Do Carmo Notes

1 Curves

1.1 Elementary Definitions

A curve is defined with a map. In particular:

DEFINITION: A *parameterized differentiable curve* is differentiable map $\alpha : I \to \mathbb{R}^3$ of an open interval $I = (a, b)$ of the real line $\mathbb{R}$ into $\mathbb{R}^3$.

We care primarily about curves that are consistent with our idea of "smoothness", giving the following definition:

DEFINITION: A parameterized differentaible curve $\alpha : I \to \mathbb{R}^3$ is said to be *regular* if $\alpha'(t)$ for all $t \in I$.

ARC LENGTH: The arc length of a regular curve from a point t_0 can be defined by the integral: $s(t) = \displaystyle\int_{t_0}^{t} |\alpha'(t)|\, dt$. This can be thought of as the limit as the length of inscribed polygons.

1.2 Local Theory of Curves

FRENET FORMULAS:

$$t' = kn$$

$$n' = -kt - \tau b$$

$$b' = \tau b$$

Where *curvature* $k = |\alpha''(s)|$ is the length of the *normal vector* $n = \dfrac{\alpha''(s)}{|\alpha''(s)|}$ and the *binormal vector* is $b(s) = t(s) \wedge n(s)$, the length of the derivative of this vector is the *torsion* of the plane and measure how fast the curve pulls out of the plane spanned by $n(s)$ and $t(s)$ called the *oscullating plane*. Thus torsion is given by $\tau(s) = |b'(s)| = t(s) \wedge n'(s)$.

FUNDAMENTAL THEOREM OF THE LOCAL THEORY OF CURVES: given functions $\tau(s)$ and $k(s)$ for torsion and curvature there is a unique curve up to rigid motion such that s is arc length, $\tau(s)$ is its torsion and $k(s)$ its curvature.

THE LOCAL CANONICAL FORM: Locally a curve can be written as

$$x(x) = s - \frac{k^2 s^3}{6} + R_x$$

$$y(s) = \frac{k}{2}s^2 + \frac{k' s^3}{6} + R_y$$

$$z(s) = -\frac{k\tau}{6}s^3 + R_z$$

. This is found through the third degree Taylor expansion of $\alpha(s)$ at a given point.

1.3 Global Theory of Curves

CURRENTLY OMITTED

2 Regular Surfaces

2.1 Differentials

Do Carmo treats this as though one's coming from something like Spivak but later uses it as though it hasn't been seen, so I chose to include it. Also included the inverse function theorem because it's really cool.

DEFINITION: Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable map. To each $p \in U$ we associate a linear map $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ which is called the *differential* of F at p and is defined as follows. Let $w \in \mathbb{R}^n$ and let $\alpha : (-\epsilon, \epsilon) \to U$ be a differentiable curve such that $\alpha(0) = p$, $\alpha'(0) = w$. By the chain rule, the curve $\beta = F \circ \alpha : (-\epsilon, \epsilon) \to \mathbb{R}^m$ is also differntiable. Then

$$dF_p(w) = \beta'(0)$$

INVERSE FUNCTION THEOREM: Let $F : U \subset \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable mapping and suppose that at $p \in U$ the differentiabl $dF_p : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Then there exists a neighborhood V of p in U and a neighborhood W or $F(p)$ in $\mathbb{R}^n$ such that $F : V \to W$ has a differentiable inverse $F^{-1} : W \to V$,

2.2 Regular patches

Rather than define a surface as a map, we shall define them as a set and use the concept of a regular patch to make keep them manageable.

DEFINITION A subset $S \subset \mathbb{R}^3$ is a *regular surface* if, for each $p \in S$, there exists a neighborhood V in $\mathbb{R}^3$ and a map $\mathbf{x} : U \to V \cap S$ of an open set $U \subset \mathbb{R}^2$ onto $V \cap \mathbb{R}^3$ such that:

1 $\mathbf{x}$ is C^∞

2 $\mathbf{x}$ is a homeomorphism.

3 For each $q \in U$, the differential $d\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-one.

The mapping $\mathbf{x}$ is called a *regular patch*, or as Do Carmo likes to call them, a *parameterization.*

PROP 1: If a function is a map from an open set in $\mathbb{R}^2$ to $\mathbb{R}^3$ then it's graph is a regular surface. i.e. the subset of $\mathbb{R}^3$ given by $(x, y, f(x, y))$ is a regular surface.

PROP 2: If $f : U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value of f, then $f^{-1}(a)$ is a regular surface in R^3. Prop 2 is basically

just saying that the fiber of a function from $\mathbb{R}^3$ to $\mathbb{R}$ is a regular surface so long as it's not the fiber of a critical point.

PROP 3: There exists a neighborhood V of a point p in regular surface S in $\mathbb{R}^3$ such that V is the graph of a differentiable function where one variable is a function of the two others.

PROP 4: condition 2 of the definition of a regular surface is superfluous is $\mathbf{x}$ is one-one.