Volatility Modeling and Trading

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Financial Mathematics

- Financial quants are not really concerned with *valuation* of transactions

- All we do is about risk management and the *hedging* of transactions

- In order to hedge transactions, we must compute their *sensitivities* to market variables

- Hence we need to produce a value as a function of these market variables

- Valuation is a by-product of risk management, not a goal in itself

Volatility

•Part of Financial Mathematics deals with *volatility*

•This is our focus here

•For clarity, we remove all other sources of complexity and/or noise

•In particular (unless explicitly stated otherwise) we always consider

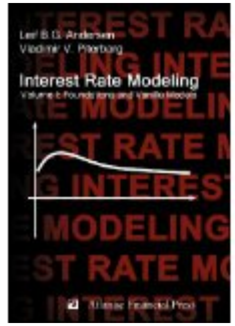
- One well defined underlying asset
- No rates, spreads, dividends, repos, credit, etc.

Volatility Modeling and Trading

Module 1: Black-Scholes

Antoine Savine

References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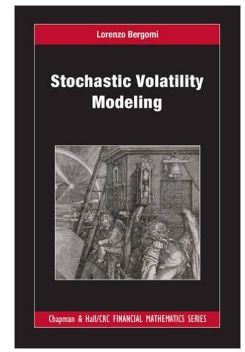
Lorenzo Bergomi
Stochastic Volatility Modeling
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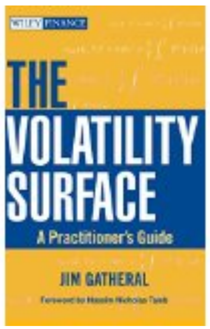
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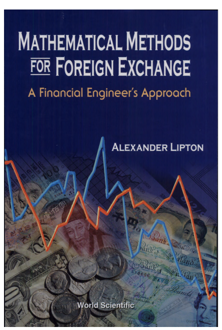
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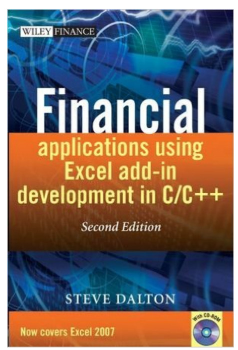
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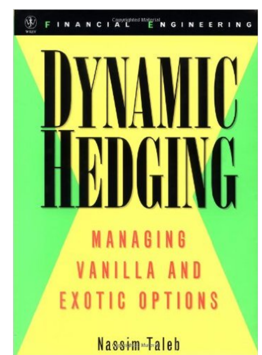
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Overview

•Derivatives mathematics are primarily concerned with the risk management of transactions and books

•Valuation is a by-product not a primary goal

•Since Black & Scholes founded modern finance in 1973, financial quants main challenges have been:

- To identify hedge instruments and strategies and value complex transactions accordingly
- To analyze residual risks and adjust models, hedges and values accordingly
- To produce generic, fast, stable algorithms implementing the above
- More recently, to incorporate regulatory requirements

Overview (2)

•Derivatives quants spend their days (and nights) working on:

- **Volatility**:
  design models on the market variable dynamics and implement risk management of options

- **Underlying assets**:
  model non-trivial underlying assets such as
  - More than one underlying asset: baskets, correlation, …
  - Interest rates: the underlying “asset” is a continuous curve of rates of different maturities with consistent multi-factor dynamics
  - Credit: jointly model default events and credit spreads
  - Hybrids: consistently model underlying assets belonging to different asset classes
  - And “exotic” underlying assets such as commodities, energy, inflation, even weather, real estate, fine arts, etc.

- **Numerical methods**:
  the practical implementation of all of the above

- More recently, **regulatory calculations**

Overview (3)

•These different fields raise specific challenges that are typically unrelated to one another

Ex: *when we build an interest rate model with stochastic volatility,*
we find that the problem of the joint evolution of rates is hardly different with stochastic volatility,
whereas the additional difficulties of stochastic volatility is no different in the case of rates

•Hence, it is appropriate to teach them separately

•We focus solely on volatility and do not cover the rest

•In particular (unless explicitly stated otherwise) we always consider

- One underlying asset
- Zero rate, spread, dividend, repo, etc.
- No credit

Overview (4)

•We do cover

- ➢ Module 1: Black & Scholes
  approached from the point of view of the dynamic hedging of options

- ➢ Module 2: the Market Implied Volatility Smile and related models
  local volatility, stochastic volatility and jumps

- ➢ Module 3: Exotics
  rationale, structuring, risk management and modeling

- ➢ Module 4: Volatility Products
  variance swaps, log-contracts and VIX related transactions

What is expected of students?

•Access to my public dropBox

•Ability to use Excel, program in C++ and export C++ to Excel

- Have a Windows 7+ computer with Excel 2007+ 32bit and Visual Studio (ideally 2015) setup for C++ development
- Have working knowledge of C++ and Excel
- Complete the tutorial for exporting C++ to Excel, including exercises, and start your own C++ library exporting to Excel
  my public dropBox folder Vol/xlCpp/ document ExportingCpp2xl.pdf

•Complete assignments

- Assignments will be given in my public dropBox, folder Vol/assignments
- They are to be returned by email only volatility@asavine.com
- Written assignments and exercises are to be returned as pdf files only
  the name of the file must be vol[Num][FullName].pdf, for example vol1AntoineSavine.pdf
- Additional programming assignments are to be returned as zip files under the name vol[Num][FullName].zip, containing
  - The Release xll
  - The Excel workbook(s)
  - Only the .cpp and .h files relevant to the assignment, excluding Excel wrappers

Black-Scholes 1973

•The $E = MC^2$ of Finance and one of the most famous formulas in history $C = SN(d_1) - KN(d_2), d_{1/2} = \dfrac{\log\left(\dfrac{S}{K}\right) \pm \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}}$

•Earned a (late) Nobel in 1997

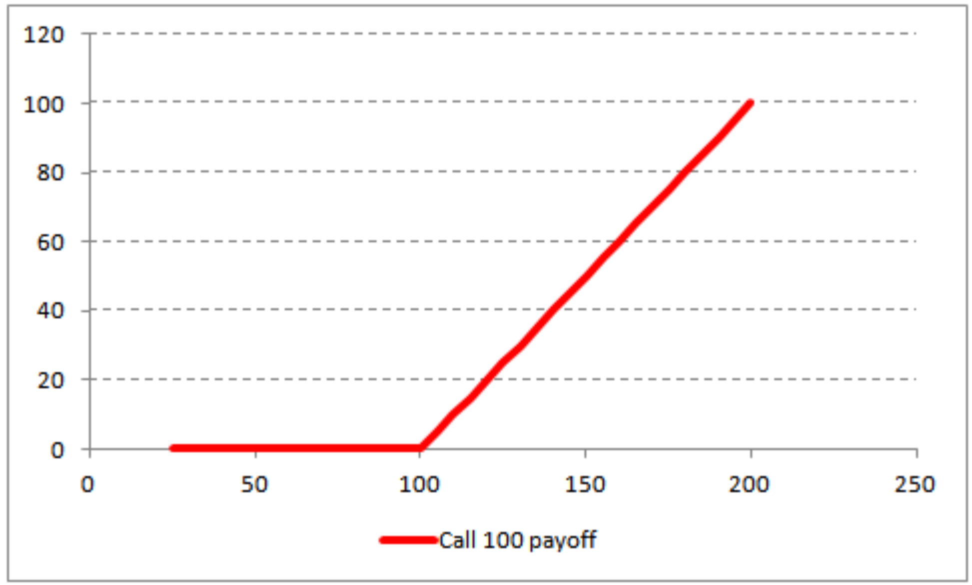
•A change of paradigm that changed the face of Finance

- Pioneered the use of advanced mathematics in Finance
- Produced the notion that options are *hedged* or *replicated* by trading the *underlying* asset
- Showed that option prices are independent of anticipations other than *volatility*
- Demonstrated that option prices are independent of *risk aversion*

European Call

•Right (but no obligation) to purchase a given asset or *underlying*
at a future date –called *expiry* or *maturity*-
for a price agreed today –called *strike*

•*Holder* will *exercise* at maturity
if underlying price > strike (*payoff* = S-K),
if not, the option expires *out of the money* (payoff=0)
hence altogether payoff = max (S-K, 0)
or in compact notation $(S-K)^{+}$



Trivial properties

- Simplified context: no rate/dividends, perfect liquidity, transparent short selling, no default
- Call price depends on *spot price (distance to strike)* and *time (residual maturity)* $C = C(S,t)$
- *Intuitively,* call price mainly (only?) depends on distance to strike/maturity $C \approx C(S-K, T-t)$
  This is actually inaccurate, but "almost" right, and helps build intuition
- Option prices are positive since the holder has rights no obligations
- Option prices must be decreasing in strike ➔ **why?**
  ➔ Therefore we also expect call price to increase in spot $\text{"}delta\text{"} = \Delta = \frac{\partial C}{dS} > 0$
- When spot price >> strike, the option is bound to be *exercised*, its price converges to *S-K*
- When spot price << strike, the option is bound to expire *out of the money*, its value converges to *0*

Less trivial properties

- The option is always worth more than its *intrinsic value max(S-K,0)* ➔ **can you tell why?**
  - Intrinsic value (IV) = value if exercised today $IV = (S_0 - K)^+$
  - Obviously, at maturity, *value = IV = max(S-K,0)*

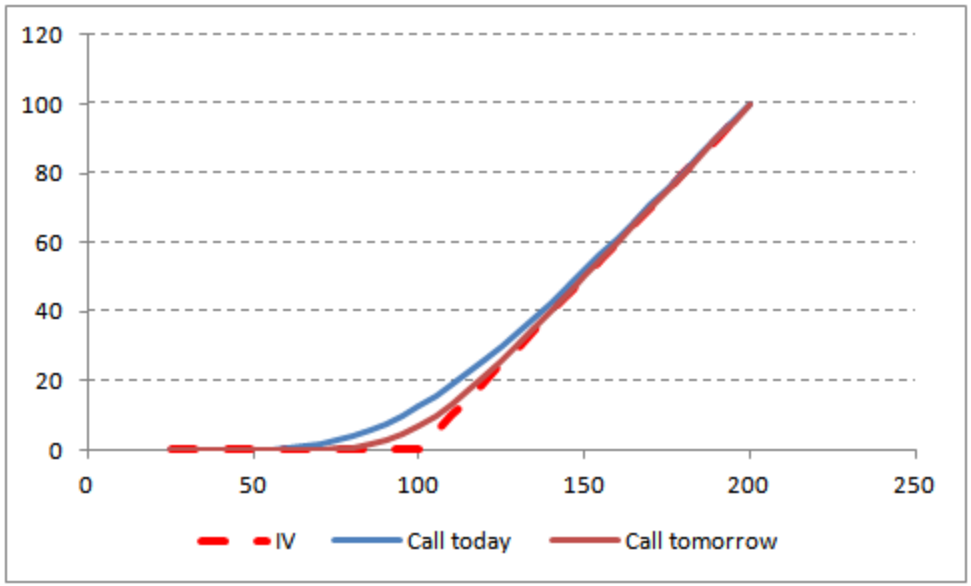
- The difference of price to IV is called *time value* (TV)
- Obviously, TV reduces to 0 as time converges to maturity

- Option price must increase with maturity ➔ **can you tell why?**
- ➔ Hence we expect call value *decreases* with time $"theta" = \vartheta = \frac{\partial C}{dt} < 0$

- Option value must be **convex** in strike ➔ **can you tell why?**
- ➔ Therefore we expect call value to be convex in spot $"gamma" = \Gamma = \frac{\partial^2 C}{dS^2} > 0$

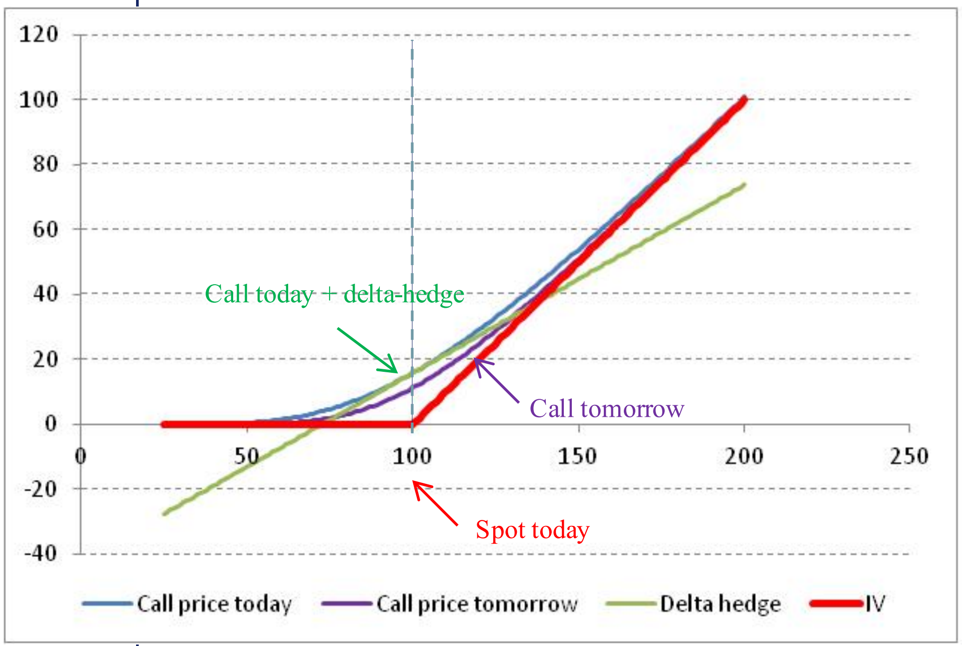
Putting it all together

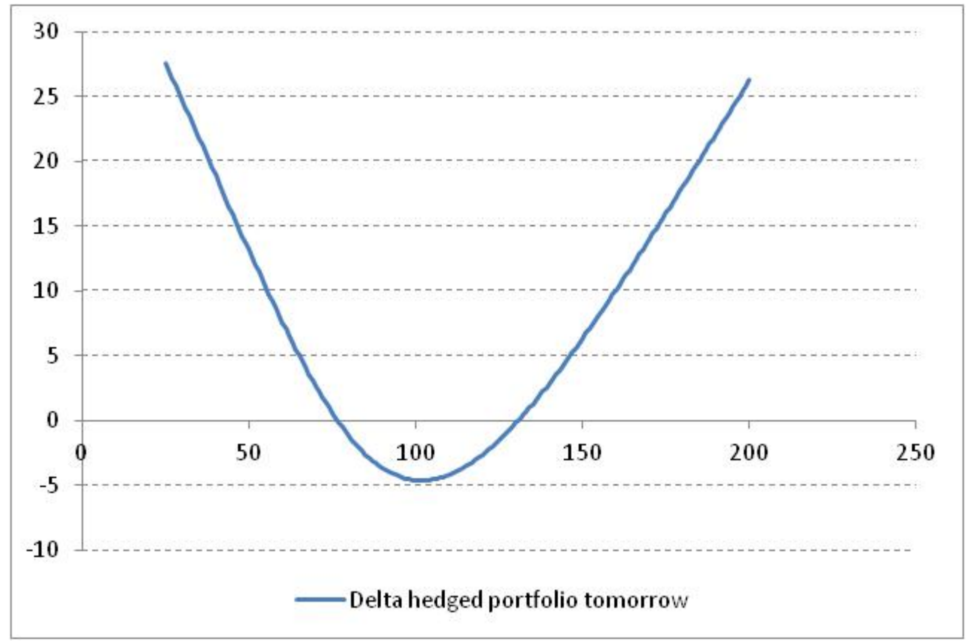
Altogether, the shape of call prices as a function of spot, today and tomorrow, must look like



P&L on delta-hedged call

Today long call, short delta stocks ➔ What is the PnL tomorrow?





P&L on delta-hedged call

- Over a (short) re-hedge period we can ignore third and more orders, so change of call price:

$$\Delta = \frac{\delta C}{\delta S}$$

$$\Delta C = \vartheta \Delta t + \Delta \cdot \Delta S + \frac{1}{2}\Gamma \Delta S^2$$

change in call value $\qquad \vartheta = \frac{\delta C}{\delta t}, \vartheta \Delta t = decay \qquad \Gamma = \frac{\delta^2 C}{\delta S^2}$

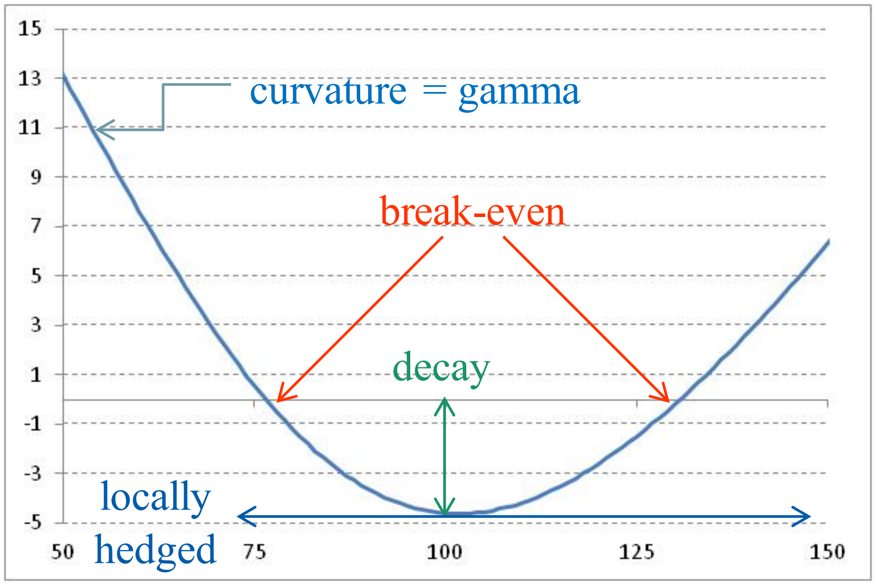
- After delta-hedge, overall delta is 0 so:

$$\boxed{PnL = \vartheta \Delta t + \frac{1}{2}\Gamma \Delta S^2}$$

- Note we need an implicit assumption of continuity
  - If we assume that the underlying price process is *continuous*
  - Then we can increase hedge frequency so our 2nd order expansion remains valid even in nervous markets
  - But if prices can jump, then the 2nd order expansion is invalid even over small intervals

Break-even

$PnL = \vartheta\Delta t + \frac{1}{2}\Gamma\Delta S^2$ depends on size but not direction of price move, the bigger the move the higher the PnL



Gamma trading

- When long gamma (long options)
  - Start the day negative due to decay
  - Benefit from spot moves during the day
  - Game = compensate decay and more trading intraday moves know when to lock intraday move
  - Often preferred by equity gamma traders
- When short gamma (short options)
  - Start the day positive
  - Lose on spot moves during the day
  - Game = keep (part of) decay know when to stop-loss on intraday moves
  - Often preferred by forex gamma traders

Break-even: the size of the move you need to exactly compensate decay $Pnl = 0 \Leftrightarrow \frac{\left(\Delta S / S\right)^2}{\Delta t} = \frac{-2\vartheta}{\Gamma S^2}$

volatility (square)

Volatility

- Measures the size (not the direction) of the moves in price per time unit
  - Variance = variance of returns per time unit
  - Volatility = square root of variance

$$R=\frac{\Delta S}{S},\sigma^2=\frac{Var(R)}{\Delta t}=\frac{E\left[(R-ER)^2\right]}{\Delta t},\sigma=\sqrt{\sigma^2}=\frac{Std(R)}{\sqrt{\Delta t}}$$

- Historical volatility = statistically estimated
  - Given data on spot prices S_i
  - Produce series of returns $R_i=\frac{S_{i+1}-S_i}{S_i}\ or\ \log\left(\frac{S_{i+1}}{S_i}\right)$
  - Compute their variance $\bar{\sigma}^2\Delta t=\frac{1}{n}\sum_{i=1}^{n}R_i^{\ 2}-\left(\frac{1}{n}\sum_{i=1}^{n}R_i\right)^2\ or\ \frac{1}{n}\sum_{i=1}^{n}R_i^{\ 2}$
  - Annualize $\bar{\sigma}^2=\frac{\bar{\sigma}^2\Delta t}{\Delta t}$
  - Take the square root as volatility $\bar{\sigma}=\sqrt{\bar{\sigma}^2}$
- Excel example

Volatility and option pricing

•If we know/estimate/assume the volatility $\sigma = \sqrt{\frac{\left(\Delta S / S\right)^2}{\Delta t}}$

•Then the option is fairly valued *against this volatility* if it makes the spot land *on* the break-even

$$\underbrace{\frac{\left(\Delta S / S\right)^2}{\Delta t}}_{\text{size of spot move = volatility (square)}} = \underbrace{\frac{-2\vartheta}{\Gamma S^2}}_{\text{on break-even}} \Rightarrow \vartheta = -\frac{1}{2}\Gamma S^2 \frac{\left(\Delta S / S\right)^2}{\Delta t} \Rightarrow \boxed{\vartheta = -\frac{1}{2}\Gamma S^2 \sigma^2}$$

•This is the Black and Scholes PDE (Partial Differential Equation)

- So Black & Scholes proved that the call price $C_t = C(S_t, t)$ given volatility σ must satisfy the PDE $\vartheta = -\frac{1}{2}\Gamma S^2 \sigma^2$
- And trivially, $C(S_T, T) = (S_T - K)^+$ provides a terminal *boundary condition* to this PDE
- It is known that such PDEs have a unique solution

- **And so our job is done**

Demonstration complete?

•All that remains is to *solve* the PDE

•Black & Scholes proved that it may be solved analytically under the assumption of a constant volatility

•But then we already know that PDEs can be resolved numerically and sometimes analytically

•This is not the important point

•The ground breaking contribution of Black & Scholes was to *derive* the PDE from arbitrage arguments...

•... And establish a "universal law of finance" that exceeds by far the simplistic assumptions of their model:

convexity has value
because it *monetizes* the volatility of the underlying
when we delta-hedge

$\frac{1}{2}\Gamma S^2\sigma^2$ is the PnL generated by convexity
when *realized* volatility is σ

$$\frac{1}{2}\Gamma S^2\sigma^2 = -\vartheta$$

when valuation is fair
you pay for convexity through decay
time value = expected cumulated decay

decay corresponds to the expected PnL
given an anticipated –or *implied*- volatility

An important comment

- We demonstrated Black & Scholes without any
  - ➢ Probabilistic model or advanced mathematics
  - ➢ Mention of Brownian Motions, Stochastic Calculus or Ito's lemma

- Advanced maths come in only for *solving* the PDE

- It follows that we don't need to assume that prices are driven by Brownian Motions to trade with Black-Scholes

- Returns don't have to be normally distributed and increments don't need to be independent

- We say that Black-Scholes is *robust* against these (unnecessary) assumptions

- The one assumption we *do* need is continuity, otherwise the 2^{nd} order expansion is invalid and our construction falls like a house of cards

- We demonstrate later another *even more crucial* robustness of Black-Scholes: robustness against the assumption of a known/constant/*deterministic* volatility

Solving the PDE: Feynman-Kac theorem

- Theorem

- Given a PDE of the form $f(x,y), \frac{\delta f}{\delta y} = -\frac{1}{2}\frac{\delta^2 f}{\delta x^2} h(x,y)^2$
- And some boundary condition $f(x,Y) = g(x)$
- Then this PDE has a unique solution, and this solution may be written as an expectation $f(x,y) = E\left[g(X_Y)/X_y = x\right]$
- Where the expectation is taken under the following process on *X*: $dX_t = h(X_t,t)dW_t$

- Application to Black and Scholes:

- Immediately, $\boxed{C(S_0,0) = E\left[(S_T - K)^+\right], \frac{dS}{S} = \sigma dW}$
- The expectation is computed under the dynamics $\frac{dS}{S} = \sigma dW$
- We call this dynamics *risk-neutral* and the related probability distribution *risk-neutral probability* or *risk-neutral measure*

Derivation of the Black and Scholes formula: high school maths

•Easy but long and inelegant solution

- We know that $\frac{dS}{S} = \sigma dW$ resolves into $\boxed{S_T = S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma W_T\right)}$
- Hence its probability distribution is $\boxed{S_T = S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma\sqrt{T}N\right)}$ where N is a standard Gaussian variable
- And $C = E\left[\left(S_T - K\right)^+\right] = E\left\{\left[S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma\sqrt{T}\mathrm{N}\right) - K\right]^+\right\} = \int \left[S_0 \exp\left(-\frac{\sigma^2}{2}T + \sigma\sqrt{T}x\right) - K\right]^+ \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$
- This integral can be computed with “high school” techniques: change of variable, integration by parts

Derivation of the Black and Scholes formula: Using Girsanov's theorem

•Fast, elegant solution using Girsanov's theorem

- $C = E\left[\left(S_T - K\right)^+\right] = E\left[\left(S_T - K\right)1_{\{S_T > K\}}\right] = E\left[S_T 1_{\{S_T > K\}}\right] - KP\left(S_T > K\right)$
- Immediately $P\left(S_T > K\right) = P\left(N < d_2\right) = N\left(d_2\right), d_2 = \dfrac{\log\dfrac{K}{S_0} - \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}}$
- And $E\left[S_T 1_{\{S_T > K\}}\right] = S_0 E\left[\varepsilon_T 1_{\{S_T > K\}}\right], \varepsilon_t = \exp\left(-\dfrac{\sigma^2}{2}t + \sigma W_t\right)$
- So $E\left[S_T 1_{\{S_T > K\}}\right] = S_0 E^{Q_S}\left[1_{\{S_T > K\}}\right] = S_0 Q_S\left[S_T > K\right]$
- Where Q_S is the probability measure defined by its Radon-Nykodym derivative ε_T with respect to the RN measure
- By Girsanov's theorem, $\tilde{W}_t = W_t - \sigma t$ is a Brownian Motion under Q_S
- Hence $Q_S\left(S_T > K\right) = Q_S\left(\sigma W_T < \log\dfrac{K}{S_0} - \dfrac{\sigma^2}{2}T\right) = Q_S\left(\sigma \tilde{W}_T < \log\dfrac{K}{S_0} + \dfrac{\sigma^2}{2}T\right) = N\left(d_1\right), d_1 = \dfrac{\log\dfrac{K}{S_0} + \dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}}$
- And finally we have Black and Scholes formula: $C = SN\left(d_1\right) - KN\left(d_2\right)$

Black and Scholes formula: remarkable properties

- Fast and easy implementation

> ➢ Did you already implement it in C++ to Excel? If not, do it

- $N(d_2)$ is the (risk-neutral) probability to end *in the money*

- $N(d_1)$ is the delta!

> ➢ Optional exercise: prove it either by differentiating Black-Scholes directly (long) or by differentiating $E\left[(S_T - K)^+\right]$ under the expectation (fast and elegant but requires differentiation of distributions and Girsanov's theorem).

- Replication strategy for the call: hold $N(d_1)$ units of the underlying, borrow $KN(d_2)$ cash

the value of the replicating portfolio = option value $SN(d_1) - KN(d_2)$

$$d_1 = \frac{\log \frac{S_0}{K} + \frac{\sigma^2}{2} T}{\sigma\sqrt{T}} = N^{-1}(\Delta) \qquad d_2 = \frac{\log \frac{S_0}{K} - \frac{\sigma^2}{2} T}{\sigma\sqrt{T}} = P^{\frac{dS}{S} = \sigma dW}(S_T > K) = d_1 - \sigma\sqrt{T}$$

Risk-Neutralization

Initial assumption:
drift is undefined, volatility is assumed

$$\frac{dS}{S} = \mu dt + \sigma dW$$

Hedge arguments lead to PDE

$$\vartheta + \frac{1}{2}\sigma^2 S^2 \Gamma = 0$$
$$C(S_T, T) = (S_T - K)^+$$

Feynman-Kac tells PDE *solution* is

$$C(S_t, t) = E\left[(S_T - K)^+ / S_t\right]$$

Where the expectation is taken
under the *modified* dynamics

$$\frac{dS}{S} = \sigma dW$$

- This process is called *risk-neutralization*
- The pricing *probability measure* is called *risk-neutral probability*

=probabilistic assumptions under which expectations are calculated as prices
- The risk-neutral probability preserves volatility (it is *equivalent* to the real-world measure) but removes the drift

- Remember: option prices are NOT expectations

they are PDE solutions calculated as expectations under dynamics that refer to parameters of the PDE

Risk-Neutralization: Example

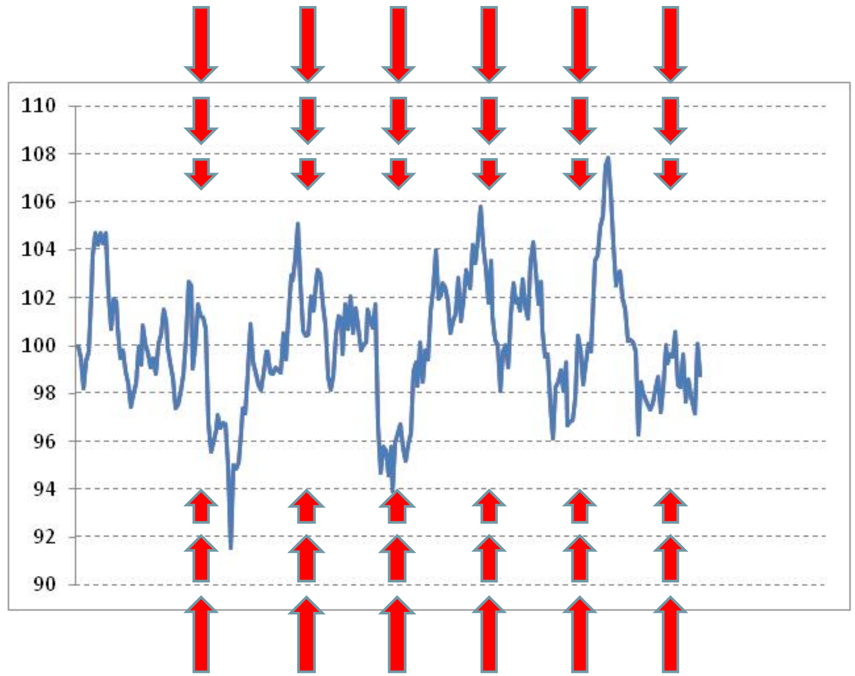
•Real-world asset dynamics is *mean-reverting*

$$dS = -\lambda(S - S_0)dt + \sigma dW$$

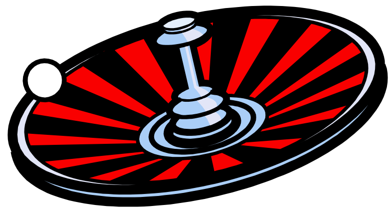
Vasicek process ➔ mean-reversion is in the *drift*

•If volatility is 25 and MR is 50% (quite extreme), then the 1y distribution has standard deviation ~20 (basic Vasicek maths)

•Question: what is the fair price of a 1y call?
Is it BS(25) or BS(20)?



Interview question



•In a casino, you realized the roulette is biased
so black comes out 70% of the time

•The odds are still 50/50,
that is you may bet 1$ on black/red, make 1$ if your color comes out, loose your 1$ otherwise

•There is no 0, the casino is managed by philanthropists

•You build a business in the casino, trading tickets that pay 100$ when red comes out (0 otherwise)

•What is the fair price of the ticket and how is it covered?

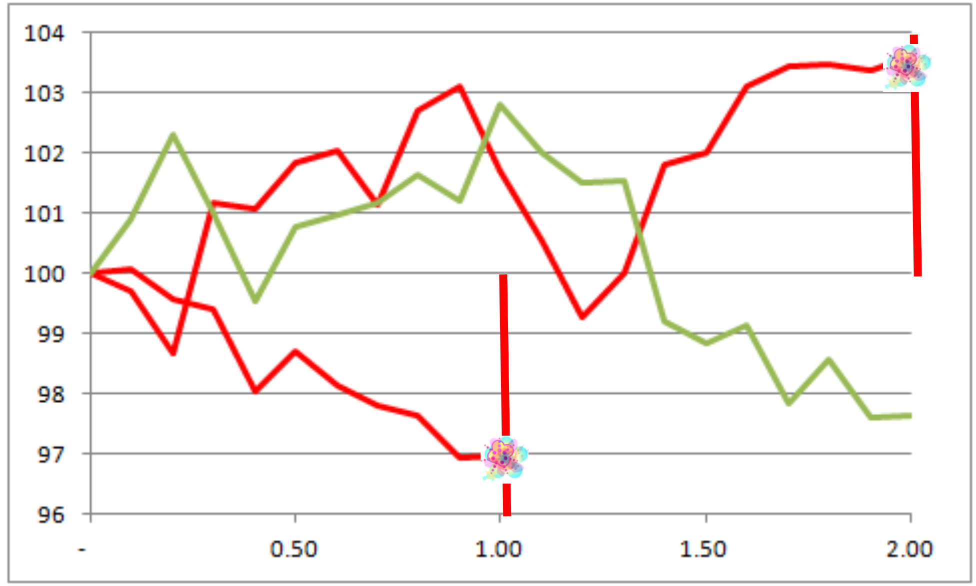
•Conclude

Interview question

•What is the price of a “Slalom option”?

•Pays 100$ if fix above 100 in 1y
and below 100 in 2y

•Spot is 100, no rates, no dividends

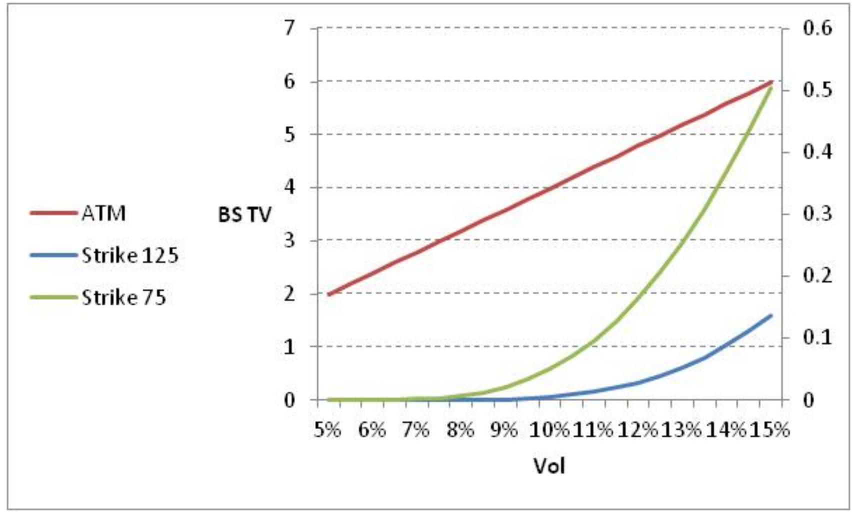


•Assume Bachelier model (also called Black-Scholes Normal): $dS = \mu dt + \sigma dW$

Implied volatility

•Option price depends on observables
and volatility

•Option price is increasing in volatility
➔**Prove it?**



•In liquid active option markets
(say S&P500 options) we can deduce
the **implied** volatility from the market prices

•Implied volatility = volatility such that BS price matches market price

•Black & Scholes ➔ (non-linear) (bilateral) translation of price into volatility
We are going to see next that this is much more than just a change of unit

•You should have it already implemented in C++ Excel

Gamma trading

•**Robustness of Black-Scholes**: remember the break-even formula $PnL = \vartheta \Delta t + \frac{1}{2}\Gamma \Delta S^2$

•If we risk risk manage with Black-Scholes with *implied* vol $\hat{\sigma}$, then valuation and risk are subject the PDE

$$\vartheta + \frac{1}{2}\Gamma \hat{\sigma}^2 S^2 = 0$$

$\hat{\sigma}$: *implied* or *management* volatility, the one used to produce daily value and risk sensitivities

•So $\boxed{PnL = \frac{1}{2}\Gamma S^2 \left[\underbrace{\frac{\left(\Delta S / S \right)^2}{\Delta t}}_{= \sigma_r^2} - \hat{\sigma}^2 \right] \Delta t}$

σ_r : *delivered* or *realized* volatility, the one that is effectively experienced during the hedge period

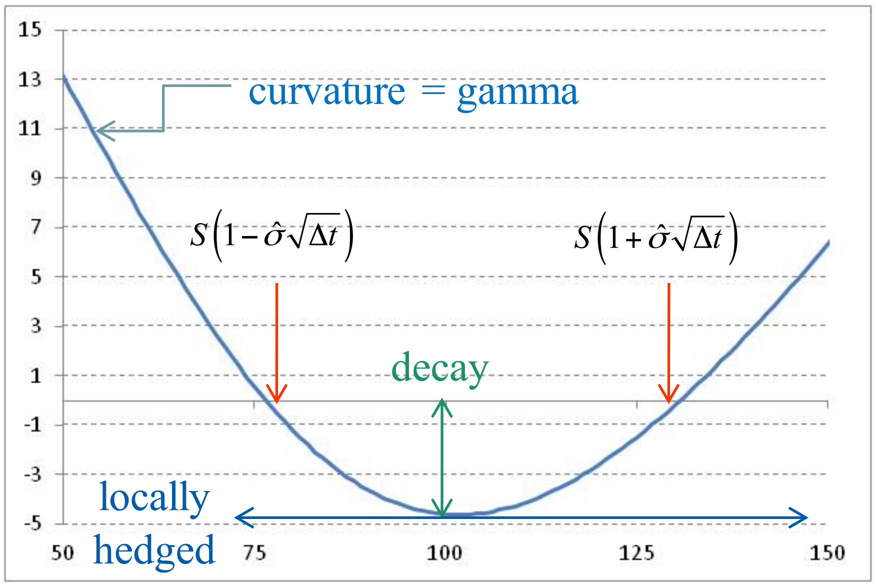
•Trading with Black-Scholes means exchanging *implied* for *delivered* vol

•Note the multiplier: $\frac{1}{2}\Gamma S^2$ sensitivity to *delivered* vol comes from gamma (calculated with *implied* vol)

•Hence, market making and risk management of (short term) options is often called “Gamma Trading”

Graphical representation: back to break-even

$$PnL = \frac{1}{2}\Gamma S^2 \left[\frac{\left(\Delta S / S\right)^2}{\Delta t} - \hat{\sigma}^2 \right] \Delta t$$



Gamma mathematics: putting it all together

- Suppose the real-world dynamics is $\frac{dS}{S} = \mu dt + \sigma_r dW$ where drift and volatility are general and may be stochastic

- We value and manage an option with Black-Scholes and some implied volatility $\hat{\sigma}$

- By Ito applied to the Black & Scholes function:

$$dBS^{\hat{\sigma}}(S,t) = BS_S^{\hat{\sigma}} dS + \left(BS_t^{\hat{\sigma}} + \frac{1}{2} BS_{SS}^{\hat{\sigma}} S^2 \sigma_r^2 \right) dt$$

- After integrating, re-arranging and naming the Greeks we get: $BS^{\hat{\sigma}}(S_T,T) = BS^{\hat{\sigma}}(S_0,0) + \int_0^T \Delta^{\hat{\sigma}} dS + \int_0^T \left(\vartheta^{\hat{\sigma}} + \frac{1}{2} \Gamma^{\hat{\sigma}} S^2 \sigma_r^2 \right) dt$

- Black-Scholes theta and gamma satisfy the PDE: $\vartheta^{\hat{\sigma}} = -\frac{1}{2} \Gamma^{\hat{\sigma}} S^2 \hat{\sigma}^2$ with the *implied* vol

- Which we inject in the previous equation to get: $\boxed{\int_0^T \Delta^{\hat{\sigma}} dS = \underbrace{BS^{\hat{\sigma}}(S_T,T)}_{=(S_T-K)^+} - BS^{\hat{\sigma}}(S_0,0) - \frac{1}{2} \int_0^T \Gamma^{\hat{\sigma}} S^2 \left(\sigma_r^2 - \hat{\sigma}^2 \right) dt}$

Delta-hedge:
Zero-cost (self financing) hedging strategy

Payoff:
What we want to replicate

Todays price:
The cash we need initially to achieve the replication

Mis-replication term:
Depends on realized vs. implied vol, weighted by gamma
Null when $\hat{\sigma} = \sigma$

Gamma mathematics (2)

•We proved ("Fundamental Theorem of Option Trading") that

$$\underbrace{BS^{\hat{\sigma}}\left(S_0,0\right)}_{\text{BS price}}+\underbrace{\int_0^T \Delta^{\hat{\sigma}}dS}_{\text{zero-cost replication strategy}} = \underbrace{BS^{\hat{\sigma}}\left(S_T,T\right)}_{\text{the payoff we want to replicate}} - \underbrace{\frac{1}{2}\int_0^T \Gamma^{\hat{\sigma}}S^2\left(\sigma_r^{\,2}-\hat{\sigma}^2\right)dt}_{\substack{BS\ misreplication \\ =\text{realized-implied variances weighted by BS gamma}}}$$

•Now, taking risk-neutral expectations under the "real world" dynamics we get:

$$\underbrace{BS^{\hat{\sigma}}\left(S_0,0\right)}_{BS\text{ price when vol is (incorrectly) anticipated to a constant }\hat{\sigma}}+\underbrace{E\left[\int_0^T \Delta^{\hat{\sigma}}dS\right]}_{=0,S:martingale} = \underbrace{E\left[BS^{\hat{\sigma}}\left(S_T,T\right)\right]}_{\substack{=E\left[\left(S_T-K\right)^+\right] \\ \text{"true" price when vol is correctly anticipated}}} \underbrace{-\frac{1}{2}E\left[\int_0^T \Gamma^{\hat{\sigma}}S^2\left(\sigma_r^{\,2}-\hat{\sigma}^2\right)dt\right]}_{\text{replication bias = expected }BS\text{ replication error}}$$

•Or rearranged:

$$\underbrace{E\left[BS^{\hat{\sigma}}\left(S_T,T\right)\right]}_{\substack{=E\left[\left(S_T-K\right)^+\right] \\ \text{"true" price when vol is correctly anticipated}}} = \underbrace{BS^{\hat{\sigma}}\left(S_0,0\right)}_{BS\text{ price when vol is (incorrectly) anticipated to a constant }\hat{\sigma}} \underbrace{+\frac{1}{2}E\left[\int_0^T \Gamma^{\hat{\sigma}}S^2\left(\sigma_r^{\,2}-\hat{\sigma}^2\right)dt\right]}_{\text{replication bias}}$$

•The "correct" price is the Black-Scholes price plus the expected replication error when hedging with BS

•This replication bias is the accumulation of realized – implied variances weighted by the *BS* gamma

An important comment

•In English the fundamental theorem states that:

➢ *The difference in value between 2 models is the expected PnL of delta-hedging with one in the world described by the other*

➢ *This, in turn, is the cumulated difference in variances weighted by (a half of) the management gamma*

•One consequence is (base model = real world):

If we trade with a mis-specified and/or mis-calibrated model, we slowly bleed PnL (not blow) as realized vol is biased vs implied

•Another important consequence is the *robustness* of Black-Scholes in the face of volatility

➢When vol is not constant (and it never is) BS remains valid → explains PnL as realized-implied vol

➢But then, we take a risk on volatility: if realized vol is different than planned, we bleed (or make) money as we delta-hedge

•The question is: is that volatility risk acceptable?

➢If yes, Black-Scholes remains valid in the face of stochastic volatility, just have a good prediction of the realized vol

➢If not (as generally accepted in Investment Banks) volatility risk must be hedged

Hedging volatility risk

- How can we hedge volatility risk?
  - We must trade instruments that depend on volatility
  - Options are hedged with options
  - If we hedge vol, we need to model the behaviour (and vol) of the vol
  - Then (and only then) we need more complex model that describe the vol process: local vol, stochastic vol, ...

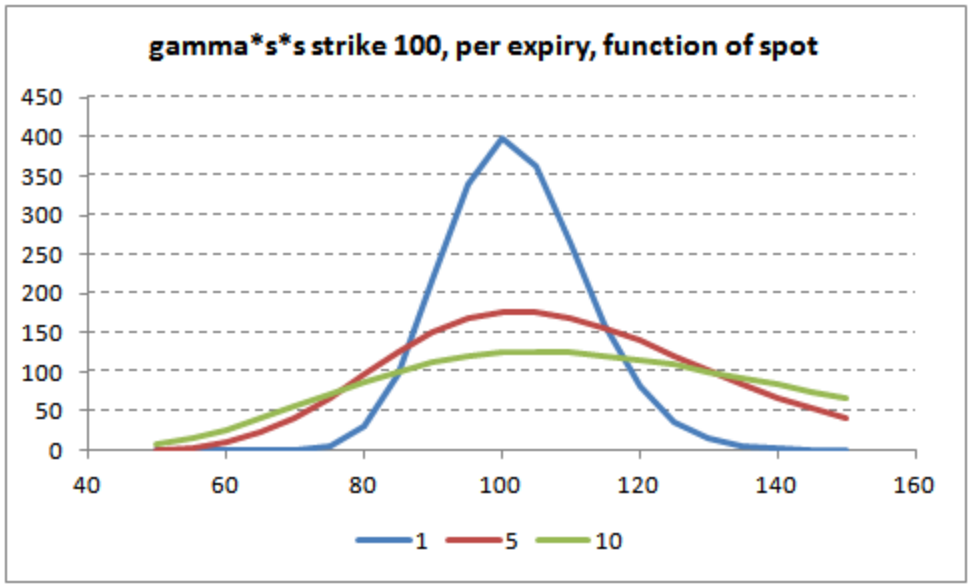
- Hence, the choice is:
  - Trade with Black-Scholes and accept the volatility risk
  - Or hedge volatility risk with more complicated volatility models

- Hence:
  - We need complicated models *not* because Black-Scholes assumptions are unrealistic
  - But because (and only because) we want to cover volatility risk

    by trading liquid options (typically ATM) to cover less liquid options (OTM and exotics)

Gamma

•In BS $\Gamma S^2 = S\frac{n(d1)}{\sigma\sqrt{T}}$



•Most gamma is experienced around the strike and near expiry

•Far from the strike and/or long time to expiry → little gamma → little sensitivity to *realized* vol

Interview questions

•I buy an ATM 1y call for an implied vol of 15% and hedge it in BS(15) until maturity

•The realized volatility over the hedge period ends up 17.5%

•Then I do my accounting and realize I lost money?? How is that possible?

•For xMas I receive an OTM call as a gift
(say a 1y 2200 S&P call when S&P is worth 2000)

•I am grateful but do not believe S&P will rise this year

•My option is worth 120 with a 15% vol
but I have no access to options markets and can't sell it

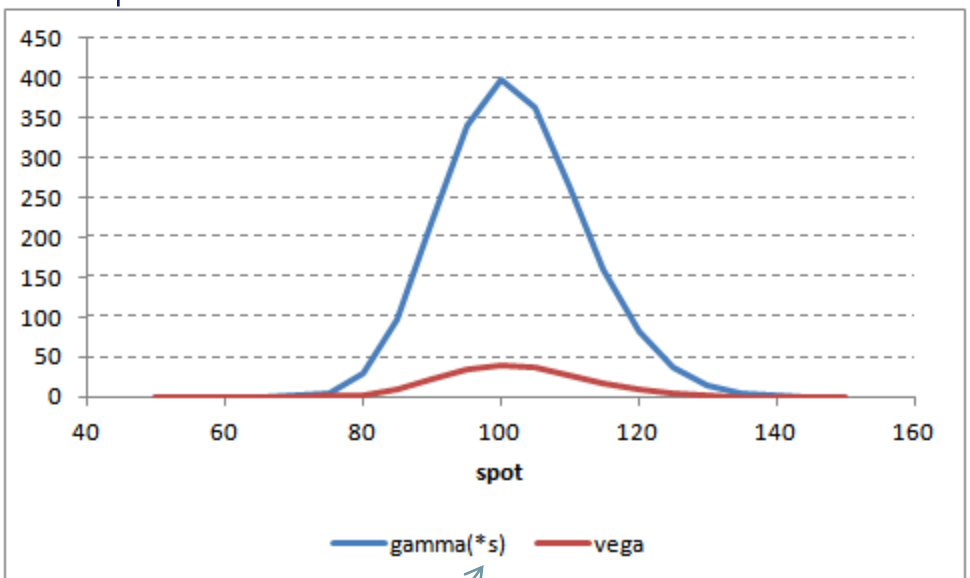
•So I decide to monetize it by delta hedging to replicate the $120 +/- realized vol

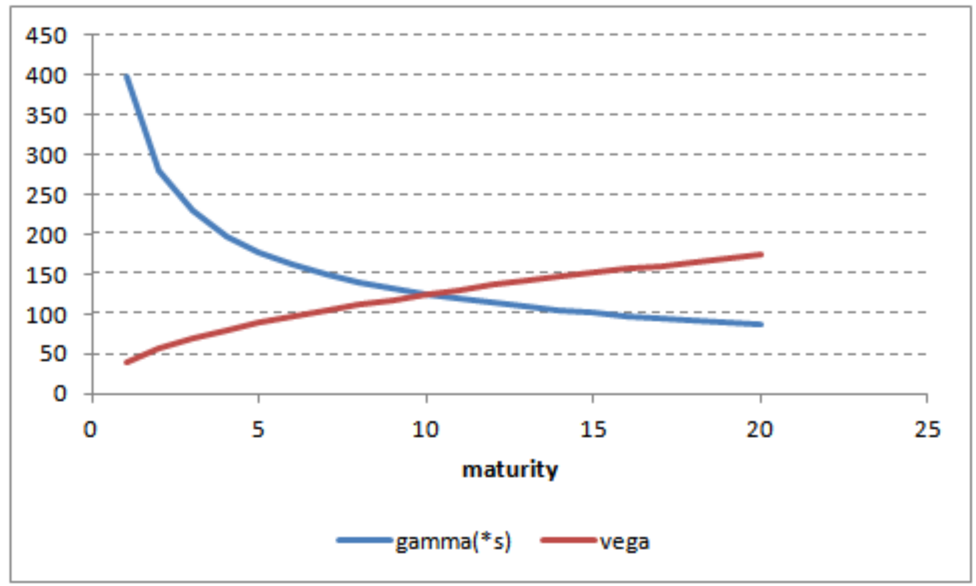
•Is it possible that I end up loosing money overall?? With a free option?

Vega trading

•Vega = sensitivity to *implied* vol $vega = \frac{\partial C}{\partial \sigma} = Sn\left(d1\right)\sqrt{T}$

•For a 1Y strike 100 with 10% vol:





$$\Gamma S^2 = S\frac{n\left(d1\right)}{\sigma\sqrt{T}}$$

Vega trading

- Long term options have little gamma
  - Delivered volatility not important
  - Bulk of risk explained by daily moves of *implied* vol
  - Requires whole different skill set and market expertise

- Most trading houses separate long term option trading into “vega desks”

- Gamma Trading
  - Maturity <3-5y
  - Focus on daily spot moves
  - Strong expertise in spot markets

- Vega Trading
  - Long maturity
  - Focus on moves of implied vol
  - Strong expertise in options and exotics markets and how they impact market price of volatility

Appendix 1:
A more rigorous demonstration of Black & Scholes

•Our approach was more intuitive than rigorous

•See "Call prices are convex in strike so they should be convex in spot as well"

•We made important points by approaching the problem from the point of view of option trading

•And from there we showed that prices must satisfy the PDE $\frac{\delta C}{\delta t}+\frac{1}{2}\frac{\delta 2C}{\delta S^2}\sigma^2 S^2=0, C(S_T,T)=(S_T-K)^+$

•The solution of the PDE is given by Feynman-Kac's theorem $C_t=E^{RN}\left[(S_T-K)^+\right], RN:\frac{dS}{S}=\sigma dW$

•And the expectation can be computed analytically $C_t=SN(d_1)-KN(d_2)$

•Now we present a more rigorous demonstration

Demonstration of positive gamma

•Black & Scholes assumption: $\frac{dS}{S} = \mu dt + \sigma dW$ where the volatility is constant and the drift is undefined

•Also *assume* $C = C(S,t)$

•By Ito: $dC = \frac{\delta C}{\delta S} dS + \frac{\delta C}{\delta t} dt + \frac{1}{2} \frac{\delta 2C}{\delta S^2} \sigma^2 S^2 dt$

•So the PnL of a delta-hedged long position reads $d\pi = \left(\frac{\delta C}{\delta t} + \frac{1}{2} \frac{\delta^2 C}{\delta S^2} \sigma^2 S^2 \right) dt$

•This is deterministic, has no risk, so *by arbitrage* it must be *0*: $\frac{\delta C}{\delta t} + \frac{1}{2} \frac{\delta 2C}{\delta S^2} \sigma^2 S^2 = 0$

•Theta negative (shorter maturity, all the rest constant) ➔ gamma positive ➔ demonstration complete

•However we assumed $C = C(S,t)$, to demonstrate this we must approach the problem from the other end

A more rigorous demonstration

•More general context: $\frac{dS_t}{S_t} = \mu_t dt + \sigma_t d\tilde{W}_t, \sigma_t = \sigma(S_t, t), \mu_t$ undefined

•We define: $\lambda_t \equiv \frac{\mu_t}{\sigma_t}$ the "risk premium", then $\frac{dS_t}{S_t} = \sigma_t \left(d\tilde{W}_t + \lambda_t dt \right) = \sigma_t dW_t$

Where $dW_t = d\tilde{W}_t + \lambda_t dt$ is a Brownian Motion under an *equivalent* probability defined by Girsanov's removal of the the drift
– since the drift corresponds to the risk premium, we call it "risk-neutral probability" and denote it Q

•We consider a general *payoff* $X = f(S_t)_{0 \le t \le T}$ a functional of the path of the spot

•Now we ***define***: $C_t \equiv E_t^Q[X]$

•And prove that this is indeed the price of the option that pays *X* on expiry

•Since this is obviously a martingale under Q, that completes the demonstration

The case of a European option

•When $X = f(S_T)$ what can we say of C_t ?

1. The spot process is *Markov* under Q, hence $C_t \equiv E_t^Q[X] = h(S_t, t)$ is a function of S_t

   and by Ito: $dC_t = h_S dS_t + \left(h_t + \frac{1}{2} h_{SS} \sigma_t^2 S_t^2\right) dt$

2. Obviously (C_t) is a martingale under Q as is (S_t) so $h_t + \frac{1}{2} h_{SS} \sigma_t^2 S_t^2 = 0$ and $C_T = C_t + \int_t^T h_S dS_u$

3. Hence, $C_T = X = \underbrace{C_t}_{\text{amount known at t}} + \underbrace{\int_t^T h_S dS_u}_{\text{zero cost strategy}}$ so by arbitrage, (C_t) is *indeed* the price of the option that pays *X*

Therefore the price is the (conditional, risk-neutral) expectation of the payoff $C_t = E_t^Q[C_T] = E_t^Q[X]$

The case of path dependence

•More generally when $X = f(S_t)_{0 \le t \le T}$

•Then $C_t \equiv E_t[X] = h\left(S_t, t, (S_u)_{0 \le u < t}\right)$ is also a function of the path before *t*

•Theorem: the "Ito" formula $dC_t = h_S dS_t + h_t dt + \frac{1}{2} h_{SS} S_t^2 \sigma_t^2 dt$ still holds

Bruno Dupire, 2009, "Functional Ito Calculus", *a contribution from Finance to Mathematics*

•Since (C_t) is a martingale, the demonstration is complete

How did we do before Dupire's Functional Ito Calculus?

•*X* is F_T measurable so it satisfies the Martingale Representation Theorem:

$$X = \int_0^T a_t dt + \int_0^T b_t dW_t = \int_0^T a_t dt + \int_0^T \frac{b_t}{S_t \sigma_t} dS_t = \int_0^T a_t dt + \int_0^T c_t dS_t$$

where *a* and *b* are *adapted* processes

•So for $C_t \equiv E_t[X] = \int_0^t a_u du + \int_0^t c_u dS_u + E_t\left[\int_t^T a_u du\right]$ we have $dC = a_t dt + c_t dS_t - a_t dt = c_t dS_t$ hence $X = C_T = C_t + \int_t^T c_u dS_u$

•Therefore (C_t) is still the price of the option and it is still a *Q* martingale so the demonstration is complete

•To properly prove that the hedge is the still the delta and that the theta/gamma relation holds we need FSC

Appendix 2: what if we do have rates, dividends etc?

•Black (without Scholes) provided the (rather trivial) extension:

$$V = DF(0,T)\,BS(F,\sigma_F)$$

discount factor to maturity

replace spot by forward to maturity

vol that matters is vol of forward
(different when rates are stochastic)

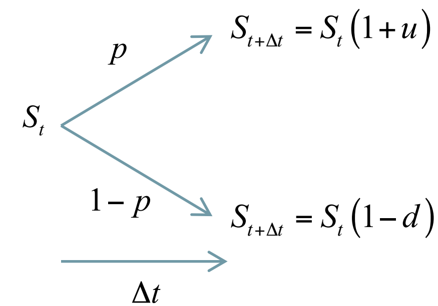
- Work with the forward, not the spot, and discount the end result $(S_T - K)^+ = (F_T - K)^+$
- Proof: we work under the "forward-neutral" martingale measure associated to the numeraire *DF(.,T)*
  Under this measure the forward is a martingale and so is the option price and the result follows

$$C = DF(0,T)\left[FN(d_1) - KN(d_2)\right], P = DF(0,T)\left[KN(-d_2) - KN(-d_1)\right]$$

•What about the put?

- Right without obligation to sell the underlying for the strike price, payoff $(K - F_T)^+$
- Immediately, at maturity, $C_T - P_T = (F_T - K)^+ - (K - F_T)^+ = F_T - K$
- Hence, by arbitrage, we have the *call/put parity* $C - P = DF(0,T)(F_T - K)$
- Note **the call-put parity is model independent**: it holds in all models (local vol, stochastic vol, …) not only Black-Scholes
- In Black-Scholes, it follows immediately that $P = DF(0,T)\left[KN(-d_2) - FN(-d_1)\right]$

Workshop: the binomial tree

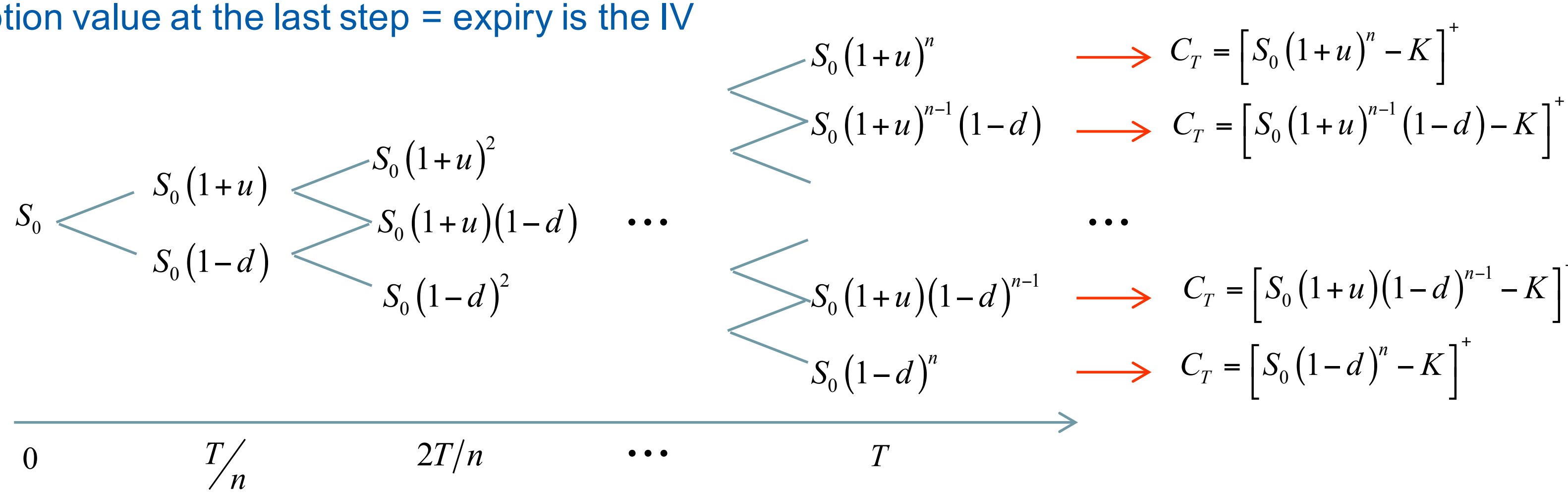


•Over a small time step, the asset price return has 2 possible outcomes: *up* and *down*

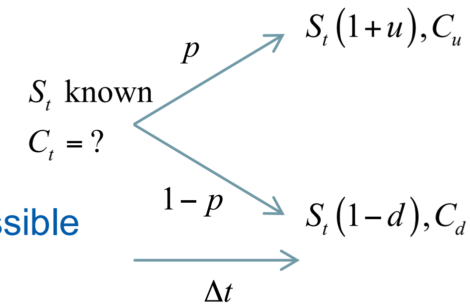
•*u* and *d* are known, *p* is undefined

•This assumption is consistent with Black & Scholes : $\left(\frac{dS}{S}\right)^2 = \sigma^2 dt \Rightarrow " \left|\frac{dS}{S}\right| = \pm\sigma\sqrt{dt} "$

•The option value at the last step = expiry is the IV



Binomial tree (2)



•Now for a given node, if we know the option values for the 2 possible outcomes on the step, what is the value at that node?

•I can replicate the option value in both outcomes by holding an amount in cash and some units of the asset

$$\begin{cases} \alpha + \delta\left[S_t(1+u) - S_t\right] = \alpha + \delta u S_t = C_u \\ \alpha - \delta d S_t = C_d \end{cases} \Leftrightarrow \begin{cases} \alpha = \dfrac{d}{u+d} C_u + \dfrac{u}{u+d} C_d \\ \delta = \dfrac{C_u - C_d}{(u+d) S_t} \end{cases}$$

•The value is the value of the replication portfolio, that is the amount of cash we need $C_t = \alpha = q_u C_u + q_d C_d, q_u = \dfrac{d}{u+d}, q_d = \dfrac{u}{u+d}$

•We note that $0 \le q_u, q_d \le 1$ and $q_u + q_d = 1$ so the *q*'s are probabilities and in regards of these probabilities $C_t = E_t^q\left[C_{t+\Delta t}\right]$

•Option prices at all nodes = conditional expectations of their values on the 2 possible next nodes and we note:

- We never used the "real" probability *p*, it is irrelevant, we only use the "risk-neutral" probability $q = d/(u+d)$
- We found the result by pure replication, we only noted after hand that it may be seen as an expectation
- Under the probability *q* $E^q{}_t\left[S_{t+\Delta t} - S_t\right] = 0$ so *S* is a *martingale* under *q*
- Replication values are expectations of future values under the probability where the asset is a martingale

Binomial trees and American options

•We illustrated risk-neutralisation by replication and arbitrage in a simplified, discrete, intuitive context

•We can compute today's option value by a backward sweep through the tree

•American options are similar to Europeans except they can be exercised at any time before expiry

•(Easy) exercise: show that in the absence of rate, dividends, etc.
the holder never exercises before expiry ➔ American options are in fact European options

•With rates and/or dividends, American options may be optimally exercised before expiry, hence extra value

•American options cannot be analytically computed in the Black & Scholes model

•However, they can be computed in the binomial tree:

- On each node I know the value of exercising = IV = f(spot)
  and the value of not exercising = option value = conditional expectation under *q* of the future value in the 2 possible outcomes
- I assign the option value for the node as the maximum of the two, and continue the recursion

•Since the Binomial tree is a correct discretization of the Black & Scholes model when $u,d = \sigma\sqrt{T}$
this may be seen as a numerical implementation of Black-Scholes that can handle Americans

•It is also an arbitrage-free model on its own right (contrarily for instance to discrete PDEs)

Notes on binomial trees

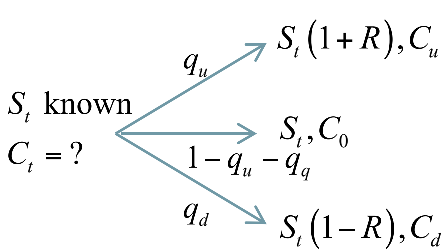
•First introduced by Cox, Ross and Rubinstein in 1979 and used to value transactions with early exercises

•Now completely deprecated:

- More sophisticated models cannot be implemented in binomial trees, ex. local volatility ➔ tree no longer *recombines*
- Emergence of more performant (by order of magnitude) numerical methods based on Finite Differences

•However the model keeps its pedagogical value as it exposes us directly
to replication arguments leading to the construction of the risk-neutral probability in a simplified intuitive setting

•Final comment: consider now the so-called trinomial tree (3 outcomes for each node). Replication arguments no longer hold (cannot replicate 3 outcomes with 2 control variables) but say (for the sake of the argument) that we can still price by expectation under the equivalent probability where the asset is a martingale:



- So that S is a martingale with vol / $\sigma^2_t\Delta t = V\left[\frac{\Delta S}{S}\right]$ we have $q_u = q_d = \frac{1}{2}\frac{\sigma^2}{R^2}\Delta t$
- Then $C_t = E^q{}_t\left[C_{t+\Delta t}\right] = \left(1-\frac{\sigma^2}{R^2}\Delta t\right)C_0 + \frac{1}{2}\frac{\sigma^2}{R^2}\Delta t\left(C_u + C_d\right)$ ie. $\frac{C_t - C_0}{\Delta t} = \frac{\sigma^2}{2}S_t^2\left(\frac{C_u + C_d - 2C_0}{S_t^2 R^2}\right)$
- The LFS is a discrete -theta, the term between () in the RHS is a discrete gamma
  ➔ we get the (discrete) PDE $2\vartheta = -\Gamma S^2\sigma^2$ so we have a discrete Feynman-Kac

Programming Assignment 1:
Gamma trading in action

•Use daily S&P data from the vol estimation sheet and your implementation of BS and Greeks in C++ Excel

•Pick a strike and a management volatility = implied volatility at the start. Place these numbers in a cell

•We purchase at the 1st date of a call with your strike for a price = BS(management vol)
and delta hedge throughout the period with the same management vol

•Compute the daily PnL both actual and theoretical (use daily gamma) and the cumulated PnL. Draw charts

•Change the management vol. What happens? What management vol gives a final PnL of 0?
This is an alternative historical estimate called ”break-even vol”.
How does it compare to the standard statistical estimate of the vol?
What would be the break-even vol if we used a portfolio of options with constant gamma?

•Change the strike. What happens? Can you make sense of it?
Write a report with all the results and charts and your thoughts in light of the theory

Written Assignment 1

•Resolve the exercise on the same pdf document as the report from the programming assignment

•Manage long call position in Black & Scholes, vol = 15.81%

•Daily hedge - 250 evenly spread business days in a year

•Daily break-even vol = 15.81% / sqrt(250) = 1%

•If the spot rallies by 1.2% every day on a straight line (0.05% per hour),
What is the realized volatility? What is the realized drift?

•Are we making or loosing money? Why is the answer a paradox?

•Now we decide to hedge twice daily. Are we making or loosing now?
What is we hedge four times a day? What if we hedge continuously?

•Conclude

Volatility Modeling and Trading

Module 2: The market implied volatility smile

Antoine Savine

Extensions to Black-Scholes

•The main result that (in absence of rates, repo, dividends, etc.) we have:

$$C = E\left[Payoff\right], \frac{dS}{S} = \sigma dW, \Delta = \frac{\delta C}{\delta S}$$

•Holds:

1. For arbitrary payoffs, not only European calls or puts ➔ Exotics, module 3
2. **When vol is not constant**: may depend on time, spot, or may be stochastic itself on its own right

•Remember:

- Option values are NOT expectations
  But they can be computed as expectations under dynamic assumptions where all *drifts are removed* ("risk-neutral")
- But whatever you assume on vol in the real world *holds in the pricing (risk-neutral) world*

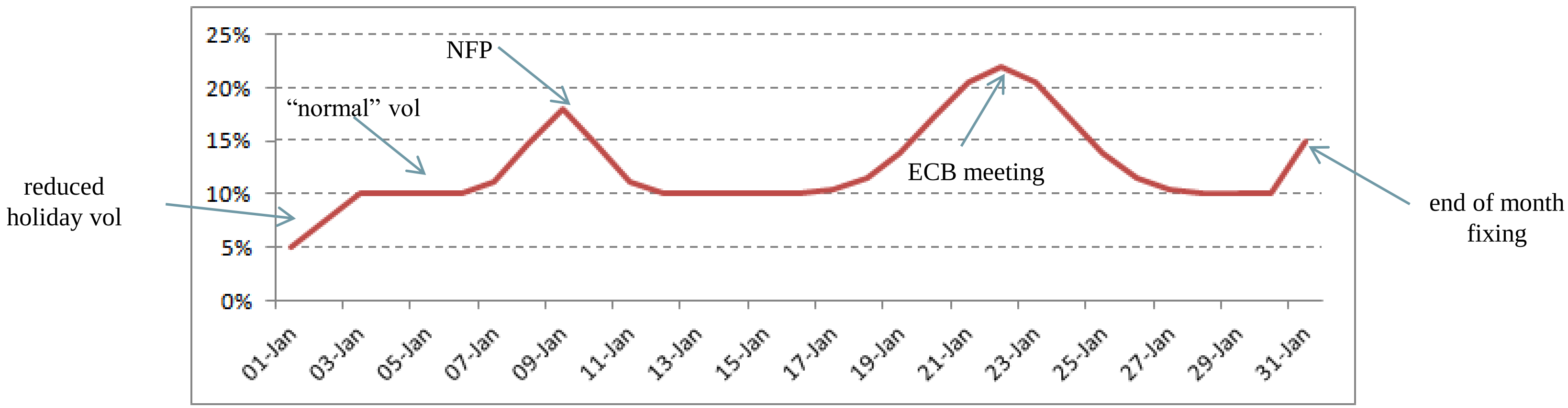
Real world assumption	$\frac{dS}{S} = \underbrace{\mu dt}_{\text{scrapped}} + \underbrace{\sigma dW}_{\text{identical}}$
Pricing dynamics	$\frac{dS}{S} = \sigma dW$

Warm-up: Black-Scholes with time-dependent volatility

•Extension of Black-Scholes where volatility depends on time but is still known from the onset –or *deterministic*:

$$\frac{dS}{S} = \sigma(t)\,dW$$

•Used to model increased volatility around important events happening at a known date: company results, NFP, central bank meetings, … Below typical January for EUR/USD



•Also used to explain the term structure of implied volatility

Pricing with time-dependent volatility

- In the extended Black-Scholes model: $\frac{dS}{S} = \sigma(t)\,dW$
- The price of European options is given by: $C = BS\{vol = \hat{\sigma}\}$
- Where the *implied* vol: $\boxed{\hat{\sigma}(T)^2 = \frac{1}{T}\int_0^T \sigma^2(t)\,dt}$ is the (quadratic) average of *local* vols $\sigma(t)$
- Hence, *Black and Scholes's formula holds*, but with a vol that is the (quadratic) average of local vols
- Proof: in the extended model we have $S_T = S_0 \exp\left[-\frac{1}{2}\int_0^T \sigma^2(t)\,dt + \int_0^T \sigma(t)\,dW\right]$

And $\int_0^T \sigma(t)\,dW_t$ has probability distribution $\sqrt{\int_0^T \sigma^2(t)\,dt}\,N$

So the final spot has probability distribution $S_T = S_0 \exp\left[-\frac{1}{2}\int_0^T \sigma^2(t)\,dt + \sqrt{\int_0^T \sigma^2(t)\,dt}\,N\right]$

And we know that $C \underset{\text{risk neutralization holds}}{=} E\left[(S_T - K)^+\right] \underset{\text{calculus}}{=} BS\left\{spot = S_0, vol = \sqrt{\frac{1}{T}\int_0^T \sigma^2(t)\,dt}\right\}$

Piecewise constant local vol

•A (very) frequent particular case is where local vols $\sigma(t)$ are constant between times $T_0 = 0, T_1, \ldots, T_n$

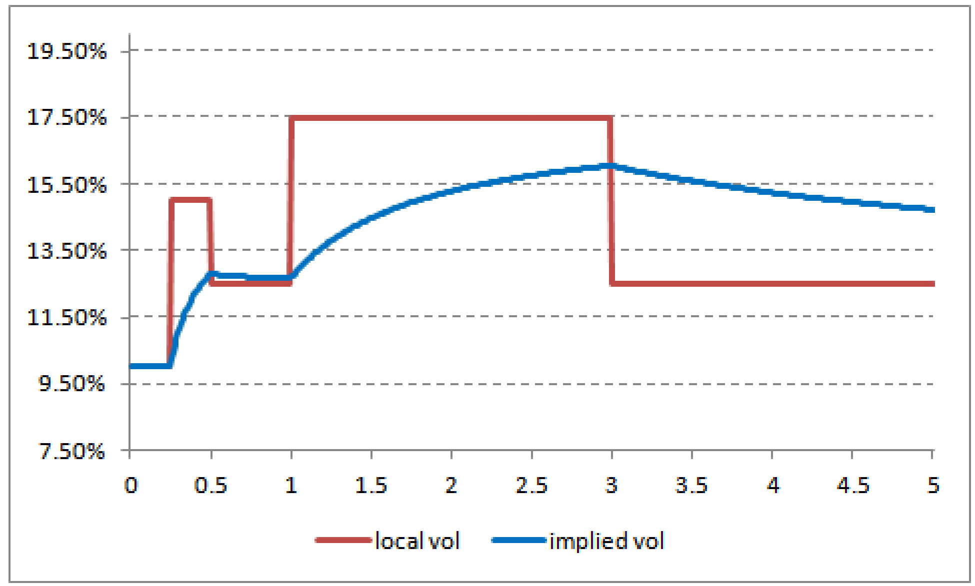
•In this case formulas reduce to

➢ local vol → implied vol

$$\hat{\sigma}^2(T_i) = \frac{\sum_{j=1}^{i} \sigma_j^2 (T_j - T_{j-1})}{T_i}, \hat{\sigma}^2(T_i < T \le T_{i+1}) = \frac{\hat{\sigma}^2(T_i) T_i + \sigma_{i+1}^2 (T - T_i)}{T}$$

➢ reversely, implied vol → local vol

$$\sigma_j^2 = \frac{\hat{\sigma}^2(T_j) T_j - \hat{\sigma}^2(T_{j-1}) T_{j-1}}{T_j - T_{j-1}}$$



•Also called *forward* volatility between 2 consecutive expiries

Assignment 2: Time-Dependent Volatility

•Assume “normal” (i.e. customary or *daily*) EURUSD volatility is σ

•Now = Monday morning in Asia

•ECB meeting Thursday 2pm in London. Volatility expected twice normal that day.
No other significant events expected.

•What should be the implied overnight volatility (expiry Tuesday 4pm in London)?

•What should it be for Friday 4pm expiry?

•Market quotes 12.65% for the Friday expiry.
What is the fair implied for the 4pm Tuesday? Wednesday? Thursday?

•Market quotes 10% for Tuesday, 12.65% for Friday.
Bank XYZ is known to linearly interpolate volatilities, without event weighting.
What are XYZ’s quotes for Wednesday and Thursday?
What forward volatility do they imply for Wednesday to Thursday?
How do you arbitrage XYZ?

Market implied volatility "smile"

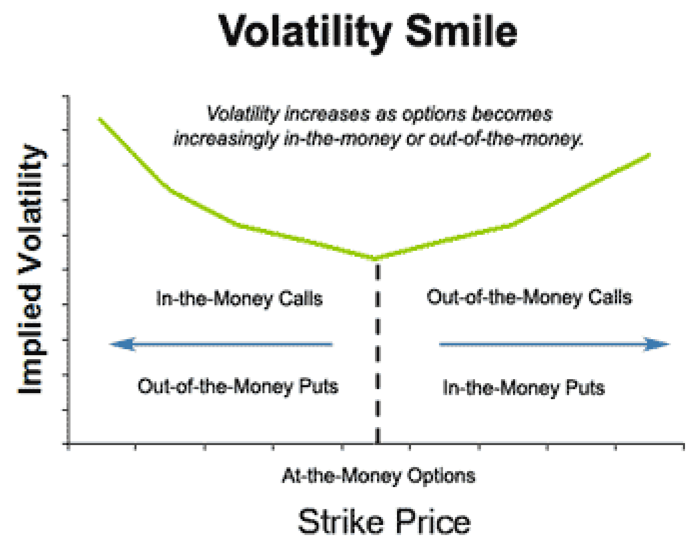
•Markets quote options of several strikes per expiry

•Market options prices ➔ implied volatilities

•Implied volatilities different across strikes!

•OTM options (or *wings*) typically more expensive

•Typical shape of a volatility curve function of strike looks like a "smile" ➔ the name stuck



•In addition, equity and other indices typically have a smile that is *skewed* towards lower strikes (OTM puts more expensive than OTM calls)

•The slope of the smile –by how much OTM puts are more expensive than OTM calls- is often called *skew*

•Its curvature –by how much OTM calls and puts are more expensive than ATM- is often called *kurtosis*

Pricing with smile: European options

To price a call of strike *K* for some maturity *T*:

•Pick the implied vol for *K* $\hat{\sigma}(K)$ on the implied volatility curve (smile) for expiry *T*

•Use BS to price the option: $C(K) = BS\left[K, \hat{\sigma}(K)\right]$

Note we need a continuous IV surface as a function of maturity and strike

•The market provides (at best) market prices of a discrete number of maturities and strikes

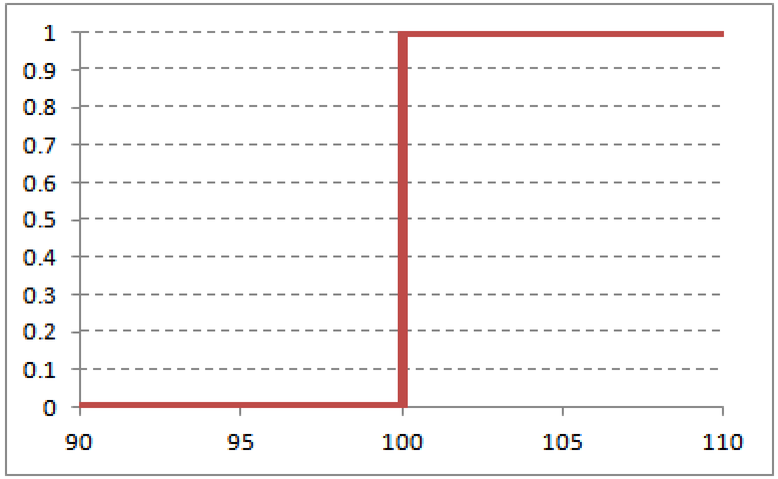
•So we need some *interpolation* scheme to produce the full IV surface

•This is (much) harder than it sounds:

- Interpolation must not break arbitrage constraints on option prices: increasing in maturity, decreasing and convex in strike
- Must be smooth so models can calibrate to it (module 3)
- (Bit) more in module 3
- In all that follows we assume a full IV surface is *given*

Pricing with smile: Digital options

•Digital or *Binary* option: pays 1$ when asset price terminates above (Digital call) or below (Digital put) strike



•Very popular, in particular for retail and private banking and with hedge funds

•Offers direct "bets", price seen as "odds"

•Market makers act as "bookmakers"

•But not really:

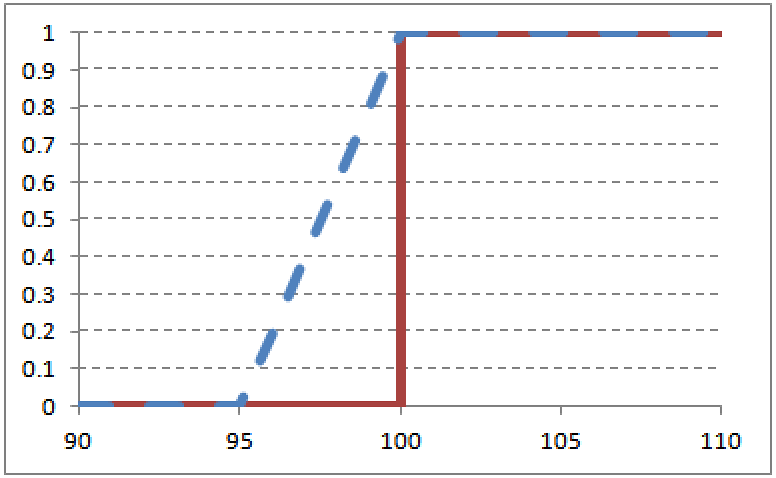
- Bookmakers quote odds as a consensus among clients, equilibrium price reflects clients views, bookmakers hold no risk
- Whereas the Digital is an option that is replicated (or hedged)
- Hence its price only depends on volatility assumptions, not trend or views or risk aversion
- Therefore the fair price may deviate from consensus, providing trading opportunities

•(Frequent) interview question : what is the price of a digital?

- In Black-Scholes?
- With smile?

Digital options: solution

•In Black-Scholes: $P^{RN}\left(S_T > K\right) = N\left(d_2\right)$



•Digital: approximated by a call spread

long call low strike, short call high strike

$$D(K) \approx \frac{C(K-\varepsilon) - C(K)}{\varepsilon} = -\frac{\delta}{\delta K} C(K)$$

•And since with smile $C(K) = C\left[K, \hat{\sigma}(K)\right]$

•We get $D(K) = -\frac{\delta}{\delta K} C\left[K, \hat{\sigma}(K)\right] = -\delta_1 C\left[K, \hat{\sigma}(K)\right] - \hat{\sigma}'(K)\,\delta_2 C\left[K, \hat{\sigma}(K)\right] = N\left(d_2\right) - \hat{\sigma}'(K) \cdot vega$

BS price

smile adjustment = BS vega x slope of IV at K

Digital options: example

•1Y ATM digital call

•Spot = 100, Vol = 15%

•Skew = 0.25%: vol up 2.50% when strike down 10 points, for instance IV for strike 90 is 17.50%

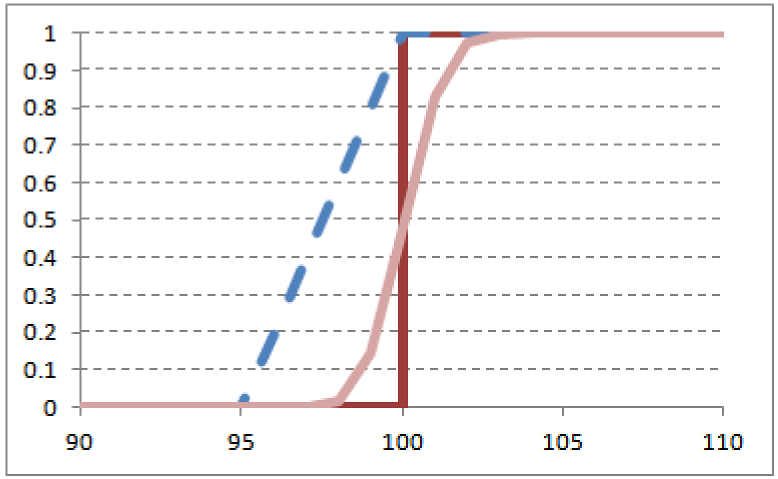
•Exercise

1. Give a ball park figure for the 1Y ATM Digital in Black-Scholes
2. Calculate the exact value in Black-Scholes, validate the approximation
3. Compute the smile adjustment using the formula in previous slide, what is the full price with skew?
4. On Excel with the BS formula coded in VB, compute the price by replication with a call spread, validate the formula
5. Comment on the size of the skew adjustment

•Solution: ~50% in Black & Scholes, ~10% skew adjustment, very significant in a realistic scenario

Digital options: risk management

•Risk management conundrum:
close to expiry around the strike, delta grows to infinity!



•This scenario is more common than one may think due to *feedback effects*

➔Delta-hedging pushes markets towards the strike at expiry

•Solution:

Instead of Digital,

Sell, book and hedge a *super-replicating* call spread

➔No more infinite delta

➔Good “surprise” on expiry when Digital expires OTM and call spread yield non-zero payout

➔Best practice: investment of (part of) profit margin into risk management

Digital options: conclusion

- Looks and feels like an *exotic*, but in fact combinations of Europeans
- *Static* European hedge
- Price = direct function of market smile, model independent

$$\boxed{D(K) = -\frac{\delta}{\delta K} C\left[K, \hat{\sigma}(K)\right] = -\delta_1 C\left[K, \hat{\sigma}(K)\right] - \hat{\sigma}'(K)\,\delta_2 C\left[K, \hat{\sigma}(K)\right] = N(d_2) - \hat{\sigma}'(K) \cdot vega}$$

- Part of the family of *statically hedged* – or *model independent* exotics
- These exotics have the same value and hedge in all models calibrated to the same smile
- This is a particularly interesting family of products - another case of interest is the *variance swap* (module 4)

What causes the volatility smile?

•Volatility smile may be seen as an aberration:
how can the same underlying exhibit different volatility depending on the strike of options written on it??

•Textbook answer:
volatility smile is the market's way of contradicting Black-Scholes assumption of a deterministic volatility and more generally correcting for BS's simplifying description of reality

•More specifically:
In what ways exactly does reality contradicts BS's assumptions and cause a smile?

1. **Local vol**: volatility depends on the spot price
2. **Stochastic vol**: volatility itself is unknown and moves randomly
3. **Jumps**: time series of asset prices are not continuous

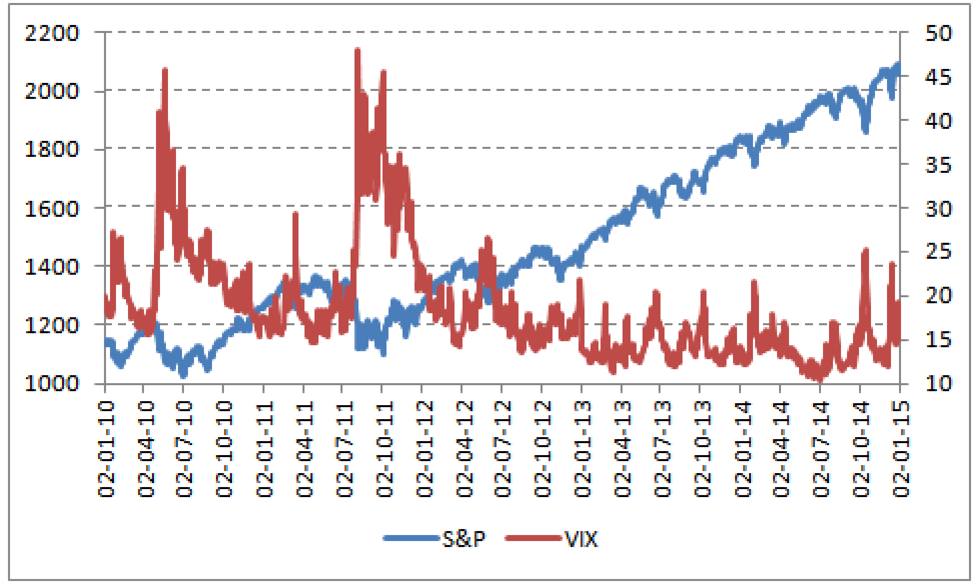
Local volatility

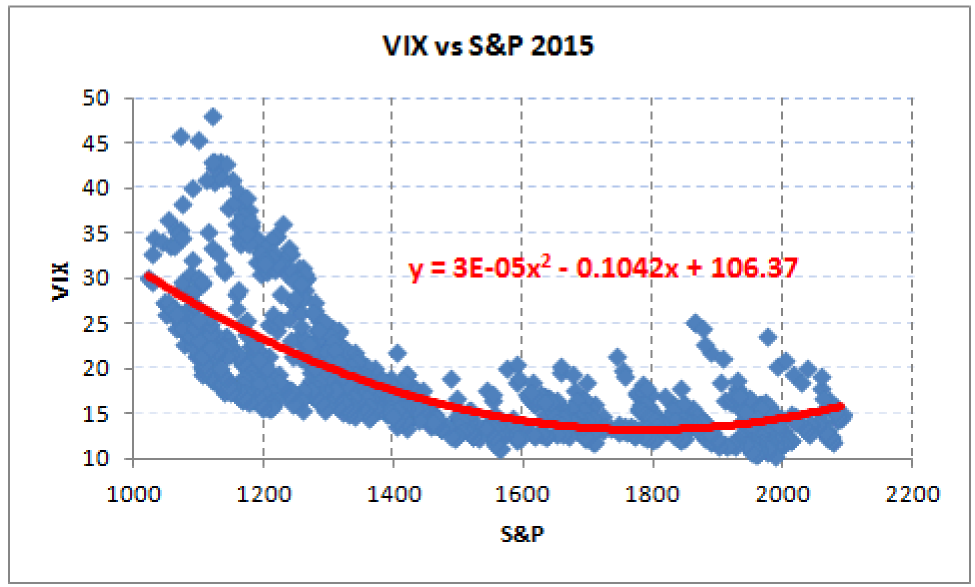
•Bear markets generally more nervous
➔Volatility higher when spot is lower

•This relationship is not linear
➔Volatility higher than expected when spot is "very low" or "very high"

•**Volatility is a function of spot**

•Bruno Dupire (1992) first theorecised the volatility smile and local volatility in a model that remains a strong market standard today (module 3)





Pricing with local volatility

•We extend the Black-Scholes assumptions with local volatility: $\frac{dS}{S} = ... + \sigma(S,t)\,dW$

•We know options prices are risk-neutral expectations of their payoff

•Under the risk-neutral dynamics: $\frac{dS}{S} = \sigma(S,t)\,dW$

•Problem: no formula for European options

•Numerical solutions:

- Solve the extended Black-Scholes PDE with a finite-difference scheme $\frac{\partial C}{\partial t} = -\frac{1}{2}\frac{\partial^2 C}{\partial S^2} S^2 \sigma^2(S,t)$
- Solve the forward (Dupire) PDE with FDM to price all options in one sweep (see module 3) $\frac{\delta C(K,T)}{\delta T} = \frac{1}{2}\frac{\delta^2 C(K,T)}{\delta K^2} K^2 \sigma^2(K,T)$
- Implement Monte-Carlo simulations

•Numerics lack intuition about how local vols combine to produce implied vols
➔ For that we have Dupire's Sigma-Zero framework

Local volatility and implied volatility

Local Volatility Function

$\sigma(S,t)$

Function of Spot and Time

Black-Scholes assumptions extended to

$$\frac{dS}{S} = \ldots + \sigma(S,t)\,dW$$

Local Volatility:
Parameter to the extended model,
Assumed vol as depending on spot

Calibration

Parameterize the model
to hit market prices

Resolved by Dupire (1992)
(module 3)

Pricing

Price options given model
parameters

"Sigma-Zero" formula
(also by Dupire)

Implied Volatility Smile

$\hat{\sigma}(K,T)$

Function of Strike and Maturity

Standard Black-Scholes assumptions
for each strike and maturity

$$\frac{dS}{S} = \ldots + \hat{\sigma}(K,T)\,dW$$

Implied Volatility:
Expression of market option prices

Dupire's sigma-zero formula

- We assume real-world dynamics is local vol parameterized by a given LV $\frac{dS}{S} = ... + \sigma(S,t)$

- We model a trader hedging a call with Black-Scholes using an implied vol $\hat{\sigma}$

- We try to quantify what she misses and deduce a fair price in the LV model

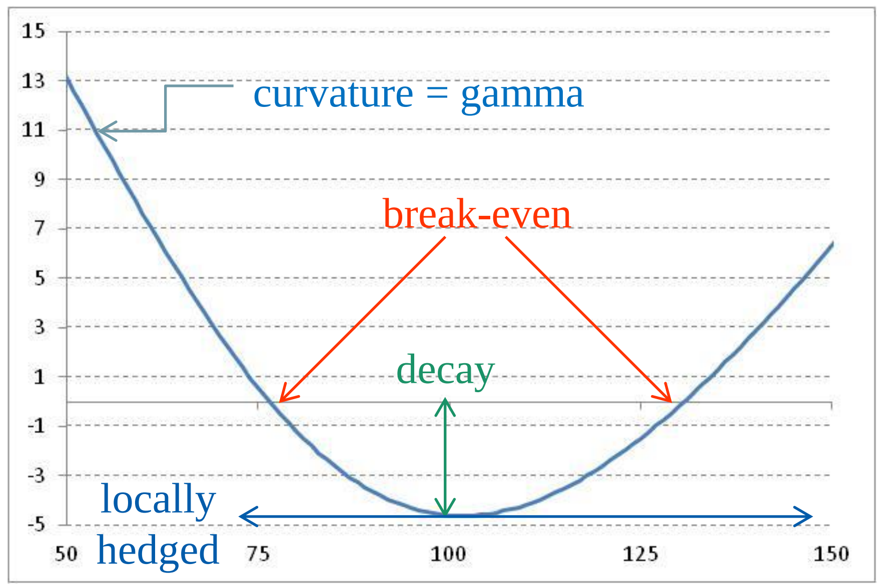
- We know her daily PnL $\Delta PnL = \vartheta \Delta t + \frac{1}{2}\Gamma \Delta S^2$

- Where greeks come from $BS(\hat{\sigma})$

- And moves ΔS^2 come from the "real world" (LV) so

$$\frac{dS}{S} = ... + \sigma(S,t) \longrightarrow E_t^{LV}\left[\Delta S^2\right] = S^2 \sigma^2 (S,t) \Delta t$$

- Where E is the *risk-neutral* expectation operator under the real-world (LV) volatility



Sigma-Zero

- Cumulated expected PnL $E^{LV}\left[PnL\right]=E^{LV}\left\{\int_0^T\left[\vartheta_{BS(\hat{\sigma})}+\frac{1}{2}\Gamma_{BS(\hat{\sigma})}S^2\sigma^2\left(S,t\right)\right]dt\right\}$

- Further we know that Black-Scholes greeks are linked by the Black-Scholes PDE: $\vartheta_{BS(\hat{\sigma})}=-\frac{1}{2}\Gamma_{BS(\hat{\sigma})}S^2\hat{\sigma}^2$

- Hence, $E^{LV}\left[PnL\right]=\frac{1}{2}E^{LV}\left\{\left[\int_0^T\Gamma_{BS(\hat{\sigma})}S^2\left(\sigma^2\left(S,t\right)-\hat{\sigma}^2\right)\right]dt\right\}$

- The option is priced fairly, that is LV is in line with IV, when expected PnL is 0

- And re-arranging the equation we get Dupire's Sigma-Zero formula

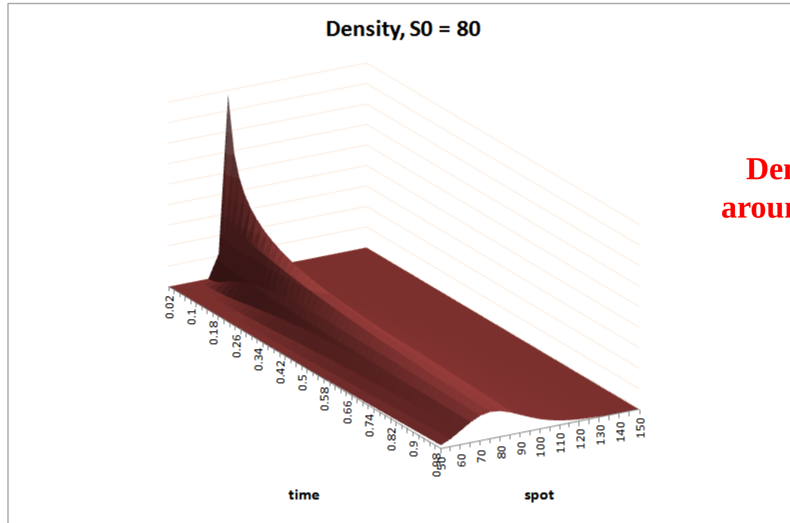
$$\hat{\sigma}^2=\frac{E^{LV}\left\{\left[\int_0^T\Gamma_{BS(\hat{\sigma})}S^2\sigma^2\left(S,t\right)\right]dt\right\}}{E^{LV}\left\{\left[\int_0^T\Gamma_{BS(\hat{\sigma})}S^2\right]dt\right\}}\xrightarrow[\substack{\gamma_{LV}\left(x,t\right)=dens\left(S_t=x\right)\\ \text{(RN) in the LV model}}]{}\hat{\sigma}^2=\frac{\int_0^T\int\gamma_{LV}\left(S,t\right)\Gamma_{BS(\hat{\sigma})}\left(S,t\right)S^2\sigma^2\left(S,t\right)dSdt}{\int_0^T\int\gamma_{LV}\left(S,t\right)\Gamma_{BS(\hat{\sigma})}\left(S,t\right)S^2dSdt}$$

- IV is a weighted (quadratic) average of LVs, in both time and spot spaces, with weights

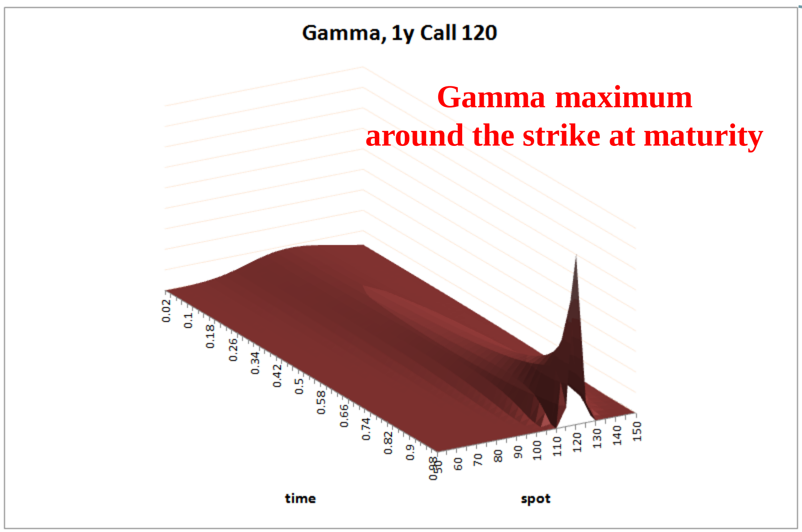
$$w\left(S,t\right)=\underbrace{\gamma_{LV\left[\sigma(S,t)\right]}\left(S,t\right)}_{LVdensity}\underbrace{\Gamma_{BS(\hat{\sigma})}\left(S,t\right)}_{BSgamma}\underbrace{S^2}_{\text{because we look at lognormal vols}}$$

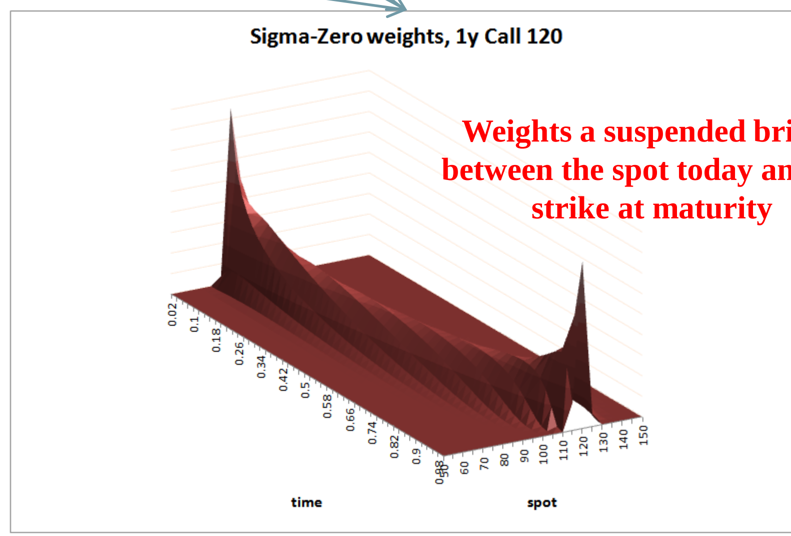
Sigma-Zero weights

$$w(S,t) = \underbrace{\gamma_{LV[\sigma(S,t)]}(S,t)}_{LVdensity}\underbrace{\Gamma_{BS(\hat{\sigma})}(S,t)}_{BSgamma}\underbrace{S^2}_{\text{because we look at lognormal vols}}$$



Density maximum
around the spot today





Sigma-Zero weights: approximations

•Sigma-zero weights are concentrated around the spot today and the strike at maturity

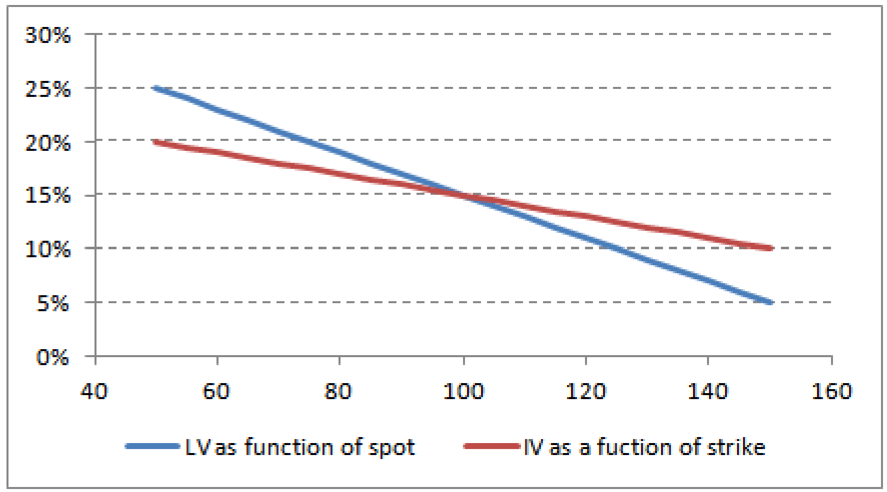
•(Very) roughly $\hat{\sigma}^2(K,T) \approx \frac{\sigma^2(S_0,0)+\sigma^2(K,T)}{2}$ that is $\hat{\sigma}(K,T) \approx \sqrt{\frac{\sigma^2(S_0,0)+\sigma^2(K,T)}{2}}$

•With time-homogeneous LV (depends on spot alone) this result simplifies into: $\hat{\sigma}(K) \approx \sqrt{\frac{\sigma^2(S_0)+\sigma^2(K)}{2}}$

These are very rough approximations
to be used with extreme care
however the following consequences are accurate:

1. ATM IV = LV $\hat{\sigma}(K)\big|_{K=S_0} \approx \sigma(S_0)$

2. ATM IV skew = half of LV slope $\hat{\sigma}'(K)\big|_{K=S_0} \approx \frac{\sigma'(S_0)}{2}$



•Remember these results! Proof: exercise

Sigma-Zero: comments

•Formula not directly usable:

- Implicit = BS IV appears on the LHS and the RHS
- LV densities $\gamma_{LV}(S,t)$ must be computed numerically

$$\hat{\sigma}^2 = \frac{\int_0^T \int \gamma_{LV}(S,t)\Gamma_{BS(\hat{\sigma})}(S,t)S^2\sigma^2(S,t)\,dSdt}{\int_0^T \int \gamma_{LV}(S,t)\Gamma_{BS(\hat{\sigma})}(S,t)S^2\,dSdt}$$

•Yet very powerful

- Important result to understand how LVs combine to produce IVs
- Is important of not only the result but also the reasoning: trading arguments → mathematical result
- May be used to demonstrate complex results, for instance Dupire's fomulas from module 3

•(Advanced) Generalization :
we prove with the same reasoning that
if we have 2 models producing values V_1, V_2
for some option with volatilities σ_1, σ_2
where $\sigma_2(S,t)$ is a local vol
and σ_1 is general (local, stochastic, …) then ➔

$$V_1 - V_2 = \frac{1}{2}E^1\left\{\left[\int_0^T S_t^2\Gamma_2(S_t,t)\left(\sigma_{1t}^2 - \sigma_2^2(S_t,t)\right)\right]dt\right\}$$

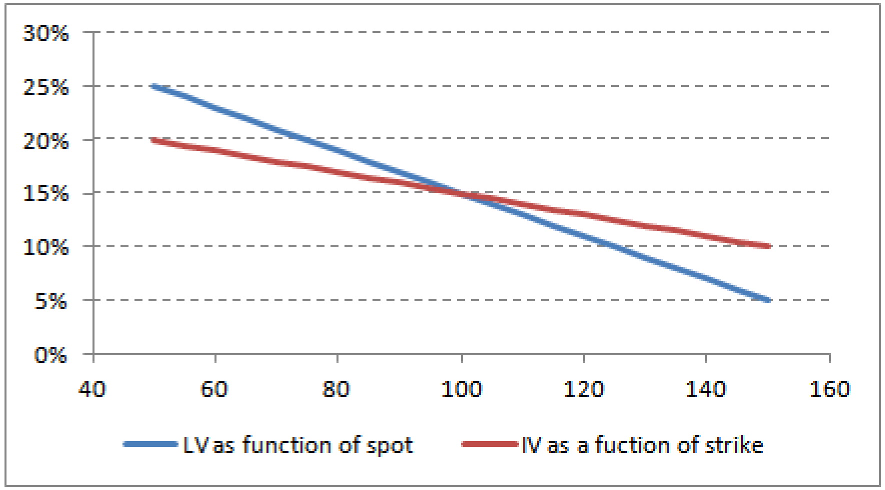
$$= \frac{1}{2}E^1\left\{\left[\int_0^T S_t^2\Gamma_2(S_t,t)\left\{\left[E^1\left(\sigma_{1t}^2 / S_t\right)\right] - \sigma_2^2(S_t,t)\right\}dt\right]\right\}$$

$$= \frac{1}{2}\int_0^T \int \gamma_1(S,t)S^2\Gamma_2(S,t)\left\{\left[E^1\left(\sigma_{1t}^2 / S_t = S\right)\right] - \sigma_2^2(S_t,t)\right\}dSdt$$

Note a LV process is a Markov diffusion hence $C_t, \Delta_t, \Gamma_t = f(S_t)$

•(Advanced) Exercise: in a LV model, show how [*microbuckets* = sensitivities to local vols] relate to [*gamma profile* = sensitivities to future gammas across scenarios (S_t, t)]

Assignment 3



•Learn to use the formulas for IV ATM and skew:

ATM IV = LV $\hat{\sigma}(S_0) \approx \sigma(S_0)$

ATM IV skew = half of LV slope $\hat{\sigma}'(K)\big|_{K=S_0} \approx \frac{\sigma'(S_0)}{2}$

•Assume smile around ATM is linear $\hat{\sigma}(K) = \hat{\sigma}_{ATM} + skew \cdot (K - S_0)$

•We use a local volatility model *calibrated* to this smile

- What is the LV?
- When spot moves by ΔS, meaning tomorrow $S_1 = S_0 + \Delta S$ what is the new ATM, that is the new IV for strike S_1 according to the LV model?
- What is the new IV for the strike S_0, that was ATM prior to the move?
- Deduce that the delta of the ATM options in the LV model is $\Delta_{LV} = \Delta_{BS} + vega_{BS} \cdot \underbrace{\frac{\delta\hat{\sigma}(K)}{\delta K}\bigg|_{K=S_0}}_{\text{ATM skew}}$

Local Volatility Models

•Dupire's general LV model is investigated in module 3

•For now, we review 3 particular cases

- Bachelier = "Normal Black-Scholes"
- CEV = Constant Elasticity Volatility = "Black-Scholes Beta"
- Displaced Lognormal = "Shifted Black-Scholes"

•These 3 are well known and widespread models

- Simplified LV models: no time dependence and dependence in spot is of some particular parametric form
- Closed formulas for European options
- Allow quick estimation of impact of skew
- Also heavily used in the context of interest rate models where general LV is not the norm

Normal Black-Scholes: Bachelier (1900) (!!)

- Bachelier's assumption: first differences, not returns, are Gaussian with constant variance λ^2 : $dS = ... + \lambda dW$
- In lognormal (proportional) terms, $\frac{dS}{S} = ... + \frac{\lambda}{S} dW$
- Model also called "Normal Black-Scholes" , local vol model with LV $\sigma(S) = \frac{\lambda}{S}$
- Risk-neutral dynamics: $dS = \lambda dW$
- Hence, $S_T = S_0 + \lambda\sqrt{T} N$
- Its RN probability distribution is Gaussian with mean *0* and variance $\lambda^2 T$
- So the call price is found as a simple integral $E\left[(S_T - K)^+\right] = E\left[\left(S_0 + \lambda\sqrt{T} N - K\right)^+\right] = \int \left(S_0 + \lambda\sqrt{T} x - K\right)^+ n(x)\, dx$
- Not difficult if somewhat painful to compute, and the result is $(S_0 - K)N(d) + \lambda\sqrt{T} n(d), d = \frac{S_0 - K}{\lambda\sqrt{T}}$
- ATM, $d = 0$ and the result collapses to $\sqrt{\frac{T}{2\pi}}\lambda \approx 0.40\lambda\sqrt{T}$ "0.4 standard devs"

Bachelier's model

•73 years before Black-Scholes, Bachelier released a model that is:

➢ Simpler, no Ito involved (good thing since Ito was born 15 years later in September 1915)

➢ Somewhat more realistic, (lognormal) vol increases when spot drops

•Of course these comments are highly unfair

➢ Black and Scholes's greatest contribution is the hedge/replication paradigm, not formula or vol spec

➢ Bachelier's skew is extreme and explodes for low strikes, (lognormal) vol goes to infinity

•Bachelier's smile

•ATM IV $\hat{\sigma}_{ATM} = \hat{\sigma}(S_0) \approx \sigma(S_0) = \frac{\lambda}{S_0} \Leftrightarrow \lambda = \hat{\sigma}_{ATM} S_0$

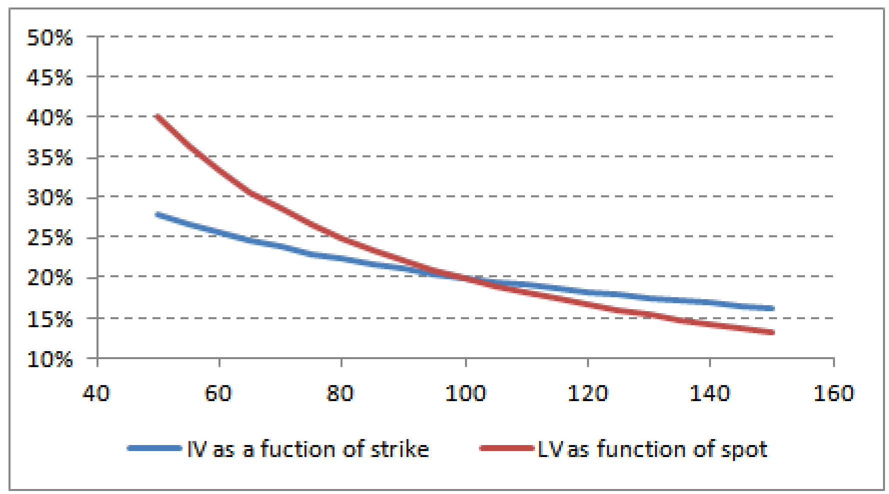
•Skew $\hat{\sigma}'(K)\big|_{K=S_0} \approx \frac{\sigma'(S_0)}{2} = -\frac{\lambda}{2S_0^{\,2}} = -\frac{\hat{\sigma}_{ATM}}{2S_0}$

•Chart:

$S_0 = 100, \lambda = 20$

$ATM = 20\%$

$skew = 0.10$ vol points per strike unit



Local Vol: CEV (Constant Elasticity Volatility) model

•"Power" LV model under RN: $\frac{dS}{S} = \lambda S^{\beta-1} dW$

•Family of models parameterized by β and that contains:

- Black-Scholes $\beta = 1$
- Bachelier $\beta = 0$
- And anything in between

•Popular with traders

- We have 1 parameter β that controls the skew from extreme $\beta = 0$ to flat $\beta = 1$
- We have the ATM IV approximation $\hat{\sigma}(S_0) \approx \sigma(S_0) = \lambda S_0^{\beta-1}$
- Hence we hit ATM vol given β by setting $\lambda = \hat{\sigma}_{ATM} S_0^{1-\beta}$
- So ultimately IV is parameterized with a known market data $\hat{\sigma}_{ATM}$ and an intuitive parameter β that controls the skew

•The formula was cracked by Andersen and Andreasen in 1998

- Involves a displaced chi-square distribution that requires numerical implementation
- *There is another way*

Local Vol: Displaced-Lognormal Model

•“Affine” LV model under RN: $\frac{dS}{S} = \frac{(a+S)b}{S} dW$

•Also a parameterized family:

- Contains Black-Scholes $a = 0$
- “Almost” contains Bachelier $a \to \infty, b = \frac{\lambda}{a}$
- Hence, *a* controls skew, move skew from flat to extreme (Bachelier) by increasing *a* from *0* to *infinity*
- While *b* is mapped to the ATM vol given *a*: $\hat{\sigma}_{ATM} \approx \frac{(a+S_0)b}{S_0} \Leftrightarrow b = \frac{S_0}{a+S_0} \hat{\sigma}_{ATM}$

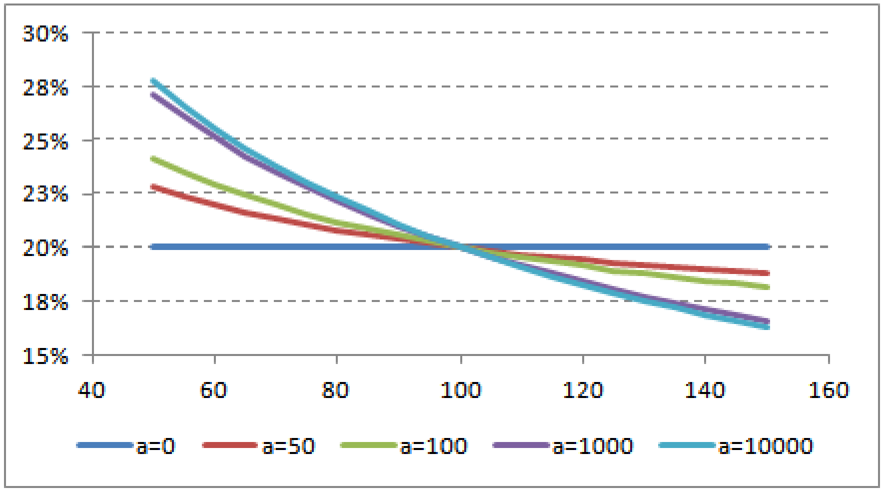
•This model (also called Shifted Black-Scholes) resolves extremely easily with a trivial change of variable

- Denote $X = S + a$, immediately $\frac{dX}{X} = bdW$ so *X* is a Black-Scholes process with volatility *b*
- But $(S-K)^+ = (X-a-K)^+ = [X-(K+a)]^+$
- A call *K* on *S* is a call *K+a* on *X*, and so the price of the call *K* on *S* in our model is the price of call *K+a* on *S+a* in BS

$$BS\{spot = S+a, strike = K+a, vol = b\}$$

Mapping Displaced-Lognormal and CEV models

- DLM smile



- DLM-CEV mapping

	ATM	Skew
DLM	$\frac{(a+S_0)b}{S_0}$	$\frac{-ab}{2S_0^{\ 2}}$
CEV	$\lambda S_0^{\ \beta-1}$	$\frac{(\beta-1)\lambda S_0^{\ \beta-2}}{2}$

equating ATM and skew in both models

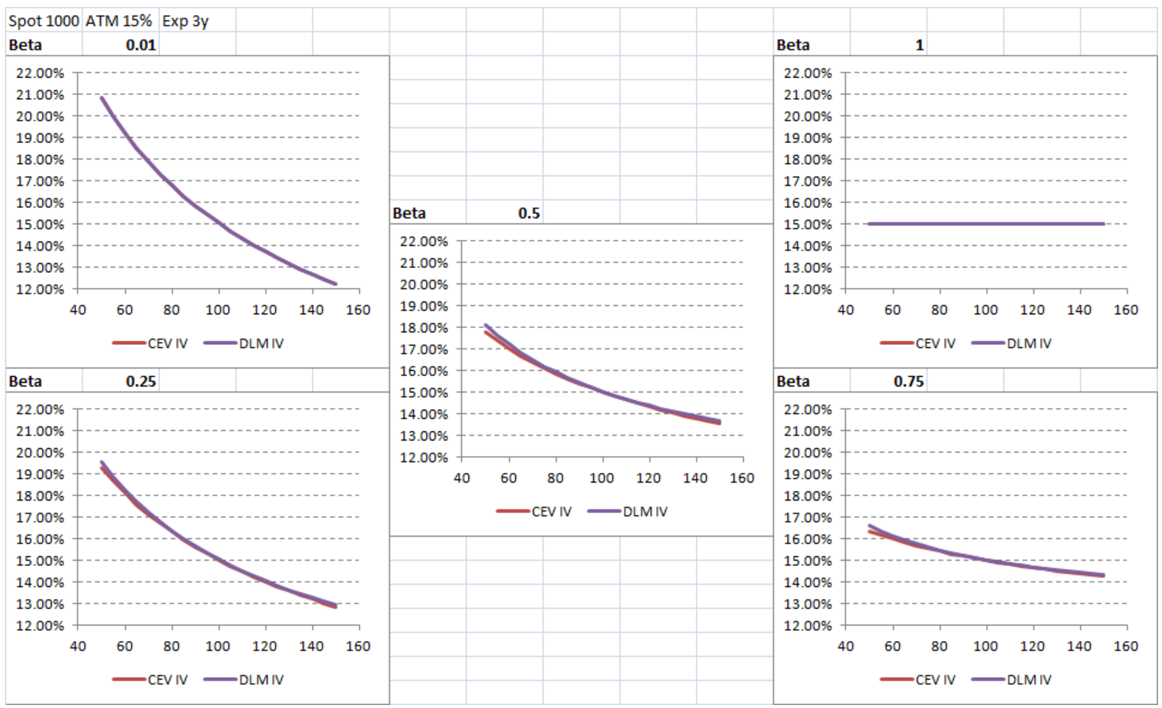
and then we can use the DLM formula as a proxy to CEV

$$a = \frac{1-\beta}{\beta} S_0$$

$$b = \lambda \beta S_0^{\ \beta-1}$$

Displaced-Lognormal: mapping quality

- This displacement technique easily adds a local vol component
- Applies to a wide variety of models: Black-Scholes, Heston, …
- A popular method that trivially resolves problems that otherwise could be very hard
- And it works remarkably well – below we compare “real” CEV smiles with mapped DLM counterparts:



Assignment 4 and one interview question

•Assignment 4

- ➢ Implement Bachelier's formula in C++ Excel
- ➢ Implement the DLM formula as a function of the ATM vol and a CEV beta parameter
- ➢ Chart generated smiles for several sets of parameters

•Interview question

You are talking to an option trader

- "How much is an ATM 1y call in % of the spot? 10% vol?"
- "4%"
- "That was fast. How about 2y?"
- "5.6%"
- "20% vol?"
- "1y = 8%, 2y = 11.2%"

You implement Black-Scholes and confirm her answers are indeed correct.

You stand impressed.

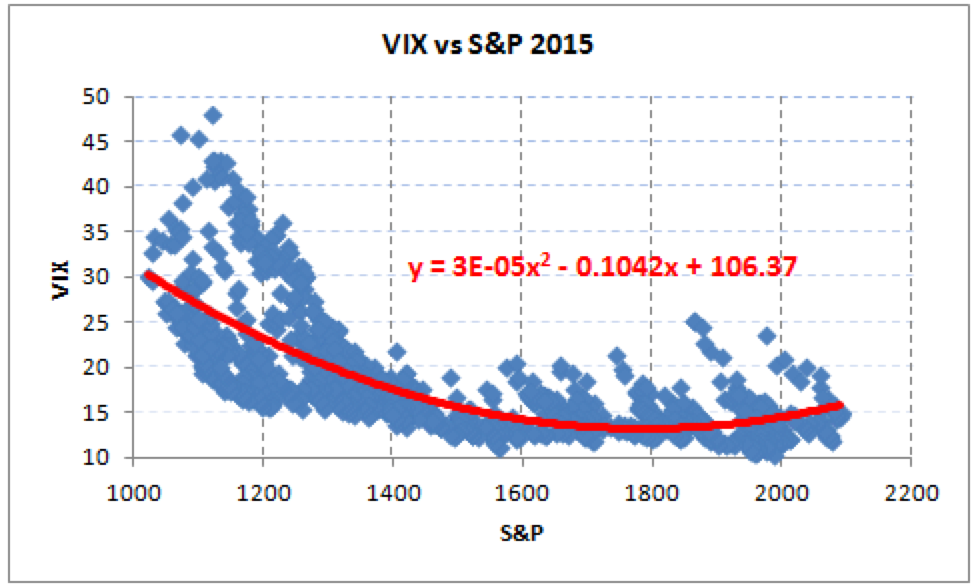
And then you realize she applies $\frac{C}{S} \approx 0.4\sigma\sqrt{T}$

How do you justify this approx?

Stochastic Volatility

•In real life volatility is stochastic

- Vol moves are not completely explained by spot moves
- Vol is also random on its own right



•But does it matter?

- Not all real life phenomena matter for option pricing
- For example, real life returns are often auto-correlated
  we can prove that this does *not* matter
  because all daily hedges are independent (Bergomi, 2016)
- First SV models introduced as early as 1987 (Hull-White)
  markets ignored them for over 10 years
- Markets started using SV suddenly in the early 2000s
  instantly adopted as a market standard for rates and forex
  meaning SV must be important **but why exactly?**

- When we say “Important”, “matters”
- We mean: impacts the PnL after hedge, means we must correct prices for impact, means correct arbitrage situations

Introducing some new “Greeks”

•We already know some “Greeks” = sensitivities of option prices to market and model parameters

- “Delta”= sensitivity to the current spot, for a call in Black-Scholes $\Delta = \delta C / \delta S = N(d_1)$

- Since spot is stochastic, we know since Ito that the second derivative matters
  - “Gamma” measures convexity in the spot and monetizes its realized volatility
  - In Black-Scholes, $\Gamma = \delta^2 C / \delta S^2 = n(d_1) / \sigma S \sqrt{T}$

- We also met the decay “theta” that measures the fair price for holding the option by time unit
  - “Theta” neutralizes arbitrage by balancing the costs/benefits of hedging/replicating options
  - In Black-Scholes, $\theta = -(1/2)\Gamma S^2 \sigma^2 = -n(d_1)\sigma S / 2\sqrt{T}$

•Similarly, when volatility is stochastic, we introduce 1st and 2nd order sensitivities to the vol

- Recall “vega”= 1st order sensitivity to *implied* vol, for a call in Black-Scholes $vega = \delta C / \delta \sigma = Sn(d_1)\sqrt{T}$

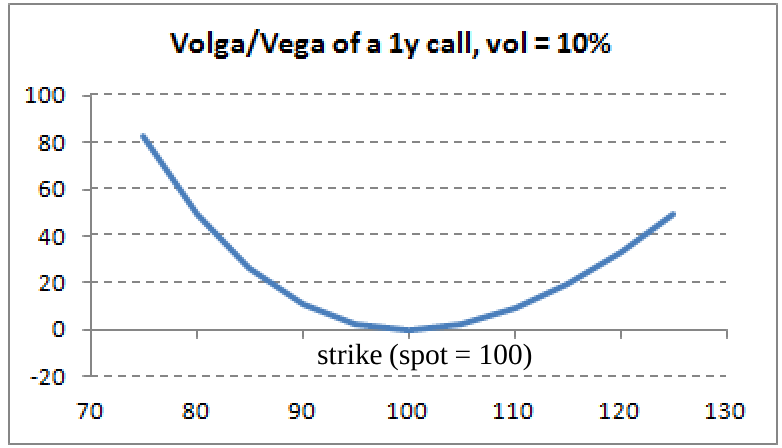
- We call “volga” the 2nd order sensitivity, in Black-Scholes $\text{volga} = \delta^2 C / \delta \sigma^2 = \delta vega / \delta \sigma = (d_1 d_2 / \sigma) Sn(d_1)\sqrt{T}$

- And “vanna” the cross sensitivity to spot and vol, in Black-Scholes $\text{vanna} = \delta^2 C / \delta \sigma \delta S = \delta vega / \delta S = \delta \Delta / \delta \sigma = n(d_1) d_2 / \sigma$

Volga and vanna / vega

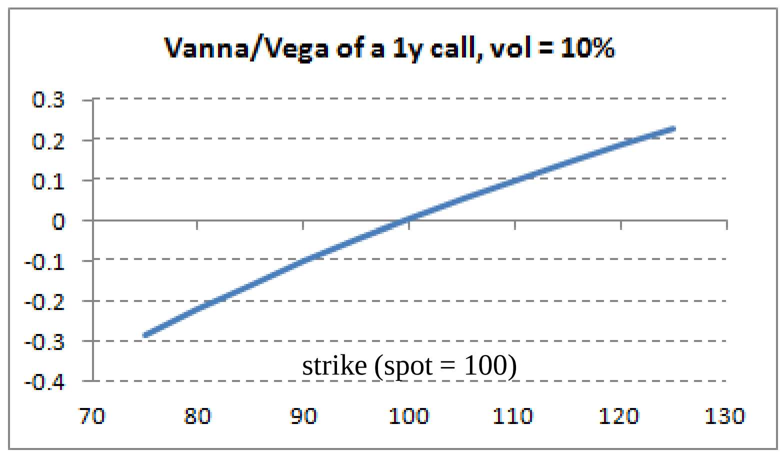
•Volga/vega (why we look at ratios / vega will be clarified)

- ATM options are virtually linear in vol, hence no volga
- OTM options have positive volga (can you explain why?)
- We note that $\text{volga}/vega = d_1 d_2/\sigma \approx d^2/\sigma, d = \log(K/S)/\sigma\sqrt{T}$
- Volga (over vega) increases (quadratically) with *log-moneyness*
- Also note volga/vega inversely proportional to maturity
- Further OTM options have higher volga (over vega)
  - Further OTM options monetize moves of volatility to a higher amount
  - We should expect SV models to value them higher



•Vanna/vega

- ATM options are virtually linear in vol, hence no vanna
  - High strike options have positive vanna
    and monetize joint moves of spot / vol *in the same direction* (explain why?)
  - Low strike options have negative vanna
    and monetize joint moves of spot / vol *in reverse directions* (explain why?)
  - SV models that expect spot and vol to move in the same direction
    should value high strikes higher, and vice-versa
- We note that $\text{vanna}/vega = \left(1 - d_1/\sigma\sqrt{T}\right)/S \approx d/\left(\sigma S\sqrt{T}\right)$
- Vanna (over vega) increases linearly with log-moneyness
  and is inversely proportional to maturity



Vega hedging

•Vega hedging = neutralize the vega of an option book by trading liquid ATM options

•Why must we hedge vega?

- Suppose we are long a 3y 120 call, volatility trades at 15%, our position is worth ~4.04
- Our delta is 0.28, meaning if spot drops 2% to 98, we lose 0.56 or 14% of our value!
- Obviously, we hedge deltas to avoid such scenarios

- Similarly, we have a vega of 0.59, meaning that if implied vol drops 2 points to 13%, we lose 1.08, or 29% of our value!!
- Even if we were allowed to ignore mark to market of volatility and keep delta hedging with a vol of 15%, that is a **bad** idea
  - The market's best estimate of the volatility to be realized over the next 3 years is now 13%
  - We will bleed daily PnL with a decay calculated with moves 15% annually / ~1% daily, where actual moves will be 13% annually / 0.85% daily
  - The expected cumulated amount of bleeding PnL during 3 years is 0.59, so in average we lose the same amount
  - In addition, we compute delta with the wrong volatility, creating random noise in our PnL
  - We are better off taking the loss up front

- The (not so surprising) conclusion is that vega must be hedged just like delta if volatility moves (and it does)
- In this case, the ATM 3y call has 0.69 vega so we neutralize our vega by selling 0.59 / 0.69 = 0.86 ATM calls
- Obviously, we thereafter hedge the resulting delta in our book incl. vega hedge (-0.19) by buying 0.19 underlying stocks

Stochastic Volatility Models: Rationale

•Why is it a good idea to price options with Stochastic Volatility models?

•In a nutshell:

To ignore stochastic volatility exposes to being arbitraged, as many banks realized in the mid-late 1990s

•A simple (stochastic) volatility arbitrage with 3 options of the same maturity:

- Assume the market actively trades the ATM option and values OTM options in Black-Scholes with the same ATM vol
  This is pretty much the state of the interest rate options market in the mid 1990s

- Then I can build a position with 3 options of the same expiry but different strikes that has 0 vega, 0 vanna and positive volga
- That position has no gamma (gamma proportional to vega in Black-Scholes) and no decay (theta also prop. to gamma in BS)

- Say the 3y ATM vol currently trades at 15%
- Then we summarize the Black-Scholes value and sensitivities of the ATM and 125 calls and the 75 put in the table below

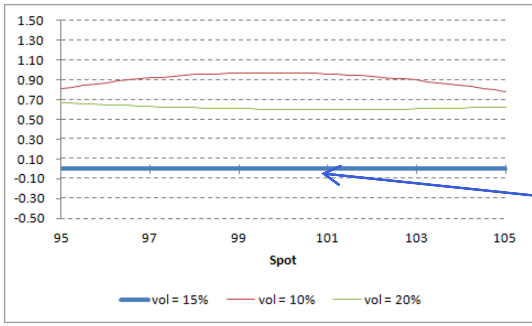
- A quick solve tells us that a combination of **1** C125 + **1.33** P75 vega hedged with **-1.40** ATM satisfies the requirements

- We take the position, delta hedge it and come back a week later

	value	delta	vega	vanna	volga
Call 125	3.13	0.23	0.53	0.020	0.025
Put 75	1.51	-0.11	0.32	-0.016	0.026
ATM	10.34	0.55	0.69	0.003	-0.001

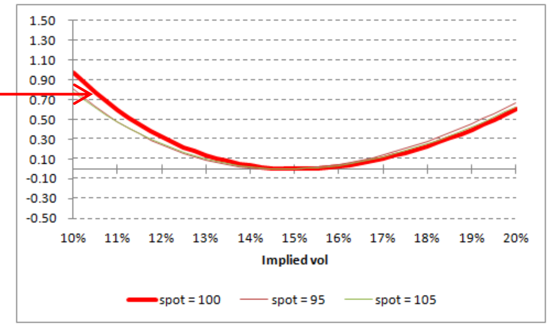
(Stochastic) volatility arbitrages

•A week later, what is our PnL as a function of the spot and implied vol at that time?



no delta
no gamma
no decay
➔ no spot exposure

no vega
no decay
positive volga
➔ volatility arbitrage



•We see that

- ➢ We have no sensitivity to spot.
  We are delta hedged, and, since gamma is proportional to vega in Black-Scholes, we have no gamma either.
- ➢ We have no decay since theta is also proportional to gamma in Black-Scholes.
- ➢ We are also vega and vanna hedged so (locally) no sensitivity to moves in implied vol or its joint moves with the spot.
- ➢ Thanks to our positive volga, we make money whenever vol moves up or down.

•We identified an arbitrage

- ➢ This is the same arbitrage Paribas realized on Interest Rate Options markets in the late 1990s
- ➢ We could buy convexity in vol for free (without paying decay for it) because the market model failed to give it value
- ➢ To correctly value convexity in vol and prevent this volatility arbitrage, we need stochastic volatility models

Stochastic volatility models and volatility arbitrage

•Recall in Black-Scholes, decay pays for expected gamma PnL given a volatility assumption:

Arb Free PDE

$$\vartheta = -\frac{1}{2}\frac{\delta^2 C}{\delta S^2}\frac{(dS)^2}{dt} = -\frac{1}{2}\frac{\delta^2 C}{\delta S^2}S^2\sigma^2$$

decay
cost of holding the option

expected gamma PnL
given vol assumption

Feynman-Kac

$\Longleftrightarrow$

Risk-neutral Pricing

$$value = RN - martingale = E^{RN}\left[payoff\right]$$

$$\frac{dS}{S}^{RN} = \sigma dW$$

•From there, it is clear that the time value only pays for convexity in spot

•In a stochastic volatility model, decay pays for:

- Expected gamma PnL, this is unchanged
- Additionally, the expected volga and vanna PnLs given assumptions on volatility of volatility and its covariance with spot

Arb Free PDE / decomposition of theta

$$\vartheta = \underbrace{-\frac{1}{2}\frac{\delta^2 C}{\delta S^2}\frac{(dS)^2}{dt}}_{\text{BS decay, pays for gamma}} \underbrace{-\frac{1}{2}\frac{\delta^2 C}{\delta \sigma^2}\frac{(d\sigma)^2}{dt}}_{\text{volga decay,}(d\sigma)^2=\sigma^2\alpha^2} \underbrace{-\frac{\delta^2 C}{\delta S\delta\sigma}\frac{d\sigma dS}{dt}}_{\text{vanna decay,} d\sigma dS = S\sigma^2\rho\alpha}$$

$$\vartheta = -\frac{1}{2}\frac{\delta^2 C}{\delta S^2}S^2\sigma^2 - \frac{1}{2}\frac{\delta^2 C}{\delta \sigma^2}\sigma^2\alpha^2 - \frac{\delta^2 C}{\delta S\delta\sigma}S\sigma^2\rho\alpha$$

Feynman-Kac

$\Longleftrightarrow$

Risk-neutral Pricing

$$value = RN - martingale = E^{RN}\left[payoff\right]$$

$$\frac{dS}{S}^{RN} = \sigma dW, d\frac{\sigma}{\sigma}^{RN} = \alpha dZ, \langle dW, dZ\rangle = \rho dt$$

Stochastic Volatility Models: Specification

•Now we know why we need a stochastic volatility model we can begin defining one

•We start with a direct extension of Black-Scholes, *directly written under the risk-neutral measure since we started with the PDE*

$$\begin{cases} S_0, \sigma_0 \\ \dfrac{dS}{S} = \sigma_t dW \\ \dfrac{d\sigma}{\sigma} = \alpha dZ \\ \langle dW, dZ \rangle = \rho dt \end{cases}$$

•We must immediately answer 3 fundamental (and related) questions:

1. Option prices are the cost of their replication ➔ how exactly can we replicate/hedge when volatility is stochastic?
2. To what extent is it correct to set the risk neutral drift of the vol to 0?
3. What is the current value σ_0 of the vol?

•And then, the (comparatively secondary) question:

➢ How can we compute option prices (expectation of payoffs) in this model?

➢ (If nothing else, we can run Monte-Carlo simulations or 2-dimensional finite-difference schemes on the associated PDE)

•Finally:

➢ What kind of smiles does this model produce and through what mechanics?

SV models are not pricing models

•Black & Scholes and LV models are pricing models

- In a market with no active options
- Estimate volatility or local volatility function/surface for underlying asset
- Risk-neutralize = remove drift
- Price, trade, hedge, risk manage options in the risk-neutral model

•Can we do that with in a Stochastic Volatility context?

- **No**
- 2 sources of uncertainty: asset price and its volatility
- Only 1 hedge asset ➔ cannot hedge all risks, cannot replicate option payoff: we call such markets *incomplete*
- Cannot risk-neutralize ➔ risk-neutral drift is *undefined* - can be set to whatever without violating arbitrage
- Higher drift → higher volatility → higher option prices ➔ option prices are also undefined
  can be whatever between IV and spot (even infinity if negative spot is possible)

•SV models are unusable unless they are *completed*

- We need another hedge instrument additional to the underlying, one that depends on volatility
- If we consider one or more options ("primary" options) as hedge assets additional to the spot
- Then we can hedge all sources of risk, risk-neutralize the model and price/hedge/replicate all *other* options

SV models are *relative value* models

•SV models require option prices to produce option prices

- “Options are hedged with options”
- Produce price/hedge/replication strategies of all options *given* some primary options
- All option prices will be consistent with primary option prices
- And assumptions on the volatility of the implied volatility of primary options (and its correlation with spot)
- Hence, prevent volatility arbitrage by pricing accordingly to convexities in volatility and vol of vol

•Practical usage of SV models

1. **Market making**: produce prices for all European options *given the most liquid ones* (typically, ATM) and SV parameters
2. **Volatility arbitrage**: when model price disagrees with market price, take the position and hedge with primary options
3. **Exotics** (module 3): value/risk of exotics given the market prices of some/all European options fed to a dynamic SV model

•In what follows

- We pick a maturity: it is customary for European Options market making to use different model parameters for each expiry
- We consider the ATM option as a primary hedge instrument
- We use our SV model to price all other strikes for that maturity

Parenthesis: another approach to risk neutralization

•Risk-neutralize Black-Scholes with non-zero (but constant) interest rates *r* and dividend yields *q*

- By Ito $dC(t,S) = \vartheta dt + \Delta dS + \frac{1}{2}\Gamma\left(dS^2\right)$
- Going long the option and delta hedging it yields the PnL

$$dPnL = \underbrace{dC}_{\text{change in C}} \underbrace{-rCdt}_{\text{financing of premium}} \underbrace{-\Delta}_{\text{short }\Delta\text{ stocks}} \left(\underbrace{dS}_{\text{change in S}} \underbrace{-rSdt}_{\text{financing of stock}} \underbrace{+qSdt}_{dividend} \right) = \vartheta dt + \frac{1}{2}\Gamma\left(dS^2\right) + (r-q)Sdt - rCdt = \left[\vartheta + \frac{1}{2}\Gamma\sigma^2 S^2 + (r-q)S - rC\right]dt$$

- PnL is deterministic, hence it must be 0, leading to the PDE $\vartheta + \frac{1}{2}\Gamma\sigma^2 S^2 + (r-q)S - rC = 0$
- And by Feynman-Kac

$$value / bankAccount = RN - martingale = E^{RN}\left[e^{-rT} payoff\right]$$

$$\frac{dS}{S} \overset{RN}{=} (r-q)dt + \sigma dW$$

•This is the clean, proper approach to risk neutralization

- Unfortunately, it is not always doable
- With SV models, we run into intractable problems
- We show another approach, one that is less proper, but a lot more practical
- That approach is heavily used in practice
- Especially for SV models where it is the only practical one

Risk neutralization through *calibration*

•Risk neutralization of Black-Scholes through *calibration*

- We start by *assuming* a process on the spot under the RN probability, under which all discounted asset prices are martingales

$$\frac{dS}{S} \overset{RN}{=} g(t)\,dt + \sigma dW$$

- Where g is some deterministic function of time

- Before using the model to price options, we *calibrate* it to ensure it correctly prices less complicated assets
- If a model is to be used to price options, it must as least correctly price forwards (which are options with strike 0)

- We know through basic arbitrage that a forward of maturity *T* must be equal to $F(T) = Se^{(r-q)T}$ (can you prove that?)

- So we have the *calibration* equation $\forall T, E^{RN}[S_T] = F_T = Se^{(r-q)T}$ which immediately gives the RN drift of *S* $g(t) = r - q$

- Hence, $\frac{dS}{S} \overset{RN}{=} (r-q)\,dt + \sigma dW$
- And the solution for a strike *K* is: $C = E^{RN}\left[e^{-rT}(S_T - K)^+\right] = e^{-rT}\left[F(T)N(d_1) - KN(d_2)\right], d_{1/2} = \frac{\log\frac{F(T)}{K} \pm \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}$

- We note that this price (and therefore Greeks too) depend only on the forward *not on the spot*

Risk neutralization through initial value

•For European options of a given maturity *T*

- (Recall we use different models (and forwards) for each maturity)
- In this case, our real hedge instrument is the forward of maturity *T*, *not* the spot
- That means we can *cheat,* and use a *driftless* model: $\frac{dS}{S}^{RN} = \sigma dW$
- But use the forward as our initial value instead of the spot $S_0^{\text{model}} = F(T) = S_0^{market} e^{(r-q)T}$
- Here, we risk-neutralized the model by changing the initial value instead of the drift
- And calibrated the initial value so that the hedge instrument (the forward) is correctly priced: $E^{RN}[S_T] = F(T)$
- This is the Black model (without Scholes, 1976)
- (It works with stochastic interest rates, although in this case, we must also change the volatility to the *volatility of the forward)*

Risk neutralization of stochastic volatility models

- We apply this methodology to our stochastic volatility model

$$\begin{cases} S_0, \sigma_0 \\ \dfrac{dS}{S} = \sigma_t dW \\ \dfrac{d\sigma}{\sigma} = \alpha dZ \\ \langle dW, dZ \rangle = \rho dt \end{cases}$$

  - This model is written under the risk-neutral measure
  - The price of all options in this model is $E\left[\left(S_T - K\right)^+\right]$
  - We can compute this price as $E\left[\left(S_T - K\right)^+\right] = f_{K,T}\left(S_0, \sigma_0, \alpha, \rho\right)$ (how exactly is not important for now)

- For a given expiry T

  - We use the ATM option as a hedge instrument
  - Hence, we calibrate the model to the ATM option by setting σ_0 such that $C^{\text{model}}\left(K = S_0, T\right) = f_{S_0,T}\left(S_0, \sigma_0, \alpha, \rho\right) = C^{\text{market}}\left(ATM, T\right)$
  - After that, the model is risk neutralized and may be used to price options of different strikes (but same expiry!) by applying *f*
  - Note that (maybe contrarily to intuition) it is *not* correct to estimate σ_0 with statistics, σ_0 must come from a hedge instrument

Comments on the risk-neutralization of SV models

1. The SV parameters α and ρ are the volatility and correlation of the *instantaneous* volatility

- What matter and can be observed/estimated are the volatility and correlation of the ATM implied volatility
- In our simple model they are essentially the same
- In more complex models (like Heston)
  this is not the case and SV parameters must be *calibrated* so that the model produce correct vol and correl for the ATM implied (more on that later)

2. Another (heavily used) approach is to calibrate not only σ_0 but also α and ρ

- In this case we calibrate to 3 (goal seek) or more (minimization of errors) options, all of the same maturity
- And use the model to value all other strikes
- In this case, the model is used as an interpolation/extrapolation tool
- But one with dynamic meaning: the calibrated α and ρ are the *market implied* volatility *and correlation* of the ATM implied vol

3. For exotics, we can't pick a maturity: exotics are path-dependent and depend on many/all maturities

- In this case, we can use a model with a drift, for instance deterministic
- Calibrate σ_0 to the shortest expiry ATM
- Calibrate the drift *g* (bootstrap) to other expiries ATM
- (Assuming or estimating SV parameters or calibrating to minimize errors on smiles of various expiries)
- And use the model to price the exotic
- More in section 3

$$\begin{cases} s_0, \sigma_0 \\ \dfrac{dS}{S} = \sigma_t dW \\ \dfrac{d\sigma}{\sigma} = g(t) + \alpha dZ \\ \langle dW, dZ \rangle = \rho dt \end{cases}$$

Pricing with stochastic volatility

- We have the calibrated / risk neutralized model (for some expiry T): $\begin{cases} s_0, \sigma_0 \\ \dfrac{dS}{S} = \sigma_t dW \\ \dfrac{d\sigma}{\sigma} = \alpha dZ \\ \langle dW, dZ \rangle = \rho dt \end{cases}$

- We want to compute option prices $E\left[\left(S_T - K\right)^+\right] = f_{K,T}\left(S_0, \sigma_0, \alpha, \rho\right)$
- The bad news is there is no Black-Scholes like closed form formula in this model
- The good news is we have a closed-form *approximation*, first computed by Hagan in 2002 using *small noise expansions*
- (The formula is given later as a particular case of the SABR model)
- This formula is approximate, hence subject to inaccuracy, noise and even statically arbitrageable prices
- It is however fast, easy to implement, and correct in most cases, hence its tremendous success
- We will not cover expansions here
- But Andreasen and Huge published a paper on expansions and how to use them to produce this formula and many others:
  ZABR: Expansions for the Masses, 2011
  This is a highly recommended reading

Stochastic volatility smiles

•Recall the SV PDE: $$\vartheta dt = \underbrace{-\frac{1}{2}\underbrace{\frac{\delta^2 C}{\delta S^2}}_{\text{"gamma"}}\underbrace{(dS)^2}_{\text{variance of spot}}}_{\text{BS decay, pays for gamma}} \underbrace{-\frac{1}{2}\underbrace{\frac{\delta^2 C}{\delta \sigma^2}}_{\text{"volga"}}\underbrace{(d\sigma)^2}_{\text{variance of vol}}}_{\text{volga decay},(d\sigma)^2=\sigma^2\alpha^2} \underbrace{-\underbrace{\frac{\delta^2 C}{\delta S \delta \sigma}}_{\text{"vanna"}}\underbrace{d\sigma dS}_{\text{covariance vol/spot}}}_{\text{vanna decay},d\sigma dS=S\sigma^2\rho\alpha}$$

- The decay, hence the time value, is the addition of a Black-Scholes part and 2 extra terms related to the convexity in volatility, volga (and the cross convexity in vol/spot, vanna) and the vol of vol (and the correl vol/spot)
- These extra terms produce an extra time value compared to Black-Scholes that we call *SV surprime*
- For a given expiry *T*, we calibrate ATM so there is no surprime ATM, and the surprime for a strike *K* is the value of that option in the SV model minus BS with the ATM vol: $SP_V(K) = V_{SV}(K) - BS(K, \hat{\sigma}_{ATM})$

- The smile produced by the SV model is, by definition, the collection for all strikes of the Black-Scholes volatilities $\hat{\sigma}(K)$ implied from the SV model prices

- By a first order expansion of the surprime formula, we immediately find that: $\hat{\sigma}(K) = \hat{\sigma}_{ATM} + \dfrac{SP_V(K)}{vega(K, \hat{\sigma}_{ATM})}$

- And since the surprime is linked to volga and vanna (see PDE), the smile is linked to ratios volga/vega and vanna/vega

- We know that volga/vega = 0 ATM, quadratic in log-moneyness (and inversely proportional to maturity) vanna/vega = 0 ATM, linear in log-moneyness (and inversely proportional to maturity)
- We therefore expect smiles to follow the same pattern, in particular we expect quadratic smiles

SV smile: the independent case (Hull & White, 1987)

•We approximate the smile produced by our SV model, following the footsteps of Hull & White (1987) In the case where the vol and spot processes are independent (rho = 0)

•Then we know that the value of a call is: $C = E\left[\left(S_T - K \right)^+ \right]$

•We call *V* the (quadratic) average vol from now to maturity $V = \sqrt{\dfrac{\int_0^T \sigma_t^2 dt}{T}}$

•*V* is a random variable with some distribution, we call $\bar{\sigma}$ its expectation and υ its standard deviation

•Using conditional expectations: $C = E\left\{ E\left[\left(S_T - K \right)^+ / \left(\sigma_t \right)_{0 \le t \le T} \right] \right\}$

•By independence, conditioning does not modify the spot's Brownian Motion:

$$S_T / \left(\sigma_t \right)_{0 \le t \le T} = S_0 e^{-\frac{1}{2}\int_0^T \sigma_t{}^2 dt + \int_0^T \sigma_t dWt} \overset{dist}{=} S_0 e^{-\frac{V^2}{2}T + V\sqrt{T}N} = S_T / V$$

•Hence (see section on time-dependent vol) $\underbrace{E\left[\left(S_T - K \right)^+ / \left(\sigma_t \right)_{0 \le t \le T} \right] = BS(V)}_{\text{Black-Scholes' formula with} V \text{ as vol}}$ so finally $\boxed{C = E\left\{ BS(V) \right\}}$

SV smiles in the independent case: approximation

- Using a second order expansion in BS $BS(V) \approx BS(\bar{\sigma}) + vega \cdot (V - \bar{\sigma}) + \frac{1}{2} \cdot \text{volga} \cdot (V - \bar{\sigma})^2$
- We get $C \approx E\left[BS(V)\right] = BS(\hat{\sigma}) + vega \cdot \underbrace{E\left[(V - \bar{\sigma})\right]}_{=0} + \frac{1}{2} \cdot \text{volga} \cdot \underbrace{E\left[(V - \bar{\sigma})^2\right]}_{=Var[V]=\upsilon^2} = BS(\bar{\sigma}) + \frac{1}{2} \cdot \text{volga} \cdot \upsilon^2$
- An ATM option has (almost) no volga, hence $C_{ATM} \approx BS_{ATM}(\bar{\sigma})$ and so $\bar{\sigma} \approx \hat{\sigma}_{ATM}$
- For all other options of that expiry: $C(K) \approx BS(K, \hat{\sigma}_{ATM}) + \frac{1}{2} \cdot \text{volga}(K, \hat{\sigma}_{ATM}) \cdot \upsilon^2$
- Or in implied vol terms: $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{1}{2} \cdot \frac{\text{volga}(K, \hat{\sigma}_{ATM})}{vega(K, \hat{\sigma}_{ATM})} \cdot \upsilon^2$
- We finally get (an aproximation of) the implied vol for all strikes K: $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{1}{2} \cdot \frac{\text{volga}(K, \hat{\sigma}_{ATM})}{vega(K, \hat{\sigma}_{ATM})} \cdot Var[V]$
- Further, in our model, $\frac{d\sigma}{\sigma} = \alpha dZ$ we can show that: $Var[V] \approx \frac{1}{3}(\alpha\bar{\sigma})^2 T = \frac{1}{3}(\alpha\hat{\sigma}_{ATM})^2 T$
- And finally we get an approx. expression for the SV smile: $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{1}{6} \cdot \frac{\text{volga}(K, \hat{\sigma}_{ATM})}{vega(K, \hat{\sigma}_{ATM})} \cdot (\alpha\hat{\sigma}_{ATM})^2 T$

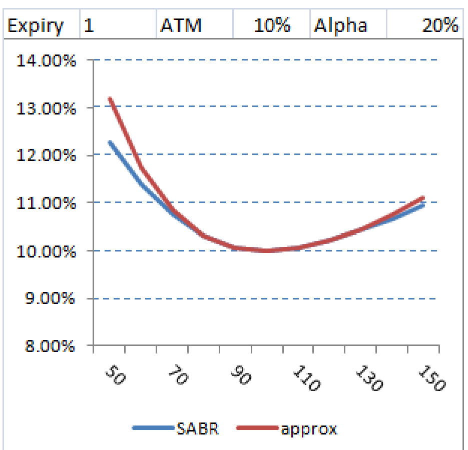
Approximation of the SV smile: testing

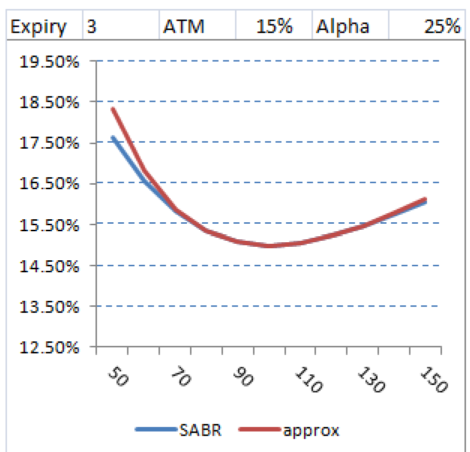
•How good is our approximation? We compare with Hagan's formula in a variety of contexts

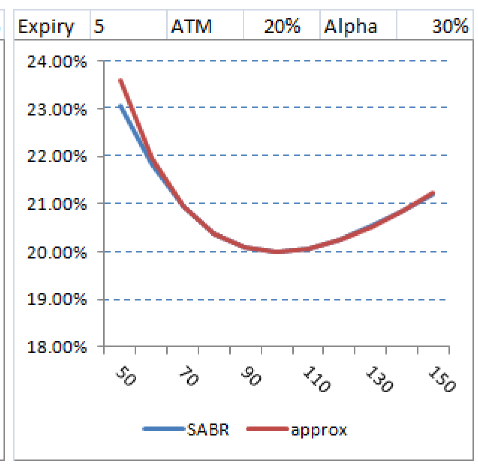
•Note: since BS volga is not exactly *0* ATM, we used the (slightly better) approx for SV smile =

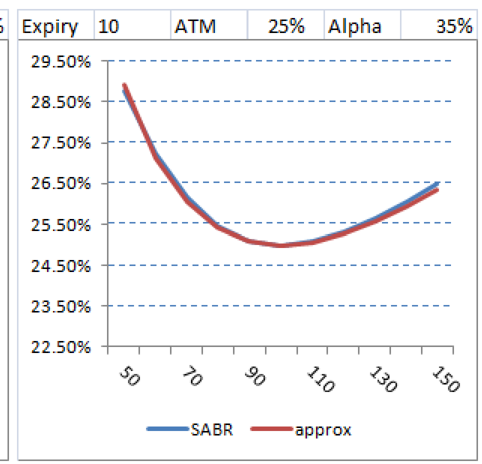
$$\frac{1}{6}\cdot\left[\frac{\text{volga}}{vega}(K)-\frac{\text{volga}}{vega}(ATM)\right]\cdot\left(\alpha\hat{\sigma}_{ATM}\right)^{2}T$$

•Also note: we always recalibrate σ_0 in Hagan's formula to the ATM implied









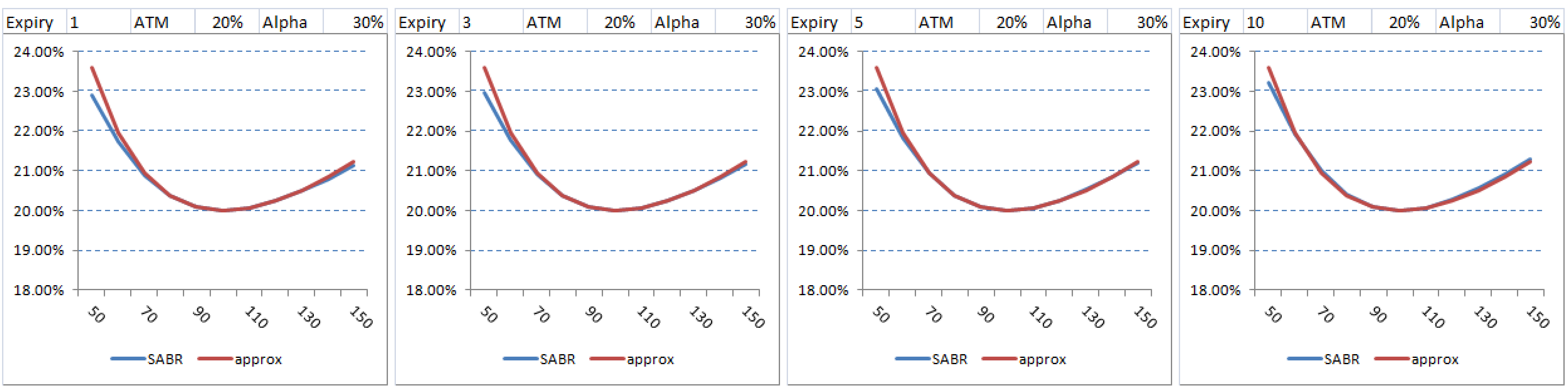
Approximation of the SV smile: conclusions

•Our approximation behaves very decently for such a simple formula $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{1}{6} \cdot \underbrace{\frac{\text{volga}(K, \hat{\sigma}_{ATM})}{vega(K, \hat{\sigma}_{ATM})}}_{\substack{\text{depends on strike} \\ \text{inversely prop. to T}}} \cdot \underbrace{(\alpha \hat{\sigma}_{ATM})^2 T}_{\text{cumul. variance of ATM volatility}}$

•In addition, it makes clear that the additional SV value is directly related (here proportional) to volga

•Further, recall that $\text{volga}/vega = d_1 d_2/\sigma \approx d^2/\sigma, d = \log(K/S)/\sigma\sqrt{T}$, hence $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{\alpha^2}{6\hat{\sigma}_{ATM}} \log(K/S)^2$

- ➢ Smiles are quadratic in log-moneyness
- ➢ The quadratic coefficient is proportional to the variance of vol
- ➢ Smile shape is independent from maturity: same for all maturities! (with same SV parameters)

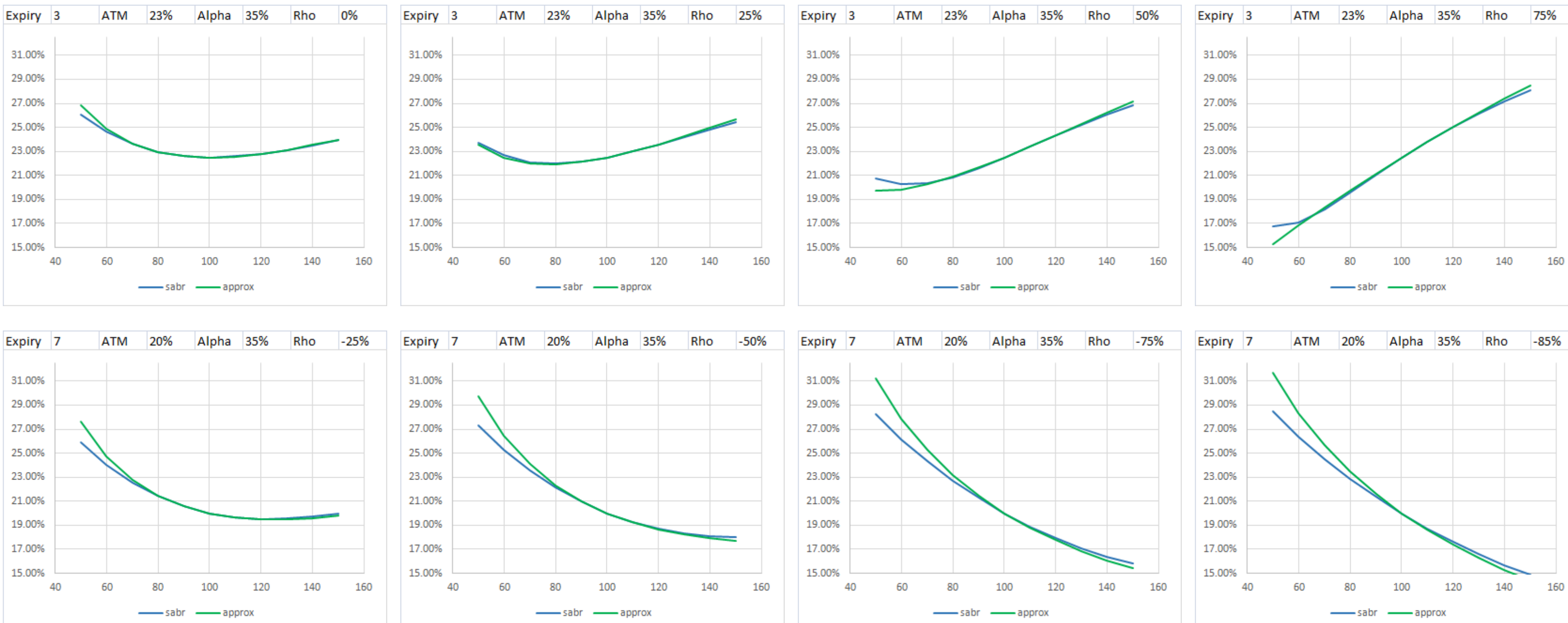


SV smile: the correlated case

- We extend our approximation $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{1}{2} \cdot \frac{\mathrm{volga}(K, \hat{\sigma}_{ATM})}{vega(K, \hat{\sigma}_{ATM})} \cdot Var[V]$ so it works with non zero correlation
- The extended result is: $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{vanna_K(S_0, \hat{\sigma}_{ATM})}{vega_K(S_0, \hat{\sigma}_{ATM})} \cdot \mathrm{cov}(S_T, V) + \frac{1}{2} \cdot (1-\rho^2) \cdot \frac{volga_K(S_0, \hat{\sigma}_{ATM})}{vega_K(S_0, \hat{\sigma}_{ATM})} \cdot Var[V]$
- Besides we have the (approximate) results for $V = \sqrt{\frac{\int_0^T \sigma_t^2 dt}{T}}$: $E[V] \approx \hat{\sigma}_{ATM}, Var[V] \approx \frac{1}{3}(\alpha \hat{\sigma}_{ATM})^2 T, \mathrm{cov}(V, S_T) \approx \frac{1}{2} \rho \alpha \hat{\sigma}_{ATM}{}^2 S_0 T$
- Hence, $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + \frac{\rho \alpha \hat{\sigma}_{ATM}{}^2 S_0}{2} \frac{vanna_K(S_0, \hat{\sigma}_{ATM})}{vega_K(S_0, \hat{\sigma}_{ATM})} T + \frac{(1-\rho^2) \alpha^2 \hat{\sigma}_{ATM}{}^2}{6} \frac{volga_K(S_0, \hat{\sigma}_{ATM})}{vega_K(S_0, \hat{\sigma}_{ATM})} T$
- The mathematical validation of this formula is still a work in progress (!!)
- Experimental validation is provided on the next slides by comparison to Hagan's formula
- We remind that $\mathrm{volga}/vega \approx \log(K/S)^2 / (\sigma^2 T), \mathrm{vanna}/vega = \log(K/S)/(\sigma^2 S T)$
- Hence the formula is quadratic in log-moneyness: $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + c \log(K/S)^2 + d \log(K/S)$, c, d constant in K, T

SV smile in the correlated case: results

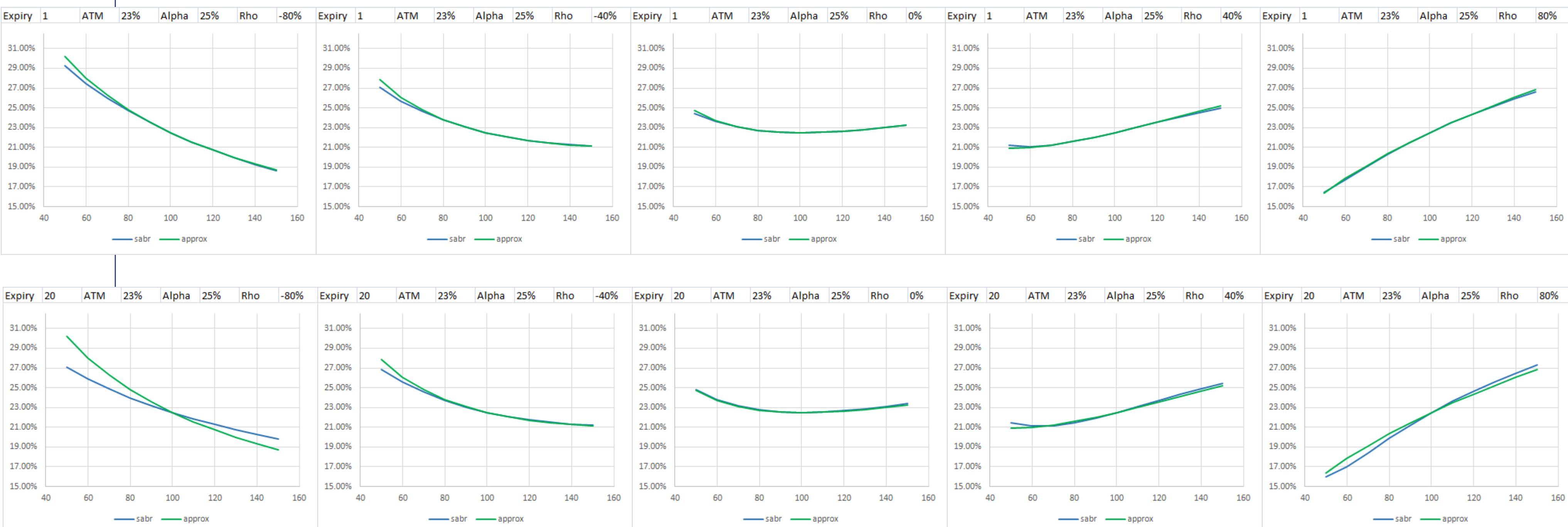
•As before, we subtract the ATM volga and vanna and calibrate Hagan’s formula ATM



SV smile in the correlated case: results (2)

- We note that Hagan’s formula (as the exact SV smile) is not exactly constant in maturity

- Hence, our approximation deteriorates for long maturities, especially with high correlation



What is really approximate in our approximation?

- The (exact) PDE states that *decay at time t* is Black-Scholes + a . “vanna”(t) + b . “volga”(t)
- Our approximation states that *total time value* is BS + a . vanna(0) + b . volga(0)
- We computed the *entire time value* using *today’s* vanna and volga
- Hence, our approximation boils down to freezing today’s vanna and volga all the way to expiry
- This is a violent approximation, somewhat of a leap of faith
- Even though tests validate this approximation for European options of a given expiry
- But it is extremely dangerous to try extend it to other expiries or exotics

What is the point of our approximation?

- We went through a lot of efforts to produce an approximation for SV pricing:

$$C_K = BS_K\left(\hat{\sigma}_{ATM}\right) + a \cdot vanna_K\left(S_0, \hat{\sigma}_{ATM}\right) \cdot T + b \cdot volga_K\left(S_0, \hat{\sigma}_{ATM}\right) \cdot T \text{ or } \hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + a \frac{vanna_K\left(S_0, \hat{\sigma}_{ATM}\right)}{vega_K\left(S_0, \hat{\sigma}_{ATM}\right)} T + b \frac{volga_K\left(S_0, \hat{\sigma}_{ATM}\right)}{vega_K\left(S_0, \hat{\sigma}_{ATM}\right)} T$$

$$a = \frac{\operatorname{cov}\left(S_T, V\right)}{T} = \frac{1}{2} \rho \alpha \hat{\sigma}_{ATM}{}^2 S_0, b = \frac{1-\rho^2}{2} \frac{Var[V]}{T} = \frac{1-\rho^2}{6} \left(\alpha \hat{\sigma}_{ATM}\right)^2$$

- We already have Hagan’s much more accurate expansion formula for that, so what’s the point?

1. Understanding what SV pricing is about: SV models attribute value to volga and vanna and nothing else
2. Qualify SV smiles: quadratic with slope = covariance and curvature = variance
3. Reason about models and their parameters (see Heston later on)
4. Quickly approximate impact of SV

VVV: Vega-Volga-Vanna

•This formula $\hat{\sigma}(K) \approx \hat{\sigma}_{ATM} + a\frac{vanna_K(S_0,\hat{\sigma}_{ATM})}{vega_K(S_0,\hat{\sigma}_{ATM})}T + b\frac{volga_K(S_0,\hat{\sigma}_{ATM})}{vega_K(S_0,\hat{\sigma}_{ATM})}T$

is sometimes called VVV (Vega-Vanna-Volga)
a method established by Forex Option traders in the mid 1990s and still *heavily* used today

•The coefficients a and b are seen as the *unit prices* of vanna and volga

•We calculated their *fair* value given SV dynamics: $a = \frac{\mathrm{cov}(S_T,V)}{T} = \frac{1}{2}\rho\alpha\hat{\sigma}_{ATM}{}^2 S_0, b = \frac{1-\rho^2}{2}\frac{Var[V]}{T} = \frac{1-\rho^2}{6}(\alpha\hat{\sigma}_{ATM})^2$

•Trading desks are more interested in their *market* value
which can be implied by solving $C(K) \approx BS_K(\hat{\sigma}_{ATM}) + a \cdot vanna_K(S_0,\hat{\sigma}_{ATM})T + b \cdot volga_K(S_0,\hat{\sigma}_{ATM})T$
in *a* and *b*, for at least 2 OTM strikes for maturity *T*

•In both cases, *a* and *b* are reused to price all other options of that maturity given their volga and vanna

•Sometimes, trading desks even use VVV to value options of another expiry, and even exotics

The attractiveness and dangers of VVV

•Forex Option traders typically like, trust and use VVV because

- ➢ The process of extracting market prices of volga and vanna out of liquid option prices and then apply it to other options fits traders mindset
- ➢ Recall that the VVV formula boils down to a quadratic formula in log moneyness
  Hence, this process of extracting *a* and *b* from 2 OTM boils down to a 3 point (incl. ATM) quadratic fit of the smile
- ➢ VVV is particularly well suited to *how Forex Options are traded* (see next slide)

•However it is dangerous because

- ➢ The formula is really quite approximate, OK to build intuition and quickly estimate impact of SV, not OK for production
- ➢ VVV = 3 points quadratic fit ➔ It is OK to interpolate implied volatilities, it is not OK to *extrapolate* them
- ➢ To apply the formula to exotics, where volga and vanna may change fast, is plain wrong

•For these reasons:

- ➢ We generally encourage the use of VVV to better understand SV and quickly estimate the impact of SV
- ➢ But strongly advise against its use in production, “real” models (next) being a much better choice
- ➢ Especially for exotics, where SLV (section 3) should be the go-to tool

Parenthesis: how Forex Options are traded

•Forex Option traders quote strikes in *delta*

- In BS $\Delta = N\left(\dfrac{\log\dfrac{S}{K}+\dfrac{\sigma^2}{2}T}{\sigma\sqrt{T}}\right)$, we can convert K to Δ and vice versa (with implementation of inverse N, like *normSinv()* in Excel)
- Delta is the hedge in spot, and it is customary in the interbank FXO market to exhange delta as the option is traded
- Delta is also the probability to end in the money, under the martingale measure associated to the numeraire S

•G11 Forex Option markets actively trade 3 strikes per maturity:

- ATM
- 25-delta call
- 25-delta put = 75-delta call

•Hence, ATM, 25DC and 25DP are 3 strikes for which the market provides implied volatilities

•However, Forex Options tend to quantify and trade the smile in terms of

- ATM straddle = ATM call + ATM put ➔ trades/quantifies ATM vol (~no vanna or volga)
- 25D risk reversal = 25D put – 25D ➔ trades/quantifies skew (essentially vanna only)
- 25D butterfly = 25D put + 25D call – 2 ATM calls ➔ trades/quantifies kurtosis (essentially volga only)

SABR: Sigma-Alpha-Beta-Rho

- SABR is a stochastic volatility model written in terms of 4 parameters: sigma, alpha, beta, rho (hence sabr)

$$\frac{dF}{F} = \sigma F^{\beta-1} dW^F, \frac{d\sigma}{\sigma} = \alpha dW^\sigma, correl\left(dW^F, dW^\sigma\right) = \rho dt$$

- This is essentially our simple model with the addition of a CEV parameter beta

- And Hagan cracked a very precise approximation for pricing Europeans in this model (from Wikipedia):

$$\sigma_{\text{impl}} = \alpha \frac{\log(F_0/K)}{D(\zeta)} \left\{1 + \left[\frac{2\gamma_2 - \gamma_1^2 + 1/F_{\text{mid}}^2}{24} \left(\frac{\sigma_0 C(F_{\text{mid}})}{\alpha}\right)^2 + \frac{\rho\gamma_1}{4} \frac{\sigma_0 C(F_{\text{mid}})}{\alpha} + \frac{2-3\rho^2}{24}\right]\varepsilon\right\},$$

$$F_{mid} = \sqrt{F_0 K} \qquad \zeta = \frac{\alpha}{\sigma_0}\int_K^{F_0} \frac{dx}{C(x)} = \frac{\alpha}{\sigma_0(1-\beta)}\left(F_0^{1-\beta} - K^{1-\beta}\right), \quad \gamma_1 = \frac{C'(F_{\text{mid}})}{C(F_{\text{mid}})} = \frac{\beta}{F_{\text{mid}}},$$

$$C(F) = F^\beta \qquad \varepsilon = T\alpha^2 \qquad D(\zeta) = \log\left(\frac{\sqrt{1-2\rho\zeta+\zeta^2}+\zeta-\rho}{1-\rho}\right). \quad \gamma_2 = \frac{C''(F_{\text{mid}})}{C(F_{\text{mid}})} = -\frac{\beta(1-\beta)}{F_{\text{mid}}^2}.$$

SABR: Formula

•The result is an implied Black-Scholes vol, to be fed into BS to obtain a price

•Formula somewhat messy but extremely easy to implement

•Also extremely fast and remarkably accurate

•Even though accuracy deteriorates for long maturities and high alpha, resulting in statically arbitrageable prices

•Historically, SABR was

- Developed by Pat Hagan in New-York for Banque Paribas when IR options were priced without a smile
- Paribas took massive wing positions and made substantial profits, catching the market's attention
- The model was published in 2002 and instantly became a market standard – still is to an extent
- After the publication, IR options started trading with a smile, often quantified in SABR parameters!

•SABR vs Heston

- Heston (see next) is a superior model
  - Mean-reversion parameter makes it usable for multiple expiries/exotics what SABR can't do
  - Exact solution means option prices don't suffer instabilities or static arbitrage
- But is is far more complicated to implement
- And way slower, even with a correct implementation

SABR: Beta and the Delta controversy

•We consider a SABR model calibrated to a market smile for some maturity T

•We will show in module 3 that for a wide class of models including SABR
and in the sense of a short expiry expansion (also see “ZABR” by Andreasen and Huge)

$$(1)\ \text{ATM vol}\quad \hat{\sigma}_{ATM} \equiv \hat{\sigma}(S_0,K)_{K=S_0} \underset{general}{=} \left\| d\log S \right\|_{S=S_0} \underset{SABR}{=} \sigma S_0^{\beta-1}$$

$$(2)\ \text{ATM skew}\quad \text{skew} \equiv \frac{\delta}{\delta K}\hat{\sigma}(S_0,K)_{K=S_0} \underset{general}{=} \frac{1}{2}\frac{\delta}{\delta S} E_t\left[d\hat{\sigma}_{ATM}/dS\right]_{\substack{t=0\\ S=S_0}} \underset{SABR}{=} \frac{1}{2}\left[(\beta-1)\frac{\hat{\sigma}_{ATM}}{S_0} + \frac{\rho\alpha}{\hat{\sigma}_{ATM}} S_0^{2\beta-2}\right]$$

•From eq 2 the skew depends on both ρ and β - these parameters are redundant for calibration
hence β can be chosen freely and ρ set to hit the skew given β

•Also from (1) and (2) the ATM delta in SABR, obtained by keeping SABR parameters constant, is equal to

$$\Delta_{SABR}^{ATM} = \underbrace{\Delta_{BS}^{ATM}(\hat{\sigma}_{ATM})}_{\Delta \text{ with IV unchanged}} + \underbrace{vega_{BS}^{ATM}(\hat{\sigma}_{ATM})}_{\text{add. } \Delta \text{ due to vol change}} \left(\underbrace{\underbrace{(\beta-1)\frac{\hat{\sigma}_{ATM}}{S_0}}_{\text{change in ATM vol}} \underbrace{-skew}_{\text{no longer ATM when spot moves}}}_{\text{change in implied vol when spot moves, SABR constant}} \right) = \underbrace{\Delta_{BS}^{ATM}(\hat{\sigma}_{ATM}) - vega_{BS}^{ATM}(\hat{\sigma}_{ATM})\cdot skew}_{\text{only depends on todays ATM and skew}} + \underbrace{vega_{BS}^{ATM}(\hat{\sigma}_{ATM})\cdot\frac{\hat{\sigma}_{ATM}}{S_0}}_{\text{only depends on todays ATM and skew}} \cdot \underbrace{(\beta-1)}_{\text{dependency on } \beta}$$

•SABR’s delta, given the market, has no dependency on ρ, only β

Delta controversy: Hagan's argument

- One of SABR's strongest selling arguments back in the early 2000s
- And one key reason for SABR's tremendous success
- The "redundant" parameters ρ and β are actually a major strength of the model
- Set β to the desired ATM delta, for instance from a statistical regression
- Then given β, calibrate ρ to the market skew
- Hence, this model may produce a realistic or desired delta while remaining completely MTM
- Contrarily to LV models, where delta is fully determined by the skew
- Instant and universal love from option trading desks, for obvious reasons
- But…

Delta controversy: Andreasen & Dupire argument

- The “real” delta is the “minimum variance delta” that is $MVD = \frac{\delta}{\delta dS} E\left[dC/dS\right]$

- For an ATM option, that is

$$MVD = \underbrace{\Delta_{\mathrm{BS}}^{\mathrm{ATM}}\left(\hat{\sigma}_{\mathrm{ATM}}\right)}_{\text{with ATM unchanged}} + vega_{BS}^{ATM}\left(\hat{\sigma}_{ATM}\right)\left(\underbrace{\frac{\delta}{\delta dS} E\left[d\hat{\sigma}_{ATM}/dS\right]}_{\substack{\text{expected move in ATM when spot moves}\\ \text{from eq.2 = 2 skew}}} - \underset{\text{no longer ATM}}{skew}\right) = \Delta_{BS}^{ATM}\left(\hat{\sigma}_{ATM}\right) + vega_{BS}^{ATM}\left(\hat{\sigma}_{ATM}\right)\cdot skew$$

- The MV delta depends only on skew, irrespectively of the LV/SV split

- The delta is fully determined by the market

- We cannot artificially split LV/SV to produce a reasonable delta

- But if the delta implied by the market is “unreasonable”, we can take an arbitrage position

- This is still a heated debate today → Where do you stand?

Heston (1992)

•A model somewhat similar to SABR $\frac{dS}{S}=\lambda\sqrt{v}dW^S, dv=-k\left(v-v_\infty\right)dt+\varepsilon\sqrt{v}dW^\sigma, v_0=1, correl\left(dW^S, dW^\sigma\right)=\rho dt$

•λ is a constant that represents the instantaneous vol and controls the short term ATM IV

•*v* is a multiplicative stochastic process so that the vol of the spot is $\lambda\sqrt{v}$ with $v_0=1$

- v_∞ is the long term stationary vol and it controls the ratio of long term to short term ATM IV
- v_∞ is often set to *1*

•ε is the (sqrt) vol of *v* and controls the vol of vol, hence the kurtosis (curvature) of the smile

•Note that ε is the (sqrt) vol of the variance, the (lognormal) vol of vol is of order *ε/2*

•ρ the correlation spot/vol and impacts their covariance (also through ε and λ), hence controls the skew (slope)

•*k* is the mean-reversion of vol and controls the speed of decay of ATM IV from λ to λv_∞

•Importantly, *k* also controls the flattening of the skew and the kurtosis of the smile with expiry

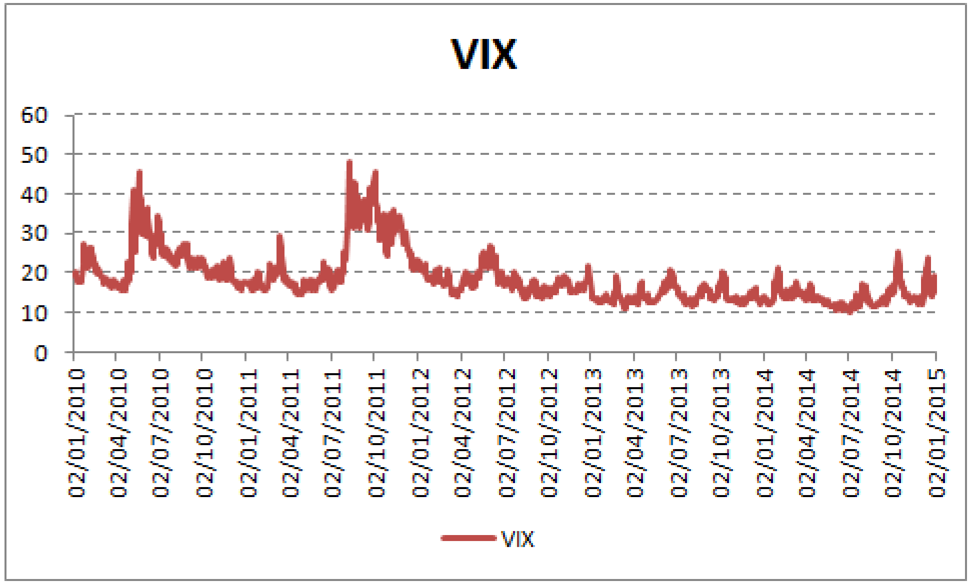
Mean-reversion

•Realistic: volatility tends to oscillate “in a tube”
→ characteristic of a mean-reverting process

•Controls how fast volatility reverts from λ to λv_∞
→ can fit or best fit a term structure of ATM IVs

•Provides numerical stability:
Feller’s condition states that
v is guaranteed to remain positive as long as $2k > \varepsilon^2$

•But: by far the most important impact is
to produce a decreasing term structure
of skew and kurtosis



Mean-reversion and the term structure of skew and kurtosis

•Recall some important results from our initial investigations:

1. The BS implied vols produced by SV models are roughly

$$\hat{\sigma}_{SV}(K) \approx \hat{\sigma}_{ATM} + c_1 \cdot \left(\log\frac{K}{S}\right)^2 \cdot \left\|\hat{\sigma}^{ATM}\right\|^2 + c_2 \cdot \left(\log\frac{K}{S}\right) \cdot \rho \cdot \|S\| \cdot \left\|\hat{\sigma}^{ATM}\right\|$$

2. Hence the skew and kurtosis are roughly constant in *T* and proportional on the vol/var of the ATM vol

$$skew(T) \sim \rho \cdot \lambda \cdot \left\|\hat{\sigma}^{ATM}(T)\right\|, kurt(T) \sim \left\|\hat{\sigma}^{ATM}(T)\right\|^2$$

3. And the ATM is roughly the expected quadratic average of instantaneous vol to maturity

$$\hat{\sigma}_{ATM}(T) \approx E\left[\sqrt{\frac{\int_0^T \left\|\log S_t\right\|^2}{T}}\right] \underset{Heston}{=} \lambda E\left[\sqrt{\frac{\int_0^T v_t dt}{T}}\right]$$

Mean-reversion and the volatility of the ATM implied

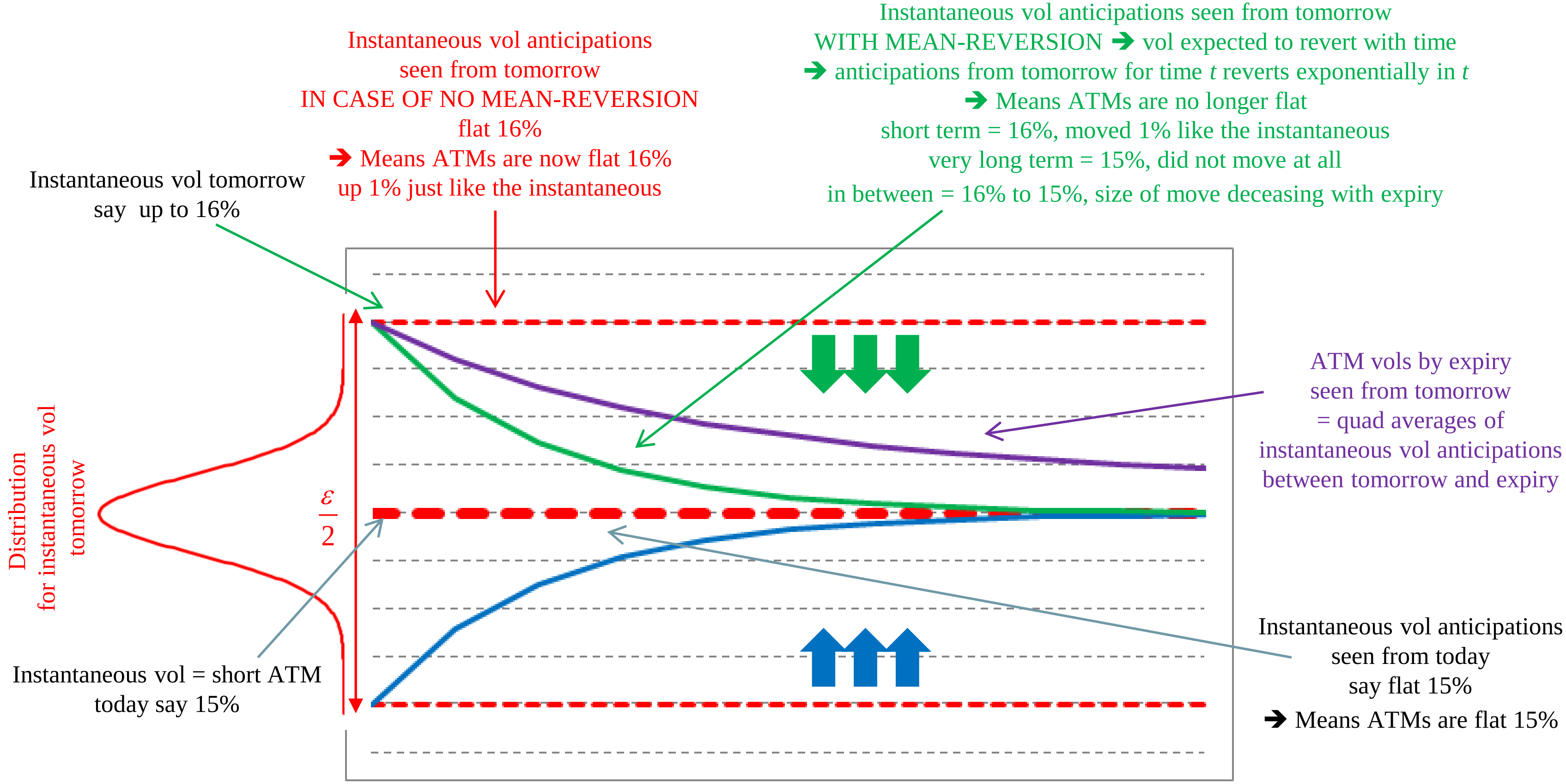
- So for an expiry T
- The skew (resp. Kurtosis) are proportional to the vol (resp. Variance) of the ATM vol of maturity T
- And we have the following (approx) result in Heston (and other SV models with mean-reversion):

1. Without mean-reversion, ATM vols of all expiries have roughly the same volatility
   Hence the skew and kurtosis are ~constant in expiry
2. In Heston, without mean-reversion, the volatility of all ATM vols is roughly $\frac{\varepsilon}{2}$
3. With mean-reversion, the volatility of a very short term ATM vol is still $\frac{\varepsilon}{2}$
4. But then, the volatility of longer term ATM vols of expiry T are (exponentially) decreasing in T
   They are also exponentially decreasing in the mean-reversion κ
   Hence the skew and kurtosis decrease ~exponentially with maturity (and mean-reversion)

- This result can be demonstrated with approximations and tedious calculations

Mean-reversion and the volatility of the ATM implied (2)

•We rather provide intuition and graphic illustration



Mean-reversion and the term structure of skew and smile

•Hence the (risk-neutral) mean-reversion of vol means that

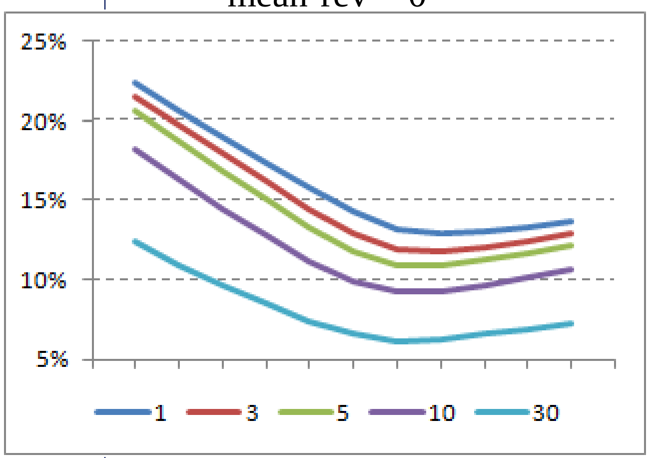
Longer term ATM move less than shorter term ATM

(Just as in mean-reverting rate models, LT rates move less than ST rates)

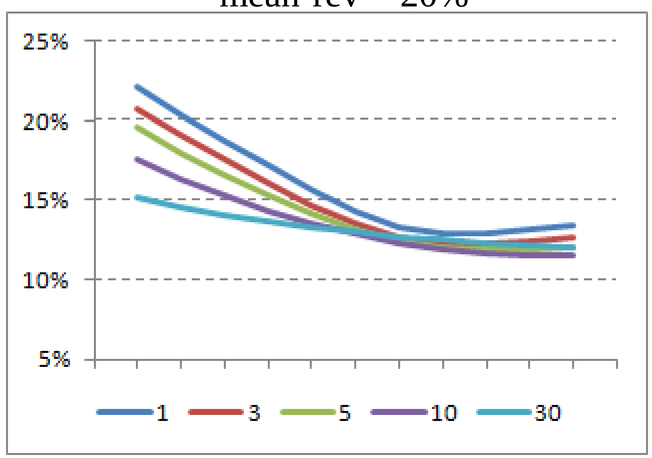
•For that reason (alone) the skew and kurtosis flatten (exponentially) with expiry and mean-reversion

•Below Heston smiles of different expiries, $\lambda = 15\%$, $\varepsilon = 80\%$, $\rho = -50\%$

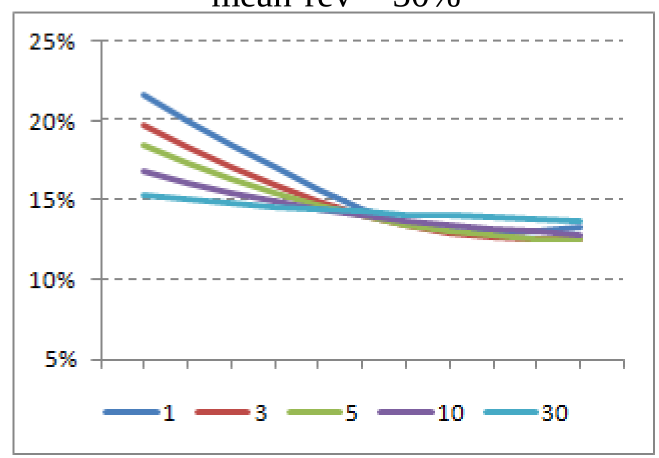
mean-rev = 0



mean-rev = 20%



mean-rev = 50%



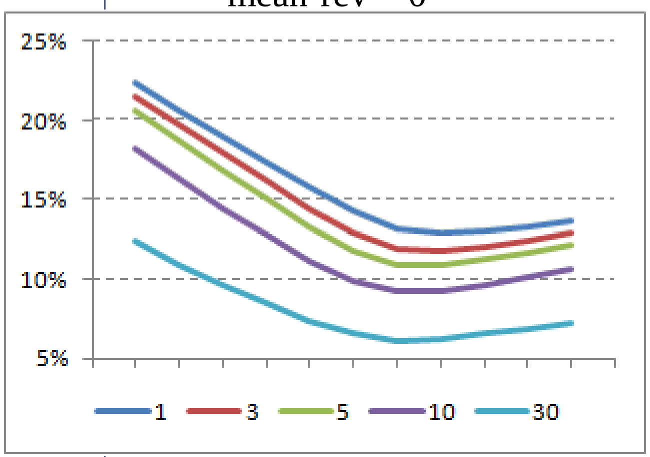
Mean-reversion: comments

•Without MR, Heston ATM volatility reduces with expiry

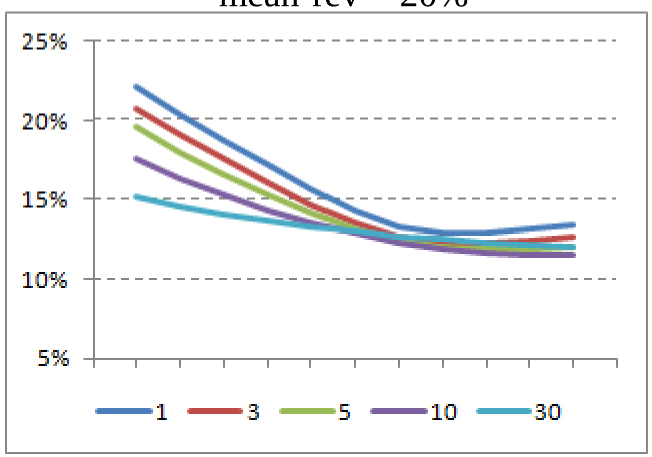
- ➢ Can you tell why?
- ➢ It does not matter, we re always re-calibrate to ATM → use higher lambdas for longer expiries

•Kurtosis (~var of ATM) dies faster than skew (~vol of ATM)

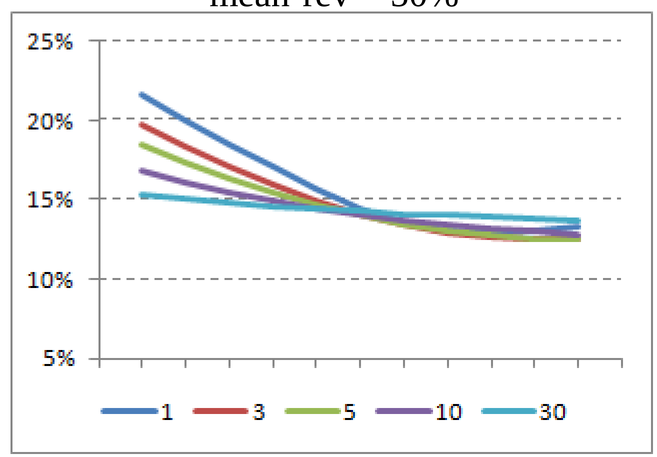
mean-rev = 0



mean-rev = 20%

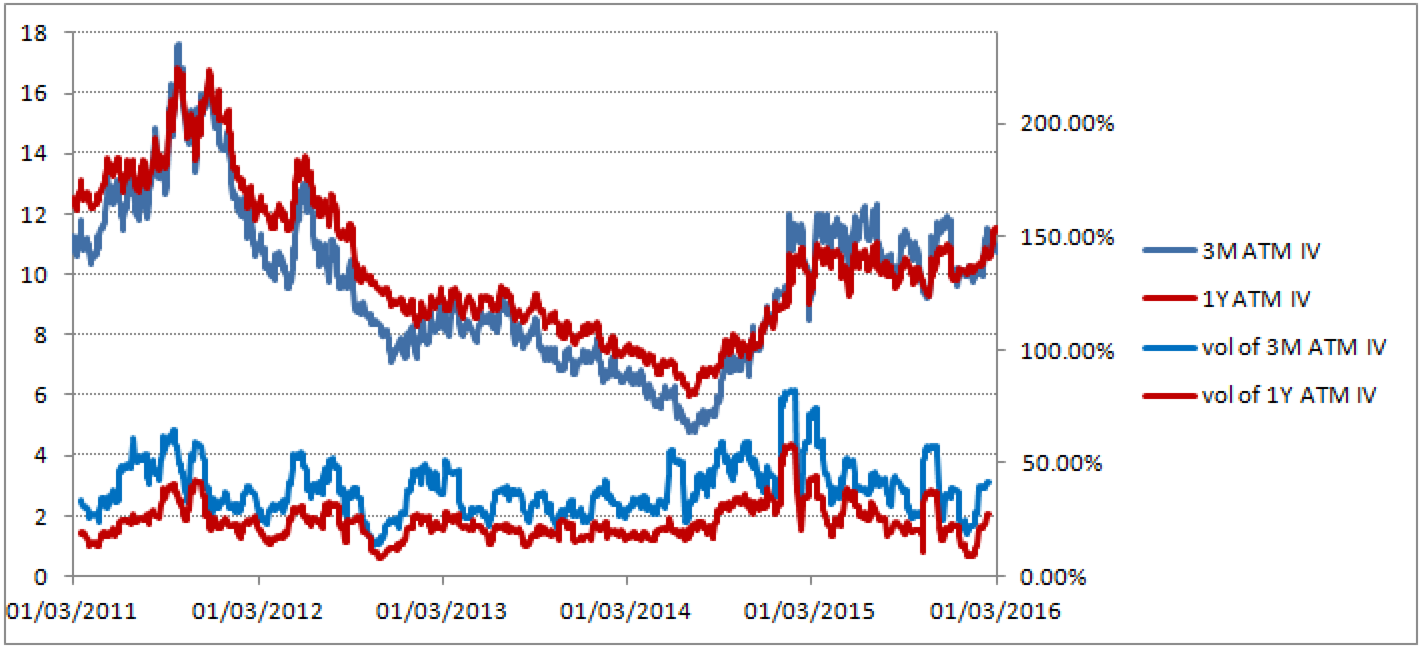


mean-rev = 50%

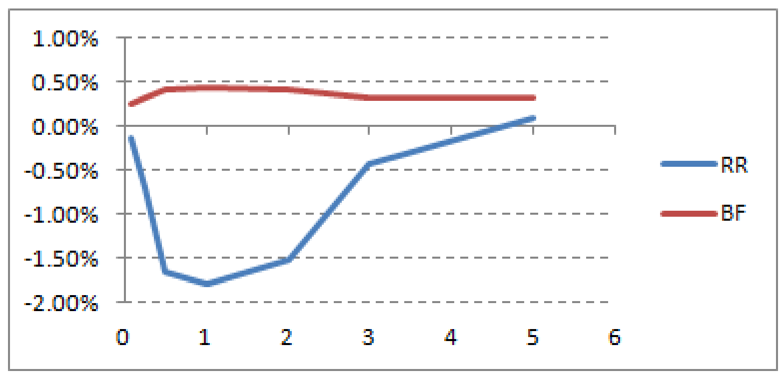


Is this realistic? Yes

•Historically,
LT IVs are more volatile
5Y history
3M and 1Y ATM IVs
on EURUSD
and their estimated vols



•In implied terms,
skew and kurtosis flatten with expiry
EURUSD skew (risk-reversal) and kurtosis (butterfly)
per expiry as of 2 March 2016
(BFLY ~flat in delta,
hence decreases in sqrt(T) in absolute terms)



Why mean-reversion really matters

•The fact that volatilities are mean-reverting in reality → historical mean-reversion - is *irrelevant*

•The MR *implied* by the ratio of the vol of the LT ATM to the vol of the ST ATM is relevant for arbitrage

•The mean-reversion *implied* by the decrease in today's market skew and kurtosis is relevant for market making

•Nice to have for Europeans ➔ stabilization of parameters across expiries

- No mean-reversion
  - Decreasing ε and ρ for (sliced) expiries
  - Means for a fixed expiry, ε and ρ increase every day
  - Pollutes risk-management
- With mean-reversion
  - Flatter ε and ρ by expiry → flattening of smile explained by MR, not decreasing parameters
  - Less parameter carry → more consistent risk management

•Crucial for exotics ➔ stationary smiles

Stationary smiles

•Skew/Kurtosis are function of the *residual* expiry, not the absolute one

•Means that the 1y skew/kurtosis predicted in 4y time is similar to the 1y skew today, not the 5y

•This matches observation:

- Long term skew/kurtosis tend to be consistently flatter than short term
- Skew/kurtosis are function of remaining maturity not absolute maturity
- This means that for a fixed absolute expiry, skew/kurtosis increase as the term approaches

•Thanks to mean-reversion

- Heston can match today's flattening term structure of smile
- While producing future skew/kurtosis identical to today's in residual maturity

•This is not (too) important for Europeans but crucial for some exotics

•Exercise: forward starting ATM digital

- Pays 100$ if spot in year 2 > spot in year 1
- How much is it worth in 1Y?
- What crucial feature is required of a model to price it correctly today?

Solving Heston

- No analytical Black-Scholes style solution even for European options

- Numerical solutions apply but more difficult than usual:
  - FDM but not standard FDM → dimension is 3 and the cross term requires a corrector-predictor scheme as in Craig-Sneyd
    See Andersen-Piterbarg, Volume 1, Chapter 2
  - Monte-Carlo simulations are complicated by the possibility that variance goes negative in discrete time: see assignment 5

- For Europeans, a near analytical fast solution exists: Fourier integration
  - We want to compute $E\left[\left(S_T - K\right)^+\right]$
  - We can't because the distribution of the final spot is unknown
  - But we can compute analytically all its moments
  - Hence, we know its Fourier transform
  - And the problem boils down to the computation of complex integrals
  - It is further complicated by some nasty numerical issues
  - All details in Andersen-Piterbarg, Volume 1, Chapter 8

Shifted Heston

- Heston doesn't have a LV component like SABR's β
- This is easily remedied by changing the spot dynamics to $\frac{dS_t}{aS_t + (1-a)S_0} = \lambda\sqrt{v}dW^S$
- Then we have a LV parameter *a* similar to SABR's β

$$a = 1 \rightarrow \text{lognormal dynamics}, a \rightarrow 0 \rightarrow \text{normal dynamics}, 0<\text{a}<1 \rightarrow \text{dynamics "in between"}$$

- The price of an option in the shifted model is $E\left[\left(S_T - K\right)^+\right] = E\left\{\left[\left(S_T + \frac{1-a}{a}S_0\right) - \left(K + \frac{1-a}{a}S_0\right)\right]^+\right\}$
- And $X_T = S_T + \frac{1-a}{a}S_0$ is a standard Heston process with vol λa

so the shifted price is just $Heston\left(S_T + \frac{1-a}{a}S_0, K + \frac{1-a}{a}S_0, \lambda a\right)$

- If you are on Hagan's side of the delta debate, you may use *a* to control your delta
- In any case, *a* helps fit strong short term skews that may be hard to fit with pure SV dynamics

Heston in practice: European options

•An alternative to VVV or SABR for sliced expiries
•“Real” model → does not generate arbitrageable option prices unlike VVV or SABR

•We set $v_0 = v_\infty = 1$

•We calibrate λ ATM → that means find λ such that the Heston value for the ATM option hits the target

•The mean-reversion *k* is set so we have:
- A reasonable value to provide numerical stability → Feller condition $2k > \varepsilon^2$
- More stable parameter values across expiries → Flattening smile with *k*, less need to decrease ε

•We are left with ε that controls vol of vol and kurtosis similarly to SABR’s α → but is of order $\alpha/2$
•And ρ that controls correl spot/vol and hence skew (together with ε) similarly to SABR’s ρ

•In theory, ε and ρ can be estimated – in practice, they are most often fitted to the market skew and kurtosis
- Use 2 options, for example 25d RR and BFLY
- Or best fit to 3+ options

Heston in practice: multiple expiries

• Improvement for Europeans: stable parameters across expiries

• Requirement for Exotics

- By definition, Exotics have multiple fixings → not just one expiry
- Consistent model across expiries → One single set of Heston parameters fitting smiles of multiple or all expiries
- We can use λ and v_∞ to exactly calibrate 2 ATMs or best fit 3+ ATMS or calibrate 1 ATM and best fit others
- We can use ε and ρ to exactly fit 1 skew/kurtosis then use *k* to best fit others
  or use *k*, ε and ρ to best fit a number of skew/kurtosis
- In all cases we are obviously limited, we have 5 parameters to fit *3n* market values

• Solution 1: make λ time-dependent

- Like in time-dependent BS, λ remains deterministic but moves with time
- Fourier inversion still works (but moments no longer analytical → solutions of simple ODEs, see Andersen-Piterbarg, Chp 9)
- Calibrate a time dependent λ to *all* ATMs
- But still only 1 skew/kurtosis or best fit

• Solution 2: make λ time *and spot*-dependent

- Use LV to "fit the gaps" left by Heston
- This is exactly the "stochastic-local vol" (SLV) model introduced in module 3

Heston in practice: multiple expiries (2)

•Solution 3: make *all* Heston parameters time-dependent

•In theory, could calibrate to ATM, skew and kurtosis of multiple expiries

•In practice, calibrates only to a restrictive subset of possible (non-arbitrageable) smiles

•On the contrary, LV (and hence SLV) calibrates to any non-arbitrageable surface ➔ SLV a better solution

•Anyhow, to resolve a fully time-dependent Heston

- Does not change Monte-Carlo or FDM
- But Fourier inversion no longer possible as such
- Solution: parameter averaging → Full story in Andersen-Piterbarg, Volume 1, Chapter 9
  - Call the standard Heston with “averaged” parameters between today and expiry
  - Compute “averages” with moment matching techniques
  - Similar to BS with time-dependent vol but not exact in this case

Heston in practice: short term

- Heston (as other SV models), cannot fit typical short term skew and kurtosis
  - Short term (<1m) skew and kurtosis typically too large to fit with reasonable SV parameters
  - Sometimes require vol of vol in thousands (!!)
  - Which questions the credibility of the model
  - And cause instabilities in numerical implementations
  - SV models cannot fit ST and LT at the same time and (at least) 2 sets of parameters must be maintained

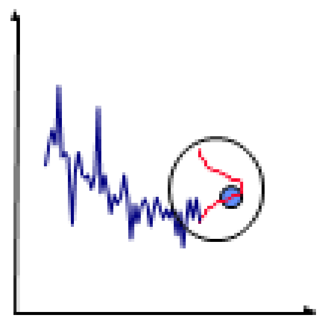
- ST skew and kurtosis are due to LV and jumps not SV

➔ Attempt to fit them with SV causes
  - Unrealistic parameters
  - Instabilities
  - Inconsistencies ST/LT

- What can we do about this?
  - 2-factor SV ➔ still unrealistic
  - Add LV on top of SV
    - Displaced Heston, reuse Heston with displaced strike and spot ➔ resolves ST skew but not kurtosis
    - Full general LV + SV ➔ SLV, see module 3
  - Add jumps, especially since they are the main cause of ST skew and kurtosis

Assignment 5

The assignment is to implement SABR and Heston simulations in C++ Excel



Follow questions and instructions on the document:

myPublicDropBox/Vol/assignments/volatility2/assignment5.pdf

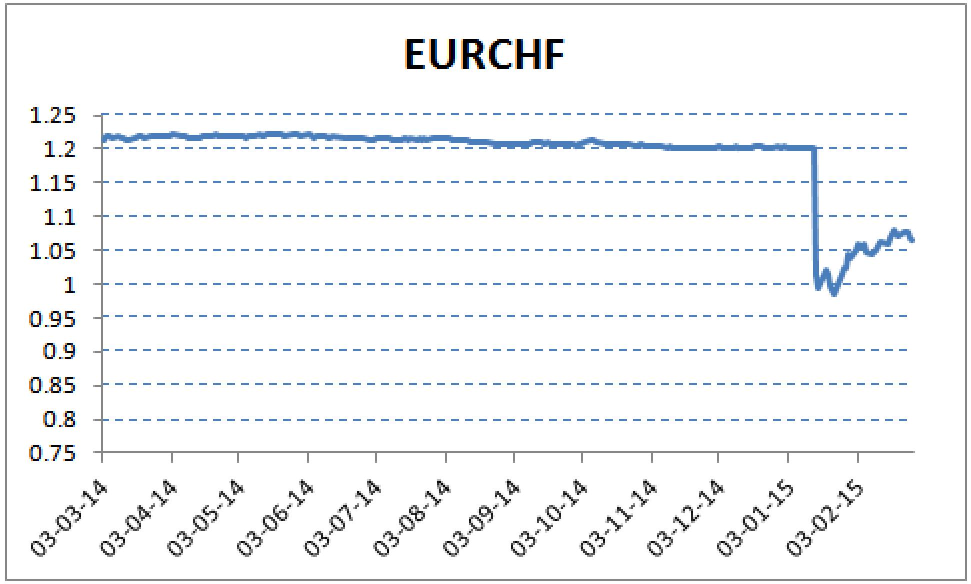
Jumps

•All models so far assume continuous prices however in reality prices sometimes jump

Example:
SNB withdraws EURCHF floor on 15Jan2015,
EURCHF drops from 1.20 to 0.85 within instants!

•What is it that defines jumps
and differentiates them from extreme volatility:

- **When we cannot trade the spot during the move**
- Our fundamental analysis to the 2nd order breaks
- And the whole theory falls apart



Short option position over a jump

•Remember the fundamental formula for the Pnl between 2 re-hedges (from module 1): $PnL = \vartheta \Delta t + \frac{1}{2}\Gamma \Delta S^2$

•In a short option position, theta is positive, gamma is negative

•Over a jump, ΔS^2 is massive and the loss may be catastrophic

•This is trivial and expected and predicted by the formula $PnL = \vartheta \Delta t + \frac{1}{2}\Gamma \Delta S^2$
However the real story is in the *unexplained* PnL

•Remember you cannot trade during the jump, by definition

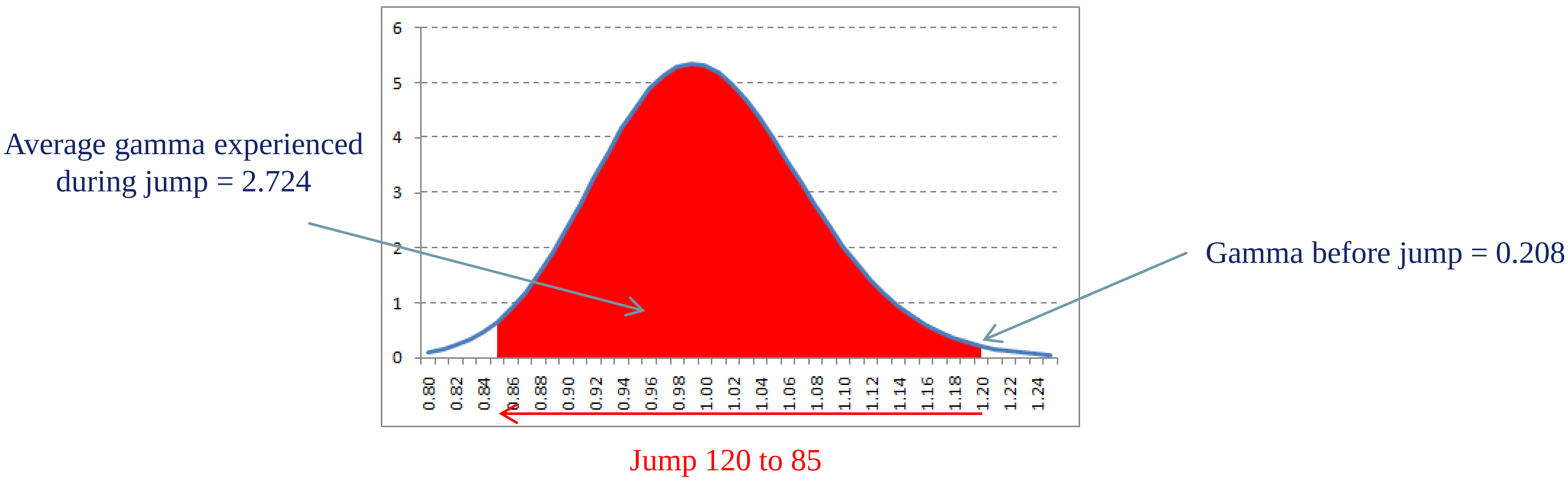
•So the 2nd order analysis is no longer valid – higher order moments account for the bulk on the *actual* PnL

•The actual PnL may differ by orders of magnitude from the PnL predicted with the formula $PnL = \vartheta \Delta t + \frac{1}{2}\Gamma \Delta S^2$

When a jump moves *towards* the strike

•Say you were short a 1.00 strike on EURCHF over the SNB, maturity 3m, implied vol = 15%

•This is what your gamma looks like as a function of the spot:

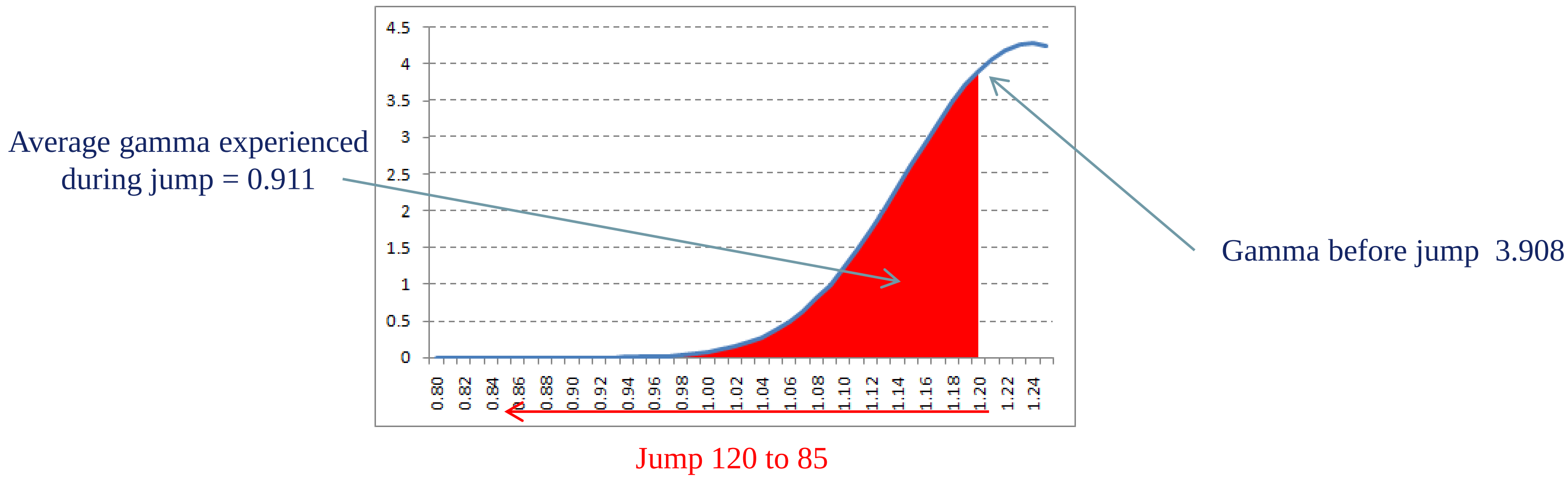


- ➢ Because we crossed the strike – region of maximum gamma
- ➢ The gamma experienced during the jump was more than 10x the initial gamma
- ➢ Hence, the actual loss exceeded the loss predicted by the 2nd order expansion **by around 10x**

When a jump moves *away from* the strike

•Say now you were short a 1.25 strike over SnB (still 3m, IV = 15%)

•This is what your gamma looks like:



- ➢ Now, the gamma experienced during the jump was less than initial
- ➢ We still loose money, jumps are never good news to short gamma positions
- ➢ But we loose much less than predicted

How jumps affect option positions

•Jumps -like high realised volatility- bring losses to short gamma positions and profits to long option positions as predicted by the 2nd order expansion

•Therefore, anticipation that jumps may occur must *increase* the overall level of option prices, hence implied vol

•In addition, unexplained losses affect short positions when jumps move *towards* the strike

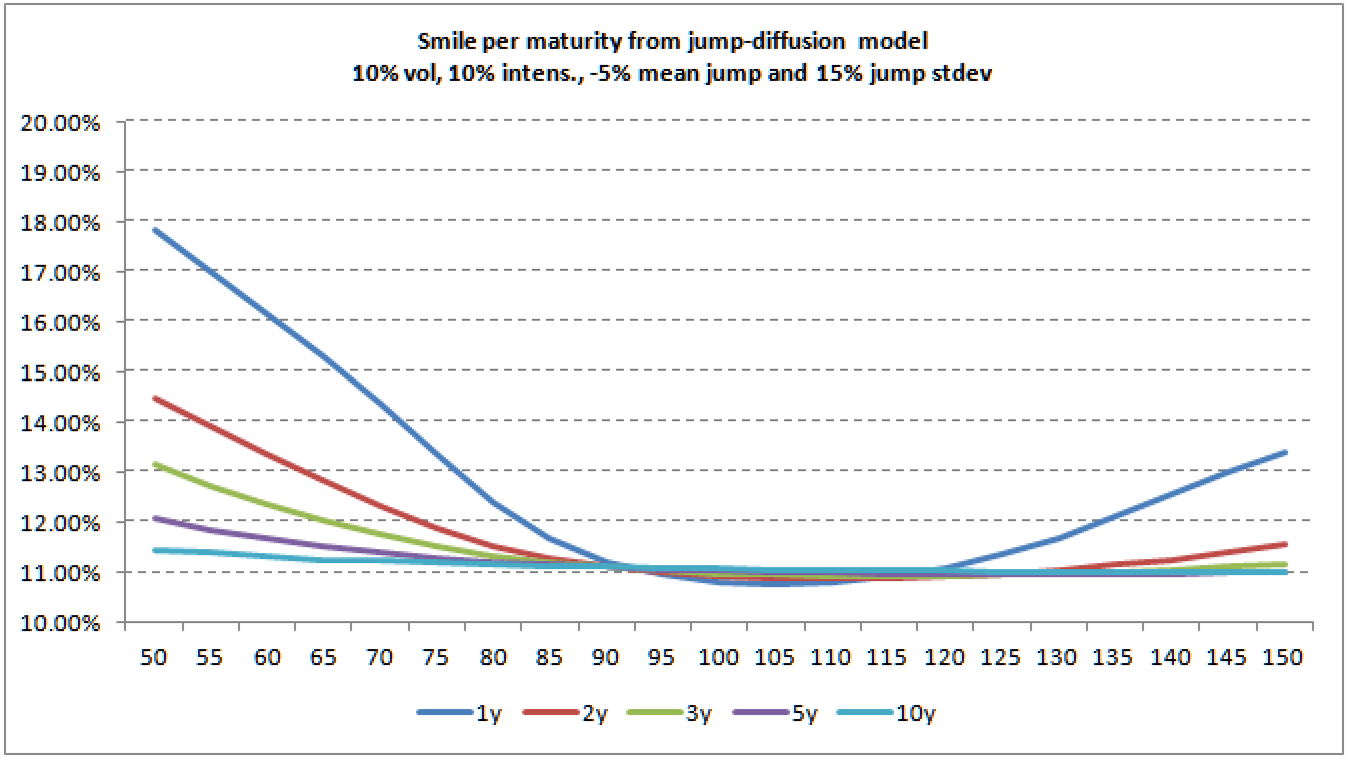
•Therefore, the “jump surprime” must raise most those strikes that are likely to experience jumps *towards* them

•OTM strikes are more likely to experience such jumps than ATM ➔ therefore jumps should create a smile

•Furthermore, if there is anticipation that jumps tend to occur downwards
Then low strikes are more likely to experience jumps towards them than high strikes
➔ therefore *asymmetric* jumps also create a skew

•Finally, jumps impact positions through gamma
So they mainly impact short term options
➔ therefore jumps produce a pronounced short term smile and skew that flatten for longer maturities

Smiles from a jump model



Modeling jumps: Poisson process

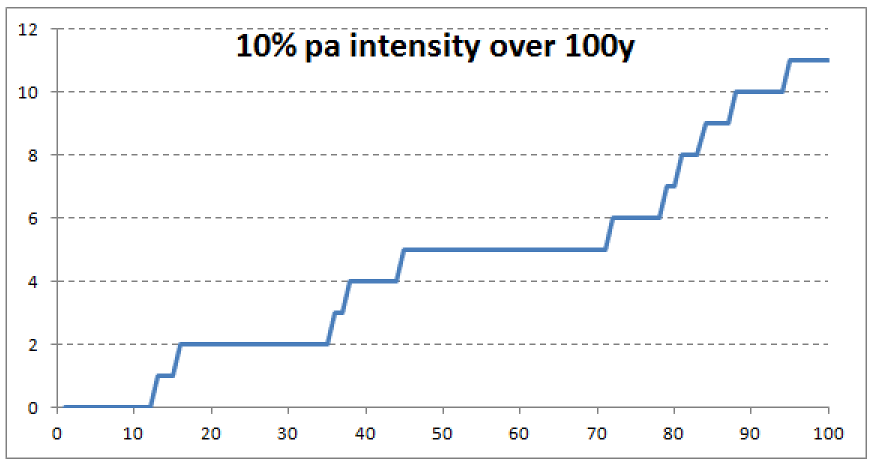
- Jumps are modeled with a Poisson process (N_t) counting the number of jumps between today and *t*

- $N_0 = 0$ and $dN_t = \begin{cases} 1 \text{ with proba } \lambda \cdot dt \\ 0 \text{ with proba } 1\text{-}\lambda \cdot dt \end{cases}$

- λ is the probability of jump per unit of time and is called jump *intensity*

- The number of jumps within a period between today and *T* follows a *Poisson* distribution:

$$P(N_T = n) = \exp(-\lambda T)\frac{(\lambda T)^n}{n!}$$



Pricing with jumps: Merton (1976) specification

•Black-Scholes risk-neutral dynamics is extended with jumps: $\frac{dS_t}{S_t} = \sigma dW + J_t dN_t - comp \cdot dt$

•Where:

- ➢ (N_t) is a Poisson process, independent from the Brownian Motion, with intensity λ
- ➢ (J_t) are the jumps, modeled as a collection of iid variables independent with the rest such that $\log(1+J_t) \approx J_t \to N(m,v)$
- ➢ Why? So $d\log(S_t) = \sigma dW - \left(comp + \frac{\sigma^2}{2}\right)dt + dN_t \log(1+J_t)$ is Gaussian conditionally to *N*
- ➢ Roughly, J_t is the size of the jump occurring at *t,* m is the average jump size, v is the variance of the jump size
- ➢ $comp$ ensures that the risk-neutral dynamics remain arbitrage free, that is (S_t) remains a *martingale*, that is $E_t[dS_t] = 0$
- ➢ In order for $E_t[dS_t] = 0$ we must set $comp = \lambda\left[\exp\left(m + \frac{v}{2}\right) - 1\right] \approx \lambda m$

Pricing with jumps: Merton (1976) solution

- Conditionally to having *n* jumps between today and expiry, $\log(S_T)$ is normally distributed

with mean $\log(S_0)+nm-\left(comp+\frac{\sigma^2}{2}\right)T$ and variance $\sigma^2 T+nv$

- Hence, the price conditional to *n* jumps is given by Black-Scholes

$$E\left[(S_T-K)^+/N_T=n\right]=BS\left\{spot=S_0\exp\left[n\left(m+\frac{v}{2}\right)-comp\cdot T\right],vol=\sqrt{\sigma^2+\frac{nv}{T}}\right\}$$

- And final price is the sum of conditional prices weighted by probability of *n* jumps:

$$E\left[(S_T-K)^+\right]=\sum_{n=0}^{\infty}E\left[(S_T-K)^+/N_T=n\right]P(N_T=n)=\sum_{n=0}^{\infty}BS\left\{spot=S_0\exp\left[n\left(m+\frac{v}{2}\right)-comp\cdot T\right],vol=\sqrt{\sigma^2+\frac{nv}{T}}\right\}\frac{\exp(-\lambda T)}{n!}(\lambda T)^n$$

- The price is a weighted average of BS prices
- The probability distribution of $\log(S_T)$ is a *mixture of Gaussian distributions* with different means and variances
- Due to the factorial, weights converge to *0* very quickly and typically only few terms need be computed

Jumps models: benefits and drawbacks

- Realistic and sensible explanation for the smile
  - Historically, traders started quoting a smile after the famous crash on “Black Monday” 19Oct 1987
  - Market makers suddenly realized that jumps happen, hurt short option positions, especially when the ove is towards strike, and incorporated a jump premium to option values accordingly
  - Short term equity smiles usually fit well jump models with reasonable parameters, contrarily to LV/SV models that require unreasonable parameters to fit
- Tractability
  - Fast, analytical pricing for Europeans
  - Efficient grid algorithm for exotics, see for instance Andreasen/Andersen (1999)
  - May be efficiently combined with local volatility and even stochastic volatility
- Stationarity
  - Stationary dynamics means model predicted future smiles have same shape as present smile
  - For example, the 1y smile has high kurtosis, the 10y smile is quite flat
The model predicts that in 9y time, the 1y smile (10y seen from today) will have same kurtosis as today's 1y smile
  - This is a desirable realistic property that is not shared by all models
In particular local vol models predict that 1y smiles will flatten with time

- However limited success with market practitioners
  - Unclear how to actually hedge the occurrence of a jump
  - Controversy about what constitutes a jump as opposed to vol peak

Assignment 6

- Implement Merton's formula in C++ Excel
  - ➢ Price options of different expiries and strikes and imply the Black-Scholes volatility
  - ➢ Play with the jump parameters, see for yourselves how the generated smile is impacted, comment and conclude

Volatility Modeling and Trading

Module 3: Exotics

Antoine Savine

Extensions to Black-Scholes

•The main result that (in absence of rates, repo, dividends, etc.) we have:

$$C = E\left[Payoff\right], \frac{dS}{S} = \sigma dW, \Delta = \frac{\delta C}{\delta S}$$

•Holds:

1. **For arbitrary payoffs ➔ exotics**
2. When vol is not constant: may depend on time, spot, be itself stochastic, etc.

•Remember:

- ➢ Option values are NOT expectations
  But they can be computed as expectations under dynamic assumptions where all *drifts are removed* (“risk-neutral”)
- ➢ But whatever you assume on vol in the real world *holds in the pricing (risk-neutral) world*

Real world assumption	$\frac{dS}{S} = \mu dt + \sigma dW$
	scrapped — identical, can be of general form
Pricing dynamics	$\frac{dS}{S} = \qquad \sigma dW$

Exotics

•Options with payoff different from simple call or put

- Digitals

•Options that may die before expiry when some conditions are met

- Barriers

•Options with strikes that fix in the future

- Cliquets or Ratchets

•Options with payoff that depends on the whole path of the underlying asset price from trade date to expiry

- Path-dependent family: Asian options (options on average), Lookback options (options on min/max), Etc.
- Volatility and variance swaps (module 4)

•Options that can be exercised before maturity

- American options
- Callable transactions

•And many more

Recent History of Exotics

• Before 1992: mostly an academic notion

- ➢ Researchers published specific formulas (Barriers) or methods (binomial tree for Americans) for pricing some exotics under standard Black-Scholes assumptions
- ➢ But exotics remained very limited as a business

• 1990s: emergence

- ➢ Creation of models designed specifically for exotics:
  Dupire (1992) on equity/forex and the Heath-Jarrow-Morton (1992) family on interest rates
- ➢ Development of valuation and risk-management platforms for exotics:
  Monte-Carlo and PDE engines, Calibration, Scripting Languages, Vega Buckets, …
- ➢ Banks grew structuring groups → structured products and exotics grew together as a new product line
- ➢ But exotics remained a marginal business compared to underlying assets and European options

Recent History of Exotics (2)

•2000-2008: the golden age

- Trading of exotics grew exponentially and accounted for a large part of Investment Banks unprecedented profits
- Profits on exotics often exceeded those on the underlying assets and European options
- Banks fought an arms race, developing more and more sophisticated models and platforms to increase their market share
- Exotics grew in complexity with the invention of products like Callable Power Reverse Duals or Callable CMS Spread Floaters
- But exotic dealers ended-up trading massive volumes in only a few product lines
- So they all piled the same, massive, complex, sometimes illiquid risks that could not be easily unwound

•Since 2008: a dying business?

- Exotics did nothing to contribute to the crisis, but they suffered tremendously from it
- Dislocated, illiquid markets exacerbated feedback effects, hurting exotic dealers hedging exotic positions
- The concentration of identical volatility and exotic risks among dealers further dislocated markets and hurt marks to market
- Exotic desks lost billions and sometimes more than profits cumulated in the 2000s
- Investors in exotics (typically long risk and short vol trough exotics) also lost massive amounts
- So, enthusiasm for exotics waned somewhat
- But exotics far from dead, still trading in decent volumes…
- … Hopefully less subject to former excess, both in volume and concentration
- Development of exotic models no longer a priority for banks, resources are drained towards regulatory calculations now

Why the hype?

- Who trades exotics on the buy side? (sell side = exotic desks in investment banks)
  - Hedge funds: leverage views, exploit market inefficiencies, and sometimes pricing discrepancies between banks
  - Institutional and private/retail investors: diversify assets and risk, seeking high returns and carry trades
  - Corporates hedging specific risks, such as insurers buying CMS caps to protect against rise in long term rates

- Why so attractive? Exotics exploit divergence between
  - The expected payoff (under "real" probability) that depends on trends, consensus, views, …
  - And price (under risk-neutral probability) that only depends on volatility/correlation

- And unfortunately also sometimes
  - Hide real transaction motives behind unnecessary complexity
  - Mislead investors into underestimating risks

- But the golden age was mainly due to persistently low interest rates and universal appetite for yield
  - Investors desperate for yield were happy to sell (apparently benign) options implicit in exotics and structured products
  - And banks were happy to oblige

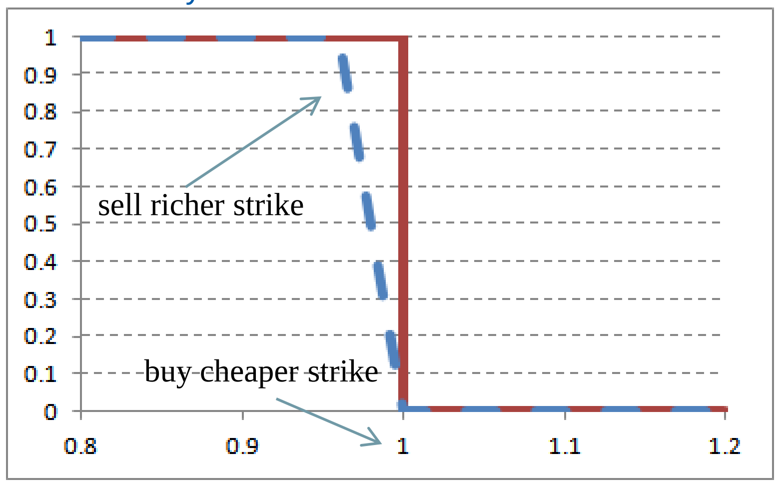
Digitals (again)

- EUR/USD as of March 2015
  - EUR: weak fundamentals, high risks, massive dilution by ECB, negative rates
  - USD: impressive recovery, rates set to rise
  - EUR/USD chart: bearish free fall with parity just around the corner
  - Strong consensus, way above 50% probability to break parity by then end of 2015
  - Goldman Sachs forecast March 16^{th} = 1.00 in September, continued fall to 0.95 by March 2016

- Yet the price of a Digital 1.00 EUR/USD put Dec 2015
  - With spot = 1.0850 and vol = 13.50% ➔ Dig = 26%
  - With 25% skew at parity (vol up 2.50% when strike down 0.1)

The digital is worth 19%!!

$$D(K) = -\frac{\delta}{\delta K} C\left[K, \hat{\sigma}(K)\right]$$
$$= -\delta_1 C\left[K, \hat{\sigma}(K)\right] - \hat{\sigma}'(K)\,\delta_2 C\left[K, \hat{\sigma}(K)\right]$$
$$= \underbrace{N(d_2)}_{26\%} - \underbrace{\hat{\sigma}'(K)\cdot vega}_{\text{skew adjustment} = 7\%}$$



- Illustrates opportunities in Exotics arising from discrepancies between:
  - Real world: consensus, views, trends ➔ 50%+
  - Risk-neutral (pricing) world: volatility, skew ➔ 20%-

Barriers

- Barriers and Reverse Knock-Out (RKO)
  - Barrier: Call or Put that dies if the underlying asset price hits some level (barrier) at any time *before* expiry
  - Most popular type = RKO when the barrier is in the money, for example a 110 call with a 120 knock-out
  - Why? Because this is where the barrier feature shaves most value

- Pricing
  - In Black-Scholes ➔ analytical formula from reflection principle
  - Problem: what volatility shall we pick in presence of a smile? ATM? At the strike? At the barrier?
  - One solution: displaced Black-Scholes, 2 parameters, can fit 2 vols, for example at the strike *and* at the barrier
  - This is not sufficient, see next slide
  - Best solution ➔ model that fits *all* IV for all strikes and expiries *then* tells what Europeans to use as a hedge

- Why are RKOs so popular?
  - RKOs are popular because they provide *massive* leverage
  - Say, a hedge fund is bullish on some asset worth 100, believes that spot will rise above 110-115 within a year…
  - But does not believe the spot may hit 120
  - What are their options? The table 2 slides down uses Black-Scholes with 20% vol and maturity 1

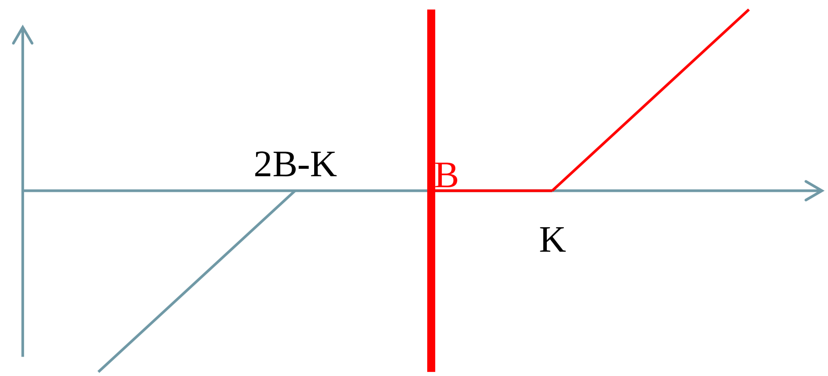
Peter Carr's static hedge of barriers

•Assumption: smile is and remains symmetric as in a call strike K>S is worth same as put strike 2S-K

•Variants with symmetric smile in log space (Black-Scholes) and more...

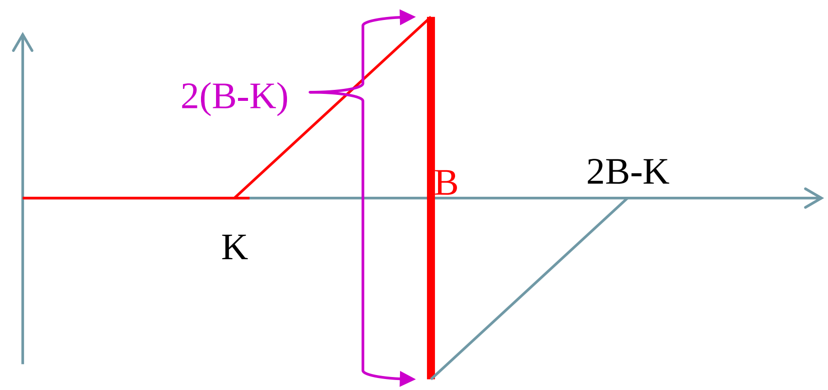
•For a barrier out of the money (down and out call)

- Hold the call K, short put 2B-K
- Barrier never hit
  → replicates payoff at maturity
- When barrier hit
  → symmetric position → unwind at zero cost
- Interestingly
  European strikes for hedge = K and 2B-K, not K and B

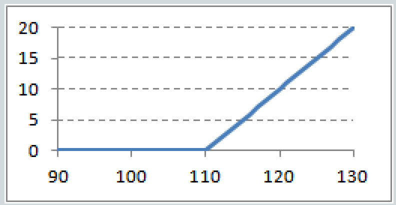
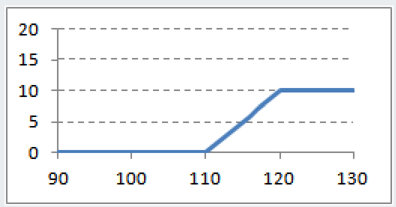
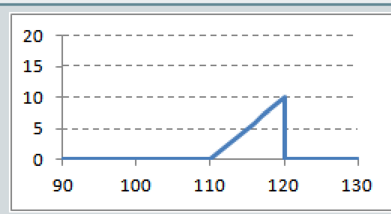
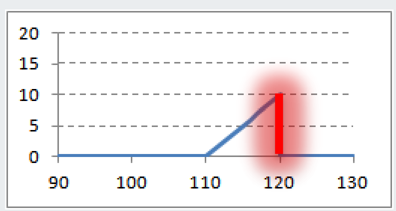


•For a barrier in the money (up and out call)

- Hold the call K, short put 2B-K
  + short 2(B-K) digital calls B
- Also depends on strikes K and 2B-K
- And IV *and skew* on strike B



Barriers and leverage

Position	Profile	Price	Leverage at 118.50
Call 110		4.29	x2
CS110-120		2.14	x4
CS110-120 -10Dig120		0.59 (Dig=16%)	x14
RKO 110-120		**0.13**	**x65**

Barrier markets

•Barrier options trade actively, sometimes as liquid as Europeans

- Equity: banks offer single stock and index barriers to their clients, including retail and private
- Fx: in addition, large, liquid OTC interbank market where barriers are exchanged with volumes approaching Europeans

•Barrier Types

- RKO, Knock-In, Knock-Out, Double Knock-Out, …
  - Note that Knock-In + Knock-Out = European
- Digital Barriers: OneTouch, NoTouch, DoubleNoTouch, …
- Exotic barriers such as partial barriers, etc.

•Quiz

- One interesting variation, trades on retail markets in Europe and Asia, is the “turbo call”
- This is a call (usually just barely) in the money, with a knock-out barrier at the strike
- Example: spot is 100, some bank issues a turbo call warrant with strike and down-and-out barrier both at 95
- How much is this worth and what is the hedge?

Some of the other popular exotics

•Interest Rates

- American (Bermudan) callable swaps, with variants involving amortized notional and/or floating capped/floored coupons
- CMS and CMS spread floaters, with capped/floored, digital and/or callable variants
- Barrier notes that pay (possibly exotic!) coupons and terminate automatically on
  - Barrier: a barrier being hit
  - TARN: the sum of all coupons exceeding a predefined target

•Equity/Fx

- Variance and Volatility Swaps, see module 4
- Autocallables: deliver high coupons in return for loss in principal (put) when the underlying underperforms over long period
- Worst-of basket: linked to the worst performing asset in a basket, sometimes with risk of loss in principal

•This is list is (highly) incomplete

- Strange payoffs such as “convex forwards” (paying ~1/spot) have been traded (in massive volumes)
- Exotic deals may be custom tailored to match a client’s risk or view precisely
- Sometimes they are structured to unwind a bank’s existing risks while creating value for the client
- High volume structured products have been traded in the *hybrid* space (involving both rates and Fx for example)
- “Passport options” and “Strategy Derivatives” were traded, Etc. No limit other than imagination…

Structured Products: Callable Power Reverse Duals

•CPRDs

- ➢ Long term (typical 30y) callable notes with leveraged (hence “power reverse”) coupons indexed on an Fx rate (hence “dual”)
- ➢ Structured in order to:
  - Pay high coupons while the Fx remains unchanged
  - An in this case be cancelled (called) rapidly
  - Hence, the transaction performs (for the investor) as long as the world remains “static” ➔ this is a carry trade
- ➢ Immensely successful with yield starved Japanese retail, massive notionals USDJPY traded 2000-2008

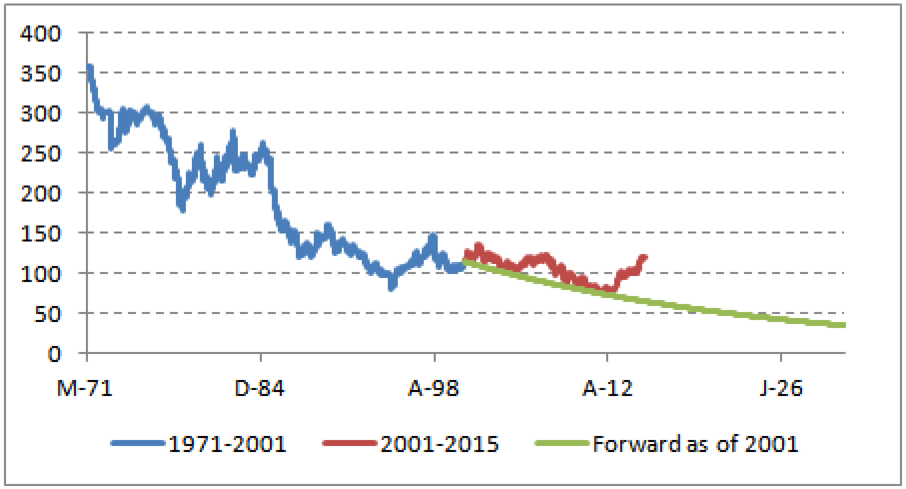
•A market mover

- ➢ 2000-2008, CPRD positions were so massive that CPRD dealers owned a major part of the USDJPY volatility market
- ➢ And CPRD hedges even became major movers to the USDJPY spot market!
- ➢ In addition, all CPRD dealers had the same positions and the same risks
- ➢ And such risk concentration was unlikely to end well should markets become shaky

CPRDs: Rationale

•Rationale: exploit interest rate differential between USD and JPY

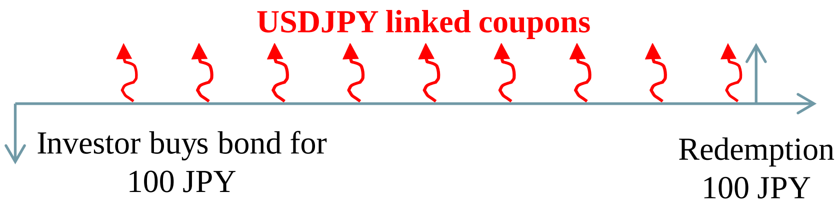
- Another application of the risk-neutral vs. real world discrepancy, this time in the rates/forward Fx space
- In 2001, USDJPY trades between 115-130 with volatility under 10%
- The 30y (2031) forward with spot=120, USD rate = 6% and JPY rate = 2% is worth 35! 2x lower than the lowest since WW2
- Fundamentals at the time favor strong USDJPY and nobody believes USDJPY may hit such levels
- And rightfully so: even 2008 crisis and the subsequent monetary QE policy in the US failed to send USDJPY below 70
- However, the forward is derived from rate differentials, not consensus or views or trends
- And this comes from inattackable, inescapable arbitrage and replication arguments: $FFx(T) = Fx(0) \cdot e^{(r_{dom} - r_{for})T}$
- …Creating a strong discrepancy between the real and the implied (pricing, risk-neutral) world…
- …Providing trade opportunities with attractive risk reward



CPRDs: Structuring

•Goal: sell long term Fx forward to the investor

•Solution: issue a domestic (JPY) bond with coupons linked to USDJPY



•Fx linked coupons

- Dual note: $C_i = \alpha \dfrac{Fx_{(T_i)}}{Fx_{(0)}}$
- The PV of the coupons is linked to the Fx forward curve and decreases rapidly
- Hence, it takes a high α to put the bond on par
- As long as the Fx remains unchanged, the coupons are equal to α
- Formally, for a par bond, $\alpha = S_{JPY} \dfrac{A_{JPY}}{A_{USD}}$ where S is the JPY par swap rate (~2%), and A is the annuity
- For a 30y bond with USD rate =~6%, $\alpha = 3.31\%$, that is 1.31% above the JPY rate – OK but nothing to get excited about

CPRDs: Structuring (2)

•Goal: increase the carry = expected coupons when Fx remains unchanged

•Solution: leverage the coupons

- Power Reverse Dual note: $C_i = \alpha \frac{Fx_{(T_i)}}{Fx_{(0)}} - \beta$
- Now we have $\alpha = \left(S_{JPY} + \beta\right)\frac{A_{JPY}}{A_{USD}}$
- In our example, $\alpha = 1.655\left(2\% + \beta\right)$ and the coupon, assuming Fx unchanged, is $c = 3.31\% + 0.655\beta$
- We can leverage the coupon as much as we want by increasing β
- In practice β was often set to 10%+, producing a carry coupon of 10%+ compared to 30y JPY rate = 2% (!!)
- Hence the success

CPRDs: Structuring (3)

- A bond cannot pay negative coupons
  - Final PRD note: $C_i = \left(\alpha \frac{Fx_{(T_i)}}{Fx_{(0)}} - \beta \right)^+$
  - Note the PRD ends up as a stream of Fx calls
  - Rising challenge number 1: in 2001, USDJPY volatility quotes actively up to 5y, how do we mark the 30y vol?

- In addition: we make the whole note callable at par after the first couple of years
  - Why?
    - Mitigate the decrease in coupons due to the zero floors
    - If Fx remains unchanged, the note is called rapidly and the investor achieves a high return in a short time
  - Hence, callable **C**PRD note = American (Bermudan=discrete American) option on a sum of options
  - Highly exotic option, requires *at least* a 3-factor (Fx, JPY rates, USD rates) model and sophisticated numerical methods
  - Rising challenge number 2
  - Early 2000s, banks raced to develop accurate, fast, stable models for CPRDs to earn share in this immensely profitable market

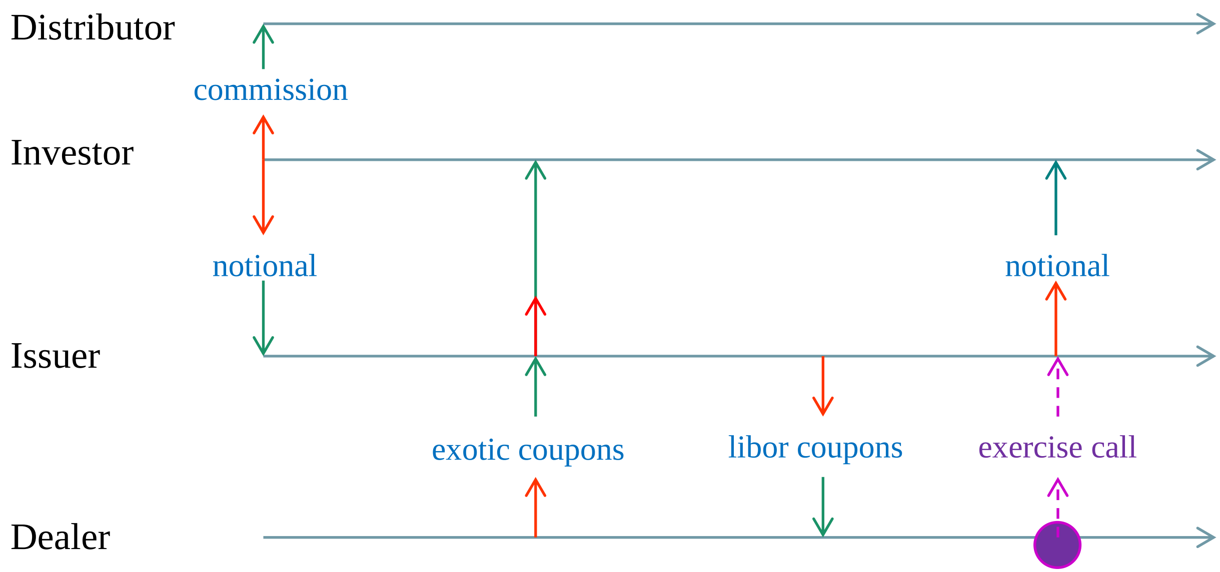
- The solutions developed are a family of models called *hybrid models* and are out of scope for this course

CPRDs: Structuring (4)

•The players

- The investor
  - JPY cash long, yield starved Japanese investor
  - Pays for other players implicit and explicit commissions…
  - …By buying above par a note with fair value below par
- The distributor
  - Sells the packaged bond to investors via local banks
  - Is compensated with commissions
- The issuer
  - A name with high credit, necessary for investment over 30y
  - Pays interest below normal funding rate – for example an issuer with funding Libor-10 may target Libor-50
  - Pays redemption to the investor at maturity or when the exotic dealer decides to call
- The exotic dealer
  - Enters in a swap with the issuer
  - Receives the issuer's interest coupons (e.g. Libor – 50)
  - Pays the issuer the exotic coupons that the issuer reverts to the investor
  - Holds the early termination option
  - Is compensated implicitly by entering an exotic transaction with positive PV
  - Uses quantitative models and platforms to risk manage the exotic and (hopefully) lock the margin

CPRDs: Structuring (5)



CPRDs: Investor Risks

- USDJPY dropped slowly before 2008 and rapidly after 2008, reaching a low of 75
  - Interestingly, followed the forward curve against expectations and forecasts
  - As a result, coupons fixed at 0 and CPRDs were not called
  - Investors ended up with a longer than expected investment, up to 15y
  - With realized interest of basically 0 as opposed to 2% risk free rate
  - But they suffered no loss in capital (unless they restructured)

- USDJPY recovered since 2012, so most CPRDs may finally be called and notionals repaid

- In fine, what was meant to be a killer trade for investors turned out bitter due to adverse market moves

- In addition, it is suspected that CPRD hedges precipitated the USDJPY drop after 2008

- The story contributed to the poor public perception for exotics

CPRDs: Dealer Risks

•CPRD dealer risk

- ➢ Overall short (illiquid) long term Fx vol due to floored coupons
- ➢ Short skew: long high strikes through the right to call (exercise when Fx is high), short low strikes (floor when Fx is low)
- ➢ Additional exotic risks such as correlation between rates and Fx

•Concentration risk

- ➢ All dealers have the same risk, and it is massive
- ➢ Hedges push markets adversely ➔ feedback effect
  - To hedge my LT vol risk I buy LT vol making it more expensive
  - I sell high strikes, buy low strikes, exacerbating the USDJPY skew I am short of
  - Attempts to hedge exotic risks with products like correlation swaps produced adverse correlation marks, everyone wants the same direction

  ➔ Due to MTM rule, I mark my CPRDs adversely, making a marking loss, having again contributed to move the market adversely!

•Smarter dealers unwound risks by structuring exotics with opposite risks with other counterparties

•But another unforeseen operational risk emerged

- ➢ CPRDs underperformed investors expectations so investor appetite quickly dried
- ➢ CPRDs were not called so the dealers had to manage their CPRD books until called (finally after 2012) or expired (up to 2031!!)
- ➢ Banks had to maintain expensive desks with skilled people for years on a business that was no longer producing any profit!

Valuation and risk management of exotics

- What do we need from our exotic model?
  - ➢ Consistency with Europeans
    - Trivial example: up and out call must converge to call as we increase barrier
    - Less trivially, Carr's static hedges show the strong relationship of barriers to Europeans
    - We want the European *hedge*: sensitivities of the exotic to a move in the European option prices
  - ➢ Dynamics that correctly represents phenomena important for the exotic: vol of vol, forward skew, kurtosis, etc.

- What's wrong with Black-Scholes?
  - ➢ Closed formulas for barriers, lookbacks, approximations for Asians, …
  - ➢ Simple, fast one factor numerical methods for others

- Yes but:
  - ➢ BS can fit one vol per expiry (two if shifted, see m2) – which one(s) do we pick?
    - See Carr's barriers → intuitions may be wrong
    - We want the model to tell us what strikes to hedge with
    - So it needs to calibrate to *all* of them
  - ➢ We already know that BS is wrong on Digitals (by large, see m2) – how wrong can it be on complex exotics?
  - ➢ BS ignores effects like vol of vol, skew, kurtosis, forward smiles, etc – some exotics are very sensitive to these effects

Exotic models

- We really need a model that:
  - Fits *all* Europeans and produces sensitivities to all IVs while correctly representing phenomena relevant to the exotic
  - BS and extensions in module 2 can't do that, they are *vanilla* models, we need an *exotic* model
  - All models seen so far are vanilla models, except perhaps Heston that sits somewhere in between…

- What is an exotic model? a parameterized set of assumptions on the future dynamics of market variables
  - Example: Dupire (1992) ➔ extension of Black-Scholes for LV dynamics $\frac{dS}{S} = ... + \sigma(S_t, t)\, dW$
  - Parameterized by its LV function $\sigma(S_t, t)$

- Popular exotic model families:
  - Equity/Fx: Dupire (1992), SLV (Dupire + SV, also by Dupire), Dupire/SLV+Jumps
  - Interest rates: model the dynamics of all rates of all maturities consistently and without arbitrage
    - HJM (1992) and Libor Market Model (1997) families
    - Extensions to multi-factor (correlation between rates of different maturities)
    - SV extensions: Heston volatility dynamics, possibly with some LV (typically displacement) and time-dependent parameters
  - Hybrids (ex. CPRDs): many models brought together and made consistent and arbitrage free

What model for what exotic?

- Exotic models are independent of the products they are pricing
- But models may be or not be relevant to some exotics
  - As a trivial example, Black-Scholes is irrelevant for Digitals because it ignores skew
  - A one-factor model is irrelevant for exchange/basket options because it believes all assets are 100% correlated
  - A time-homogeneous model is irrelevant for Ratchets because it thinks expected volatility is constant in time
- Some products are forgiving should you use a model without SV
  - Digitals are priced the same by all models as long as they agree on the smile at maturity
- And some are not
  - Tight Double No Touches are know for their extreme volga (convexity in vol)
  - Models without SV may severely underestimate their price
  - Even 2 models that fully agree on all European Prices of all strikes and maturities…
    - That means 2 models which parameters have been *calibrated* to the entire surface of implied volatilities
    - Say, 1 Dupire and 1 SLV
  - ... May produce *significantly* different prices for these exotics
- Hence,
  - Deep understanding of the exotic
  - And ability to choose and calibrate model accordingly
  - Is an *essential* skill for *all* exotic professionals

Exotic parameters

•Exotic models have *2* families of parameters

•Volatility parameters

- Control the volatility of the underlying market variables
- These parameters must be *calibrated*, i.e. solved so that model correctly fits prices of European options observed in the market

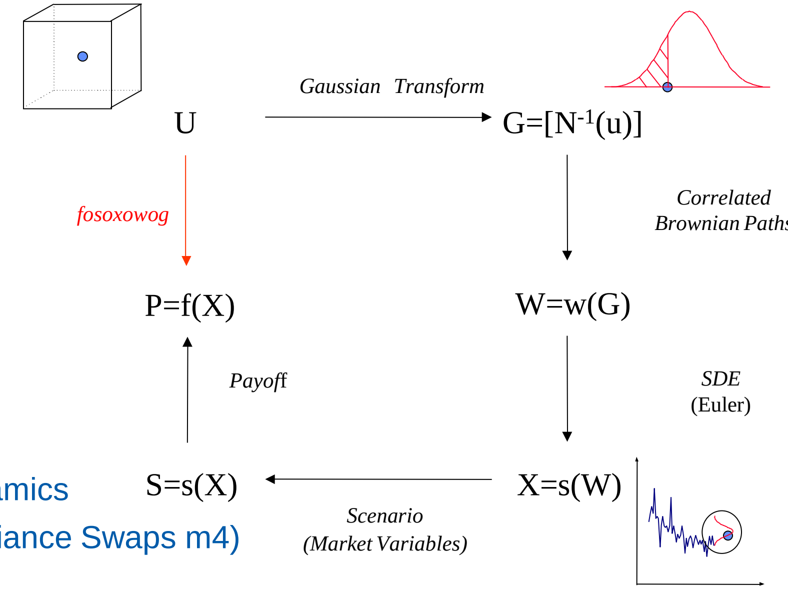
•Exotic parameters

- Exogenous paramaters such as correlations, vol of vol, jump frequency …
- These parameters must be *estimated*, for instance from historical data
- *Before* the model is calibrated, in other words volatility params are calibrated *given* exotic params

Example: Dupire (1992)

- Parameterized by a local vol function $\sigma(S_t, t)$
- No exotic parameters
- Fits the entire implied volatility surface by *analytical* fit of $\sigma(S_t, t)$ to all implied vols $\hat{\sigma}(K, T)$
- Note part of the attractiveness of the model is absence of exotic parameters
- But really particular case of a family of models (SLV) that do have exotic parameters (vol of vol) and are calibrated numerically

Numerical methods: Monte-Carlo



•Once the model is calibrated, it is used to value exotics

- Price of the exotic = risk-neutral expected payoff under *calibrated* dynamics
- No analytical formula, with extremely rare exceptions (Digitals m2, Variance Swaps m4)
- We use numerical methods to compute the expected payoff

•Monte-Carlo simulations

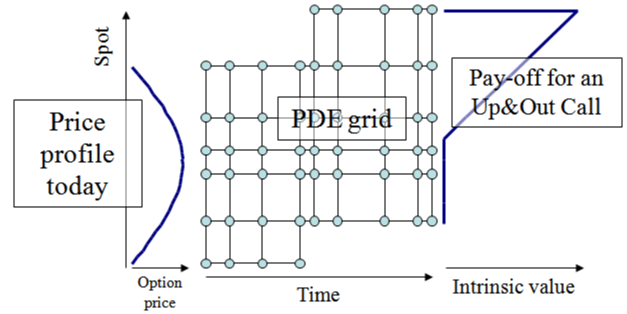
- Randomly simulate future evolutions of the world according to calibrated dynamics
- Compute payoff in each simulation
- Estimate price as the average payoff among simulations

- Exploits the Law of Large Numbers and the Central Limit Theorem
- Slow running time and convergence (in square root of number of simulations)
- But relatively “easy” implementation

- Relevant for wide variety of exotics and models
- Immediately applicable to path-dependent exotics
- Applicable to callable exotics with advanced techniques such as Longstaff-Schwartz (2001)

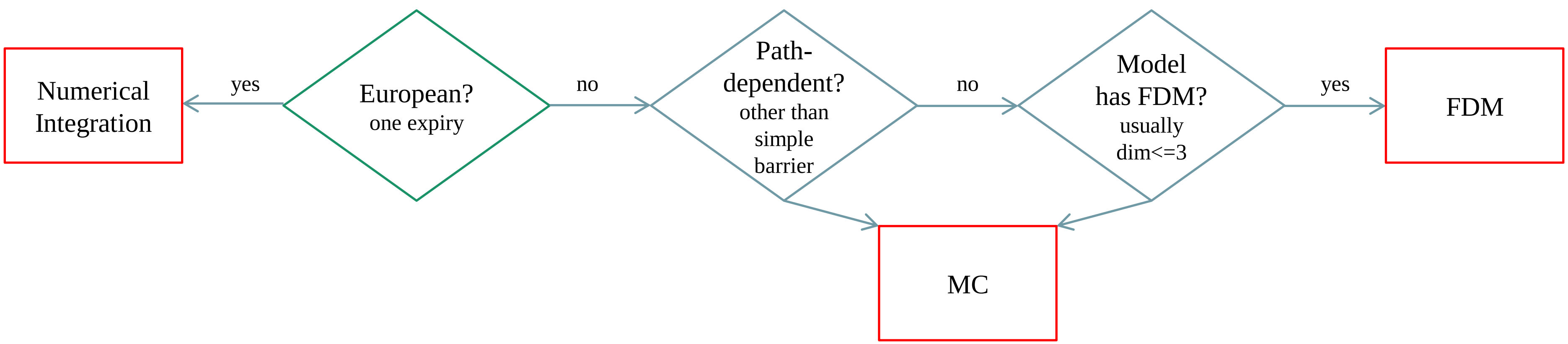
Numerical methods: PDE

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\frac{\partial^2 C}{\partial S^2} S^2 \sigma^2 (S,t)$$



- Finite Difference Methods (FDM)
  - Exploits Feynman-Kac's theorem stating that the (conditional) expectation is the solution of a Partial Differential Equation (PDE)
  - FDM consists in solving a discrete version of the PDE where derivatives are replaced by finite differences
  - Extremely fast and accurate, converges in the square of the number of nodes in space/time
  - Directly applicable to callables by applying boundary PV=max(PV,value when called) on each node
  - Directly applicable to barriers by applying boundary PV=0 on barrier nodes
  - Somewhat harder to implement when dimension (number of state variables) is more than 1
  - Unpractical with path-dependant payoffs or when dimension is more than 4
  - Hence, limited to low dimensional models and exotics

- Do FDM if you can, MC otherwise



Scripted exotics

- Numerical methods and exotic transactions
  - Most of the steps in numerical methods are *independent* of the transaction being valued
    - Generation of the Monte-Carlo simulations
    - Backward stepping in the FDM
  - The cash flows are independent of the model and come in as *boundary* conditions
    - Evaluate payoff along a Monte-Carlo simulation
    - Apply boundary condition on the FDM nodes

- Numerical algorithms are implemented generically, once and for all

  And they communicate with some representation of exotic cash flows to apply boundary conditions

- Most advanced exotic systems use scripting languages
  - Purposed made "programming" languages for the description of cash-flows
  - Cash-flows are internally "compiled" into a form understandable by the numerical scheme
  - And turned into boundary conditions to value the transaction

- Exotics professionals can describe exotic cash-flows at run time and see price and risk in real time

- Live demo

Risk management of exotics

•Recap

- Exotic parameters are estimated
- Volatility parameters are *calibrated* to European option prices
- The *calibrated* model is used to *numerically* value the exotic transaction
- In all, we have that **exotic_value = f (market_variables, implied_vols, exot_parms)** given a model type and a transaction

•Risk sensitivities

- We compute the sensitivities of **f** to inputs, generally by *bumping*
  - Bump inputs one by one by a small amount
  - Recalibrate, recalculate
  - Bit clumsy, we do better than that now (see AAD, well out of scope here)
- Sensitivities to market variables give the delta
- Sensitivities to IVs give the “vega buckets”
- Sensitivities to exotic params are exotic risks
- These risks are aggregated across (possibly thousands of) exotic transaction in exotic books

Risk management of exotics (2)

•As usual, deltas tell how many underlying assets should be traded to hedge the 1st order sensitivity of the book

•Vega buckets

- Spread exotic volatility risks across European options of different strikes and maturities
- Aggregated with the risk of European options (1 bucket for each European) and globally hedged in the *vanilla* option market

•Exotic risks

- Also aggregated across exotic transactions in a book (vanillas have no exotic risk)
- Cannot be easily hedged
- Often reserved against and limited by policy
- Smarter dealers structure and push exotics with opposite exotic risks

•Example: CPRDs

- Dealers are short long term vol, short low strikes, long high strikes
- Ideal opposite position: buy long term convex forwards = exotics paying ~ 1/spot
- Buy at very attractive price from someone who can afford the risk and monetizes their risk tolerance

Exotics: what can go wrong?

•Exotics are a very profitable business

- Large volumes
- Big margins

•However dealing exotics takes

- Skill and skilled and expensive professionals: traders, quants, structurers
- Risk tolerance: things can go wrong

•What can go wrong?

- Mishedge
  - Books may not be as well hedged as the trader believes
  - May cause books to blow, see Variance Swaps, module 4
- All kind of operational and human risks, see for instance CPRDs for an unexpected one
- **Model misspecification/miscalibration**
- **Concentration affecting mark to market**
- **Feedback effects**

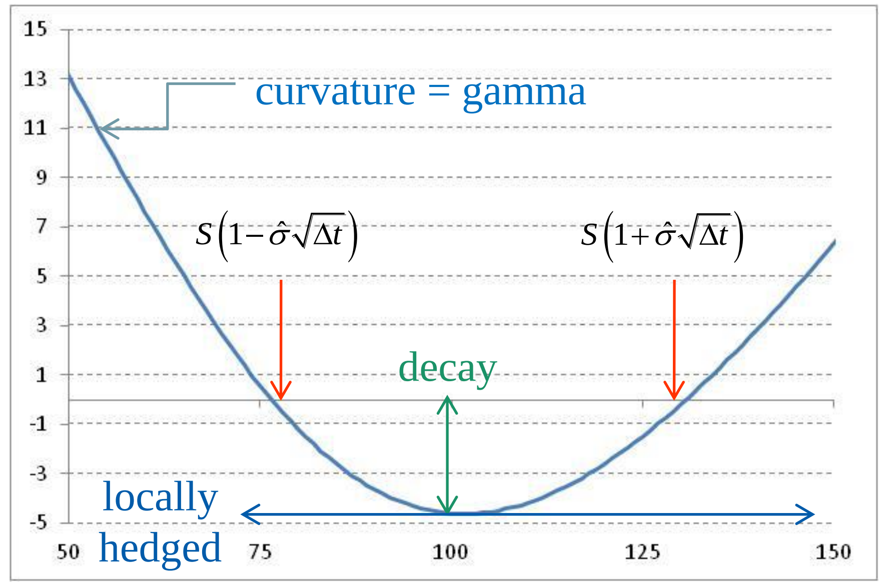
Poorly specified or calibrated models

- It's all down to the break even analysis (module 1):

$$E_t\left[dPnL\right]=\frac{1}{2}\Gamma_t S_t^{\,2}\left[\sigma_r^{\,2}-\hat{\sigma}^2\right]dt$$

delivered volatility
the actual size of the move

model implied volatility
the size of the move
as forecasted by the model



- So if the model is mis-parameterized
  - Books don't blow
  - But they bleed everyday as market moves underperform (long gamma) or overperform (short gamma) model implied moves
  - Interestingly, this effect is (almost) always against the dealer
    - When mis-specification favors the dealer's position, the price is bad and the dealer doesn't trade
    - When it is against the dealer, the price is (too) good and the dealer trades in size
    - Dealers know that and become suspectful when winning too much business
    - Rare exception when 1 dealer has a better model producing worst prices, it wins no business while others bleed
    - Solution: publish the better model!
  - When the model is recalibrated, the expected bleeding is monetized immediately as a (sometimes significant) loss in PV

Concentration risks

•When all dealers sell the same exotics in massive sizes

➔All dealers are long or short the same exotic or illiquid variables

➔The “market” values for these variables evolve adversely

➔Forcing dealers to mark their exotics adversely and post negative *marking* Pnl

•Example: CPRDs made every dealer short long term illiquid vol

- Say the fair value, of the 30y USDJPY vol, as given by some robust model, is 17.50%
- Say that it trades in a narrow, biased OTC interbank market around 20%
- If you believe in your model, you can sell the 30y vol at 20%, however:
  - Unless the market reverts, and it has every reason not to, it can take 30y to realize any gamma profits
- Whereas if you buy vol at 20%, you pay too much and you may bleed the difference over 30y, however:
  - You will bleed gamma, but not now, a 30y option has virtually no gamma
  - You free some vega limits so you can trade more CPRDs and make big profits **now**
  - You free some dearly penalized illiquid LT vega reserves, again, making profits **now**
- Evidently, everyone keeps buying so the vol keeps rising to, say, 22.50%, in complete dislocation with fair value
- And this may continue for a long, long time
- And mark to market principle forces to mark CPRDs at the “market” level of 22.50%
- If yesterday was 20%, you have (massive) negative *marking* PnL

Feedback effects

•Feedback effect = hedge affecting the underlying market in a way that hurts the hedge

•Example: say selling 90 puts is in fashion with buy side firms

- So dealers are long the 90 strike - overall, the market is balanced but the **buy side do *not* hedge**, so not so balanced
- Initially, dealers are short delta and they buy the stock as a hedge, pushing the market up
- But thereafter, for as long as the stock remains above 90, the dealers delta reduces ("delta decay") to 0 at maturity
- Hence, dealers are on the offer every day, pushing the market down towards 90
- As we hit 90, dealers are maximum long gamma, meaning higher delta when stock raises, lower delta when stock drops
- So as to keep their delta flat, dealers buy the stock when it drops, sell when it rises
- By doing so, they contribute "pinning" the stock to the 90 strike
  - Reducing realized vol
  - While keeping the market where low vol hurts most (maximum gamma)

•How strong feedback is depends on size of hedges relative to overall market

- Exacerbated with concentration of similar risks between dealers
- Related to liquid market variables, affects hedging PnL, not marking PnL
- Hurts dealers all the same

Distribution maths

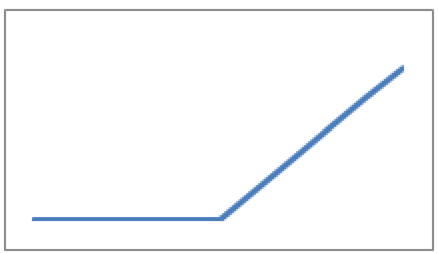
•We closely follow the lines of my 2001 paper

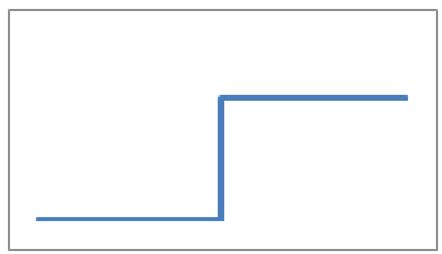
•We want to apply differential, integral and stochastic calculus to functions like

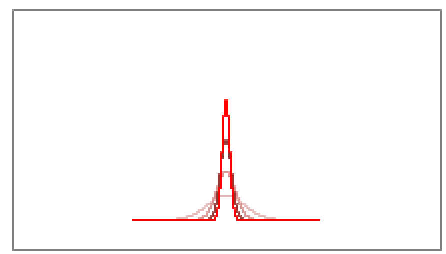
➢ "call" $c_k(x) = (x-k)^+$

➢ "digital" or "Heaviside" $d_k(x) = 1_{\{x>k\}}$

➢ "butterfly" or "Dirac mass" $\delta_k(x) = \lim_{\sigma \to 0^+} dens(N(k,\sigma) = x)$







•These functions are not differentiable in the traditional sense

➔But they are differentiable in the sense of *distributions* (Laurent Schwartz, Medal Fields in 1950)

Distribution theory

•Very roughly

- Whenever some function f is the limit of a series of smooth differentiable functions $f(x) = \lim_{n \to \infty} f_n(x)$
- Then f is differentiable, at least in the sense of distributions, and its derivative is given by $f'(x) = \lim_{n \to \infty} f_n'(x)$

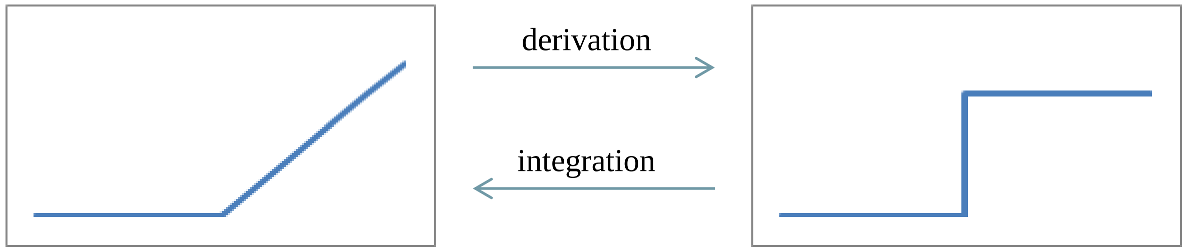
•Importantly, the rules of differential, integral and stochastic calculus apply to distributions (with some technical restrictions)

•Even Ito can be applied in this sense

•Ito’s lemma applied to distributions is known as “Tanaka’s formula”

Distributions: deriving calls

•Calls and digitals



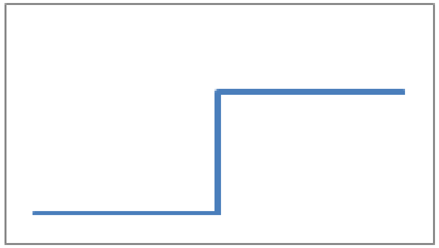
$c_k(x) = (x-k)^+$ $\qquad$ $d_k(x) = 1_{\{x>k\}}$

•That should be pretty intuitive, let's show it by applying distribution theory

- Denote $B_k^\sigma(x) = E\left(N(x,\sigma)-k\right)^+$ the Bachelier (m2) formula with spot x and strike k with *standard deviation* σ
- Then $c_k(x) = \lim_{\sigma \to 0^+} B_k^\sigma(x)$ - price converges to payoff as $std = vol \cdot spot \cdot \sqrt{T}$ decreases to 0
- Hence, in the sense of distributions, $c_k{}'(x) = \lim_{\sigma \to 0^+} B_k^\sigma{}'(x)$
- We know that $B_k^\sigma{}'(x)$ is Bachelier's delta, worth $B_k^\sigma{}'(x) = N\left(\frac{x-k}{\sigma}\right)$
- And so $c_k{}'(x) = \lim_{\sigma \to 0^+} N\left(\frac{x-k}{\sigma}\right) = 1_{\{x>k\}}$ limit of a normal distribution when variance decreases to 0

Distributions: deriving digitals

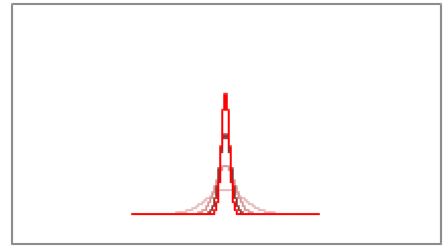
- Digitals and Diracs



$$d_k(x) = 1_{\{x>k\}}$$

derivation →

← integration



$$\delta_k(x) = \lim_{\sigma \to 0^+} dens\left(N(k,\sigma) = x\right)$$

- Applying distribution theory

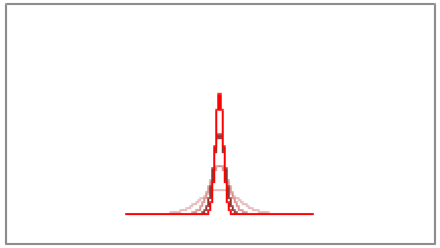
  - We know that $d_k(x) = 1_{\{x>k\}} = \lim_{\sigma \to 0^+} N\left(\frac{x-k}{\sigma}\right)$
  - So $d_k'(x) = \lim_{\sigma \to 0^+} N'\left(\frac{x-k}{\sigma}\right) = \lim_{\sigma \to 0^+} n\left(\frac{x-k}{\sigma}\right) = \delta_k(x)$

- An important property of Dirac masses if that
(since they're limit of Gaussian densities)
For any function g (satisfying some technical conditions) we have

$$\boxed{\int_x g(x)\delta_k(x)dx = g(k)}$$

Distributions: deriving Diracs

•Diracs and ?



$$\delta_k(x) = \lim_{\sigma \to 0^+} dens\left(N(k,\sigma) = x\right)$$

derivation →

← integration

•Applying integration by parts

➢ We know that $\int g(x)\delta_k'(x)\,dx = -\int g'(x)\delta_k(x)\,dx = -g'(k)$

➢ Hence, the derivative of a Dirac has (and can be defined) by the following property:

For any suitable function g, $\boxed{\int_x g(x)\delta_k'(x)\,dx = -g'(k)}$

➢ And it follows that $\int_x g(x)\delta_k^{(n)}(x)\,dx = (-1)^n g^{(n)}(k)$

Application 1: Risk-neutral distributions from smile

•Denote q the risk-neutral density for the spot price at time T, we know that todays call value C is

$$C(K)=E\left[\left(S_T-K\right)^+\right]=\int_{S_T}\left(S_T-K\right)^+ q\left(S_T\right)dS_T$$

•Differentiating twice with respect to K in the sense of distributions, we have:

$$\frac{\partial^2\left(S_T-K\right)^+}{\partial K^2}=\left(S_T-K\right)^+_{KK}=\delta_{S_T}\left(K\right)=\delta_K\left(S_T\right)$$

•Hence, $C_{KK}=\int_{S_T}\left(S_T-K\right)^+_{KK} q\left(S_T\right)dS_T=\int_{S_T}\delta_K\left(S_T\right)q\left(S_T\right)dS_T=q(K)$

•**The risk-neutral density is the second derivative of call prices with respect to strikes**

•Perhaps unsurprisingly, the RN density is the price of a tight butterfly

•The risk-neutral density and the market implied volatility smile are equivalent and carry the same info

In particular,
to calibrate a model to the market implied volatility smile
is the same as setting the parameters of its dynamics so it fits the probability distributions implied in the smile

Application 2: Stop-loss hedge strategy

- What if, to hedge an option
  - Instead of bothering with Black-Scholes, delta, gamma, etc.
  - I just use common sense and hold 1 stock above the strike, sell it below the strike
  - This is called “stop-loss strategy” - What happens then?

- Applying Tanaka (Ito for distributions):
  - We have: $d\left(S_t - K\right)^+ = 1_{\{S_t > K\}} dS_t + \frac{1}{2}\delta_K\left(S_t\right)S_t^{\,2}\sigma^2 dt$
  - Or integrated over time:

$$\overbrace{\left(S_T - K\right)^+}^{\text{payoff}} = \overbrace{\left(S_0 - K\right)^+}^{\text{intrinsic value}} + \overbrace{\int_0^T 1_{\{S_t > K\}} dS_t}^{\text{SL strategy}} + \overbrace{\frac{1}{2}\int_0^T \delta_K\left(S_t\right)S_t^{\,2}\sigma^2 dt}^{\textbf{extra unexpected term LT}}$$

- What is LT?
  - Mathematicians call it “local time” spent on the strike from now to maturity
  - Mishedge from the SL strategy
  - Linked to variance at the strike, intuition:
    - If I never cross the strike, the SL strategy works fine
    - The more I cross the strike, the more I leak by buying high and selling low, depending on variance
    - This is **not** a transaction cost effect, merely the speed of oscillations that is too fast to buy/sell spot on strike

Stop-loss hedge strategy

- Note that $E[LT] = \frac{K^2}{2}\int_0^T q_t(K)E\left[\sigma^2 / S_t = K\right]dt$ in general, and $E[LT] = \frac{K^2\sigma^2}{2}\int_0^T q_t(K)\,dt$ in Black-Scholes

- And since $E\left[(S_T - K)^+\right] = (S_0 - K)^+ + E\left[\int_0^T 1_{\{S_t > K\}}\,dS_t\right] + E[LT]$ and $E\left[\int_0^T 1_{\{S_t > K\}}\,dS_t\right] = 0$ we have the result that:

- The time value of an option *is* the expected local time $E\left[(S_T - K)^+\right] - (S_0 - K)^+ = E[LT]$

- Consequences:
  - The (risk-neutrally) expected leak from the SL strategy corresponds to the time value
  - So delta-hedge and SLS produce the same *expected* PnL
  - But LT is non-deterministic, it has variance, depending on the time spent around the strike
  - Whereas DH has (theoretically) 0 variance
  - Hence, with SLS, we have the same expected PnL but (much much) higher variance

- Assignment 7:
  - Go back to the delta-hedge simulation from module 1
  - Implement the SLS, compute its PnL, compare to DH PnL
  - Move the strike and analyze SLS PnL depending on the time spent around the strike

Application 3: The Fokker-Planck formula

•Fokker-Planck

- Assume a general diffusion of the form $dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t$
- We want to know the probability density q for X at time t
- Fokker-Planck tells us that q satisfies the PDE: $\frac{\partial q_t(k)}{\partial t} = -\frac{\partial}{\partial k}\left[q_t(k)\mu(k,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial k^2}\left[q_t(k)\sigma^2(k,t)\right]$

•Proof: left as exercise ➔ You can do it !! Just use Tanaka and distributions

•What's the point?

- Very useful for calibration
  - Can price all calls of all strikes and expiries in one single FDM sweep
  - Remember: call prices are double integrals of distributions
- Originally used by Bruno Dupire to derive his famous result
  - We now have a better way

Dupire's model

- Dupire's formula (1992, the 2nd most famous formula in finance)

  - A general local volatility model or the form (under RN measure) $\frac{dS}{S} = \sigma(S,t)\,dW$
  - Fits all observed European options prices of all strikes and expiries $C(K,T)$
  - Or, equivalently, all BS implied volatilities $\hat{\sigma}(K,T)$
  - If and only if the local vol is set to $K^2\sigma^2(K,T) = \dfrac{2\dfrac{\partial C(K,T)}{\partial T}}{\dfrac{\partial^2 C(K,T)}{\partial K^2}}$

- In short, a local vol model is calibrated to all call prices when it is parameterized as

$$\boxed{K^2\sigma^2(K,T) = \frac{2C_T(K,T)}{C_{KK}(K,T)}}$$

  - The numerator is the market price of 2 calendar spreads
  - The denominator is the market price of a butterfly

Dupire's formula: proof

- By Tanaka: $d\left(S_t-K\right)^+ = 1_{\{S_t>K\}}dS_t+\frac{1}{2}\delta_K\left(S_t\right)S_t{}^2\sigma^2\left(S_t,t\right)dt$

- Using the risk-neutral expectation operator: $dE\left[\left(S_t-K\right)^+\right]=E\left[1_{\{S_t>K\}}dS_t\right]+\frac{1}{2}E\left[\delta_K\left(S_t\right)S_t{}^2\sigma^2\left(S_t,t\right)\right]dt$

- Then we know that:

➢ $E\left[\left(S_t-K\right)^+\right]=C\left(K,t\right)$

➢ $E\left[1_{\{S_t>K\}}dS_t\right]=0$

➢ $E\left[\delta_K\left(S_t\right)S_t{}^2\sigma^2\left(S_t,t\right)\right]=\int\delta_K\left(S_t\right)S_t{}^2\sigma^2\left(S_t,t\right)q_t\left(S_t\right)dS_t=K^2\sigma^2\left(K,t\right)q_t\left(K\right)$

➢ Where *q* is the risk-neutral density, remember that $q_t\left(K\right)=C_{KK}\left(K,t\right)$

➢ And it follows that $C_t\left(K,t\right)=\frac{1}{2}K^2\sigma^2\left(K,t\right)C_{KK}\left(K,t\right)$

Dupire’s formula: simple extensions

•With rates, repo and a dividend yield, following the same steps we prove that:

$$\sigma(K,T)=\sqrt{2\frac{C_T(K,T)+(r-\mu)C(K,T)+\mu K C_K(K,T)}{C_{KK}(K,T)K^2}}$$

Where r is the risk-free rate
and μ is the risk-neutral drift,
that is stock financing rate – dividend yield for equities
and domestic-foreign rate for fx

•As a direct function of implied volatilities, we prove by differentiation of the Black-Scholes formula that:

- Since $C(K,T)=BS\left(K,T,\hat{\sigma}(K,T)\right)$
- We find C_T and C_{KK} by differentiation of Black-Scholes and, with zero rates and dividends, Dupire’s formula becomes:

$$\sigma^2=\frac{\hat{\sigma}^2+2T\hat{\sigma}\hat{\sigma}_T}{\left(1+Kd_1\hat{\sigma}_K\sqrt{T}\right)^2+\hat{\sigma}K^2T\left(\hat{\sigma}_{KK}-d_1\hat{\sigma}_K{}^2\sqrt{T}\right)}$$

Can we invert Dupire's formula?

•Dupire's formula provides an analytical solution for the calibration of local volatilities to implied volatilities

•Can we design an inverse expression for implied volatilities function of local volatilities?

•Short answer is no:
we need to price options numerically in the LV model and then imply their Black-Scholes volatility

•However in M2, we saw the (exact) Sigma-Zero formula (also derived by Dupire)

$$\hat{\sigma}^2 = \frac{E\left\{\left[\int_0^T \Gamma_{BS(\hat{\sigma})} S^2 \sigma^2(S,t)\right] dt\right\}}{E\left\{\left[\int_0^T \Gamma_{BS(\hat{\sigma})} S^2\right] dt\right\}}$$

- Intuition on how local volatilities combine to produce implied volatilities but not a direct formula, numerical methods still apply
- From Sigma-Zero, we derived the rough approximation $\hat{\sigma}(K,T) \approx \sqrt{\frac{\sigma^2(S_0,0)+\sigma^2(K,T)}{2}}$
- But it is very approximate, except for short maturities close to the money
- Its usefulness is hence limited to the ATM volatility and skew in time homogeneous LV models: $\hat{\sigma}(S_0) \approx \sigma(S_0), \hat{\sigma}'(S_0) \approx \frac{\sigma'(S_0)}{2}$

Can we invert Dupire's formula? (2)

- If we want a direct expression for IVs outside the money in an LV model, we need to do a lot better

- Remember Dupire's formula as a function of implied volatilities: $\sigma^2 = \dfrac{\hat{\sigma}^2 + 2T\hat{\sigma}\hat{\sigma}_T}{\left(1+Kd_1\hat{\sigma}_K\sqrt{T}\right)^2 + \hat{\sigma}K^2T\left(\hat{\sigma}_{KK} - d_1\hat{\sigma}_K{}^2\sqrt{T}\right)}$

- We perform a short expiry expansion, meaning we drop the terms in T:

$$\sigma^2 \approx \frac{\hat{\sigma}^2}{\left(1+K\log\left(\frac{S}{K}\right)\frac{\hat{\sigma}_K}{\hat{\sigma}}\right)^2} \rightarrow \sigma \approx \frac{\hat{\sigma}}{1+K\log\left(\frac{S}{K}\right)\frac{\hat{\sigma}_K}{\hat{\sigma}}}$$

- We note that $\hat{\sigma}(S) = \sigma(S)$ and rearrange the equation: $\dfrac{1}{\sigma} = \dfrac{1}{\hat{\sigma}} + K\log\left(\dfrac{S}{K}\right)\dfrac{\hat{\sigma}_K}{\hat{\sigma}^2} = \dfrac{1}{\hat{\sigma}} - K\log\left(\dfrac{S}{K}\right)\left(\dfrac{1}{\hat{\sigma}}\right)_K$

- Hence, the function $f(K) = \dfrac{1}{\hat{\sigma}(K)}$ satisfies the Ordinary Differential Equation $f(K) - K\log\left(\dfrac{S}{K}\right)f'(K) = \dfrac{1}{\sigma(K)}$

- With boundary $\hat{\sigma}(S) = \sigma(S)$ and the solution is: $\dfrac{1}{\hat{\sigma}(K)} = \dfrac{\int_K^S \frac{1}{u\cdot\sigma(u)}du}{\log S - \log K}$,hence $\boxed{\hat{\sigma}(K) = \dfrac{\log S - \log K}{\int_K^S \frac{1}{u\cdot\sigma(u)}du}}$

Assignment 8

- The approximation $\hat{\sigma}(K) = \dfrac{\log S - \log K}{\int_K^S \frac{1}{u \cdot \sigma(u)} du}$ is very accurate

- Verify for yourselves on Excel with Bachelier and Displaced-Lognormal models

- Use the formula for the CEV model, and compare with DLM parameter matching (see M2)

- Besides, the formula validates our M2 approx around the money: $\hat{\sigma}(S) \approx \sigma(S), \hat{\sigma}'(S) \approx \dfrac{\sigma'(S)}{2}$

- Prove it

- Note the formula $\hat{\sigma}(K) = \dfrac{\log S - \log K}{\int_K^S \frac{1}{u \cdot \sigma(u)} du}$ remains accurate away from the money,

  contrarily to our rough approximation $\hat{\sigma}(K) \approx \sqrt{\dfrac{\sigma^2(S) + \sigma^2(K)}{2}}$

- It is also robust for long maturities: where does it start breaking for Bachelier? In strike and maturity?

Implied Volatility Surfaces

•Dupire's formula $\sigma^2(K,T) = \dfrac{2C_T(K,T)}{K^2 C_{KK}(K,T)}$ requires a continuous surface of call prices function of strike and expiry

•Or, equivalently, a continuous surface of Black-Scholes implied volatilities $\hat{\sigma}(K,T)$

•This is called "Implied Volatility Surface" and it must be:

- Smooth, we are taking derivatives
- Non-arbitrageable or we get negative local variances, that is complex local vols!
  - Positive calendar spreads Ct
  - Positive butterflies Ckk

•In practice, the information we have is:

- Equity index or G10 Forex: market prices for a set of strikes and expiries
- Less liquid single stock, commodity or EM Fx: traders view on where the ATM/Skew/Kurtosis should be
- Or often a mix of both

•Building a IV surface out of this is a problem that is much harder than it looks

Geometric construction of IV Surfaces

- Use an interpolation scheme in strike for each maturity
  - ➢ Quadratic in strike IV(K) = ATM + skew (K-S0) + kurt * (K-S0)^2
  - ➢ Quadratic in log-strike ➔ VVV
  - ➢ Gatheral's SVI: 5 parameters
    - Control over different kurtosis for OTM calls and OTM puts and far wings extrapolation
    - See Jim Gatheral's public papers and presentations
- For each expiry, set parameters of the interpolation:
    - Equity indices: best fit to market prices
    - G10 Forex: fit exactly to 3 or 5 market prices
    - Less liquid underlying assets: directly set some parameters, for example fit market's ATM but directly input skew/kurtosis
- Interpolate parameters across expiries

- Benefits
  - ➢ Implied vols are maintained as a small set of meaningful market data per expiry: ATM, skew, kurtosis, …
  - ➢ Derivatives to strike are analytical, simplifies implementation of Dupire's formula
- Problems
  - ➢ Each expiry is maintained in isolation
  - ➢ No guarantee of no-arbitrage
    - Can generate negative butterflies
    - Can generate negative calendar spreads

Build IV Surfaces with a Model

- Alternative to a geometric construction: use a model
  - ➢ One model per expiry
  - ➢ Model parameters (best) fit to market data or input
  - ➢ Examples
    - Interest Rate Options (Caps/Floors and Swaptions): SABR with a set of parameters per expiry/underlying
    - Forex: some houses started using Heston instead of VVV

- Benefits
  - ➢ Implied vols are maintained as a small set of meaningful *dynamic assumptions* per expiry: vol of vol, correl spot/vol, …
  - ➢ Model parameters more intuitive and prone to cheap/rich analysis than raw market data
  - ➢ Models provide derivatives of option prices to strike
  - ➢ Generally arbitrage-free *on each maturity slice*
    - However approximations can create problems, see SABR
    - And no-arbitrage *across expiries* is not guaranteed

- Problems
  - ➢ Each expiry is still maintained in isolation, no guarantee of no-arbitrage across expiries
  - ➢ SV models fail to match short term skew/kurtosis with meaningful, stable paramaters, see module 2

Implied and Local Volatility Surfaces

- As an industry, we have conquered Stochastic Volatility and implemented complex models
- All these models assume that a full IV surface is given
- However the construction of the IV surface is still a work in progress
- The most promising research (Andreasen-Huge, Volatility Interpolation, Risk Quants of the Year 2012) *reverses* the problem
- Interpolation/parameterization of IV surface is hard due to all the constraints
- But on the LV surface there are no constraints other than positivity
- And Dupire's formula can be seen as a PDE on the today's call prices *given* a LV surface

$$C_T(K,T) - \frac{1}{2}K^2\sigma^2(K,T)C_{KK}(K,T) = 0$$

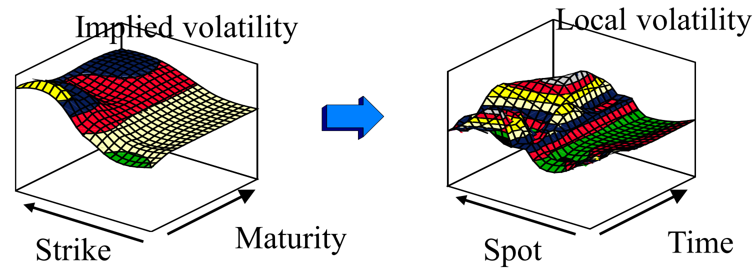
Dupire's forward PDE

- This is a *forward* PDE $C_T(K,T) - \frac{1}{2}K^2\sigma^2(K,T)C_{KK}(K,T) = 0$
- Boundaries are prices of calls expiring today: $C(K,0) = (S_0 - K)^+$
- And from there the PDE provides call prices of all strikes and expiries
- It can be implemented with FD to compute all call prices in a single sweep
- Given a parameterization/interpolation of LVs
- Then we can solve for the parameters of the LVs so as to hit the market prices of the liquid calls
- And price Exotics with a (standard backward) FDM or Monte-Carlo using the *calibrated* LV surface
- And Europeans with the forward FDM, once again with the fitted LVs
- Many flavours in literature and systems: Crank-Nicolson, one-step implicit (Andreasen-Huge), etc.

Dupire’s model in practice

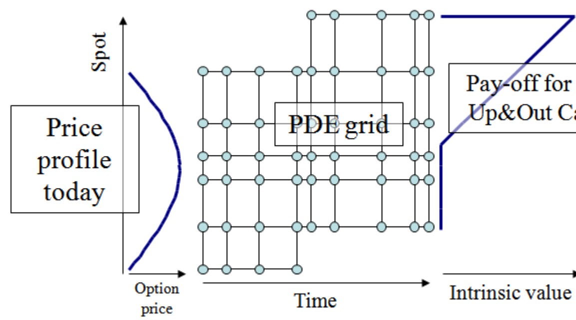
$$\sigma^2(K,T) = \frac{2C_T(K,T)}{K^2 C_{KK}(K,T)}$$

1. Run Dupire’s formula to compute LVs out of IVs



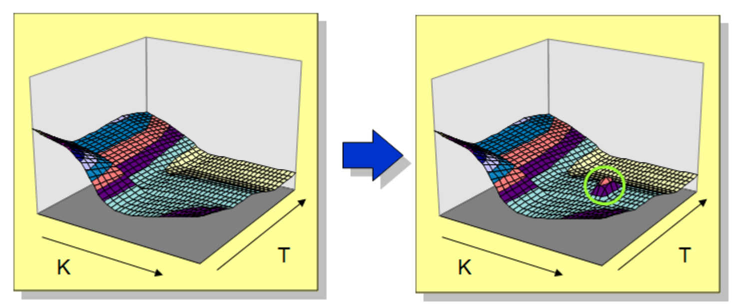
2. Run the calibrated model as a PDE or Monte-Carlo implementation to value exotics

$$\frac{\partial C}{\partial t} = -\frac{1}{2}\frac{\partial^2 C}{\partial S^2} S^2 \sigma^2(S,t)$$



3. Compute the “superbucket” risk of exotics to Europeans

- ➢ Bump IVs one by one
- ➢ Recalibrate the model
- ➢ Reprice the exotics
- ➢ Result = Sensitivities of exotics to IV = European hedge for exotics



- ➢ Charts due to Guillaume Blacher, Global Head of IR Quants at BofA

Dupire's formula for success

•The first and major model for exotics

•Fits all European options prices

•MTM: value exotics consistently with all European prices

•Risk: spread exotic vega across European strikes and expiries

•Ease of implementation

- Analytical calibration
- 1D PDE/MC for fast pricing
- The most difficult part is construction of the IV surface

•(Apparent) absence of exotic parameters

- Fitting all European option prices without exogenous parameters
- Provides (false) sense of confidence that the model fits all information necessary for pricing and hedging exotics
- Exotics given Europeans looks like a "closed" problem
- This is not quite the case

Assignment 9

- The assignment is to implement Dupire simulations
- We work around the complication of IV surfaces by building a market with Merton's jump model
- This model is fast and guaranteed arbitrage-free
- Follow questions and instructions on the document

    myPublicDropBox/Vol/assignments/volatility3/assignment9.pdf

Important consideration regarding exotic parameters

- **2 models may agree on *all* Europeans and produce (very) different prices for exotics**

- Example
  - ➢ Generate a IV surface with Heston (pure SV)
  - ➢ Calibrate it with Dupire (pure LV)
  - ➢ We get 2 very different models: one pure LV, one pure SV, that agree absolutely on all European option prices
  - ➢ However, in Heston, volatility is a lot more volatile
  - ➢ Hence, exotics with large volga (like tight Double No Touches) will have (much) higher price in Heston
  - ➢ However, prices of exotics that only depend on smiles (Digitals, Variance Swaps) are the same

- Traders point of view:
exotics are not only sensitive to European option prices, but also to complex dynamic assumptions

- Quants point of view:
exotic prices depend not only on *distributions*, but also on *processes* that generate them

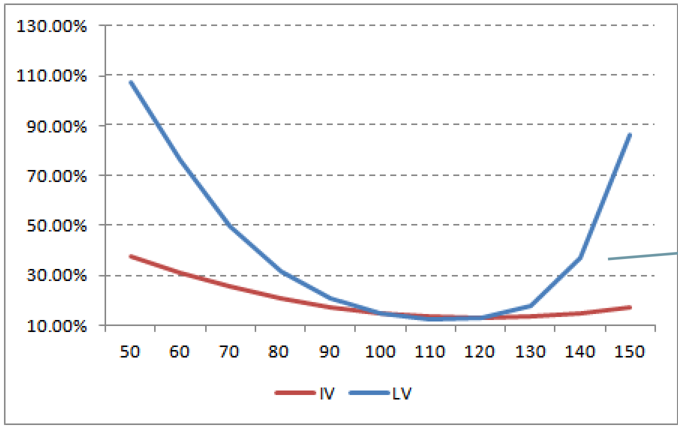
- This realization pushed practitioners to identify inaccuracies in the LV model and develop extensions

Dupire's drawbacks

•Underestimates vol of vol ➔ undervalues high volga exotics

•Short term backbone

➢ Remember from module 2, **IVs are weighted averages of LVs between spot today and strike at maturity**



1m smile in red (IV) vs fitten 1m local vols in blue (LV)
IV = weighted (quad) averages of LV between spot and strike
Therefore LV amplifies the smile
And we have the "backbone"
•ATM = f(spot) is LV
•Skew = f(spot) = is half the slope of LV

➔ ATM IV = LV at todays spot, ATM skew = twice the slope of LV at todays spot (M2)

➢ Hence, LV model predicts that
  - ATM vol sharply increases when spot moves up or down
  - Skew increases when spot drops, decreases and reverses when spot rallies

➢ LV predicted moves of ATM vol and skew exceed by far historically realized moves

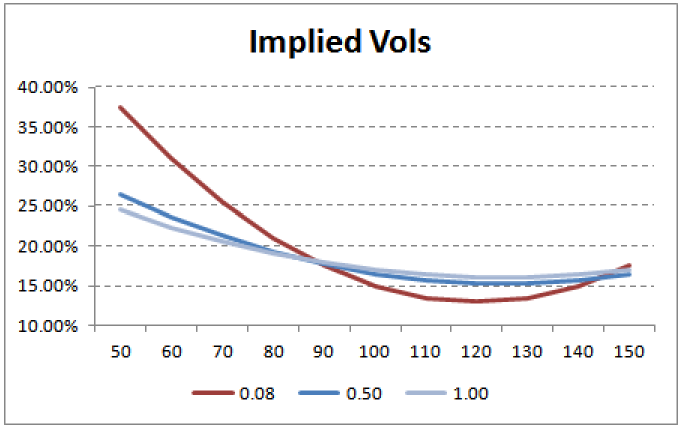
➢ And may result in mispricing some exotics

➢ Solution ➔ Add jumps, see Savine 2001 or Andesen-Andreasen 2001 for Dupire with Jumps

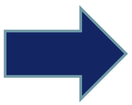
Dupire's drawbacks (2)

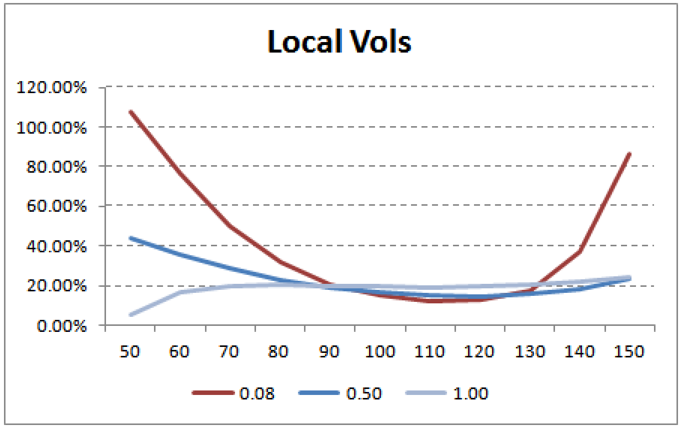
•Forward skew (and kurtosis)

- On most markets, skew (and kurtosis) flatten with maturity
- That causes LV to also flatten, faster



$$\sigma^2(K,T) = \frac{2C_T(K,T)}{K^2 C_{KK}(K,T)}$$





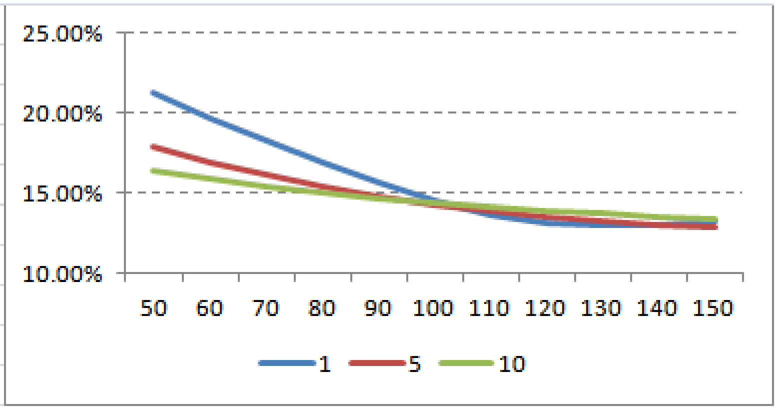
- Hence, Dupire "predicts" that the skew (and kurtosis) will flatten in the future
- In reality, term structure of skew/kurtosis tends to "slide", that is remain strong on the short end, flatter on the long end
- Causes mispricing in exotics that depend on forward skew/kurtosis, such as forward starting digitals

- Solution ➔ Add mean-reverting Stoch Vol, known to generate skew/kurtosis that flattens with expiry, but not with time

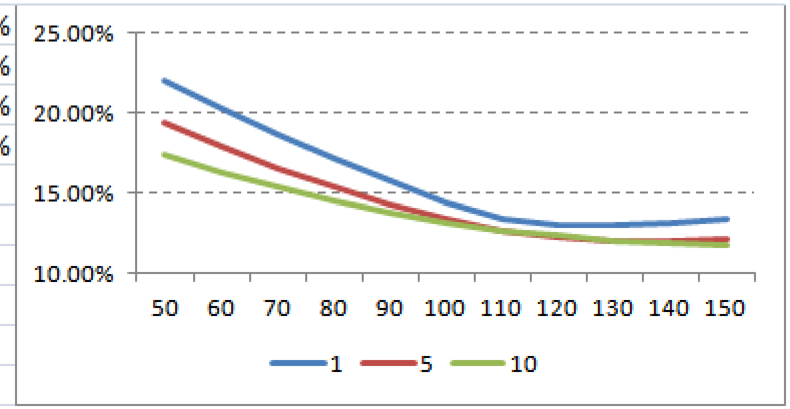
Forward skew/kurtosis with SV

•We remind (M2) that SV models also generate skew/kurtosis that decay with expiry due to mean-reversion

lambda	15%
alpha	100%
rho	-50%
k	75%

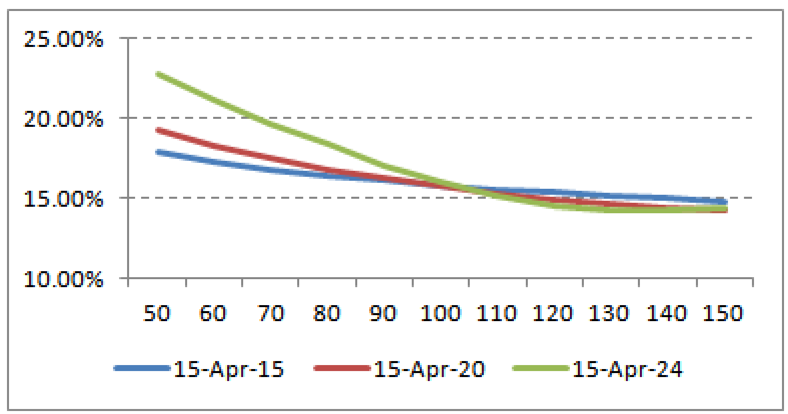


lambda	15%
alpha	100%
rho	-50%
k	25%



•But in a stationary way, meaning that the 10y skew/kurtosis is flat today, but like the 1y today in 9y time

➔ Correct forward skew/kurtosis



Mixing LV and SV

	LV – Dupire	SV - Heston
Exactly fits whole implied vol surface		
Produces vega buckets to relevant European hedge		
Correctly values convexity in vol (volga)		
Produces correct forward skew/smile (stationarity)		

Enter SLV

- The current standard for exotics

- Mix LV and SV in an attempt to combine benefits and minimize drawbacks

➢ LV $\frac{dS}{S} = \sigma(S,t)\,dW$

➢ SV $\frac{dS}{S} = \lambda\sqrt{v}dW^S, dv = -k(v - v_\infty)dt + \alpha\sqrt{v}dW^\sigma, correl(dW^S, dW^\sigma) = \rho dt$

➢ SLV $\frac{dS}{S} = \sigma(S,t)\sqrt{v}dW^S, dv = -k(v-1)dt + \alpha\sqrt{v}dW^\sigma, correl(dW^S, dW^\sigma) = \rho dt$

- Note that SLV is also due to Dupire, who nailed a general SV extension to his famous formula

- Bruno Dupire, Unified Theory of Volatility (UTV), 1996

- Our Heston specification is just one possibility, Dupire's 2nd formula works with *any* SV spec: SABR, ...

UTV (Dupire, 1996)

•The stock follows the process $\frac{dS_t}{S_t} = \sigma_t dW_t$ where volatility is completely general, local, stochastic or a mix

•By Tanaka: $d(S_t - K)^+ = 1_{\{S_t > K\}} dS_t + \frac{1}{2}\delta_K(S_t) S_t^2 \sigma_t^2 dt$

•Hence, in risk-neutral expectation : $dE\left[(S_t - K)^+\right] = \frac{1}{2} E\left[\delta_K(S_t) S_t^2 \sigma_t^2\right] dt$

•Note: $E\left[\delta_K(S_t) S_t^2 \sigma_t^2\right] = E\left[\delta_K(S_t) S_t^2 E\left(\sigma_t^2 / S_t\right)\right] = q_t(K) K^2 E\left(\sigma_t^2 / S_t = K\right) = C_{KK}(K,t) K^2 E\left(\sigma_t^2 / S_t = K\right)$

•Hence the model is calibrated to all European option prices if and only if:

$$\boxed{K^2 E\left[\sigma_T^2 / S_T = K\right] = \frac{2C_T(K,T)}{C_{KK}(K,T)}}$$

•Generalized Dupire's formula, also by Dupire in 1996

•Application to SLV as a particular case: $\sigma^2(K,T) = \frac{2C_T(K,T)}{K^2 C_{KK}(K,T) E\left[v / S_T = K\right]}$

An important expansion result

•From UTV, we can (finally) prove that (in the sense of a short expiry expansion)

•With any SLV specification

•(But works better = for longer expiries) when instantaneous vol is a martingale, like in SABR but unlike Heston

$$(1)\ \text{ATM vol}\quad \hat{\sigma}_{ATM} \equiv \hat{\sigma}(S_0, K)_{K=S_0} = \|d\log S\|_{S=S_0}$$

$$(2)\ \text{ATM skew}\quad \text{skew} \equiv \frac{\delta}{\delta K}\hat{\sigma}(S_0, K)_{K=S_0} = \frac{1}{2}\frac{\delta}{\delta dS}E_t\left[d\hat{\sigma}_{ATM}/dS\right]_{\substack{t=0\\S=S_0}}$$

•Why this result is important

1. Proves that the MV delta depends only on skew, not the model
2. Historical estimation for skew from spot and ATM vol data: in any (continuous) model
3. Arbitrage opportunities when market skew if different from estimation

•Proof in *myPublicDropBox/Vol/material/volatility3/shortMatAtmSkew.pdf*

SLV in practice

1. Provide exotic SV parameters: vol of vol, correl spot/vol, mean-reversion of vol

2. Run Dupire's UTV formula in a 2D (forward) PDE and solve for LV $C_T(K,T) = \frac{1}{2}\sigma^2(K,T)K^2 C_{KK}(K,T) E[v/S_T = K]$

- Solve for LV given SV
- LV "fills the holes" left by SV and ensures perfect fit

3. Run the calibrated model as a (2D) PDE or Monte-Carlo implementation to value exotics

4. Compute the "superbucket" risk of exotics to Europeans

- Bump IVs one by one, Recalibrate the model, Reprice the exotics
- Result = Sensitivities of exotics to IV = European hedge for exotics

5. Compute the SV risk of exotics to Europeans

- Bump exotic parameters one by one, Recalibrate the model (step 2), Reprice the exotics (step 3)
- Result = Sensitivities of exotics to SV parameters *with IV constant*

How we choose SV parameters for SLV

•We could try to estimate vol of vol, correl and mean-reversion historically

•But this is not how it is done in practice

•We first best fit a stationary pure SV model to the entire IV surface

•Then we keep correlation and mean-reversion, but scale vol of vol by a factor p, typically 0.5 or 0.6

•And add LV components solved for using Dupire’s UTV formula

•The idea behind is to allocate a fraction p of the market skew/kurtosis to SV and the rest to LV

•A more sophisticated, less widely used approach, has been suggested by Blacher in the early 2000s
➔ Fit the mix LV/SV to fit target moves of ATM, skew, kurtosis function of spot and time

Volatility Modeling and Trading

Module 4: Variance swaps

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Variance swaps

- Pays, on expiry, the realized variance from now to expiry
- Payoff = $\frac{252}{n}\sum_{i=1}^{n}\left(\frac{S_{T_i}-S_{T_{i-1}}}{S_{T_{i-1}}}\right)^2$
- (In general mean is not removed so more E2 than Var)
- Alternative (same to 1st order) = $\frac{252}{n}\sum_{i=1}^{n}\left[\log\left(S_{T_i}\right)-\log\left(S_{T_{i-1}}\right)\right]^2$
- Attractive play for hedge funds and prop desks to express direct views on volatility
- Major business for equity derivatives desks mid2000s-2008
- The payoff certainly looks path-dependent
- How to value and hedge these products?

Variance swaps: hedge

•Remember break-even analysis: when delta-hedging a (portfolio of) European options the daily PnL is:

$$\frac{1}{2}\Gamma S^2\left[\left(\Delta S/S\right)^2-\hat{\sigma}^2\Delta t\right]$$

•So the cumulated PnL *after n rehedges* = $\alpha\sum_{i=1}^{n}\Gamma_{T_{i-1}}S_{T_{i-1}}{}^2\left(\frac{S_{T_i}-S_{T_{i-1}}}{S_{T_{i-1}}}\right)^2+\beta$

•Gamma depends on the option portfolio being hedged, Ti are rehedge times

•We want to hedge $\sum_{i=1}^{n}\left(\frac{S_{T_i}-S_{T_{i-1}}}{S_{T_{i-1}}}\right)^2$ with $\alpha\sum_{i=1}^{n}\Gamma_{T_{i-1}}S_{T_{i-1}}{}^2\left(\frac{S_{T_i}-S_{T_{i-1}}}{S_{T_{i-1}}}\right)^2+\beta$

•**How can we do that?**

Variance swaps: solution

- We need $\Gamma S^2 = cste$
- And so $\Gamma = \frac{\beta}{S^2} \rightarrow \Delta = \alpha - \frac{\beta}{S} \rightarrow V = \alpha S - \beta \log S$
- In particular the payoff has to be of the form $V_T = \alpha S_T - \beta \log S_T$
- For example $V_T = \frac{S_T - S_0}{S_0} - \log \frac{S_T}{S_0}$
- We call this an *ATM log contract*
- And: $V = E^{BS}\left[\frac{S_T - S_0}{S_0} - \log \frac{S_T}{S_0} \Big/ S_t\right] = -\log \frac{S_t}{S_0} + \frac{1}{2}\hat{\sigma}^2 (T-t), \Delta = -\frac{1}{S_t}, \Gamma = \frac{1}{S_t^2}$
- So it actually works and for 1$ notional of variance we need to buy and hedge 2*252/n log-contracts

Log-contracts

•We approximate the payoff $\dfrac{S_T - S_0}{S_0} - \log\dfrac{S_T}{S_0}$ by a combination of call and put payoffs

Exercise (solve on excel)

Given a function f(S), in our case

$$f(S_T) = \frac{S_T - S_0}{S_0} - \log\frac{S_T}{S_0}$$

And a set Ki of traded strikes
(Ki<S0->Put, Ki>S0->Call)

What are the weights wi such that

$$\hat{f}(S_T) = \sum_{K_i \geq S_0} w_i (S_T - K_i)^+ + \sum_{K_i \leq S_0} w_i (K_i - S_T)^+$$

Coincides with f on strikes Ki?

•And so eventually we replicate the variance swaps by hedging a combination of Europeans

•The VS is another model-independent exotic, like digitals, with price (=log-contract) only depends on the smile

Variance swaps: conclusion

•Replication of variance swaps

- Buy 2 Log-contracts = combination of European calls and puts
- Delta-hedge in Black-Scholes
- The cumulated PnL is the realized variance

•Price of variance swaps

- Delta-hedge self financing
- Hence, Price = Price of log-contract = price of the combination of Europeans

•VS have a semi-static replication

•Price only depends on the smile at maturity

•2 different models that both match the smile at maturity give the same price

•Like digitals, but less intuitively so, VS are not “real” exotics, they are combinations of Europeans

History of variance swaps

•Bruno Dupire first established the theory of variance swaps in the early 1990s, in the intro to his paper “Arbitrage Pricing with Stochastic Volatility”

•Variance swaps became a massive and profitable business for equity derivatives desks around 10 years later in the early 2000s

•Around the same time, in 2003, the CBOE started publishing the famous VIX volatility index, a universally recognized measure of stock volatility

•The formula used by the CBOE, directly based on Dupire’s work, computes the VIX as the (square root of) the fair value of a 1m variance swaps on the S&P 500

•The enthusiasm for variance swaps somewhat waned after the crisis, when equity derivatives traders lost massive amounts

•How can we loose (billions) on a product that has a static hedge?

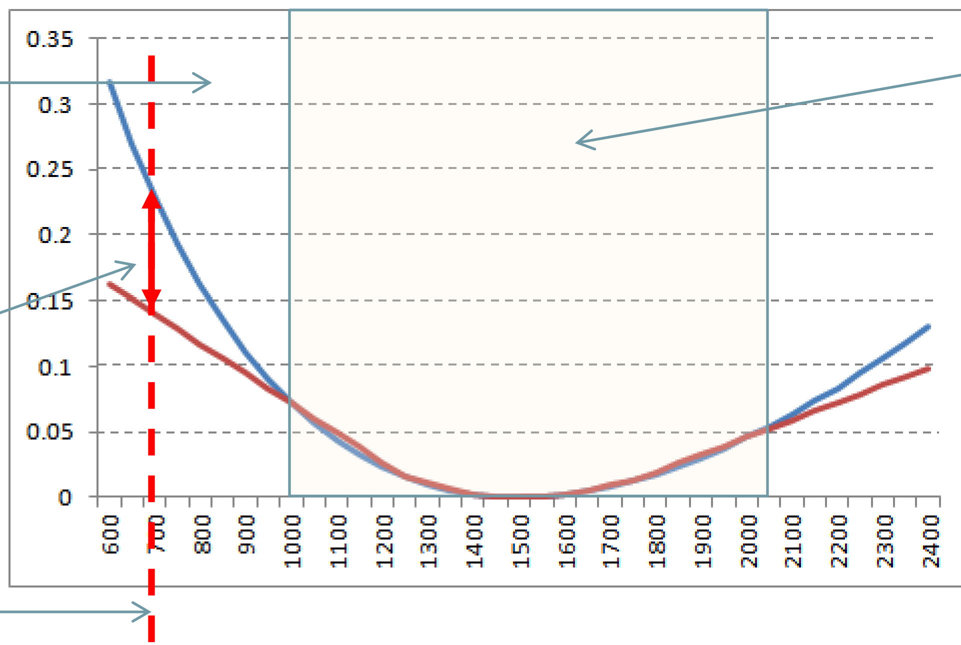
Variance Swaps and the Global Crisis

- An actual log-contract would have worked (to the 2nd order)
- But log-contracts don't trade and are replicated with sets of calls and puts
- This replication works inside the range of liquid option strikes, outside the range it is an *underhedge*
- The crisis sent the S&P way out of the range, causing massive losses

Non-tradable range
Strikes not initially traded

Liquidity range
Range with traded option strikes

- Massive misreplication
- Absence of convexity that must be replenished buying options when volatility shoots through the sky



S&P under 700
Following GFC

VIX

- Published volatility index with futures and options traded on it
- Linked to the 1m implied volatility of the S&P
- Extremely successful, seen as universal measure of equity volatility
- Formula, see white paper on https://www.cboe.com/micro/vix/vixwhite.pdf

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{RT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2$$

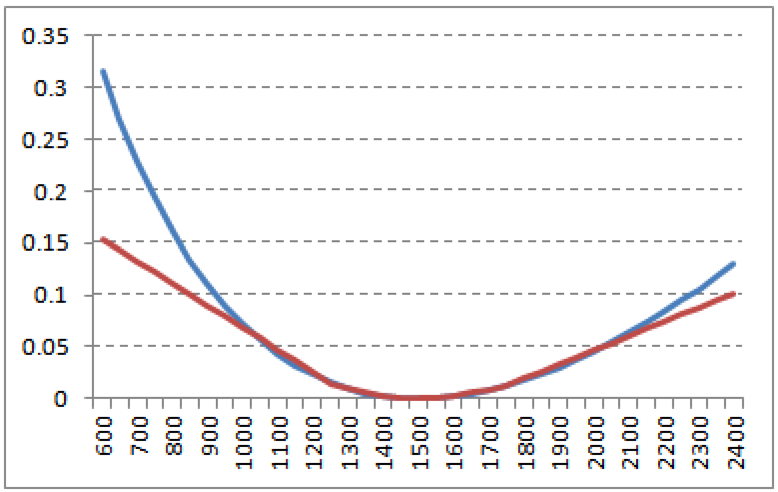
- Where Q = price of call (Ki>S0) or put (Ki<S0) of strike Ki and $\Delta K_i = \frac{K_{i+1} - K_{i-1}}{2}$
- The VIX (square) is a portfolio of options!

VIX (2)

•Note when F=S0=K0, r=0: no rates/dividends, ATM option traded, the formula simplifies into

$$\frac{VIX^2 T}{2} = \sum_{Ki \geq S_0} w_i C(K_i) + \sum_{Ki < S_0} w_i P(K_i), w_i = \frac{K_{i+1} - K_{i-1}}{K_i^2}$$

•We chart the payoff of the resulting portfolio of options, and compare to a log-contract:



What is the VIX (square)?

•Replication of a log-contract with traded options

•Price of a variance contract

•Fair value of 1m variance to be delivered

Forward variance swaps

- Delivers realized variance from T1 (>today) to T2

- Replicated by holding and delta-hedging long 2 Log-contracts maturity T2 , short 2LC expiry T1

- Total payoff: $-2\log\left(\frac{S_{T_2}}{S_{T_1}}\right)$

- Price in BS before T1: $V=-2E\left[\log\left(\frac{S_{T_2}}{S_{T_1}}\right)\right]=\sigma^2\left(T_2-T_1\right),\Delta=\Gamma=0$

- Price in BS *after* T1: $V=-2E\left[\log\left(\frac{S_{T_2}}{S_t}\right)+\log\left(\frac{S_t}{S_{T_1}}\right)\right]=\sigma^2\left(T_2-T_1\right)-2\log\left(\frac{S_t}{S_{T_1}}\right),\Delta=-\frac{2}{S_t},\Gamma=\frac{2}{S_t^{\,2}}$

- Hence, cumulated PnL from delta-hedge = realized variance

- Important note
  - At T1, the position is 2 ATM log-contracts = (T2-T1) VIX^2
  - Hence, forward T1-T2 VS = T1 forward on (T2-T1) VIX^2

Volatility swaps

•Call RV the realized variance, payoff of a variance swap

•We now know that $VS = E\left[RV\right] = \frac{2}{T} LC$

where LC is a log-contract, a portfolio of European options, which value only depends on today's smile

•A *volatility* swaps pays $\sqrt{RV}$

•By Jensen's inequality, we know that the price of a vol swap $VolSwap = E\left[\sqrt{RV}\right] < \sqrt{E\left[RV\right]} = \sqrt{E\left[RV\right]} - a$

•A vol swap is worth *less* than the square root of a var swap

•The difference is a *concavity adjustment* due to the square root, and depends on the vol of vol
➔Hence, the vol swap, contrarily to the var swap, is model dependent

Volatility swaps: a simplified model

- We know that RV is a positive variable with (risk-neutral) expectation VS

- Assume it is log-normally distributed with standard deviation α

- Then $RV = VS \exp\left(-\frac{\alpha^2}{2} + \alpha N\right)$ where N is a standard Gaussian

- And $\sqrt{RV} = \sqrt{VS}\exp\left(-\frac{\alpha^2}{4} + \frac{\alpha}{2}N\right) = \sqrt{VS}\exp\left(-\frac{\alpha^2}{8}\right)\exp\left(-\frac{\alpha^2}{8} + \frac{\alpha}{2}N\right)$

- Hence, $VolSwap = E\left[\sqrt{RV}\right] = \sqrt{VS}\exp\left(-\frac{\alpha^2}{8}\right)$ concavity adjustment, depends on var of var, hence on model

directly from the smile, model independent

- The concavity adjustment is multiplicative, directly dependent on the variance of the realized variance

VIX futures

- We know that a future T1 on VIX^2 is a T1-1m forward variance swap, and its value is

$$FVS = E\left[VIX_T^{\ 2}\right] = \frac{2}{1m}\left(LC_{T+1m} - LC_T\right)$$

- The value of this future on VIX^2 is model independent and only depends on T1 and T1+1m smiles

- The value of a VIX future is subject to a concavity adjustment:

$$VIXfut = E\left[VIX_T\right] = E\left[\sqrt{VIX_T^{\ 2}}\right] < \sqrt{E\left[VIX_T^{\ 2}\right]} = \sqrt{FVS} - a$$

- The random variable $VIX_T^{\ 2}$ is positive with (risk-neutral) expectation FVS

- Assume it is log-normally distributed with log-std $2\alpha\sqrt{T}$ that is $VIX_T^{\ 2} = FVS \exp\left(-2\alpha^2 T + 2\alpha\sqrt{T}N\right)$

- Then $VIX_T = \sqrt{FVS}\exp\left(-\alpha^2 T + \alpha\sqrt{T}N\right) = \sqrt{FVS}\exp\left(-\frac{\alpha^2 T}{2}\right)\exp\left(-\frac{\alpha^2 T}{2} + \alpha\sqrt{T}N\right)$

- And finally $VIXfut = E\left[VIX_T\right] = \sqrt{FVS}\exp\left(-\frac{\alpha^2 T}{2}\right)$

VIX futures and options

•Market quoted VIX futures are model dependent through a concavity adjustment

•Contrarily to futures on VIX^2 that don’t trade

•The market also quotes VIX options

•VIX options are listed and actively traded

•The VIX smile = implied BS volatility of VIX options per strike,
And its relation to the smile on the S&P,
Are the subjects of active research and debate

•The fact that the underlying VIX itself is model dependent gives and idea of the depth of the problem

Variance frequency swaps

- Say we want to replicate the realized *daily* variance, that is the variance of realized *daily* returns
  - Buy 2 Log-contracts
  - Delta-hedge *daily*

- If we want to replicate the realized *weekly* variance, that is the variance of realized *weekly* returns
  - Buy 2 Log-contracts
  - Delta-hedge *weekly*

- So if we want to *swap* the daily variance for the weekly variance
  - Buy 2 Log-contracts
  - Delta-hedge the long position *daily*
  - Sell 2 Log-contracts
  - Delta-hedge the short position *weekly*

- In all
  - Delta-hedge the long position *daily* and delta-hedge the short position *weekly*
  - No need for log-contracts or any options!
  - Model and *market/smile* independent! Works with whatever implied vol, even 0!
  - Zero-cost

Algorithmic volatility trading

•Exchange high frequency (say 5mins) variance for lower frequency (say 4mins) variance on a basket That exhibits higher high freq variance

•No need for options, model and market independent, zero cost

•But need to trade basket every 5 mins

•And must implement some execution algorithms to minimize transaction costs

•Undoable manually

•But a perfect fit for algorithmic trading

Assignment 10: Normal Variance Swaps

•Standard VS deliver the realized *lognormal* variance $LRV = \frac{1}{T}\sum\left(\frac{\Delta S_{T_i}}{S_{T_i}}\right)^2$

•We now look into *normal* VS that deliver the realized normal variance, or variance of first differences

$$NRV = \frac{1}{T}\sum \Delta S_{T_i}{}^2$$

1. What is the daily PnL of a delta-hedged book of options function of its Greeks in the management model and ΔT and ΔS
2. We manage the book in Bachelier (not Black-Scholes) with Bachelier implied vol $S_0\hat{\sigma}$. The risk-neutral dynamics in the management model is $dS_t = S_0\hat{\sigma}dW_t$. What is the PDE satisfied by the management Greeks?
3. Inject Bachelier's PDE into the daily PnL equation and integrate to maturity.
4. What is the condition on Bachelier's gamma of the option book so we replicate the payoff of the NVS? What is the delta? What is the value? What is the payoff? What is the portfolio? What is the full hedge strategy?

Assignment 10: Normal Variance Swaps (2)

- Assuming you can purchase the full exact portfolio of options required to replicate the NVS

- Assume 0 transaction costs

1. What can go wrong? Is the replication resilient to jumps?

2. With the same assumptions, is the standard (lognormal) VS resilient to jumps? Conclude