• In this plot, we study basic theories of Brownian motion.

• Let us start with writing down the celebrated Langevin equation that describes the Brownian motion of a particle diffusing in a fluid.

• Consider a spherical particle of radius $a$ and mass $m$ in a solvent fluid.

• Assuming the size of the Brownian particle is much larger than the size of the fluid molecules, we can treat the fluid as a continuum medium with viscosity $¥eta$.

• Let $R(t)$ be the temporal position of the particle at time $t$ and $V$ its velocity.

• To write down Newton’s equation of motion for the Brownian particle, the mass times acceleration should equal the total force acting on the particle.

• First, if a body is moving relative to a fluid, it experiences a friction force, colored in blue,

• which will be proportional to the velocity with the constant of proportionality called the friction constant $¥zeta$.

• For a spherical particle the friction constant $¥zeta$ is given by $6¥pi¥eta a$, known as Stokes law.

• And second, in addition to the friction force, we know that there must be another type of force which gives rise to the irregular motion of the Brownian particle.

• We call this the random force $F(t)$ colored in red, which represents the effects of the many collisions taking place between the Brownian particle and the fluid molecules.

• Finally, by putting all of this together, we can write down the Langevin equation shown here as Eq.(21).

• Now, let us characterize this random force in more detail.

• Assuming three-dimensional Cartesian coordinates, the random force has three components along the x, y, and z directions.

• Without loss of generality, we can assume that the average of the force along any direction is zero as shown in Eq,(22).

• If this was not the case, we could always separate the non-zero part as an extra drift force to be added separately in the Langevin equation.

• We still need to specify how the forces at different times, or along different directions are correlated.

• We note that we are interested in the dynamics of the Brownian particle at time scales much larger than the time-scales of the collisions with the fluid molecules.

• Therefore, we can assume that the successive random forces are uncorrelated on the time scale of the Brownian particle.

• This is expressed mathematically in Eq.(23) using the autocorrelation function for the random force.

• Here, $¥delta of ¥alpha¥beta$ is the Kroenecker’s delta, it is $1$ if $¥alpha = ¥beta$ and zero otherwise.

• $¥delta(t)$ is the Dirac delta function, it is zero everywhere except at the origin t=0, where it diverges.

• The crucial point is that the integral of the delta function equals to one.

• Noise which obeys equations (22) and (23) is called white noise or the Gaussian noise.

• Now, let us calculate the power spectrum of the random force $F(t)$.

• Using the Wiener-Kintchine theorem, we can write the power spectrum S\_F (¥omega) in terms of the random force autocorrelation function ¥phi\_F(t).

• Using the properties of the white noise, this autocorrelation Eq.(23) is proportional to a delta function, which kills the integral.

• Thus, we arrive at the simple result that the power spectrum is a constant proportional to D tilde, which determines the amplitude of the random force in Eq. (23).

• Let us try and see what the power spectrum and correlation functions look like.

• On the left, we plot the power spectrum of the random force, on the right the auto-correlation function.

• S\_F(¥omega) is a constant, as we have just proved, and ¥phi(t) is proportional to a delta function.

• Next, let us characterize the properties of the particle velocity $V$.

• Taking the Fourier transform of the Langevin equation Eq (1), we obtain a simple algebraic equation for V(¥omega).

• This can be easily solved, to give the following equation for V(¥omega) as a function of F(¥omega).

• Now, we are in a position to calculate the power spectrum of the velocity.

• By definition, S\_V(¥omega) is given by the square norm of the Fourier transform of the velocity.

• Writing $V(¥omega)$ in terms of $F(¥omega)$, and using Eq.(24) for the power spectrum of the random forces, we obtain Eq.(25).

• We note that the form of this equation appears so often in Physics and Mathematics that it has its own name. It is the Lorentzian function.

• Once we have computed the power spectrum, we can use the WienerKhintchine theorem to obtain the velocity autocorrelation function.

• In the last step, we have used a well known-result, that the Fourier transform of a Lorentzian is a two-sided decaying exponential.

• This means that it is an even function of time t, and therefore depends only on the absolute value of t.

• You can look this up in any table of integrals, use a computer algebra system, or do it yourself by hand using Cauchy’s integral formula.

• Let us now visualize the previously derived results for the velocity of the Brownian particle.

• On the left, we have the Lorentzian function describing the Power spectrum,

• and on the right we have the two-sided decaying exponential which gives us the auto-correlation of the velocities.

• With the results we have obtained so far, we are in a position to derive a useful relation named the fluctuation-dissipation theorem.

• Furthermore, using the equipartition theorem of classical statistical mechanics, we also know that this average should be equal to 3 times kbT divided by m.

• Solving for D tilde, we finally obtain the fluctuationdissipation theorem, which relates the amplitude of the fluctuating random forces with the magnitude of the dissipative friction forces.

• Let us now turn our attention to looking at the temporal particle positions or displacements, in order to characterize its diffusive motion.

• By definition, the displacement of the particle after some time $t$ can be expressed as the time integral of the velocity V from 0 to t.

• The mean square displacement of the particle is defined as in this equation.

• By rewriting the displacement as a time integral of the velocity,

• we are left with a double integral of the autocorrelation of the velocity, which has been calculated in Eq.(26), at two distinct times, t\_1 and t\_2.

• The integral can be performed analytically, but care must be taken to properly handle the absolute value that appears in the exponential.

• We have drawn the integration domain on the right.

• The blue outline gives the original integration limits for t1 and t2, which is represented by a square domain of side length t.

• Since the function we are integrating depends only on the absolute value of $t\_2 t\_1$,

• we can divide this domain into an upper triangular part, the red region, where t\_2 is larger than t\_1, and a lower triangular part where t\_1 is larger than t2.

• Note that the value of the integral over the upper triangular domain must be equal to the value of the integral over the lower domain,

• and it should be half of the integral over the square domain.

• Thus, we can rearrange the integration limits to go only over the red domain, where t\_2 is larger than t1.

• With this, we can get rid of the annoying absolute value from the equation, and the integrals over t1 and t2 can now be separated and easily performed.

• After doing all this, we see that the mean square displacement is increasing lineally with time t.

• At the end of the plot, we will define the self-diffusion constant of Brownian particles, as the long-time limit of the mean-square displacement divided by 6 t.

• We then obtain that the Diffusion constant is D tilde divided by ¥zeta^2 as shown in Eq.(30).

• Using the fluctuationdissipation theorem Eq.(29), we arrive at the Einstein relation shown as Eq.(31).

• Finally, from the Einstein relation, together with Stokes law, we obtain the StokesEinstein relation shown as Eq.(32),

• which gives a good estimate for the self diffusion constant of spherical particles of radius a, diffusing in a fluid of viscosity ¥eta at temperature T.