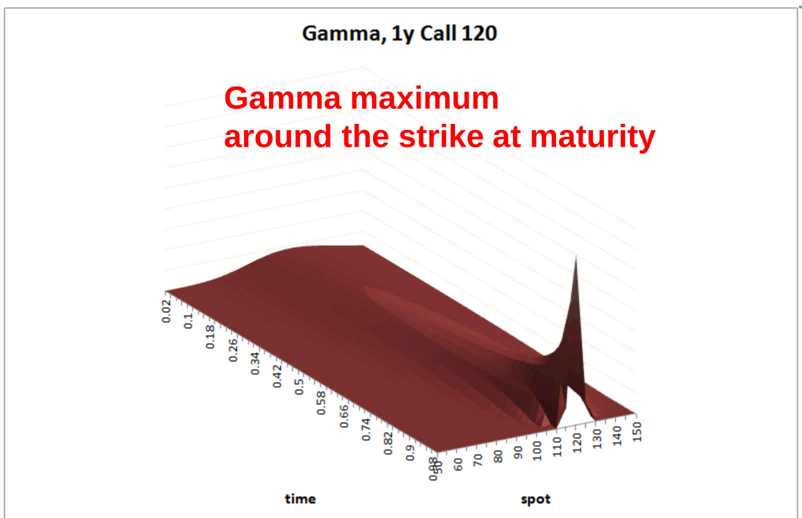
- In English: we have a ”reciprocal” Dupire formula
  Expressing the implied variance as an average of local variances in spot and time
  Weighted by probability times gamma (times spot^2 in log-normal terms)

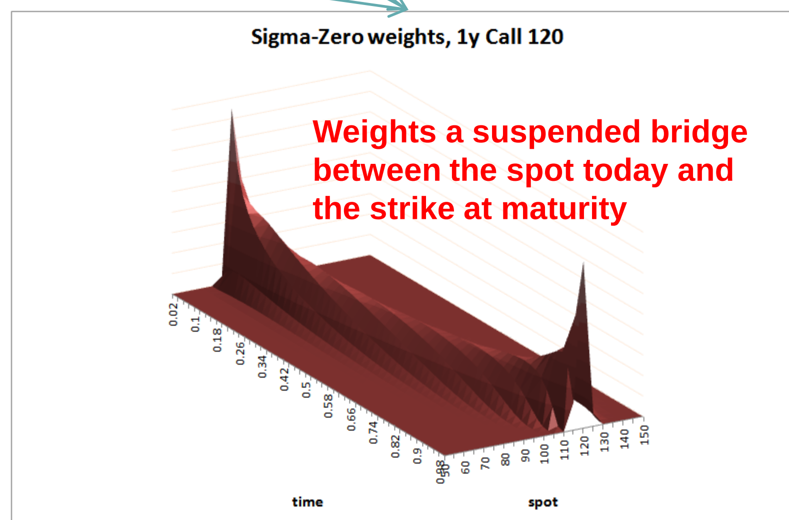
The IV formula

$$\hat{\sigma}^2 = \int_0^T \int w(x,t)\, E^Q\left[\sigma_t^2 \middle| S_t = x\right] dx\,dt$$

$$w(x,t) = \frac{q(x,t)\,\hat{\Gamma}(x,t)\,x^2}{\int_0^T \int q(x,t)\,\hat{\Gamma}(x,t)\,x^2\,dx\,dt}$$







The IV formula

- Does not lend itself to simple or efficient implementation (contrarily to the local volatility formula the other way around)
- Essential analysis tool for valuation and risk
- Leads to many useful results (including Dupire’s formulas of 1992 and 1996)
- **See Blacher (RiO 2018) for an application to calibration of complex hybrid models**
- Generally taught in programs in Finance, including
  - NYU (by Bruno Dupire himself)
  - Baruch (by Jim Gatheral, whose book Volatility Surface discusses applications of sigma-zero in deep detail)
  - Copenhagen University (where I teach volatility)

Back of the envelope calculation: spot risk vs volatility risk

- Very roughly:
  - Normal (Bachelier) approximation
  - First order analysis

- S&P ATM call
  - Delta ~ 0.5, vega ~ 0.4 * sqrt(T)
  - VIX (S&P volatility) generally between 10 and 20 outside stress periods
  - With volatility 15, maximum loss from ”forgetting” delta-hedge with 97.5% confidence ~ 2 standard devs = delta * 2 std = 0.5 * 2 * 15 sqrt(T) = 15 sqrt(T)
  - Expected loss from mis-predicting volatility by a maximum of 5 points (bounds of normal range) = 5 vegas = 2 sqrt(T)
  - volatility risk ~ spot risk / 7.5 times smaller but of same order (”similar”)

Options are hedged with options

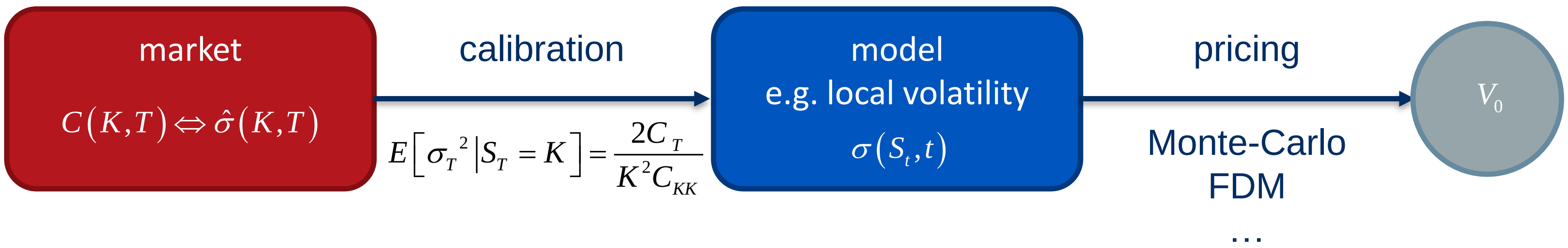
- If we don’t hedge volatility risk
  - Estimate volatility, delta-hedge
  - Run risk that volatility realizes differently than predicted
  - Risk similar to not hedging in the first place

- If we do hedge volatility, we must trade options
  - European options become hedge instruments, like additional underlying assets
  - ”Options are hedged with options”

- Our model must respect the market price of hedge assets
  - Calibrate (not estimate) volatility
  - (Should also reasonably represent their price dynamics => stochastic volatility models)

Superbuckets

- Valuation pipeline



  - Ultimately: $V_0 = h(\hat{\sigma})$ where $\hat{\sigma}$: *two*-dimensional surface

- Superbucket = two-dimensional collection of differentials $\frac{\partial V_0}{\partial \hat{\sigma}(K,T)} \Leftrightarrow \frac{\partial V_0}{\partial C(K,T)}$

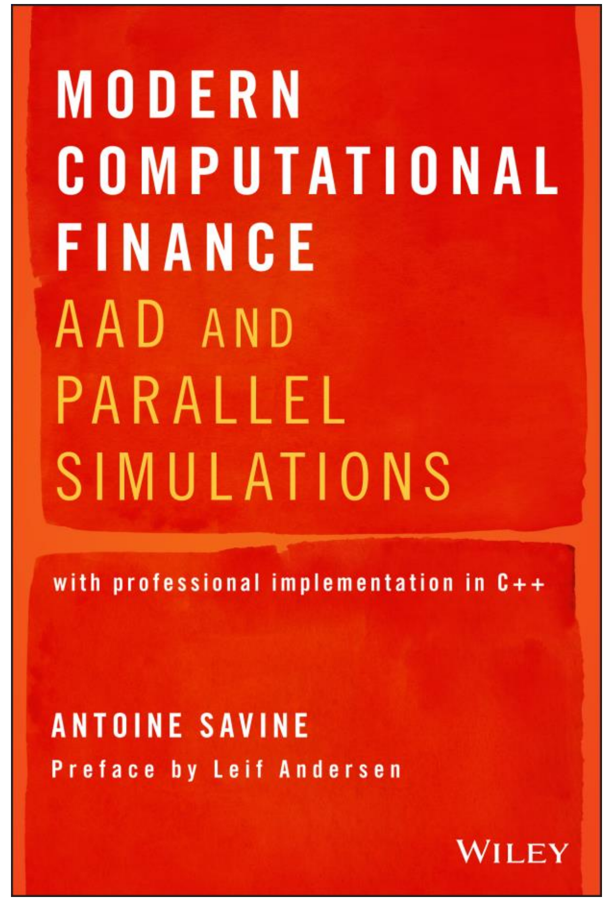
  - $\frac{\partial V_0}{\partial C(K,T)}$ = how many calls (K, T) should I sell to hedge exposure in $\hat{\sigma}(K,T)$

  - (Evidently) European options have a single non-zero superbucket since they hedge themselves
  - But the vega of exotic transactions is split over strikes and maturities and hedgeable by trading Europeans

Superbuckets in practice

- Superbuckets are famously hard in practice, especially with Monte-Carlo
  - Unstable differentiation of Monte-Carlo result
  - Unstable differentiation through calibration
  - Discontinuous profiles like barriers are not differentiable
  - Slow differentiation by finite differences
  - …

- See discussion and elements of solution leveraging automatic adjoint differentiation (AAD), parallel Monte-Carlo
  and modern C++ code in the chapter 13 of ➔



Bruno Dupire’s legacy to derivatives risk management

- Goes well beyond the local volatility model $\frac{dS_t}{S_t} = \sigma(S_t, t)\, dW \ , \ \sigma(K,T) = \sqrt{\frac{2C_T}{K^2 C_{KK}}}$
  - Rather simplistic and criticized (not least by Bruno Dupire himself) for unrealistic dynamics
  - Yet still implemented as standard in all in-house and external derivatives systems 25 years after publication
- Helped establish the universal practice of calibration
  - Approaching derivatives from the hedge persepctive and applying the FTDT
  - Leading to the idea of ”hedging options with options”
  - Itself leading to the principle of calibration: predict is good, hedge is better
- Provided a practical necesary and sufficient condition for calibration to options:

$$E\left[\left(\frac{dS_T}{S_T}\right)^2 \middle| S_T = K\right] = \sigma_f^{\ 2}(K,T) \equiv \frac{2C_T}{C_{KK}}$$

- And an ”inverse” sigma-0 formula to analyze option prices across models and understand/manage model risk

Price difference in different models $E^{Q_1}\left[C_T\right] - E^{Q_0}\left[C_T\right] = E^{Q_1}\left[\int_0^T \frac{\Gamma^0{}_t}{2}\left(E^{Q_1}\left[\sigma_t^{\ 2} \middle| S_t\right] - \sigma^0(S_t,t)^2\right) S_t^{\ 2} dt\right]$

Implied volatility = weighted average of local volatility $\hat{\sigma}^2 = \int_0^T \int w(x,t)\, E^Q\left[\sigma_t^{\ 2} \middle| S_t = x\right] dx\,dt \ , \ w(x,t) = \frac{q(x,t)\hat{\Gamma}(x,t)x^2}{\int_0^T \int q(x,t)\hat{\Gamma}(x,t)x^2 dx\,dt}$

Variance swaps and the VIX index

From Arbitrage Pricing with Stochastic Volatility (Dupire, 1992) to the VIX index

VIX index

- One month S&P volatility index published by CBOE
- Highly succesful barometer of capital markets nervousness
  - Example: Financial Times, May 22, 2018
    ”Heightened volatility struck US equities on Tuesday as developments in Italy’s political sphere fueled investor anxiety. The CBOE’s VIX index, a widely tracked measure of volatility, rose 4.4 points to 17.64.”
  - VIX futures and options trade on the CBOE in extremely large volumes
- How is the VIX calculated?
  - From CBOE’s white paper $$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0}-1\right]^2$$
  - Where F = 1m forward, Q = price of call (Ki>S0) or put (Ki<s0) of quoted strike Ki, maturity 1m and $\Delta K_i = \frac{K_{i+1}-K_{i-1}}{2}$
  - So the VIX (square) is a the price of a portfolio of European options, not an estimation of volatility
- How is this possible?
  - The answer is given in Dupire’s 1992 founding paper Arbitrage Pricing with Stochastic Volatility
  - Established theoretical grounds for Variance Swaps and a forward looking volatility index (VIX)

Variance swaps

- (Apparently) exotic option that delivers realized variance at maturity: $V_T = \int_0^T \left(\frac{dS_t}{S_t}\right)^2 = \int_0^T \sigma_t^2 dt$

- How do we replicate it and what is its value?

  - Recall the FTDT: $C_T = C_0 + \int_0^T \hat{\Delta}_t dS_t + \frac{1}{2}\int_0^T \hat{\Gamma}_t \left(\sigma_t^2 - \hat{\sigma}^2\right) S_t^2 dt = C_0 + \int_0^T \hat{\Delta}_t dS_t + \int_0^T \frac{\hat{\Gamma}_t S_t^2}{2}\sigma_t^2 dt - \frac{\hat{\sigma}^2}{2}\int_0^T \frac{\hat{\Gamma}_t S_t^2}{2} dt$

  - The term in red looks similar to the payoff, as long as $\frac{\hat{\Gamma}_t S_t^2}{2} = 1 \Leftrightarrow \hat{\Gamma}_t = \frac{2}{S_t^2}$

- We buy ~some~ option C and delta hedge it; we get: $C_T - C_0 - \int_0^T \Delta_t dS_t = \int_0^T \frac{\hat{\Gamma}_t S_t^2}{2}\left(\sigma_t^2 - \hat{\sigma}^2\right) dt$

  - If C is such that $\hat{\Gamma}_t = \frac{2}{S_t^2}$ then $\underbrace{C_T - C_0 - \int_0^T \Delta_t dS_t}_{\text{buy and delta-hedge}} = \underbrace{\int_0^T \sigma_t^2 dt dt}_{\text{desired payoff}} - \underbrace{\hat{\sigma}^2 T}_{\text{constant}}$

  - Note: we generally don't like the error term but for variance swaps, we <u>use </u>it to produce the desired payoff

Log-contracts

- Elementary integration bears: $\hat{\Gamma}_t = \frac{2}{S_t^{\,2}} \Leftrightarrow \hat{\Delta}_t = -\frac{2}{S_t} + \alpha \Leftrightarrow \hat{C}_t = -2\log(S_t) + \alpha S_t + \beta$

- Must hold at maturity T so the payoff must be: $C_T = -2\log(S_T) + \alpha S_T + \beta$

  For instance: $C_T = 2\left(\frac{S_T - S_0}{S_0} - \log\frac{S_T}{S_0}\right)$ ”ATM delta-neutral log-contract”

- Conversely: $\hat{C}_t = E^{BS(\hat{\sigma})}\left[C_T \middle| S_t\right] = 2\frac{S_t - S_0}{S_0} - 2\log\frac{S_t}{S_0} + \hat{\sigma}^2(T-t)$

- So, effectively, to buy the log-contract and delta-hedge it replicates the desired payoff

$$C_T - \hat{C}_0 - \int_0^T \hat{\Delta}_t dS_t = \int_0^T \left(\sigma_t^{\,2} - \hat{\sigma}^2\right) dt = \int_0^T \sigma_t^{\,2} dt - \hat{C}_0 \Leftrightarrow \int_0^T \sigma_t^{\,2} dt = C_T - \int_0^T \hat{\Delta}_t dS_t$$

Variance swaps in practice

- To replicate a variance swap, buy a log-contract and delta-hedge it with Black-Scholes
  It follows that the premium of the variance swap is the market price of the LC

- Log-contracts are European options that (obviously) don't trade directly
  $$C_T = 2\left(\frac{S_T - S_0}{S_0} - \log\frac{S_T}{S_0}\right)$$
  What does trade are European calls and puts of given strikes Ki

- Like any European payoff
  LCs are approximated
  as piece-wise linear payoffs,
  with a combination of calls and puts
  See e.g. Carr-Madan (1999)

- We can now reveal what VIX really is

What really is VIX?

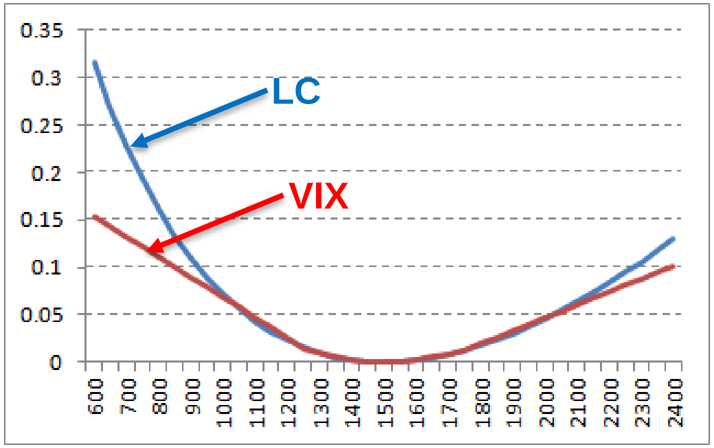
- VIX (squared) is the price of a combination of traded calls and puts of expiry 1m

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0}-1\right]^2$$

- When rates=dividends=0 and ATM option trades, F=S0=K0 and the formula simplifies

$$\frac{VIX^2 T}{2} = \sum_{Ki \geq S_0} w_i C(K_i) + \sum_{Ki < S_0} w_i P(K_i), w_i = \frac{K_{i+1} - K_{i-1}}{K_i^2}$$

- We chart the payoff of this combination on calls and puts:



- Clearly:
  - VIX portfolio = traded calls and puts that best replicate LC
  - VIX^2 is the value of a 1m variance swap on S&P

Bruno Dupire on variance swaps and the VIX

- The VIX is (the square root of) the price of a variance swap

- This explains why VIX is so succesful:
  - Not an estimation but a forward looking, tradable consensus extracted from quoted option prices
  - VIX^2 = necessary amount for delivering future realized variance modulo simple trading strategy

- Bruno Dupire laid the foundation of the multi-billion variance swap business (for which he is generally credited)

- It follows that Bruno Dupire really invented the VIX (for which the CBOE white paper ”forgets” to credit him)

Functional Ito Calculus (Dupire, 2009)

Extension to path-dependent products

Fundamental elements of risk analysis and risk management

- Delta-hedge: $dC(S_t,t)=\frac{\partial C}{\partial S}dS_t+\ldots dt$

- Condition for absence of arbitrage: $\frac{\partial C}{\partial t}=-\frac{\partial^2 C}{\partial S^2}S_t{}^2\sigma_t{}^2$

- Foundamental Theorem of Derivatives Trading: $C_T=C_0+\int_0^T \Delta^0{}_t dS_t+\frac{1}{2}\int_0^T \Gamma^0{}_t\left(\sigma_t{}^2-\sigma_0{}^2\right)S_t{}^2dt$

- Sigma-Zero formula: $E^{Q_1}\left[C_T\right]-E^{Q_0}\left[C_T\right]=E^{Q_1}\left[\int_0^T \frac{\Gamma^0{}_t S_t{}^2}{2}\left(E^{Q_1}\left[\sigma_t{}^2|S_t\right]-\sigma^0(S_t,t)^2\right)dt\right]$

- All rely on first and second order sensitivities of: $C_t=E\left[C_T|S_t\right]=C(S_t,t)$

  and stochastic calculus (Ito's Lemma): $dC(S_t,t)=\frac{\partial C}{\partial S}dS_t+\left(\frac{\partial C}{\partial t}+\frac{\partial^2 C}{2\partial S^2}S_t{}^2\sigma_t{}^2\right)dt$

What about path-dependent options?

- Path-dependent option (barrier, lookback, asian...): $C_T = h\left(S_t, 0 \le t \le T\right)$, functional of the path

- We still have risk-neutral pricing (Harrisson-Pliska, 1980): $C_t = E\left[C_T \middle| F_t\right]$

- But price is now a functional of current path: $C_t = h\left(S_u, 0 \le u \le t\right)$

- Calculus involving functionals of stochastic processes did not exist

➔ No well defined delta, no theta/gamma equation,
no PnL explain or model risk decomposition (sigma-zero)

Functional Ito Calculus (Dupire, 2009)

- Did not stop market participants to estimate sensitivities with frozen history:

$$C_t = h\left(\left\{S_u, 0 \le u < t\right\}, S_t, t\right), \Delta_t \approx \frac{h\left(\left\{S_u, 0 \le u < t\right\}, S_t + \varepsilon, t\right) - C_t}{\varepsilon}, etc.$$

- And apply delta-hedge, theta-gamma equation, PnL explain and model risk
  Even in the absence of a mathematical definition or guarantee
- Turns out this methodology is correct
- In 2009, Bruno Dupire extended stochastic mathematics to:
  - Define sensitivities of functionals of stochastic processes
  - Demonstrate that Ito’s lemma, theta-gamma equation, FTDT and sigma-zero all hold for exotics
- Functional Ito Calculus ”closed the circle”
  - Completed financial theory by extending mathematical principles of risk management to exotics
  - Reconciliated theory and practice
  - Established mathematical guarantees for market and model risk analysis and management of exotics
- **See Yuri Saporito, RiO 2018, for a (excellent) introduction to FIC**

Conclusion

A model-free thinker

- Bruno Dupire is best known for his local volatility model of 1992 which is a fraction of his work

- Dupire’s work always focused on hedging and replication
- His major contributions:
  - Calibration condition, systematic application of FTDT, sigma-zero, VS/VIX, functional calculus
  - All apply to a wide variety of models

- Bruno Dupire is not a ”one-model researcher” but a ”model-free” thinker
- Further emphasized by his less known work on
  - Mapping major martingale properties to market arbitrage (1997)
  - Tradable estimates and indicators (2015)

A vast contribution to the theory and practice of capital markets

- Bruno Dupire only published a tiny fraction of his work
  He mainly contributed through:
  - Leading research in SocGen(1991-1992), Paribas(1992-1997), Nikko(1997-1998) or Bloomberg(since 2004)
  - Internal memos like his ”Notes de Recherche” from Societe Generale of 1991-1992 (French manuscript)
  - Professional presentations and working papers
- Yet, his contribution is universally recognized and earned multiple prestigious awards
- Some of his major research remains less well known:
  - First application of artificial neural networks (ANN) to finance in 1988, 30 years ahead of current craze
  - Substantial work on numerical integration and Monte-Carlo simulations around 1995
  - Model-free arbitrage outside VS/VIX
- In addition to his scientific contribution
  Dupire trained and mentored many ”babies” who grew up to run derivatives markets:
  - 3 former partners and a number of MDs at Goldman-Sachs,
    a former Global Head of Trading and a former Global Head of Research at BNP,
    a current Global Head of Interest Rate Research at BAML, and many more

Bruno Dupire

- 60 years of existence as of November 30, 2018
- 30 years of major innovations, shaping global derivatives markets
- An outstanding thought leader, admirable human being and incredible friend

- Thank you for your attention

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